THE UNIVERSITY OF QUEENSLAND

AUSTRALIA

MATH1040
Basic Mathematics

Study Guide

Tenth edition, 2011

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Contents

Introduction

Welcome to The University of Queensland course, Basic Mathematics. If you are doing this course, you will probably be enrolled in MATH1040 (if you are an undergraduate student) or MATH7040 (if you are a postgraduate student). But don't worry: both courses cover exactly the same material, with similar assessment (MATH7040 includes some additional assessment activities).

The aim of this study guide is to provide an additional resource for students undertaking the Basic Mathematics course.

This study guide is designed as a companion to the course lecture notes, which are a complete set of lecture slides that are shown to students attending regular classes. The notes contain all of the mathematical content, including many examples and sample questions for each section.

It is essential that you obtain a copy of the course lecture notes, for the current year (as there are changes and amendments from year to year). You can purchase the notes from the university, or you can print them from the internet, or if you are a MATH7040 student, a copy will be sent to you. We will make frequent reference to those notes and you should have them on hand whenever you work from this study guide.

Recent changes

There were several big changes introduced at the start of 2008. Colour was introduced into the notes for the first time and a lot of new graphs were added, to better illustrate what is happening.

Keeping with the annual tradition of making improvements each year, the notes have once again been changed at the end of 2006. The first few sections have been cleaned up a lot, some new material added (on estimating and checking), and the order changed a bit (for example, square roots and surds moved from Section 1 to Section 2). The entire section on limits (at the start of Calculus) has been deleted.

Sections in the notes and study guide underwent a major reordering at the end of 2005, with several sections split. The new ordering means that some topics previously covered on the mid-semester exam will not be covered there from 2006 onwards, and some sections previously not covered on the mid-semester exam now will be.

The notes and study guide underwent major revisions of style and formatting at the end of 2004. The new material was very different to older versions. Previous exam papers and solutions were included in the study guide for the first time. *Matrices* were removed from the content at the end of 2004, and *circles* were introduced. Also, the sections on *probability* and *logs and exponentials* were rewritten and greatly expanded. However, set notation and probability have been removed for 2011. Some extra trigonometry has been added in 2011.

Structure of the study guide

This study guide provides two useful sources of information. The first 80 or so pages will help you work through the lecture notes, highlighting what is important, and giving some extra information. If you are working independently, or if you wish to read some material before the lectures, then use this section of the study guide to direct your work. Each section also directs you to some relevant practice questions. The final 60 or so pages of this guide give a number of mid-semester and final exam papers from recent years, with worked solutions. Your exams will be very similar to these.

Here, we describe the structure of this study guide, to help you make effective use of it.

Each section of this study guide corresponds to a section of the MATH7040 lecture notes. Rather than immediately hitting you with a whole collection of obscure mathematics, we want you first to understand why we are studying particular topics, and where they will be useful. Each section of the study guide commences by explaining the **relevance of the material**, in the following manner:

What?
What is it? A brief description of the topic covered in this section.

Who?
Who uses it? Some of the people who might use this material in their job or in everyday life.

How?
How is it used? An example of a practical application of the topic.

Why?
Why are we doing it? Some reasons the material is included in this course.

Next, each section has a list of **key learning goals**. These are the set of skills that you should have acquired on completion of the section. They will be presented in the following format:

At the end of this section you should:

- Here you will find a list of key learning goals, for example:
    - Understand the format of the study guide.
- You may find some extra examples or hints.

Next, you need to start working through the mathematical content in the lecture notes. The lecture notes are further divided into subsections, and the study guide matches this format. Each of the subsections will be discussed as follows:

Recommended reading

- You'll be referred to certain pages in the lecture notes.

- Occasionally, you'll be given a reference to somewhere containing extra information.

Key learning goals for the subsection. These may be:

- Understand some definitions.

- Apply some techniques.

- Remember some critical points.

Difficult concepts

Tips for understanding these concepts will also be given. For example:

- This introductory section contains a lot of material.

  - Read through this material several times to ensure you understand it.

Recommended exercises

These are the recommended exercises to test your understanding of the material. All exercises are at the end of this study guide, with worked solutions. **Note that every exercise comes off a previous exam paper, so gives an excellent idea of what you should expect this year.**

Previous exam papers

The last part of this study guide comprises mid-semester and final examination papers from the last few years, starting on Page 70. Each exam paper is followed by a full set of solutions.

Acknowledgments

Many thanks to Trevor Pickett, who did much of the work in this guide, including writing and drawing many of the *excellent* cartoons. Thanks also to Jane Boulton and Michael Jennings, who provided all exam papers, solutions and examples.
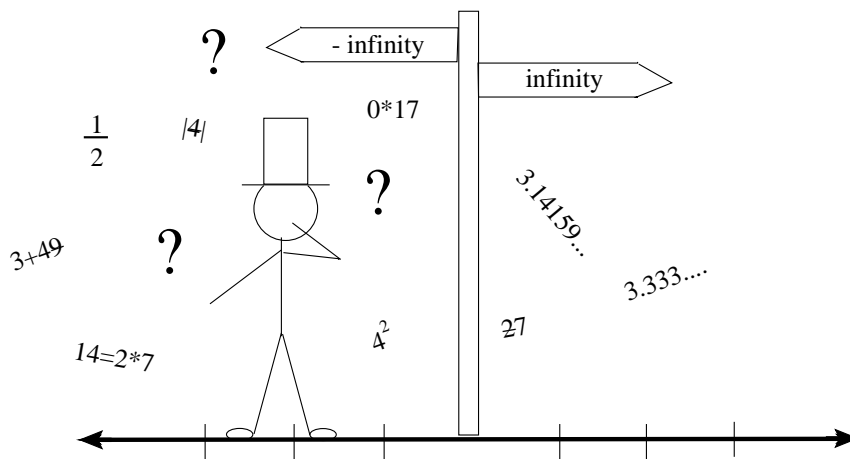
1   Numbers and arithmetic

What?
In this section we give a quick introduction to different types of numbers, and some operations that can be performed on them. Most of you will be familiar with most of this material. However, you might have forgotten some of it, and some of the concepts will probably be new. We'll cover things like number systems, absolute values, mathematical operations, precedence, prime numbers, fractions, exponentials and square roots.

Who?
Numbers are one of the most fundamental concepts in human existence, and underpin almost everything you see and do. Almost everyone will make use of these techniques, in their everyday life and in their jobs.

How?
You can use this material to do such things as work out the fuel economy of your car, create a weekly budget, complete your tax return, estimate the interest you pay on a car loan, interpret a mobile phone bill or work out your GPA.

Why?
This section introduces some of the basic skills you'll need to complete this course. For some people it will simply be a revision section, for others it will be much harder. However easy or difficult you find it, completion of this section will greatly help when you tackle the harder concepts later on in the notes. In particular, many people find numbers easy, but have a lot of trouble when it comes to using letters in algebraic manipulations. If you think about this material and become very comfortable with it then everything else you do will be much easier.

At the end of this section you should be able to:

- Appreciate the importance of thinking about your answer and checking.

- Identify the different types of numbers.

- Understand the concept of number lines and order.

- Be able to find the absolute value of a number.

- Perform simple mathematical operations in the correct order.

- Identify prime numbers.

- Find factors of integers.

- Work confidently with fractions.

- Correctly evaluate a number raised to an exponent.

- Understand basic square roots.

Work through this section as follows:

1.1 Thinking about Maths.

Read pages $8 - 10$ of the lecture notes, and work through any examples.

Thinking and checking

- The concepts and approaches presented here are very important. Don't just look at the content and then forget it.

- Whenever you solve a problem using mathematical concepts (in this course, or another course, or in your job when you enter the workforce) you should always look at the answer and see if it looks sensible.

- You should also check your working.

- Be a bit careful; if an answer looks wrong, it might still be correct. As you practice more, you will develop better judgement of what's correct and what's incorrect.

- On an exam, if your answer is wrong but you recognise that it's wrong, and explain how you know that it's wrong, then you'll receive more marks than if you just have a wrong answer.

- Being able to estimate an approximate answer is a very useful skill!

1.2   Types of numbers.

Read pages $11-12$ of the lecture notes, and work through any examples.

Number types

- Be comfortable with the definition of each type of number.

- Remember the notation:

    - $\mathbb{N}$ means Natural numbers.
    - $\mathbb{Z}$ means Integers.
    - $\mathbb{Q}$ means Rational numbers.
    - $\mathbb{R}$ means Real numbers.

- Note that:

    - every Natural number is also an Integer
    - every Integer is also a Rational number
    - every Rational number is also a Real number

- In each case, the converse is **not** true.

1.3   Number lines and order.

Read pages $12-13$ of the lecture notes, and work through any examples.

1.4   Absolute value.

Read pages $13-14$ of the lecture notes, and work through any examples.

Absolute values

- If the number is positive, then absolute value leaves the number unchanged.

- If the number is negative, then absolute value removes the negative sign.

1.5 Simple mathematical operations.

Read pages $14 - 15$ of the lecture notes, and work through any examples.

Mathematical operators

- Brackets can be written as ( ) or [ ].

- We can write $a \times b$ as $a \cdot b$ or ab.

- We can write $a \div b$ as a/b or $\dfrac{a}{b}$.

- Note what happens when you multiply or divide two negative numbers: the two negative signs cancel.

Dividing by zero.

Be careful: you **cannot** divide by zero. If ever you are asked to, your answer should be "undefined" or "not possible".

1.6 Order of operations.

Read pages $15 - 17$ of the lecture notes, and work through any examples.

BEDMAS

- Know how to apply BEDMAS (**B**rackets, **E**xponentiation, **D**ivision, **M**ultiplication, **A**ddition and **S**ubtraction).

- Remember that multiplications and divisions are performed from left to right, in the order in which they appear in the expression. Similarly, additions and subtractions are also done in the order in which they appear, from left to right.

Understanding order of operations

It is critically important that you can understand and apply operations in the correct order. Much of the work in this course, and many other courses, requires this skill. Practise until you can do it easily.

1.7 Prime numbers and factors.

Read pages 18 − 19 of the lecture notes, and work through any examples.

Factors

- The tricks for finding small factors 'in your head' are often useful. However, you don't need to remember them, and may well prefer to use a calculator.

- Understand the meaning of the terms: *prime number*, *common factor*.

1.8 Fractions.

Read pages 20 − 23 of the lecture notes, and work through any examples.

Fractions

- Understand the meaning of: *numerator*, *denominator*, *inverse*, *equivalent fractions*, *simplest form*, *cancellation*, *common denominator*.

- Be able to add, subtract, multiply and divide fractions.

Working with fractions

Many people find fractions difficult to work with. Even if you think they are easy, fractions get much harder when the denominator and/or numerator includes letters and mathematical expressions. We use fractions quite a lot in this course, so take the time to practise on some easy examples.

1.9 Introduction to exponentiation.

Read pages 25 − 26 of the lecture notes, and work through any examples.

Basic exponentiation

- Be able to evaluate basic exponential terms.

- Understand what happens to negative numbers when they are raised to odd powers (the answer is negative) versus what happens when they are raised to even powers (the answer is positive).

1.10   Square roots

Read pages $27 - 28$ of the lecture notes, and work through any examples.

Square roots

- This is just a brief introduction to the idea of square roots, and we'll cover them in much more detail later on.

- Understand why a negative number does not have a square root. (Actually, if you study more maths, you see a new type of number system, called *complex numbers*, that do allow us to find square roots of negative numbers. But as far as we are concerned, negative numbers do not have square roots.)

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Section A, Q1,2,3 Page 73.
2. Section A, multichoice, Q2,4 Page 73.
3. Section A, Q2-5 Page 78.
4. Section A, multichoice, Q4 Page 78.
5. Section B, Q1 Page 78.
6. Section A, Q2 Page 82.
7. Section A, multichoice, Q2 Page 82.
8. Section B, Q1 Page 82.
9   Q1a,b, 4a Page 121.
10. Q1b Page 127.
11. Section A, Q5 Page 87.
12. Section A, Q5 Page 92.
13. Section A, Q3 Page 97.
14. Section B, Q1, Page 93.
15. Section B, Q1, Page 98.
16. Section A, multichoice, Q5, Page 103.
17. Section B, Q1, Page 104.
18. Section B, Q1, Page 110.
19. Section A, multichoice, Q4, Page 115.
20. Section B, Q1, Page 116.
21. Q1, (c), Page 139.
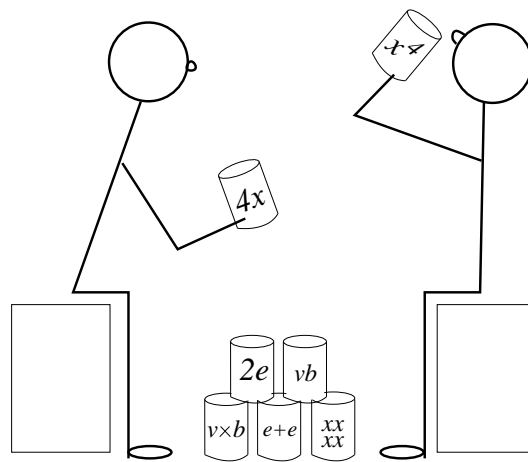
2 Algebra

What?

In this section we study expressions that involve letters. We will use the techniques similar to those we encountered in the previous section (numbers) to manipulate and simplify these expressions. Many people find this manipulation (commonly called *algebra*) to be confusing. Topics we'll cover include algebraic expressions, expanding and factorising, formulae, equations, solving problems involving absolute values, intervals, inequalities, square roots, and power rules.

Who?

Introducing letters into expressions is a logical and important step beyond only using numbers. Every computer program or spreadsheet makes use of algebraic expressions. Almost every problem that requires any significant mathematical sophistication will need to be specified using a combination of letters, numbers and operations.

How?

You can use this material to do such things as developing a spreadsheet for managing a budget, calculating total tax payable on various income levels, writing software to optimise production in a company, calculating areas and lengths of various physical quantities and predicting the outcome of movements in the world oil price.

Why?

For the rest of semester, and in your subsequent university study, you will continually encounter questions and problems which need to be formulated and solved using these techniques. It will be assumed that you are completely comfortable with this approach, so it's important that you take the time to get a firm grasp of the material.

