MATH1040 Semester 2, 2008  Solutions to Assignment 5

1. [6 marks]
(a) $125 = 5^3$ so $x = 3$.
(b) $16 = 2^4$ and $8 = 2^3$. Thus $2^4 = 2^{3x}$ so $x = \frac{4}{3}$.
(c) $\frac{1}{9} = 3^{-2}$ so $x = -2$.
(d) $16 = 2^4$ so $x = 4$.
(e) $e^7 = e^x$ so $x = 7$.
(f) $1 = e^x$ so $x = 0$.

2. [6 marks]
(a) Let $A(5730) = 50$, $Q = 100$ and $t = 5730$. Then
\[
\begin{align*}
50 &= 100e^{-5730k} \\
0.5 &= e^{-5730k} \\
-5730k &= \ln(0.5) \\
k &\approx -0.693 \\
-5730 &
\end{align*}
\]
The value of the constant $k$ for carbon-14 is approximately 0.000121.
(b) Here we know that $A(t)$ is 75% of $Q$ so
\[
\begin{align*}
\frac{A(t)}{Q} &= e^{-0.000121t} \\
0.75 &= e^{-0.000121t} \\
-0.000121t &= \ln(0.75) \\
t &\approx \frac{-0.288}{-0.000121}
\end{align*}
\]
The bone is approximately 2380 years old.

(Note: Your answers may differ slightly depending on if you used the approximate values for $\ln(0.5)$ and $\ln(0.75)$ or the full accuracy of your calculator. Marks will not be deducted for rounding (or not rounding).)

3. [5 marks]
(a) $x^2 + y^2 = 16$.
(b) In this case $x^2 + y^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2 = 4(2) + 4(2) = 8 + 8 = 16$. This satisfies the equation from part (a) so the point does lie on the circle.
(c) We have $x^2 + y^2 = 16$ for the point $(x, \sqrt{7})$. Thus
\[
x^2 + 7 = 16 \text{ so } x^2 = 9.
\]
The two possible values for $x$ are $x = 3$ or $x = -3$.

4. [6 marks]
(a) (i) $45^\circ \times \frac{\pi}{180} = \frac{\pi}{4}$
(ii) $150^\circ \times \frac{\pi}{180} = \frac{5\pi}{6}$
(iii) $-180^\circ \times \frac{\pi}{180} = -\pi$
$-\pi$ is equivalent to the angle $\pi$
(b) (i) $\frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$
(ii) $\frac{3\pi}{2} \times \frac{180}{\pi} = 270^\circ$
(iii) $-\frac{3\pi}{4} \times \frac{180}{\pi} = -135^\circ = 225^\circ$
(Please turn over)
5. [9 marks] We will use the following three triangles.

(a) From the first triangle we have \( \tan 40^\circ = \frac{x + 5.6}{15.5} \). Thus \( x + 5.6 = 15.5 \tan 40^\circ \) so \( x = 7.4060 \ldots \). Hence they scored approximately 7.4 metres to the left of the left-hand goal post.

(b) From the second triangle, we have

\[
\tan \theta = \frac{7.4}{15.5} \quad \text{so} \quad \theta = \tan^{-1} \left( \frac{7.4}{15.5} \right) = 25.52 \ldots .
\]

Thus the angle is approximately 25.5°.

(c) From the third triangle, we have that

\[
\frac{15.5}{y} = \cos 30^\circ \quad \text{so} \quad y = \frac{15.5}{\cos 30^\circ} = 17.897 \ldots .
\]

Thus the distance to the try-line is approximately 17.9 metres. As this is less than 20 metres, the goal will be scored and the Wallabies win the match.

6. [3 marks] Wherever the graph of the function \( f(x) \) has a maximum or minimum, its gradient is zero, so these values of \( x \) (\( x = -2, 0, 2, 4 \)) are where the derivative \( f'(x) = 0 \), so where the graph of \( f'(x) \) crosses the \( x \)-axis. The graph of \( f(x) \) is decreasing on the intervals \( x < -2, 0 < x < 2 \) and \( x > 4 \), so this is where the value of \( f'(x) \) is negative. The graph of \( f(x) \) is increasing on the intervals \( -2 < x < 0 \) and \( 2 < x < 4 \), so this is where the value of \( f'(x) \) is positive. A rough sketch of the derivative \( f'(x) \) is below. (As we cannot determine the actual value for \( f(x) \), except for at the points where \( f'(x) = 0 \), the maxima and minimum of \( f'(x) \) may be higher or lower than in this sketch.)

(Please turn over)
7. [9 marks]
(a) \( y' = 20x^4 - 4x \)
(b) \( y' = -12x^3 + 3 - \frac{6}{x^4} + \frac{20}{x^6} \)
(c) \( y' = 3 \left( \frac{1}{2} \right) x^{-1/2} = \frac{3}{2\sqrt{x}} \)
(d) \( y' = 6x - \cos x \)
(e) \( y' = -5\sin x \)
(f) \( y' = 2e^x - 5 \)
(g) \( y' = \frac{4}{x} \)

8. [4 marks]
(a) \( y' = 2x(e^x + 1) + x^2(e^x) \)  
(b) \( y' = -\frac{6}{x^4} \cos x - \frac{3}{x^2} \sin x \)

9. [6 marks]
(a) \( y' = \frac{(4x + 1)(x^3 - 1) - (2x^2 + x)(3x^2)}{(x^3 - 1)^2} \)
You do not need to simplify this but if you wanted to, the answer would be
\[
y' = \frac{4x^4 + x^3 - 4x - 1 - 6x^4 - 3x^3}{(x^3 - 1)^2} = \frac{-2x^4 - 2x^3 - 4x - 1}{x^6 - 2x^3 + 1}.
\]
(b) \( y' = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} \)

10. [6 marks]
(a) \( y' = 9(4x^3 - 2x)^8(12x^2 - 2) \)
(b) \( y' = -\sin(3x + x^2) \times (3 + 2x) = -(3 + 2x) \sin(3x + x^2) \)
(c) \( y' = \frac{1}{5x^4} \times 20x^3 = \frac{4}{x} \)