At the end of this section you should be able to:

- Identify algebraic expressions.

- Expand, factorise and simplify an algebraic expression.

- Transpose and solve equations and formulae.

- Solve problems involving absolute value signs.

- Solve problems involving inequalities and write your answers:

  - On a number line.
  - In interval notation.
  - In inequality form.

- Manipulate square roots.

- Apply the power rules.

Work through this section as follows:

2.1 Introduction to algebra.

Read pages $32 - 36$ of the lecture notes, and work through any examples.

Algebraic expressions

- Understand the meaning of: *variable, algebraic expression, coefficient, like term.*

- Be able to group like terms.

Many people are ok with numbers, but freeze with letters. There is no reason to be afraid. Letters are just like numbers, but we don't know their actual value.

Grouping like terms

When grouping like terms you must move the term **and the sign associated with it**. For example:

$$2x^4 + 4x^2 + 8 - 9x^4 - 3x^2 - 6 = 2x^4 - 9x^4 + 4x^2 - 3x^2 + 8 - 6$$

Arithmetic on algebraic expressions

- Be able to simplify algebraic expressions by applying BEDMAS and adding or subtracting like terms.

2.2   Expanding and factorising.

Read pages $37 - 43$ of the lecture notes, and work through any examples.

Expanding

- Understand the different ways to expand an expression involving brackets. (In the notes there are three sets of rules. Don't try to commit them to memeory: in fact, they are effectively the same rule, applied to different cases. The two diagrams in the notes should make it clear.)

- Be able to simplify an algebraic expression after expanding.

Factorising

- Understand what it means to factorise an algebraic expression.

- Be able to simplify an algebraic expression by factorisation.

2.3   Equations and Formulae.

Read pages $43 - 49$ of the lecture notes, and work through any examples.

Equations

- Be able to evaluate an equation by substituting in values.

- Be able to transpose equations.

- Understand the relationship between transposition and solving equations.

Transposing and solving equations

Transposing and rearranging equations can be quite difficult. The two rules for transposing are very important and you should remember them. They can be summarised as:

Whatever you do to one side you **must** *also do to the other side.*

2.4   Solving absolute values.

It may help to first review Section 1.4 (Absolute value).

Read pages $50 - 51$ of the lecture notes, and work through any examples.

2.5   Intervals on the real line.

It may help to first review Section 1.3 (Number lines and order).

Read pages $52 - 54$ of the lecture notes, and work through any examples.

Intervals

- Remember that

    - a solid circle or a square bracket represent $\leq$ and $\geq$.
    - a non-filled circle or a round bracket represent $<$ and $>$.
    - each pair of brackets in interval notation contains two numbers. The left-hand number is **always** smaller than the right-hand number.

2.6   Solving Inequalities.

Read pages $55 - 56$ of the lecture notes, and work through any examples.

Solving inequalities

Be careful when dividing or multiplying by negative numbers. Remember that the inequality **must** be reversed.

2.7 Square roots.

It may help to first review Section 27 (Square roots).

Read pages $57 - 61$ of the lecture notes, and work through any examples.

Square roots

- Try to understand the properties and non-properties of square roots.

Surds

- Be able to manipulate surds by:

  - simplifying
  - multiplying

Simplifying surds

Some people get confused when simplifying an expression like $4\sqrt{6} \times 2\sqrt{3}$. This expression is really $4 \times \sqrt{6} \times 2 \times \sqrt{3}$ and we already know that we can change the order as the only operations are multiplication. Hence we can multiply the co-efficients together and the surds together, giving

$$
\begin{aligned}
4 \times 2 \times \sqrt{6} \times \sqrt{3} &= 8 \times \sqrt{18} \\
&= 8 \times 3\sqrt{2} \\
&= 24\sqrt{2}
\end{aligned}
$$

2.8 Powers and Exponents.

It may help to first review Section 1.9 (Exponentiation).

Read pages $62 - 68$ of the lecture notes, and work through any examples.

Exponents

- You must know and be able to apply the power rules.

- Understand that the **bases must be the same** for the rules to work.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Section B, Qn 2, Page 104
2. Section A, multichoice, Qns 6 Page 88
3. Section B, Qn 2, Page 88
4. Section B, Qns 3, 7, 8, 9, Page 88
5. Section A, multichoice, Qn 3, Page 97
6. Section B, Qn 2, Page 98
7. Section B, Qn 3, Page 98
8. Section B, Qn 4, Page 98
9. Section A, multichoice, Qn 4, Page 103
10. Section A, multichoice, Qn 9, Page 104
11. Section B, Qn 4, Page 104
12. Section B, Qn 6, Page 105
13. Section B, Qn 7, Page 105
14. Section A, multichoice, Qn 4, Page 109
15. Section A, multichoice, Qn 9, Page 110
16. Section A, multichoice, Qn 13, Page 110
17. Section B, Qn 2, Page 110
18. Section B, Qn 4, Page 110
19. Section B, Qn 6, Page 111
20. Section B, Qn 7, Page 111
21. Section A, multichoice, Qn 9, Page 115
22. Section B, Qn 2, Page 116
23. Section B, Qn 4, Page 116
24. Section B, Qn 5, Page 116
25. Section B, Qn 6, Page 117
26. Qn 2, Page 151
27. Qn 4, Page 157
28. Qn 4, Page 169
29. Qn 4, Page 174
30. Section B, Qn 3, 116
31. Sect A, multichoice, Qn 9, Page 93
32. Section B, Qn 2, Page 93
33. Sect B, Qns 6,8,10,11, Page 93
34. Qns 1(a), 5(a), Page 145
35. Qn 7, Page 146
36. Qns 6, 8, Page 140
37. Qns 6,7,8,9,10,12,13, Page 87
38. Qns 3,4,6,7,8,9,10,11,12,13, Page 92
39. Qns 5,6,7,8,9,10,16,19, Page 97
40. Qns 4,5,6,7,8,9,10, Page 103
41. Sect A, multichoice, Qns 1,3,5-8, Page 73
42. Sect B Qns 1-4,6,7, Page 74
43. Sect A, multichoice, Qn 1,3,5,6, Page 78
44. Sect B Qns 2-4,6,7,10, Page 79
45. Sect A, multichoice, Qn 1,4,9, Page 82
46. Sect B Qns 2,3,5,7, Page 79
47. Qn 2, Page 121
48. Qns 1(a), 2,4,5, Page 127
49. Qns 1(a)(b)(c), 6(a), Page 133

You should be able to do most of the first group of questions in Part A from the mid-semester exams, missing out any that involve curly brackets (like { and }), sigma-notation (written like Σ), or functions (written like $f(x) = \ldots$).

For example, on Page 92 you can do all of the first 26 questions except for numbers 2, 14, 22, 23, 24 and 26.
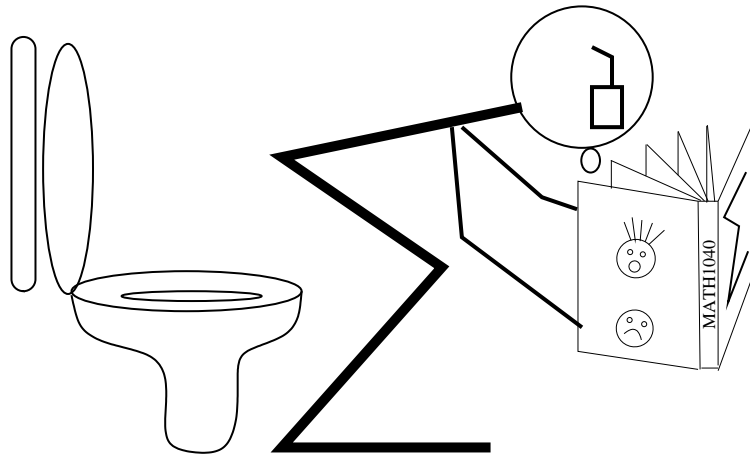
3 $\sum$ notation

What?

This section gives a basic introduction to sigma notation, which is a short-hand way of writing extended sums. We'll investigate sigma notation, seeing how to convert sums to sigma notation and expand and evaluate sums expressed in sigma notation. (That $\sum$ symbol in the title of this section is the Greek letter sigma. By convention this symbol means "to sum", and so this notation is sometimes referred to as summation notation).

Who?

People often want to write lengthy sums in a more compact form. Hence sigma notation is used by statisticians, biologists, economists and computer scientists as a quick and convenient way of writing long (or even infinite) sums.

How?

You can use this material to do such things as work out the mean value of some data (such as heights, income or marks on an exam), to calculate probabilities or to calculate the size of a growing population.

Why?

Sigma notation is used because writing out very long sums can be impractical. Some general way of expressing the sum using a formula is very important.way of

At the end of this section you should be able to:

- Express a sum in sigma notation.

- Write a sigma notation expression as an expanded sum and evaluate if required.

Work through this section as follows:

3.1   Introduction to sigma notation

Read pages $72 - 73$ of the lecture notes, and work through any examples.

Sigma notation

- Understand the meaning of the terms: *upper bound*, *lower bound*, *expression*.

3.2   Expanding sums

Read pages $73 - 75$ of the lecture notes, and work through any examples.

Sigma notation skills

- You must be able to take an expression written in sigma notation and expand it.

- Remember that the variable (for example, i) **always increases by 1 each time**.

- Remember i is a dummy variable. We can use any letter we like, so:

$$\sum_{i=1}^{4} i = \sum_{j=1}^{4} j = \sum_{k=1}^{4} k = 10$$

3.3   Reducing sums

Read pages $76 - 77$ of the lecture notes, and work through any examples.

Sigma notation skills

- You must be able to take an expanded sum and write it in sigma notation.

Checking your sigma notation expressions

When writing an expanded sum in sigma notation you should always check your answer by expanding it and seeing if you get the sum you started with.

Sigma notation

Many students have trouble with converting a sum to sigma notation. Here are some things to try if you are having trouble:

- Examine the terms in the sum and look for a pattern of numbers increasing by one each time.

- Are the terms all even? Then you need to use something like $2i$.

- Are the terms all odd? Then you need to use something like $2i + 1$.

- Are the terms powers of i? For example:

$$
\begin{aligned}
1 + 4 + 9 + 16 + \ldots &= 1^2 + 2^2 + 3^2 + 4^2 + \ldots \\
1 + 8 + 27 + 64 + \ldots &= 1^3 + 2^3 + 3^3 + 4^3 + \ldots
\end{aligned}
$$

3.4 Applications of sigma

Read pages $78 - 79$ of the lecture notes, and work through any examples.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1.	Section B, Qn 4, 5, 12, Page 88	2.	Qn 11, Page 140
3.	Section B, Qn 5, Page 98	4.	Section B, Qn 9, Page 105
5.	Section B, Qn 14, Page 105	6.	Section B, Qn 8, Page 111
7.	Section B, Qn 8, Page 117	8.	Qn 1, Page 151
9.	Qn 1, Page 157	10.	Qn 1, Page 163
11.	Qn 1, Page 169	12.	Qn 21, Page 170
13.	Qn 1, Page 174	14.	Section B, Qn 3, 4, 13, Page 93
15.	Qn 11, Page 146	16.	Qn 5,9, Page 74
17.	Qn 5,11, Page 79	18.	Qn 4,8, Page 84
19.	Qn 6,9a, Page **??**	20.	Qn 3,4, Page 133

You should be able to do any sigma-notation questions from Part A of the midsemester exams.

$y = -x$

$y = x$

$y = -1/3x + 2$

$y = 2$
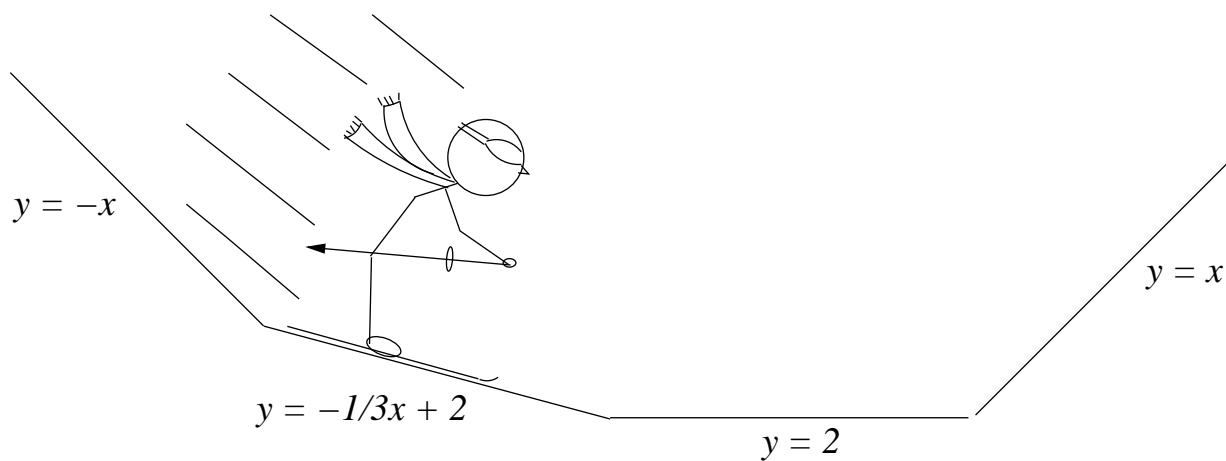
4 Straight lines and their graphs

What?
This section gives a basic introduction to graphs. We'll see how to plot data, and how to draw the graphs of straight lines. Then we investigate properties of equations of straight lines, including gradients and intercepts. We discuss the relationships between parallel and perpendicular lines, and encounter the formula for the distance between two points.

Who?
Graphs are one of the most frequent ways of representing data. They are used by many people, such as economists, statisticians, scientists, and sales people. Straight lines and linear equations are used by many people including engineers, physicists and business analysts. Most people use distance measurements every day of their lives.

How?
Here are two practical uses of this material.
Australia and India are playing in a one day cricket match. Represent the run rate for both teams visually so that a television audience can immediately understand how their favourite team is doing.
Roads are not flat: instead they slope down from the centre to the shoulder. This slope is called the crossfall. The crossfall of a particular road is to be 3%. The finished height of the centreline 18.357 metres above sea level. A point on the shoulder perpendicular to this is 3.65 metres away. At what height above sea level should this point be?

Why?
Graphs provide us with the tools we need to solve certain problems and visualise data. Rather than a confusing list of numbers, graphs provide a picture which can be more easily analysed and interpreted. Much of your later year study (for example, microeconomics) will require you to both draw and interpret graphs. We need to be able to analyse the relationship between input and output variables over a given interval. Linear (straight line) relationships are the simplest kind, so we develop tools to deal with straight lines and adapt them for use with other kinds of functions later on.

At the end of this section you should be able to:

- Identify the x-axis, the y-axis and the origin.

- Plot coordinates on a set of axes.

- Identify the four quadrants of the plane.

- Identify the dependent and independent variables.

- Given a simple equation, calculate some x and y values, plot these values and sketch the resulting graph.

- State the equation of a straight line in general form.

- Take a given straight line equation and identify the gradient, the y-intercept and the x-intercept.

- Identify the equations of lines that are parallel to one of the axes.

- Given two points, calculate the gradient of a straight line between these points and then find the equation of the line.

- Calculate the slope of a line parallel to a given line.

- Calculate the slope of a line perpendicular to a given line.

- Find the distance between two points.

Work through this section as follows:

4.1   Introduction to graphs

Read pages $83 - 84$ of the lecture notes, and work through any examples.

Introduction to graphs

- Understand the meaning of: *x-axis*, *y-axis*, *origin*.

- Points on our graph look like (x-coordinate, y-coordinate). Remember:

    - if the y-coordinate is 0, the point lies on the x-axis.
    - if the x-coordinate is 0, the point lies on the y-axis.
    - if both coordinates are 0, the point is the origin.

- Understand what sign x and y take (that is, positive or negative) if a point lies in a given quadrant.

- Know the meaning of dependent variable and independent variable and know which axis corresponds to which variable.

4.2 Sketching equations

Read pages $84 - 86$ of the lecture notes, and work through any examples.

Sketching equations

- Given a simple linear equation:

    - be able to calculate some points on the line; and
    - be able to plot these points and draw a graph.

4.3 Straight line (linear) graphs

Read pages $87 - 90$ of the lecture notes, and work through any examples.

Linear graphs

- Remember that you only need two points to plot a straight line

- Understand the meaning of the terms: *y-intercept*, *x-intercept*.

4.4 Standard form for the equations of straight lines

Read pages $90 - 94$ of the lecture notes, and work through any examples.

Straight lines

- Know that the general equation of a straight line is $y = mx + c$.

- Understand the meaning of the terms: *gradient*, *slope*.

- You must know that m is the gradient. and c is the y-intercept.

- Be able to recognise the slope and y-intercept from an equation of a straight line in standard form.

- Understand what form a graph takes with:

    - m negative, positive or equal to zero.
    - c negative, positive or equal to zero.

Gradient, slope or m?

Don't get confused by the interchanging terms gradient, slope and m. The gradient is the slope, and this is the value that m has when a straight line equation is written in the form $y = mx + c$.

4.5 Lines parallel to the axes.

Read pages $95 - 96$ of the lecture notes, and work through any examples.

Lines parallel to the axes

- You must know that lines parallel to the x-axis:

    - are horizontal and therefore have slope $m = 0$.
    - have equations of the form $y = c$.
    - are easily identified from a pair of points, since the points will be of the form (x_1, y) and (x_2, y), with the same y values.

- You must know that lines parallel to the y-axis:

    - are vertical and therefore have slope $m = \infty$.
    - have equations of the form $x = c$.
    - are easily identified from a pair of points, since the points will be of the form (x, y_1) and (x, y_2), with the same x values.

Identifying lines parallel to the axes

Being able to easily identify lines parallel to axes from a pair of points will save you a great deal of time on an exam. All you need to remember is:

- Do the points share a common y coordinate, say c? If yes, then the points lie on a line parallel to the x-axis with slope $m = 0$ and the equation for this line is $y = c$.

- Do the points share a common x coordinate, say c? If yes, then the points lie on a line parallel to the y-axis with slope m undefined and the equation for this line is $x = c$.

4.6 Finding gradients

Read pages $96 - 99$ of the lecture notes, and work through any examples.

Calculating the gradient

It is not too hard to remember that the formula for finding the slope of a straight line, given two points, is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

but some people forget which coordinate is y_1, which is y_2, which is x_1, and which is x_2. Here is a simple method to stop getting confused. Consider the points $(0, 0)$ and $(3, -3)$. We need to let one point be (x_1, y_1) and the other be (x_2, y_2). You can choose them arbitrarily: it won't make any difference. Next, write under each point whether it is (x_1, y_1) or (x_2, y_2), like so:

$$\begin{matrix} (3 \ , & -3) & (0 \ , & 0) \\ x_1 & y_1 & x_2 & y_2 \end{matrix}$$

Now we can easily see which value is substituted for which coordinate and it is a simple matter to finish evaluating the slope:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-3)}{0 - 3} = \frac{3}{-3} = -1.$$

Take care with signs

The previous example showed that you have to take care with negative signs in these calculations. For example, $1 - (-2) = 1 + 2 = 3$, and $-4 - (-1) = -4 + 1 = -3$.

4.7   Finding the equation of a straight line

Read pages $99 - 103$ of the lecture notes, and work through any examples.

Straight lines

- Be able to find the equation of a line given:

    - the gradient of the line and a point on the line,
    - two points on the line.

4.8   Parallel and perpendicular lines

Read pages $104 - 105$ of the lecture notes, and work through any examples.

Parallel and perpendicular lines

- Know that for two lines with slopes m_1 and m_2 respectively:

    - if the lines are parallel, $m_1 = m_2$.

    - if the lines are perpendicular, $m_1 \times m_2 = -1$, or equivalently $m_2 = \dfrac{-1}{m_1}$, or equivalently $m_1 = \dfrac{-1}{m_2}$.

4.9   Measuring distance

Read pages $106 - 108$ of the lecture notes, and work through any examples.

Distance

- Understand and be able to apply Pythagoras' theorem in a triangle.

- Be able to apply the distance formula given two points.

Choosing (x_1, y_1) **and** (x_2, y_2)

You may choose either point as (x_1, y_1) as long as you choose the other point as (x_2, y_2), as the following example illustrates:

Suppose you were required to find the straight line distance between the points $(12, 6)$ and $(9, 4)$. If you choose $(12, 6)$ as (x_1, y_1) and $(9, 4)$ as (x_2, y_2), you get:

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(12 - 9)^2 + (6 - 4)^2} \\
&= \sqrt{3^2 + 2^2} \;=\; \sqrt{9 + 4} \;=\; \sqrt{13}
\end{aligned}
$$

Similarly if you choose $(9, 4)$ as (x_1, y_1) and $(12, 6)$ as (x_2, y_2), you get:

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(9 - 12)^2 + (4 - 6)^2} \\
&= \sqrt{(-3)^2 + (-2)^2} \;=\; \sqrt{9 + 4} \;=\; \sqrt{13}
\end{aligned}
$$

which is the same answer.

Distance formula and gradient formula

Don't get the distance formula mixed up with the gradient formula. They are different.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Graph Qn, Page 98   (not Eqns 1,2)
2. Section B, Qn 8, Page 99
3. Section B, Qn 9, Page 99
4. Section A, multichoice, Qn 1, Page 103
5. Section A, multichoice, Qn 10, Page 104
6. Graph Qn, Page 104   (not Eqns 5,6)
7. Section B, Qn 12, Page 105
8. Section B, Qn 13, Page 105
9. Section A, multichoice, Qn 1, Page 109
10. Section A, multichoice, Qn 10, Page 110
11. Section A, multichoice, Qn 11, Page 110
12. Section A, multichoice, Qn 12, Page 110
13. Graph Qn, Page 110   (not Eqns 3,6)
14. Section B, Qn 12, Page 111
15. Section B, Qn 13, Page 111
16. Section A, multichoice, Qn 1, Page 115
17. Section A, multichoice, Qn 10, Page 116
18. Graph Qn, Page 116   (not Eqns 1,4,7)
19. Section B, Qn 12, Page 117
20. Section B, Qn 13, Page 117
21. Section B, Qn 14, Page 117
22. Qn 5, Page 151
23. Qn 12, Page 152   (not Eqns 1,2,4,8)
24. Qn 6, Page 157
25. Qn 8, Page 157
26. Qn 13, Page 158   (not Eqns 1,5,6)
27. Qn 16(a), Page 159
28. Qn 6, Page 163
29. Qn 8, Page 163
30. Qn 13, Page 164   (not Eqns 1,5,6)
31. Qn 6, Page 169
32. Qn 8, Page 169
33. Qn 16, Page 170   (not Eqns 3,5,6)
34. Qn 6, Page 174
35. Qn 15, Page 175   (not Eqns 2,5,6)
36. Section A, multichoice, Qn 6, Page 92
37. Section B, Qn 9, Page 93
38. Qn 2 (not Eqn 2),3, Page 145
39. Section A, multichoice, Qns 8, 9, Page 88
40. Section B, Qns 6, 11, Page 88
41. Qn 3, Page 139
42. Section A, Qns 13,18,19,26, Page 82
43. Section A, multichoice, Qns 5-7, Page 83
44. Section B, Qn 10, Page **??**
45. Qn 1d,e,7,8, Page 127
46. Qn 5, Page 133
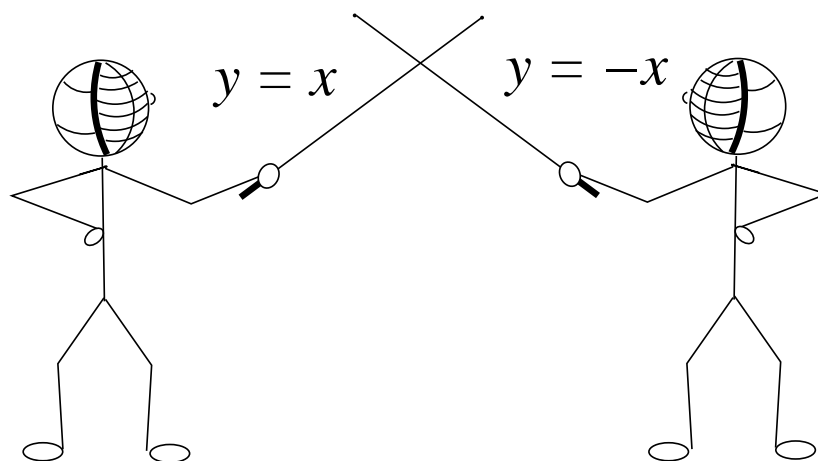
5 Intersecting lines; simultaneous equations

What?

In this section we encounter simultaneous equations, seeing what they are and how to solve them. We'll see that when there are two linear equations and two unknowns, the problem can be interpreted as finding the intersection of two straight lines. We will look at how to solve such equations. We'll concentrate on the case where there are two unknowns, although the techniques can be applied to bigger problems.

Who?

Solving simultaneous equations is one of the most important applications of mathematics, and it is regularly undertaken by engineers, physicists, economists and surveyors. Most supercomputers in the world spend a great proportion of their time solving such equations with thousands or millions of unknowns. Indeed one of the primary reasons that computers were initially developed was to solve simultaneous equations.

How?

Jim buys 3 pieces of fish and 2 serves of chips for $11.20. Mary buys 1 piece of fish and 4 serves of chips for $10.40. What does a piece of fish cost? What about a serve of chips?

Why?

We need to be able to solve pairs of simultaneous equations. The intersection point of two linear functions represents an x, y pair that satisfies both equations. We need to be able to determine if such an x, y pair exists, and if so find their values. You will apply these techniques in subsequent courses.

At the end of this section you should be able to:

- Given a pair of simultaneous equations, *algebraically* determine if they have have an infinite number of solutions, one solution, or no solution, and if a solution exists, find it by either the substitution or elimination methods.

Work through this section as follows:

5.1 Intersections of lines.

Read pages $112 - 113$ of the lecture notes, and work through any examples.

Intersections of lines

- Know that any pair of straight lines drawn on the same set of axes will intersect in either one, none or an infinite number of points.

- Know that if a pair of straight lines intersect in one point, there is precisely one pair (x_i, y_i) that satisfies the equations of both lines. We say that this pair of simultaneous equations has a **unique** solution.

- Know that if a pair of straight lines are parallel (ie. they intersect in no points), then there is no pair (x_i, y_i) that satisfies the equations of both lines. We say that this pair of simultaneous equations has **no** solution.

- Know that if a pair of straight lines overlay each other (ie. they intersect in every point, and are in fact the same line), then for every pair (x_i, y_i) that satisfies the equation of the first line, (x_i, y_i) will also satisfy the equation of the second line. We say that this pair of simultaneous equations has an **infinite number** of solutions.

5.2 Solving simultaneous equations

Read pages $113 - 120$ of the lecture notes, and work through any examples.

Algebraic solution of simultaneous equations

- Be familiar with both the substitution and elimination methods for solving simultaneous equations, and be able to choose which method is best.

- Be able to identify if there is no solution, one solution, or an infinite number of solutions.

Substitution or elimination?

Many students ask when to use elimination and when to use substitution. Unless you are explicitly asked to use one method or another you may use which ever method you prefer. There are however times when substitution is clearly easier. If, in one of the equations, x or y has a coefficient of 1, it is a simple matter to rearrange this equation and isolate the variable, so substitution is the way to go. If this is not the case, and you would rather not mess around with fractions, use elimination.

Substitution

Here are the steps for substitution:

- Choose and isolate.

  - choose an equation and isolate one of the variables to the left-hand side.
  - the choice will usually be obvious.
  - if the choice is not obvious, then play it safe and use elimination.

- Substitute.

  - take the equation with the isolated variable on the left and substitute the expression that is on the right hand side of that equation, for the relevant variable in the other equation.
  - take care when substituting. Put brackets around the substituted expression to ensure correct evaluation.

- Solve for one variable.

  - you should now have an equation with one unknown (either x or y). Solve the equation and find this unknown.

- Substitute and solve.

  - Substitute the value found in the previous step for the correct variable in either of the original equations.
  - solve this equation and find the value of the other variable.

- Substitute and check.

  - Substitute the x and y values you now have into both original equations, and check that your solution is correct.

Elimination

Here are the steps for elimination:

- Eliminate one variable.

  - add a multiple of one equation to a multiple of the other equation, thus eliminating one of the variables.
  - take care with signs when performing this step.
  - although not explicitly mentioned in the lecture notes, you can subtract one equation from another instead of adding them together. Sometimes this method is easier (of course this is the same as multiplying an equation by -1 and then adding the equations).

Elimination   *(continued)*

- Solve for one variable.

    - solve this equation to find one of the variables.

- Substitute and solve.

    - Substitute the value found in the previous step for the correct variable in either of the original equations.

    - solve this equation and find the other variable.

- Substitute and check.

    - Substitute the x and y values you now have into both original equations, and check that your solution is correct.

The difference between the substitution and elimination methods

Note that the only difference between the substitution method and the elimination method is the way in which we form the equation to find the first unknown.

- In the substitution method we **substitute** an expression for a variable, forming a new equation to find the value of the other variable.

- In the elimination method we **eliminate** one variable, forming a new equation to find the value of the other variable.

Checking your solutions

In both the elimination and substitution methods you will notice that the last step is to substitute and check your answers. You must get into the habit of checking your solutions. Mistakes (particularly with a misplaced sign) are easy to make. Substitute your solutions into both original equations and verify that your answers are correct. On your exam, if your answer is incorrect but you substitute and recognise that it is incorrect, you may not have time to recalculate your answer. You can recover some lost marks by demonstrating that you have substituted and checked your answers and you are aware that they are incorrect.

No solution; infinite number of solutions

You should work through the Questions in the lecture notes to understand what happens algebraically if the pair of simultaneous equations has no solution or an infinite number of solutions. Here are some general examples:

- If at any stage in the solution process you get something like

$$-6 = 0$$

  then this clearly is **never true**, which means the pair of equations has **no solution**.

- If on the other hand you get something like

$$5 = 5$$

  then this clearly **always true**, which means the pair of equations has an **infinite number of solutions**. This means the lines are parallel.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Question 7, Page 140

2. Section B, Question 10, Page 99

3. Section B, Question 11, Page 99

4. Section B, Question 15, Page 105

5. Section B, Question 15, Page 111

6. Section B, Question 15, Page 117

7. Section B, Question 16, Page 117

8. Question 4, Page 151

9. Question 5, Page 157

10. Question 5, Page 163

11. Question 5, Page 169

12. Question 8aiii,c, Page 122

13. Question 11, Page 128

14. Question 10, Page 134
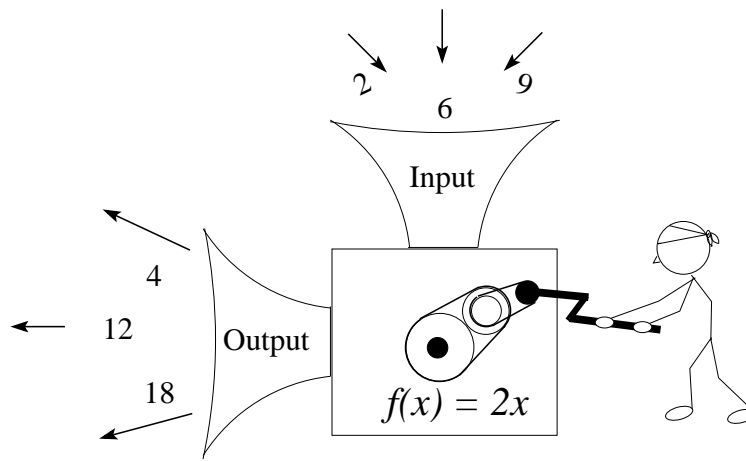
6 Functions

What?
This section gives a basic introduction to functions We'll investigate function notation, simple functions, and the input and output values (domain and range) that these functions can have. At the end of this section we'll briefly cover composition of functions.

Who?
Functions are used by scientists, economists and statisticians to describe the behaviour of complex systems such as weather prediction, share market trends and population dynamics.

How?
The population of Mexico City has been accurately recorded for the past 100 years. Create a model based on this information that predicts the population of Mexico City in 2031.

Why?
We have seen that graphs were useful for visualising data. But graphs have their limitations. It is hard to zoom in and read fractional values from a graph, and it would be impossible to completely draw a graph that goes from $-\infty$ to ∞. It is often better to use functions to determine the behaviour of systems, rather than relying on graphs. We can immediately evaluate and compare given points, regardless of how far apart they are, and we can immediately achieve whatever level of accuracy we desire. Composition of functions will be important when we start differentiation (especially the chain rule).

At the end of this section you should be able to:

- Feel comfortable using function notation.

- Evaluate functions for given x values.

- Accurately determine domain and range.

- Determine $f(g(x))$ and $g(f(x))$.

Work through this section as follows:

6.1 Functions and function notation.

Read pages $124 - 128$ of the lecture notes, and work through any examples.

Functions and function notation

- Understand that a function must have a unique output for each input.

- Understand that a function is a mathematical rule

    – we give it some valid input.
    – we extract a unique output.

- Be comfortable with the notation: it can seem tricky at first.

Understanding functions

Many students have trouble trying to understand functions and yet have been using them for years without even knowing it. A function is a mathematical rule, just like the one for finding the area of a circle, $A = \pi r^2$. We pass our function some input, in this case r, the radius of a circle, and calculate the output A, which is the area of a circle. We could rewrite this rule in function notation as $f(x) = \pi x^2$ and nothing would change, except that we now call our input x and our output $f(x)$. The output would still be the area of a circle whose radius is x.

Function notation

Many students get thoroughly confused with function notation, particularly using $f(x)$, $g(x)$ and y. Note that for each of

$$f(x) = 3x + 4 \quad g(x) = 3x + 4 \quad y = 3x + 4$$

the only difference is that the output has a different name in each case. **The value of the output does not differ**. If we let $x = 1$, the answer will always be 7. The functions $f(x)$, $g(x)$ and y are just names we give to the output. Similarly for each of

$$f(a) = 3a + 4 \quad g(u) = 3u + 4 \quad y = 3x + 4$$

a, u and x are just names for the input variable. If I let $a = 1$ in the first equation, $u = 1$ in the second equation and $x = 1$ in the third equation, the output in each case will equal 7.

6.2 Domain and range.

Domain and range are usually expressed in interval notation. It may help to revise Section 2.5 (Intervals on the real line) before proceeding.

Read pages $128 - 136$ of the lecture notes, and work through any examples.

Domain and range

- Distinguish between domain and range.

    - domain means all possible input values for a function.
    - range means all possible output values for a function.

- Be able to determine the valid domain and range for a given function and express them in interval notation.

Tips for finding the domain

Here are some tips for finding the domain of a function. Ask yourself: does the function have any square roots or fractions? If so then note that:

- the domain cannot contain any value which gives a negative inside a square root sign (you can't find the square root of a negative number).

- the domain cannot contain any value which gives zero as the denominator of a fraction (as you can't divide by zero).

Tips for finding the range

Here are some tips for finding the range of a function.

- Is the domain restricted? If so it may restrict the range.

- Are there any square root signs, squared indices or absolute values in the equation? If so then note that:

    - the square root of every number is always positive.
    - any number squared is always positive.
    - the absolute value of any number is always positive.

6.3 Composition of functions.

Read pages $137 - 138$ of the lecture notes, and work through any examples.

Composition of functions

- You must understand composition of functions. For a given $f(x)$ and $g(x)$ you should be able to find $f(g(x))$ and $g(f(x))$.

- It is important that you are able to distinguish between $f(g(x))$ and $g(f(x))$.

Composition of functions

Students always find composition of functions hard. If you follow this simple rule it will be easier: to find $f(g(x))$, all you have to do is take the term on the right hand side of $g(x)$ and put brackets around it. Now, every time x occurs in $f(x)$, replace it with the right hand side of $g(x)$ (surrounded by brackets).

For example, suppose you are told that $f(x) = -x^2 + x$ and $g(x) = x + h$ and you are asked to find $f(g(x))$. First take the term from the right hand side of $g(x)$ and put brackets around it. This gives $(x + h)$. Now, wherever there is an x in $f(x)$, you replace it with $(x + h)$. This gives:

$$f(x) = -x^2 + x$$
$$f(g(x)) = -(x + h)^2 + (x + h)$$

Note the brackets around $x + h$. We put them there for two reasons. Firstly, using brackets makes it absolutely clear what you are substituting for x in $f(x)$. Secondly, using brackets ensures there is no ambiguity if you have to expand the composite function out later on. See if you can find $g(f(x))$.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Qn 2, page 139
2. Qn 9, page 140
3. Section A, multichoice, Qn 1, Page 97
4. Section A, multichoice, Qn 2, Page 97
5. Section B, Qn 7, Page 99
6. Section A, multichoice, Qn 2, Page 103
7. Section A, multichoice, Qn 3, Page 103
8. Section B, Qn 10, Page 105
9. Section A, multichoice, Qn 2, Page 109
10. Section A, multichoice, Qn 3, Page 109
11. Section B, Qn 10, Page 111
12. Section A, multichoice, Qn 2, Page 115
13. Section A, multichoice, Qn 3, Page 115
14. Section B, Qn 10, Page 117
15. Qn 6, Page 151
16. Qn 4, Page 163
17. Qn 1(b),(c), 4, Page 145
18. Qn 5(b), 8, 12, Page 146
19. Qn 3, Page 121
20. Qn 3, Page 127
21. Qns 2,8,9, Page 133

$$x^3 + 2x^2 - x + 3$$
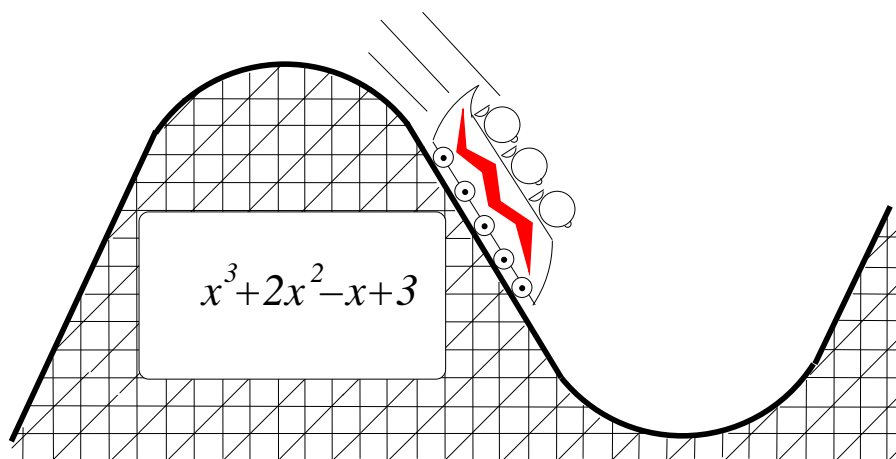
7 Quadratic equations and polynomials

What?

This section covers polynomials and, in particular, quadratic equations. Our work so far has mainly been concerned with straight lines and linear equations. We now progress a step further and look at curves and polynomial equations.

In linear equations the unknown x is only raised to the power one. With polynomials, x may be raised to any integer power, and the resulting graph is a curve. The simplest polynomials (apart from straight lines) contain x raised to the power two (that is, x^2) and are called quadratics. Much of this section will be spent dealing with quadratics.

Who?

Quadratics and polynomials are an important part of mathematics. Being able to accurately model curves is an important part of the design of everything from your mobile phone or your car, or even the Sydney Harbour Bridge. Economists, mathematicians, physicists, and biologists invest much time and money trying to find polynomial equations to model fluctuations in stockmarket trends, production schedules, and species population dynamics.

How?

Traffic flow on a major intersection is measured hourly from 5:30 am to 6:30 pm every weekday for a month. The hourly totals are averaged and plotted on a graph. Find a curve which best describes the traffic flow.

Why?

Quadratics and polynomials are important to us in this course. Learning how to plot polynomials on a graph and to solve quadratic equations is a major step toward understanding the calculus components later on in this course.

At the end of this section you should be able to:

- Recognise a polynomial by its equation and determine if it is linear, quadratic, cubic or other.

- Identify the coefficients and the constant term in a polynomial equation.

- Sketch a quadratic curve from its polynomial equation.

- Determine the number of roots of a quadratic equation.

- Find the roots of a quadratic equation using the quadratic formula or factoring.

- Solve real world problems using the quadratic formula.

Work through this section as follows:

7.1 Introduction to polynomials.

Read pages $142 - 143$ of the lecture notes, and work through any examples.

Introduction to polynomials

- Be able to recognise polynomials of degree one (linear equations), degree two (quadratic equations), and degree three (cubic equations).

- Be able to identify the coefficients and constant term of a given polynomial.

Constant terms

As for linear equations, the constant term of all polynomials is the y-intercept. For linear equations the slope was m, but for general polynomials the slope is more complicated.

7.2 Quadratics.

Read pages $143 - 145$ of the lecture notes, and work through any examples.

Quadratics

- Know:

    - that the general form of a quadratic is $f(x) = ax^2 + bx + c$
    - that a quadratic may cross the x-axis two times, once, or not at all.
    - that the quadratic crosses the x-axis only when $y = 0$.
    - that the values of x when $y = 0$ are called the roots of the equation.

7.3   Shapes of some polynomial functions.

Read pages $145 - 147$ of the lecture notes, and work through any examples.

Shapes of some polynomial functions

- Know the general shape of a quadratic with $a < 0$ and 0, 1 or 2 roots.

- Know the general shape of a quadratic with $a > 0$ and 0, 1 or 2 roots.

- You must be aware of the effect of changing the values of a and c.

- You should be able to sketch a variety of quadratic equations.

7.4   Solving quadratics using the Quadratic formula.

Read pages $148 - 151$ of the lecture notes, and work through any examples.

Solving quadratics

- You must be able to find the roots of a quadratic using the quadratic formula.

- Know that if:

    - $b^2 - 4ac > 0$ the quadratic has two distinct real roots,
    - $b^2 - 4ac = 0$ the quadratic has one real root,
    - $b^2 - 4ac < 0$ the quadratic has no real roots.

Important things to note when solving quadratics

The are some important things to note when solving quadratics:

(1) The right hand side of the equation **must** equal zero. The quadratic formula won't work if the equation is not written as $ax^2 + bx + c = 0$.

(2) Setting out can save a lot of trouble. It is a good idea to write down the values of a, b and c before substituting them into the quadratic formula. As an example, find the roots of $2x^2 = 2x + 12$.

First, rewrite the equation in standard form, so $2x^2 - 2x - 12 = 0$.

Important things to note when solving quadratics *(continued)*

Now $\quad a = 2 \qquad b = -2 \qquad c = -12.$

When we substitute them into the quadratic formula we get

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot (-12)}}{2 \cdot 2} \\
&= \frac{2 \pm \sqrt{4 - (-96)}}{4} = \frac{2 \pm \sqrt{100}}{4} = \frac{2 \pm 10}{4}
\end{aligned}
$$

$$x = \frac{12}{4} \quad \text{or} \quad x = \frac{-8}{4} \qquad \text{so} \qquad x = 3 \quad \text{or} \quad x = -2$$

(3) Take extreme care with signs when substituting into the formula. In the above example we saw that $b = -2$ so $-b = -(-2) = 2$ and $c = -12$, so $-4ac = -4 \times 2 \times -12 = 96$.

7.5 Solving quadratics by factoring.

Read pages $152 - 155$ of the lecture notes, and work through any examples.

Solving quadratics

- Understand the relationship between factors and roots.

- Be able to find some factors of a polynomial by trial and error.

7.6 Applications of quadratics

Read pages $156 - 157$ of the lecture notes, and work through any examples.

Applications of quadratics

- Be able to construct quadratic equations from wordy problems, solve the equations and relate the answers back to the original problem.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

x	$ln(x)$		x	$ln(x)$		x	$ln(x)$
1	0		4	1.3863		7	1.9459
2	0.6931		5	1.6094			
3	1.0986		6	1.7918			

8 Logarithms and exponentials

What?

In this section of the course we'll look at two special types of function: logarithms and exponentials. We'll see the general form of exponential functions, see how they lead to exponential growth, encounter formulae for compound interest, discuss exponential decay, and then encounter a special irrational number, e. Finally we'll encounter logarithmic functions and look at a few of their properties.

Who?

Logarithms and exponents play an important part in modeling biological data, particularly population modeling. They are also an important part of statistical analysis and have applications in engineering and commerce; for example, compound interest grows exponentially. Archaeologists and palaeontologists use exponential functions when carbon dating fossils and bones.

How?

The half-life (the time taken for half of a given sample to disintegrate) of Carbon-14 is 5730 years. The bones of an ancient Iceman are found to have 50% as much Carbon-14 as a living bone. When did the Iceman roam the earth?

Why?

In many other courses at university you will encounter logarithms and exponentials, particularly if you undertake any commerce, science or engineering courses, so it is important to learn the basics about these functions here. Before electronic calculators were available, logarithms were used to solve calculations involving large numbers (particularly for engineering problems) and books containing logarithm tables were consulted when solving these problems. Today, logarithmic and exponential functions are used to model things like population dynamics, ecological and financial problems and even to encrypt electronic data.

At the end of this section you should be able to:

- Identify exponential and logarithmic functions and their graphs.

- Calculate compound interest.

- Understand exponential growth and exponential decay.

- Understand the exponential function e^x.

- Identify plots of e^{kx} for k positive and k negative

- Understand natural logarithms and the notation $\ln x$.

Work through this section as follows:

8.1   Introduction to exponentials.

Read pages $161 - 162$ of the lecture notes, and work through any examples.

Introduction to exponentials

- Be able to identify an exponential function.

- Understand the differences between exponential and polynomial functions.

8.2   Exponential growth.

Read pages $162 - 166$ of the lecture notes, and work through any examples.

Exponential growth

- Understand that Ca^x with $C > 0$ and $a > 1$ represents exponential growth, and be able to plot graphs of such functions.

- Be clear on how exponential growth functions look for $x < 0$ and $x > 0$.

- Exponential growth arises in many practical problems: be able to solve them.

8.3 Compound interest

Practical problems

- Understand how the formula for compound interest arises: if P earns a rate of r per time period for 1 time period, then the new balance will be $F = P(1 + r)$ (this is easy to see. For example, if $P = 100$ and $r = 10\% = 0.1$, then the final balance will be $F = 100(1 + 0.1) = 110$.)

- After another time period it will be $(1 + r)$ times what it was, so will be $F = P(1 + r)^2$ (here, $F = 100(1 + 0.1)^2 = 121$).

- This continues for each time period, so after n time periods, $F = P(1 + r)^n$.

8.4 Exponential decay.

Exponential decay

- Understand that Ca^{-x} with $C > 0$ and $a > 1$ represents exponential decay, and be able to plot graphs of such functions.

- Be able to explain the similarities and differences between exponential growth functions, and exponential decay functions.

- Our main use for exponential decay functions is for solving practical problems involving radioactive decay, population decline and inflation.

8.5 The exponential function, e^x

The exponential function, e^x

- e is a very important number; from now on we will almost exclusively talk about exponentials in terms of e, as we can rewrite any exponential function with any base a as an exponential with base e.

- Understand the general form of exponential functions: $y = Ce^{kx}$, $k \neq 0$, $C > 0$.

- e has many important properties: one is the way it arises in problems involving continuous compounding.

- You do not need to remember the value for e, but you must be comfortable with using it in exponential functions. Remember that e is irrational (like π).

Evaluating e^x

Mostly you will not need to evaluate e^x. However, there is one case when you should evaluate e^x and that is when $x = 0$. Remembering that anything raised to the power zero equals 1, you should always rewrite e^0 as 1.

e^x is always positive

From the graphs of e^x and e^{-x}, we see that both of them are always positive (as they do not cross the x-axis). This is a very useful fact to remember: e^x and e^{-x} are never equal to 0 or negative!

8.6 Logarithms

Read pages $176 - 178$ of the lecture notes, and work through any examples.

Logarithms

- Logarithms are simply inverse functions of exponential functions: just as x^2 and $\sqrt{x}$ have opposite effects, so too do e^x and $\ln x$.

- You don't need to remember anything about logarithms to any base besides e: that is; we will look further at $\ln x$ in this course, but not at $\log_a x$ for any other base a.
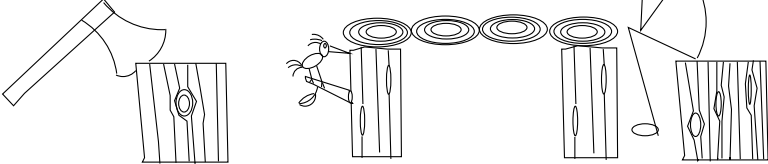
- Be aware that if $y = e^x$ then $x = \ln y$.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Qn 15, Page 153   2. Qn 1(e),(f),(g), Page 145   3. Qn 9, Page 146
4. Qn 1(d),(e), Page 139   5. Qn 10, Page 140
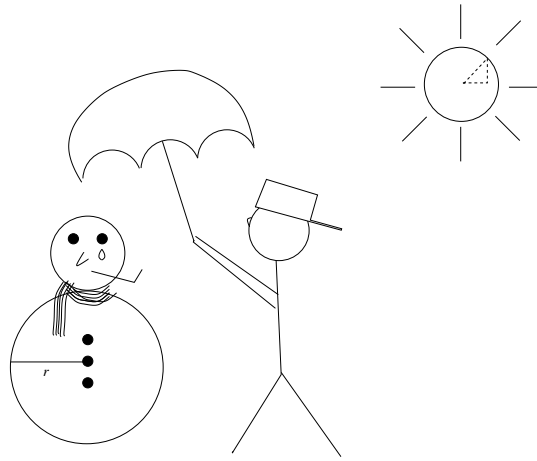
9 Miscellaneous non-linear functions

What?
This is a very short section in which we encounter a few more non-linear functions. The main topic we cover is the equations and graphs of circle.

Who?
Non-linear functions are extensively used in modelling, approximating data and designing equipment and processes.

How?
Given three points, find the equation of a circle that passes through all of the points.

Why?
The main reason we study circles is because we use them to develop triogonometry and the trigonometric functions in the next section.

At the end of this section you should be able to:

- Be able to sketch certain non-linear functions and state their domain and range.

- Recognise the equation of a circle.

Work through this section as follows:

9.1 Non-linear functions.

Read pages $182 - 185$ of the lecture notes, and work through any examples.

Non-linear functions

- Be able to recognise graphs of $y = x^3$, $y = \sqrt{x}$, $y = 1/x$, $y = |x|$ and $y = -|x|$.

- Understand how to find the domain and range of these functions.

Choosing the correct x-values

When calculating some points to plot a graph, it is often difficult to decide what values to pick for x. Here are some tips:

- Non-linear equations usually require many points.

- $x = 0$ is usually a safe and easy choice (as long as it does not make the denominator of a fraction zero).

- Choose values that give integer answers, wherever possible.

- If y-values change rapidly, choose x-values close together.

- Don't be afraid to choose negative x-values.

- Rather than just choosing x-values and calculating corresponding y-values, you can instead choose y-values and calculate corresponding x-values. For example, $y = 0$ is sometimes a sensible choice.

9.2 Circles.

Read pages $186 - 187$ of the lecture notes, and work through any examples.

Circles

- Know that a circle with radius r and centre $(0, 0)$ has equation $x^2 + y^2 = r^2$.

- Be aware that a unit circle is a circle with centre $(0, 0)$ and radius 1.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Graph Qn, Page 139
2. Graph Qn, Page 98
3. Graph Qn, Page 104
4. Graph Qn, Page 110
5. Graph Qn, Page 116
6. Qn 12, Page 152
7. Qn 17, Page 153
8. Qn 13, Page 158
9. Qn 13, Page 164
10. Qn 7, Page 121
11. Qn 10, Page 128
12. Qn 7, Page 133

By now you should be able to do all of the questions that involve matching graphs with equations.
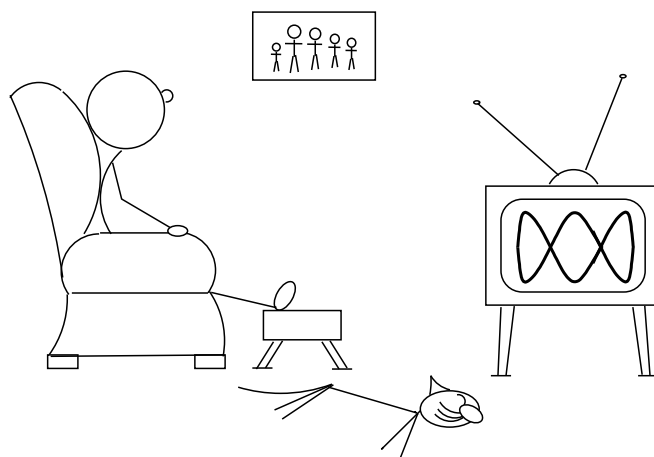
10 Trigonometry

What?
Trigonometry is the study of angles, and is one of the oldest fields of mathematical study. It has its origins in astronomy and is said to have originated in ancient Greece in the third century B.C. It is still one of the most widely used applications of mathematics today. Every man-made structure uses some trigonometry. In this course we'll encounter angles, trig functions ($\sin\theta$, $\cos\theta$, and $\tan\theta$), degrees and radians, and then see how the trig functions become periodic functions.

Who?
Trigonometry is used extensively by surveyors, engineers, sailors and military personnel. It has applications in civil engineering, such as in the construction of buildings, roads and bridges. It is also essential for navigation and in the military (such as for aiming artillery). Economists also use periodic functions (like the trig functions) to model many phenomena.

How?
Two trees on a bank of a straight river are 25 metres apart. I am standing on the other bank, directly opposite the first tree. The angle formed by the first tree, myself and the second tree is exactly $45°$. How wide is the river crossing?

Why?
A basic knowledge of trigonometry is useful to everyone. You may use it when building a fence, putting up a shelf, designing a garden at home, or perhaps sailing a yacht from Sydney to Hobart. This course provides an introduction to trigonometry, starting with some basic trigonometric rules that you may be familiar with, and moving on to harder and perhaps more unfamiliar material. Some of the material you encounter here will be useful later on in the calculus portion of this course.

At the end of this section you should:

- Understand some basic facts about angles and triangles.

- Be able to identify the sides of a right angle triangle.

- Know the definitions of: $\sin\theta$ $\cos\theta$ $\tan\theta$.

- Be able to use some special triangles to evaluate trig functions for some common angles.

- Understand the relationship between the radius of a circle and one radian.

- Be able to convert degrees to radians and radians to degrees.

- Be able to identify the graphs of $\sin x$, $\cos x$, and $\tan x$.

- Know the effect of varying a and b on the graph of $a\sin bx$.

Work through this section as follows:

10.1   Introduction to trigonometry.

Read page 191 of the lecture notes, and work through any examples.

Introduction to trigonometry

- Understand the meaning of the terms: *angle, acute angle, obtuse angle, isosceles triangle, equilateral triangle, right angle triangle.*

- Know that the sum of the angles in a triangle is $180°$.

- In a right angle triangle with a given angle θ, be able to identify the
  hypotenuse   adjacent side   opposite side.

- Know the definitions of: $\sin\theta$ $\cos\theta$ $\tan\theta$. (Note: θ is a Greek letter, pronounced 'theta', and is often used to represent an angle.)

Remembering the formulae for $\sin\theta$, $\cos\theta$, and $\tan\theta$.

There are many little rhymes for remembering the formulae for finding $\sin\theta$, $\cos\theta$, and $\tan\theta$. Most rely on substituting a word for a major component of the formulae to form a verse. Here is one:

some **o**ld **h**ippies   **c**an **a**ct **h**appy   **t**ripping **o**n **a**cid.

From the first letter in each word of this we get:

$$\mathbf{s}\text{in }\theta \;=\; \frac{\mathbf{o}\text{pposite}}{\mathbf{h}\text{ypotenuse}} \qquad \mathbf{c}\text{os }\theta = \frac{\mathbf{a}\text{djacent}}{\mathbf{h}\text{ypotenuse}} \qquad \mathbf{t}\text{an }\theta = \frac{\mathbf{o}\text{pposite}}{\mathbf{a}\text{djacent}}$$

10.2 Radians.

Read pages $195 - 196$ of the lecture notes, and work through any examples.

Radians.

- Understand the relationship between the radius of a circle and one radian.

- You should be able to convert degrees to radians.

- You should be able to convert radians to degrees.

- You should know that 2π radians $= 360°$, the number of degrees in a circle.

Calculators and degrees

Your scientific calculator should know all about degrees. It may however have a button for entering degrees, minutes and seconds. You don't need to worry about this. All degree measurements in this course will be in decimal form. If you enter values in decimal then you should obtain correct answers.

Calculators and radians

If you are using your calculator to evaluate trig functions in radians, you must ensure that the correct mode is set. If you have a scientific calculator your display will probably show "deg" if you are in degree mode and "rad" if you are in radians mode (it may even have a "grad" mode which is not used in this course). You should consult the book that came with your calculator and learn how to switch modes. Trying to solve problems in radians on a calculator set to degrees is a sure way to get incorrect answers every time.

Calculators and π

Your scientific calculator will probably have a button which gives you the value of π. Unless you are explicitly told to use some approximate value for π, (probably 3.14), you should use the π button on your calculator. Contrary to what you may have been told at school, π does **not** equal 22/7.

10.3 Angles bigger than $\pi/2$ ($90°$).

Read pages $196 - 199$ of the lecture notes, and work through any examples.

Angles bigger than $\pi/2$ ($90°$)

- Remember that we always start measuring our angles from the positive x-axis, with larger angles being further around the circle **anticlockwise**.

- Remember that $360°$ is one full (anticlockwise) rotation of a circle.

- Remember that negative angles mean the angle is measured clockwise.

- Know in which quadrants $\sin x$, $\cos x$ and $\tan x$ are positive.

10.4 Graphs of $\sin x$ and $\cos x$.

Read pages $200 - 204$ of the lecture notes, and work through any examples.

Graphs of $\sin x$ and $\cos x$.

- Know that the **range** of the graphs of $\sin x$ and $\cos x$ is $[-1, 1]$.

- Be able to identify the graphs of $\sin x$ and $\cos x$.

- Understand the meaning of the terms: *amplitude, frequency*.

- Understand the effect of varying a and b on the graph of $a \sin bx$.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Qn 7, Page 151
2. Qn 11, Page 158
3. Qn 16(b), Page 159
4. Qn 11, Page 164
5. Qn 14, Page 169
6. Qn 13, Page 174
7. Qn 4(b), Page 121

$f'(x)$

$f(x)$

$f''(x)$
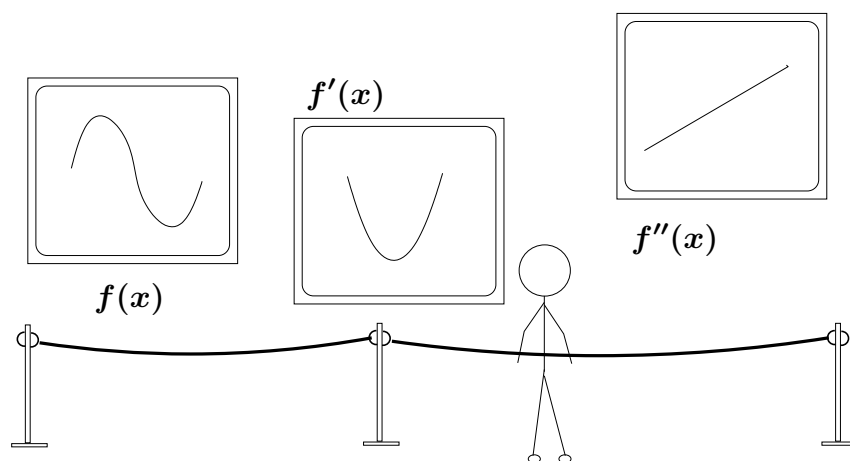
11   Derivatives and rates of change

What?

Being able to find the slopes of many functions is incredibly useful. It is so important that there is a special name for it: the slope of a function is called its *derivative*, and the process of finding a derivative is called *differentiation*. In this section we'll see how to rapidly find the derivatives of many functions, by applying some special rules. Then we'll see how to find the second derivative of a function.

Who?

Finding derivatives and calculating rates of change is an important skill for mathematicians, statisticians, hydrologists, biologists, engineers, physicists and economists. For example, in microeconomics you'll study marginal cost and marginal revenue curves. These are all derivatives.

How?

A worker in a factory works on the 3 pm to 11 pm shift. If her production is plotted on a graph it can be described by the function $p = -t^2 + 14t - 33$. At what time during the shift is the rate of production increasing the fastest? At what time is the rate of change zero? At what time is the rate of production decreasing the fastest? Give a possible explanation for these answers.

Why?

In Section 4 we looked at the slope of straight lines. In this section we learn some rules which allow us to find the slopes of a wide range of functions, including some quite complex functions.

At the end of this section you should be able to:

- Understand what the derivative represents.

- Use and understand the notation $f'(x)$.

- Correctly identify the appropriate rule for differentiation.

- Find the derivatives of a variety of functions by using a combination of the rules for differentiation.

- Find the value of the derivative of a function at a given point.

Work through this section as follows:

11.1 Differentiation and derivatives.

Read page 208 of the lecture notes, and work through any examples.

Differentiation and derivatives

- Be familiar with the terms:

  - *differentiate, differentiable,* and *differentiation*.

  - *derive, derivable,* and *derivative*.

  - $f'(x)$, $\quad , \dfrac{df}{dx}$, $\quad y', \dfrac{dy}{dx}$

- Be able to evaluate the derivative of a function at a given point.

Confusing notation

Don't be confused by all the different notation used to define the derivative of a function. This notation is all interchangeable. For example, if

$$f(x) = y = 2x^2 + 4x + 9$$

then the derivative of this function can be expressed as any of:

- $f'(x) \quad , \quad y' \quad , \quad \dfrac{dy}{dx} \quad , \quad \dfrac{d}{dx}\left(2x^2 + 4x + 9\right) \quad \text{or} \quad 4x + 4.$

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11.2 Interpreting derivatives

Read pages $209 - 212$ of the lecture notes, and work through any examples.

Interpreting derivatives

- Be quite clear on the meaning of derivatives.

- Be able to describe the graph of a function from the graph of its derivative.

- Be able to look at the function and derivative and explain the relationship between them.

Distinguishing the graph of a function from the graph of its derivative

On an exam you may be presented with a question like this:
The following set of axes contain the graph of a function and the graph of the derivative of the function. Identify which graph represents the function, and which graph represents the derivative. Explain your answer.
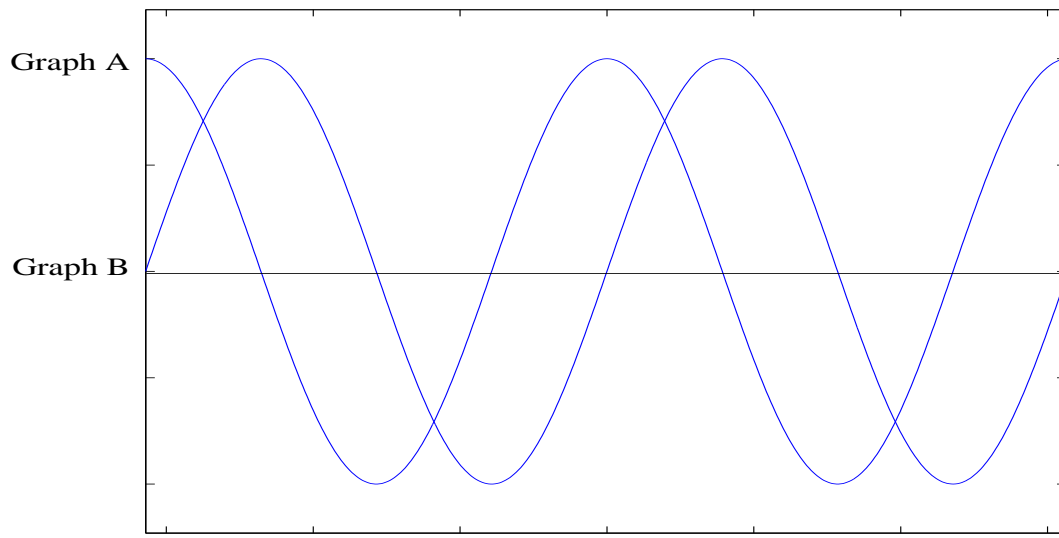
This may seem a daunting task, but if you apply what you know about derivatives, you will find it is quite easy. Remember that the derivative represents the **slope of the function**. This means that whenever the **slope** of the original **function** is **positive** (runs uphill), the **graph** of the **derivative** must be **positive** (the graph of the derivative must lie above the x-axis). Whenever the **slope** of the original **function** is **negative** (runs downhill), the **graph** of the **derivative** must be **negative** (the graph of the derivative must lie below the x-axis). Finally, whenever the **slope** of the original **function** is **zero** (changing from uphill to downhill or downhill to uphill) the **graph** of the **derivative** must be **zero** (the graph of the derivative cuts the x-axis).

Let's use this information to examine the two graphs. Start with Graph A. Graph A starts off with a negative slope, so if B were the derivative of A, then anywhere A has a negative slope B should be negative (lie below the x-axis). But it doesn't; for example look near the very left-hand edge. So B is not the derivative of A, and since you are told that one graph is the function and one graph is the derivative, then A must be the derivative of B. On an exam you could write:
"Graph B is the function. Graph A is the derivative.
Graph B is not negative whenever A has negative slope, so B cannot be the derivative of A."

But what if you started with Graph B? Then you would have found that everywhere B has positive slope A is positive, everywhere B has negative slope A is negative and everywhere B has zero slope A is zero (you should verify this by inspection), so A must be the derivative of B.

11.3   Simple differentiation

Read pages $213 - 216$ of the lecture notes, and work through any examples.

Rules for simple differentiation

- You must know and be able to apply the rules for simple differentiation.

- These rules are very important; make sure you are completely comfortable with them.

- You must become very fast at correctly differentiating polynomials.

- Given a function like $y = 6x^3 + 2x^2 - 4x + 1$, you should be able to immediately determine y', almost without thinking.

The derivative of x^n used in unexpected ways

The rule for finding the derivative of $y = x^n$ $(n \neq 0)$ is: $y' = nx^{n-1}$.
This rule is quite easy to apply for obvious cases like finding the derivative of x^4: all you need to do is find n and substitute it into the formula to get your answer. There are some cases where it is not so obvious that this rule should be used, but as we shall see, once we have determined that we should use this rule, the technique doesn't change. **Always identify n and substitute it into the formula to get your answer**.
Example 1: Find the derivative of $\sqrt{x}$.
Remembering the power rules, we can rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$, so $n = 1/2$ and we substitute into the formula to obtain our answer.

Example 2: Find the derivative of x^{-4}.
Here $n = -4$ (don't leave off the minus sign) and we can substitute -4 into the formula to obtain an answer.

Example 3: Find the derivative of $\dfrac{1}{x^4}$.

Again remembering our power laws we can rewrite $\dfrac{1}{x^4}$ as x^{-4} and we have the same situation as in Example 2.

11.4   Derivatives of some common functions

Read pages $217 - 219$ of the lecture notes, and work through any examples.

Derivatives of some common functions

- You must know the derivatives of $\sin x$, $\cos x$, $-\sin x$, $-\cos x$, e^x and $\ln x$.

Derivatives of $\sin x$, $\cos x$, $-\sin x$, and $-\cos x$

You need to remember the derivatives of $\sin x$, $\cos x$, $-\sin x$, and $-\cos x$. Fortunately this is not to difficult if you remember that the derivatives follow the rules:

> The derivative of $\sin x$ is $\cos x$,   the derivative of $\cos x$ is $-\sin x$,
> the derivative of $-\sin x$ is $-\cos x$,   the derivative of $-\cos x$ is $\sin x$,

and the process starts again. One way to remember this is:

> $\sin x$ goes to $\cos x$ goes to $-\sin x$ goes to $-\cos x$ (which goes to $\sin x$)

or you might like to try and remember this picture:
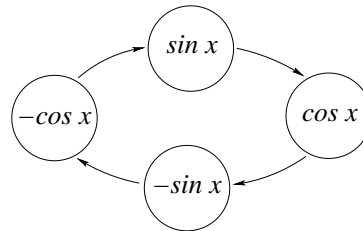
Derivative of e^x

Note that e^x is a special function, in that if you differentiate it, you get the same thing back. Many students get confused when differentiating e^x: they say to themselves 'I know that if $y = x^3$ then $y' = 3x^2$: I need to subtract 1 from the power'. This is perfectly correct, **because x is in the base**. Then they look at e^x, and say 'I still need to subtract 1 from the power'. This is completely wrong: the difference is that x **is in the power, not the base**. The **only** rule you need to use when differentiating e^x is that **you get the same thing back**: if $y = e^x$ then $y' = e^x$. This is the easiest differentiating you will ever have to do!

11.5   Product rule.

Read pages $220 - 223$ of the lecture notes, and work through any examples.

Product rule

- You must be able to apply the product rule to find the derivative of a function.

- Be able to expand a function and differentiate it to show that the product rule works.

Using the product rule

Here are the steps involved with applying the product rule.

1. Be able to identify when to use the product rule.

2. Be able to identify u and v.

3. Be able to find u' and v'.

4. Substitute 2 and 3 above into the product rule formula to find the derivative.

Using the product rule (an example)

Example: find the derivative of $y = x^2 \sin x$.

1. **Decide whether you need to use the product rule.**
   Ask yourself these two questions:

   - Does the function contain at least two terms, each containing an x?
   - Are these terms multiplied together (is there a product)?

   If the answer to both questions is yes then you should use the product rule. The example contains two terms with x in them. They are x^2 and $\sin x$. These terms are multiplied together, so you need to use the product rule.

2. **Identify u and v.**
   By convention, u is the first term of the product and v is the second term in the product. In this example you should write:

$$u = x^2 \qquad v = \sin x$$

3. **Find u' and v'.**
   Using the rules from the previous section you get:

$$u' = \frac{du}{dx} = \frac{d}{dx}(x^2) = 2 \times x^{(2-1)} = 2x$$

$$v' = \frac{dv}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

4. **Substitute into the product rule formula to find the derivative.**

$$f'(x) = u'v + v'u = 2x \cdot \sin x + \cos x \cdot x^2 = 2x \sin x + x^2 \cos x$$

11.6 Quotient rule.

Read pages $224 - 225$ of the lecture notes, and work through any examples.

Quotient rule

- You must be able to apply the quotient rule to find the derivative of a function.

Using the quotient rule

Here are the steps to using the quotient rule.

1. Be able to identify when to use the quotient rule.

2. Be able to identify u and v.

3. Be able to find $u' = \dfrac{du}{dx}$ and $v' = \dfrac{dv}{dx}$.

4. Substitute the results from Steps 2 and 3 into the quotient rule formula to find the derivative.

5. Simplify your answer if required.

Using the quotient rule (an example)

Example: find the derivative of $\dfrac{(x^2 + 5)}{x^2}$ and simplify your answer.

1. **Decide whether you need to use the quotient rule.**
   Ask yourself these two questions:

   - Does the problem have a quotient (is one term divided by another)?
   - Is there an x in both terms of the quotient?

   If the answer is yes to both questions then you should use the quotient rule.

2. **Identify u and v.**
   When applying the quotient rule, u is always the numerator and v is always the denominator. You should write down u and v. In this example you have:
   $u = x^2 + 5$ and $v = x^2$.

3. **Find $u' = \dfrac{du}{dx}$ and $v' = \dfrac{dv}{dx}$:**

$$u' = \frac{du}{dx} \;=\; \frac{d}{dx}\left(x^2 + 5\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(5\right) = 2 \times x^{2-1} + 0 = 2x$$

$$v' = \frac{dv}{dx} \;=\; \frac{d}{dx}\left(x^2\right) = 2x$$

(continued over)...

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4. **Substitute 2 and 3 above into quotient rule formula to find derivative:**

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{2x \cdot x^2 - (x^2 + 5) \cdot 2x}{(x^2)^2}$$

5. **Simplify your answer if required:**

$$\frac{2x \cdot x^2 - (x^2 + 5) \cdot 2x}{(x^2)^2} = \frac{2x(x^2 - (x^2 + 5))}{x^4} = \frac{2(x^2 - (x^2 + 5))}{x^3}$$
$$= \frac{2(x^2 - x^2 - 5)}{x^3} = \frac{2(-5)}{x^3} = \frac{-10}{x^3}$$

11.7   Chain rule.

Read pages $226 - 231$ of the lecture notes, and work through any examples.

Chain rule

- You must be able to identify when the chain rule is appropriate, then apply it in order to find the derivative of a function.

- Remember that if $y = e^{kx}$, then $y' = \dfrac{dy}{dx} = ke^{kx}$, for k a constant.

Using the chain rule

Many students find the chain rule pretty tricky. Here are the steps involved with applying the formula.

1. Be able to identify when to use the chain rule.

2. Be able to identify u.

3. Be able to find $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$.

4. Substitute the result from Step 3 into the chain rule formula to find derivative.

5. Back substitute for u to tidy up.

Using the chain rule (an example)

Example: find the derivative of $y = (x^2 + 10)^{12}$.

1. **Decide whether you need to use the chain rule.**
   Ask yourself this question:

   - Is the function I am given a composite function, that is, is it a function which contains a function? (Composite functions are covered in Section 6.3; go back and review them now if you need to.)

   If the answer is yes then you should use the chain rule.
   Here, $y = (x^2 + 10)^{12}$ is a composite function. Let $g(x) = x^{12}$ and $h(x) = (x^2 + 10)$ and note that $y = (x^2 + 10)^{12} = g(h(x))$. So you should use the chain rule.

2. **Identify u.**
   u is **always** the **inside part** of the composite function. In this case $u = (x^2 + 10)$. Once you have identified u you should write down what it is and also rewrite the original function substituting in u. In this example we have:

   $$u = x^2 + 10 \qquad y = u^{12}$$

3. **Find $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$.**
   $\dfrac{dy}{du}$ is the rewritten original function derived with respect to u (you just treat the u like an x and use the normal derivative rules), and $\dfrac{du}{dx}$ is just the normal derivative of u. For the example here you should get:

   $$\frac{dy}{du} = \frac{d}{du}(u^{12}) = 12 \times u^{12-1} = 12u^{11}$$

   $$\frac{du}{dx} = \frac{d}{dx}(x^2 + 10) = \frac{d}{dx}(x^2) + \frac{d}{dx}(10) = 2 \times x^{2-1} + 0 = 2x$$

4. **Substitute the above into the chain rule formula:**

   $$y' = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 12u^{11} \times 2x = 2x \cdot 12 \cdot u^{11} = 24x \cdot u^{11}$$

5. **Back substitute for u to tidy up.**
   From before, $u = (x^2 + 10)$ so:

   $$24x \cdot u^{11} = 24x(x^2 + 10)^{11}$$

Evaluating the derivative at a given point

Many students master the rules for differentiation but have difficulty understanding what it is they have actually found and how to use it. Remember that the derivative is just the slope, and for curves the slope is always changing. To evaluate the slope somewhere on the function (at a given point (x_1, y_1)), we find $f'(x_1)$, which is the derivative **evaluated at x_1**. We simply substitute the value of x_1 into the derivative $f'(x)$. The value of $f'(x_1)$ gives us the value of the derivative at the point (x_1, y_1), which is the slope of the function at the point (x_1, y_1).

11.8   Second derivatives

Read pages $232 - 233$ of the lecture notes, and work through any examples.

Second derivatives

- Know that the following things are all the same: *second derivative*, $f''(x)$, y'', $\dfrac{d^2y}{dx^2}$.

- Be able to evaluate the second derivative of a function at a given point.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Qn 10, Page 152
2. Qn 11, Page 152
3. Qn 16(a), (c), Page 153
4. Qn 10, Page 158
5. Qn 17(a),(c), Page 159
6. Qn 10, Page 163
7. Qn 16(a),(c), Page 165
8. Qn 10(a), Page 169
9. Qn 11, Page 169
10. Qn 12, Page 169
11. Qn 20, Page 170
12. Qn 9(a), Page 174
13. Qn 10, Page 174
14. Qn 11, Page 174
15. Qn 19(a),(c), Page 175
16. Qn 13, Page 146
17. Qn 12, Page 140
18. Qn 16, 17, Page 141
19. Qn 10, Page 122
20. Qn 13, Page 128
21. Qn 12, Page 134
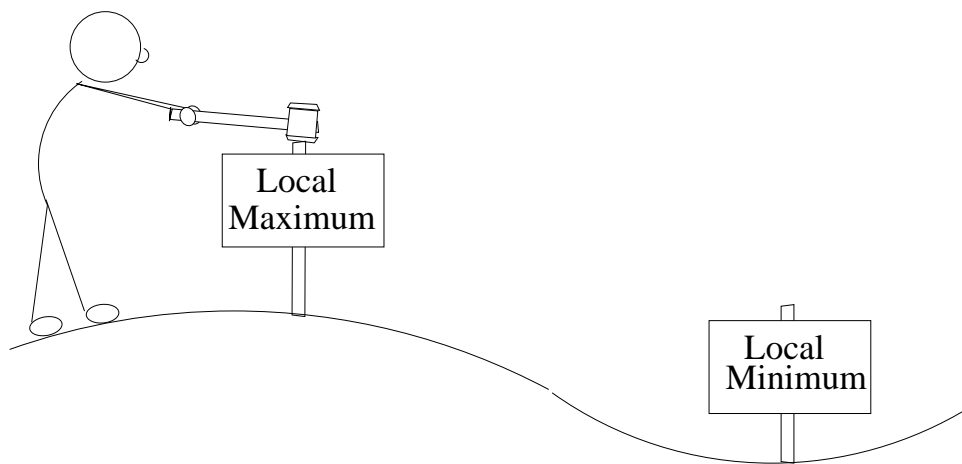
12   Applications of derivatives

What?

In the previous section we learned some basic rules for finding derivatives. Now we're going to see how derivatives can be used to solve a variety of important problems. We'll start by looking at tangent lines to functions, then we'll see how derivatives relate to displacement, velocity and acceleration. We'll see how the second derivative can help with maximising and minimising functions. Finally, we'll look at some interesting practical problems.

Who?

Derivatives are very important in almost any technical field. They enable functions to be maximised or minimised. Businesses might use derivatives in order to maximise profits or minimise costs. Vehicle designers use derivatives to analyse acceleration and deceleration. Materials engineers use derivatives to optimise use of resources.

How?

A farmer has 100 metres of fencing material. What is the largest rectangular area she can enclose?

Why?

Derivatives and their applications are probably the most important section of this course, and are the reason why most of you are made to enroll in this course. Being able to apply these techniques is very important in a wide range of areas of study.

You must have a good grasp of Section 11, (Derivatives and rates of change), before proceeding; revise that material if you need to. We also cover tangent lines, so you may find it beneficial to revise Section 4 (Graphs and straight lines) as well.

At the end of this section you should be able to:

- Find the equation of a line tangential to a curve at a given point.

- Understand that the derivative of displacement is velocity and the derivative of velocity is acceleration.

- Find and classify all critical points for a given function.

- Apply your knowledge of first and second derivatives to solving real world problems.

Work through this section as follows:

12.1 Tangent lines.

Read pages $237 - 239$ of the lecture notes, and work through any examples.

Tangent lines

- Be able to find the equation of line tangential to a curve at a given point. This requires you to differentiate, then apply the theory of straight lines.

12.2 Derivatives and motion.

Read pages $240 - 244$ of the lecture notes, and work through any examples.

Derivatives and motion

- Understand the relationship between displacement, velocity and acceleration.

- Given a function for displacement, be able to find velocity and acceleration at any time t.

12.3 Local maxima and minima.

Read pages $245 - 250$ of the lecture notes, and work through any examples.

Local maxima and minima

- Understand the meaning of the term *critical point*.

- Know that a critical point occurs when the derivative is zero.

- Understand the first derivative test.

- Be able to recall and apply the second derivative test to classify critical points.

12.4   Some practical problems.

Read pages $250 - 255$ of the lecture notes, and work through any examples.

Some practical problems

- Be able to use first and second derivatives to solve practical problems.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Qn 13, Page 140
2. Qn 18, Page 141
3. Qn 14, 17, Page 146
4. Qn 9, Page 152
5. Qn 16(b),(d), Page 153
6. Qn 9(b),(c), Page 157
7. Qn 12, Page 158
8. Qn 17(b),(d), Page 159
9. Qn 9(b),(c), Page 163
10. Qn 12, Page 164
11. Qn 16(b),(d), Page 165
12. Qn 10(b), Page 169
13. Qn 13, Page 169
14. Qn 15, Page 170
15. Qn 9(b), Page 174
16. Qn 12, Page 174
17. Qn 14, Page 175
18. Qn 19(b),(d), Page 175
19. Qn 12,13,14, Page 123
20. Qn 16,17, Page 129
21. Qn 13, Page 134

$$v(t) = 2t + 1$$
$$S(t) = t^2 + t + C$$
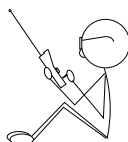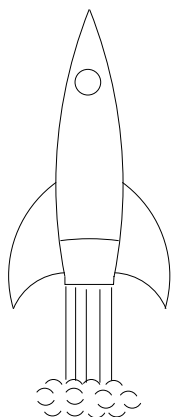
13   Integration

What?
In the last two sections we looked at differentiation. In each case we started with a function, and found its derivative. Now we look at the reverse process: starting with a derivative, can we find the original function, and is this interesting? The answer to all of these questions is yes, and the process is called integration. In this course we'll see how to integrate some simple functions, and we'll see a few ways in which this is very useful.

Who?
Even though integrals can be difficult to find, they have extremely important applications in science, engineering, and finance.

How?
A rocket takes off at time $t = 0$ with velocity $3t^2 + 2t$. At time $t = 3$, the rocket's displacement is 46 metres. Find an expression for the rocket's displacement at any time t.

Why?
Many later-year university courses, particularly in mathematics, engineering and commerce, will involve integration, so it is important that you learn the basics here.

At the end of this section you should be able to:

- Understand the terms: *indefinite integral*, *definite integral*, *antiderivative*, *constant of integration*, *limits of integration*, *initial conditions*, *area under the curve*.

- Understand the notation $\int f(x)dx$ and $\int_a^b f(x)dx$.

- Understand and apply some rules for integration.

- Find indefinite integrals of some simple functions.

- Evaluate definite integrals and find the area under the curve.

- Use integration to solve some simple physical problems.

Work through this section as follows:

13.1 Introduction to integration.

Read pages $259 - 261$ of the lecture notes, and work through any examples.

Introduction to integration

- Understand the meaning of the terms: *integration, integral, indefinite integral, antiderivative, constant of integration.*

- Understand that integrating is the reverse step to differentiating.

- Be familiar with the notation $\int f(x)dx$.

- Be able to integrate the examples in the notes (remember, these are indefinite integrals, so you need to add C, the constant of integration).

Checking your integral

You should always check any integral that you find. All you need to do is find the derivative of the integral and compare it with the function you were supposed to integrate. If they are the same then you have integrated the function successfully. (Don't forget that the derivative of C (the constant of integration) is zero.)

13.2 Rules for integration.

Read pages $262 - 265$ of the lecture notes, and work through any examples.

Rules for integration

- Be able to apply the rules for integration to find indefinite integrals.

- Note that each of these rules matches a related rule for differentiation.

Applying Rule 1

Students often have difficulty applying some of the rules for integration in the notes, in particular Rule 1. This rule says:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

This may look difficult, but it is just the reverse of the rule for differentiating x^n. All you need to do to apply this rule is to identify n and substitute it into the formula to get your answer.

13.3 Initial conditions.

Read pages 265 − 266 of the lecture notes, and work through any examples.

Initial conditions

- Be able to find the exact value of C (the constant of integration) if you are supplied with initial conditions.

13.4 Definite integrals and areas.

Read pages 267 − 272 of the lecture notes, and work through any examples.

Definite integrals and areas

- Understand the meaning of the terms: *area under the graph*, *definite integral*, *limits of integration*.

- Be familiar with the fundamental theorem of calculus.

- Understand how a definite integral differs from an indefinite integral.

- Understand what happens to the constant of integration when you evaluate a definite integral.

- Be able to apply what you know about definite integrals to finding the area under the curve for a specified interval.

13.5 Integrals and motion.

Read pages 273 − 275 of the lecture notes, and work through any examples.

Integrals and motion

- Be able to use integrals to solve physical problems.

- These relationships are simply the reverse of the relationships we saw when studying derivatives. Velocity is the derivative of displacement, so displacement is the integral of velocity. Similarly, acceleration is the derivative of velocity, so velocity is the integral of acceleration.

- Given a function for acceleration and some initial conditions, be able to find velocity and displacement at any time t.

Finally.

To reinforce the material from this section, you should attempt the following questions from previous exam papers. Solutions to each question are included after the exam paper which contained the question.

1. Qn 13, Page 153
2. Qn 14, Page 153
3. Qn 14, Page 159
4. Qn 15, Page 159
5. Qn 14, Page 165
6. Qn 15, Page 165
7. Qn 17, Page 170
8. Qn 18, Page 170
9. Qn 19, Page 170
10. Qn 16, Page 175
11. Qn 17, Page 175
12. Qn 18, Page 175
13. Qns 15, 16, 18, Page 146
14. Qns 14, 15, Page 141
15. Qn 11, Page 122
16. Qns 14, 15, Page 128
15. Qn 12, Page 134

14   Previous exams and exam techniques

This section should help you to prepare for your MATH1040 exams. In Section 14.1 we make some suggestions on how to approach the exams. Then we give some previous mid-semester exam papers and final exam papers for MATH1040. Solutions are also given for all the questions. Of course, the format and style of your exam may vary from these copies, but these will give you a great idea of what to expect. The MATH1040 lecturer should tell you exactly what your paper looks like, and how it compares to these.

In this section we have given you the following exams and solutions. Note that on the real exam papers there is much more space for your writing! Note also that the final exam often includes a formula sheet; your lecturer can show you exactly what your exam will include.

Mid-semester exams and solutions

Section	Description	Page
Section 14.2	Mid-semester exam 2010 (90 marks in total)	73
Section 14.3	Solutions	76
Section 14.4	Mid-semester exam 2009 (90 marks in total)	78
Section 14.5	Solutions	80
Section 14.6	Mid-semester exam 2008 (90 marks in total)	82
Section 14.7	Solutions	85
Section 14.8	Mid-semester exam 2007 (90 marks in total)	87
Section 14.9	Solutions	90
Section 14.10	Mid-semester exam 2006 (90 marks in total)	92
Section 14.11	Solutions	95
Section 14.12	Mid-semester exam 2005 (90 marks in total)	97
Section 14.13	Solutions	100
Section 14.14	Mid-semester exam 2004 (90 marks in total)	103
Section 14.15	Solutions	106
Section 14.16	Mid-semester exam 2003 (90 marks in total)	109
Section 14.17	Solutions	112
Section 14.18	Mid-semester exam 2002 (90 marks in total)	115
Section 14.19	Solutions	118

Final exams and solutions

In the exams and solutions we have given here, we have deleted some questions containing material which is no longer part of the course. That is why the exam papers here are worth strange numbers of marks!

14.1 Possible exam techniques

Let's talk about exam technique. Some of these points may be useful: adopt them if you like. This is all personal opinion (having sat many many uni exams). If you have a method which works for you, use it.

- Don't panic! Look at how long you have to do the paper (90 minutes mid-semester, 120 minutes final exam). This is a long time, so expect things to take awhile.

- You don't **have** to do the whole paper. Try to, but if you can't do some bits, that's not a disaster.

- Use perusal carefully. It's 10 very important, bonus minutes. You might like to

  - write down, on the blank page, some things which you have committed to memory.
  - quickly look through the paper, to see which bits you can or cannot do, and to work out where you are going to start.
  - if you have time, you could start working out some answers, **on the blank sheet**, not on the exam paper. Do questions which you are good at and which are worth lots of marks.

- I think the first 2 of those perusal actions are more important than the 3rd. Use perusal to collect your thoughts, relax, work out where you're going to start, and write down facts you're likely to forget.

- Always **start with the easy bits** (those bits you can do).

- This 'guarantees' you marks, boosts the confidence, and doesn't run the risk of spending too long on a hard question.

- You don't have to start at question 1: if you are a quadratic guru, start there. And (if there are parts) **don't** let Part A panic you. Start with Part B if you like!

- Do the easy bits of each question first. If you get to a hard part of a question, then stop, and come back to it. Don't waste lots of time getting nowhere.

- Come back to the hard bits near the end if you have time.

- Having said this, try to get something done for each question if you possibly can: you may get part marks.

- Show all working! Part marks are given for each correct and relevant thing you do.

- Don't cross out your answer and replace it with nothing. You are better with a rubbish answer than no answer at all (if you have already written it down. That is, if you do something, then deduce it's wrong but don't have time or knowledge to go back and redo it, leave what you have).

- If you know you are wrong, say so, and say how you know. You'll get back up to half of the marks you lost if you give a logical reason for how you know your answer is wrong.

- Don't panic if many are leaving early. They might not be able to do some questions. Don't panic if you want to leave and no-one else is. Maybe you're smarter than they are?

- Check any answers you can, given time.

14.2 Mid-semester exam, 2010

Part A

For each of the following 30 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. $x = 4 - 2 \times 3$

2. x is less than -5.

3. $x = 4^2 \times (9 - 3) \div 8 - 4 \times 3$

4. $x = 9^{\frac{1}{2}}$

5. $x = -\mid -2 + 3^1 \mid - \mid 4 - 2 \mid$

6. $x = \mid 7^0 \mid$

7. $A = \{3, -1\}, B = \{3, 4\}, A \cap B = \{x\}$

8. $A = \{3, -1\}, B = \{3, 4\}, A \backslash B = \{x\}$

9. $y = -\dfrac{x}{3} - 1$ given $y = -2$

10. $y = 2x - 3$, where $y = 7$

11. $xp - 7p = -p$

12. $2(x + 3) = 14$

13. $3x + 1 = -\dfrac{16}{2}$

14. $3x - 1 = 4 + 2x$

15. $x = \dfrac{\sqrt{3}}{2} \times 4\sqrt{3}$

16. $x\sqrt{6} = \sqrt{24}$

17. $x = (\sqrt{4})^2 + 1$

18. $x = \sqrt{3 \times 0 - 2 + 2}$

19. $\displaystyle\sum_{i=1}^{4} i = 2 + 3 + 4 + x$

20. $\displaystyle\sum_{j=1}^{3} jx = -24$

21. $\displaystyle\sum_{i=-2}^{0} -2x = 6$

22. Removed.

23. Removed.

24. $x^2 + 9 = (-3)^2$

25. $5^{\frac{2}{x}} = \dfrac{1}{\sqrt{5}}$

26. $\dfrac{2}{x} + \dfrac{2}{x} = 1$

27. $\dfrac{x + 3}{1 + 3} = 2$

28. $\sqrt{5}\sqrt{x} = 2\sqrt{5}$

29. $4^x = 14 + x$

30. $54^{\frac{1}{2}} = 3\sqrt{x}$

For each of the following eight multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. Two numbers that lie in the interval $[-1, 3)$ are:
   (A) -1 and 3 (B) 0 and 3 (C) 1 and 3 (D) -2 and 2 (E) -0.5 and 2 (F) 0.9 and 3.1.

2. $\dfrac{1}{2} - \dfrac{2}{3}$ equals:
   (A) $\dfrac{7}{6}$ (B) $\dfrac{3}{5}$ (C) $\dfrac{7}{5}$ (D) $-\dfrac{1}{6}$ (E) $\dfrac{1}{6}$ (F) -1.

3. $(x - 3)^2$ expands and simplifies to:
   (A) $x^2 - 9$ (B) $x^2 + 9$ (C) $x^2 - 6x - 9$ (D) $x^2 - 6x + 9$ (E) $2x + 6$ (F) $2x - 6$.

4. $\dfrac{a}{b} + \dfrac{c}{d}$ equals:
   (A) $\dfrac{ac}{bd}$ (B) $\dfrac{a + c}{b + d}$ (C) $\dfrac{a + c}{bd}$ (D) $\dfrac{ad + bc}{bd}$ (E) $\dfrac{adbc}{bd}$ (F) $\dfrac{bd}{ac}$.

5. $4a^3b + 4a^3b$ simplifies to:

    (A) $8a^3b$ (B) $8a^6b^2$ (C) $4a^6b^2$ (D) $4a^6b$ (E) $4a^9b$ (F) $4a^9b^2$.

6. $4a^3b \times 4a^3b$ simplifies to:

    (A) $8a^9b^2$ (B) $16a^9b^2$ (C) $16a^9b$ (D) $16a^6b$ (E) $16a^6b^2$ (F) $4a^6b^2$.

7. $\dfrac{3x + 12}{x + 4}$ simplifies to (A) 3 (B) 6 (C) 22 (D) $\dfrac{15}{4}$ (E) $\dfrac{15}{5}$ (F) $\dfrac{3}{0}$.

8. $\mid x^2 \mid = 16$ means: (A) $x = 4$ (B) $x = -4$ (C) $x = 4$ or $x = -4$
                                  (D) $x = 16$ or $x = -16$ (E) $x = 8$ (F) $x = 16$.

Part B

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1.

(a) Simplify $3x(5x^2 - x + 2) = 2x(x^2 + 2x - 4)$.   (2 marks)

(b) Solve for x: $\dfrac{4(x - 3)}{3} - 3 = 5$ (3 marks)

2. Solve for x: $\dfrac{x}{2} + \dfrac{2x}{3} = -14$ (2 marks)

3. Solve $|4x - 1| = 11$ (2 marks)

4. **(a)** Find all x for which $2(x - 1) \geq 3x - 2$, writing your answer in inequality
    form. (An example of inequality form is $x > \ldots$).   (3 marks)

    (b) Write your answer to part (a) in interval format.   (1 mark)

    (c) Mark your answer to part (a) on the real line.   (1 mark)

5. **(a)** Write in summation (sigma) notation:

    $4k + 7k + 10k + ...$ (2 marks)

    (b) Write in summation (sigma) notation:

    $x^2 + 2x - 1 + 2x^2 + 2x + 1 + 3x^2 + 2x + 3$ (4 marks)

6. Simplify $x^{-2}y^3 \div y^2 \times x^{-2} + (x^2y)^3 \times (xy^2)^{-2} \div \dfrac{2}{x}$ (4 marks)

7. Write an algebraic equation for each of the following and solve the problems:

    (a) Sausages are sold in packets containing s sausages. When three packets of sausages are opened and two sausages are removed there are 25 sausages. Find the number of sausages per packet. (2 marks)

    (b) You save x number of dollars per month for 8 months and receive \$25 for your birthday. You now have \$185. How much do you save each month?   (2 marks)

    (c) Julian rode to the train station which was $\frac{1}{3}$ of the way to UQ. He then traveled $\frac{3}{4}$ of his remaining journey by train, and the final 2km of the trip by bus. How far does Julian live from UQ?   (3 marks)

8. Removed.

9. Solve for x: $\sum\limits_{i=-1}^{2} (2i + 1) = \left(\dfrac{6}{x} - \dfrac{1}{x} \right) \times \dfrac{x^2}{\sqrt{2}} \div (10x) \times \sqrt{8}\, x^3$ (5 marks)

10. A wine barrel has a capacity of 800 litres. It was only $\frac{5}{8}$ full when some customers were each given 20 one litre bottles out of the barrel. Twenty two more litres of wine were then added to the barrel and the entire contents bottled. If there was only enough to give the same customers another 9 bottles each, how many customers were there? Write an algebraic equation and solve the problem. (5 marks)

14.3 Solutions to mid-semester exam, 2010

<div align="center">

Part A, Mid-semester exam, 2010

</div>

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1	-2	**2**	-6	**3**	0	**4**	3	**5**	-3	**6**	1	**7**	$-$	**8**	$-$	**9**	3
10	5	**11**	6	**12**	4	**13**	-3	**14**	5	**15**	6	**16**	2	**17**	5	**18**	0
19	1	**20**	-4	**21**	-1	**22**	$-$	**23**	$-$	**24**	0	**25**	-4	**26**	4		
27	5	**28**	4	**29**	2	**30**	6										

Working for some harder questions:

1. Here we must perform multiplication first so the question becomes $x = 4 - 6 = -2$.

4. To the power of a half is the same as the square root, so the answer is 3.

6. Anything (except zero) to the power of 0 is 1.

15. $\sqrt{3} \times \sqrt{3} = 3$, so $x = 3 \times 4 \div 2$, so $x = 6$.

20. Expanding the sum gives $1x + +2x + 3x = -24$, so $6x = -24$, so $x = -4$.

21. Expanding the sum gives $-2x - 2x - 2x = 6$, so $-6x = 6$, so $x = -1$.

25. $\sqrt{5}$ is the same as $5^{1/2}$, but as it is $\frac{1}{\sqrt{5}}$ it has to be $5^{-1/2}$. This means $x = -4$.

26. $\frac{2}{x} + \frac{2}{x} = \frac{4}{x}$, so $x = 4$.

27. $\frac{x+3}{4} = 2$ so $x = 5$.

29. Trial and error is a good way to attack this question. Four to a negative power is going to give us a fraction and four to the power of 0 is 1 which means $x = -13$ (not allowed). Four squared is 16, so $x = 2$.

30. 54 is not a square number, but it does have square factors. $54 = 9 \times 6$, so $\sqrt{54} = \sqrt{9 \times 6}$, so $x = 6$.

Multichoice: **1** (E) **2** (D) **3** (D) **4** (D) **5** (A) **6** (E) **7** (A) **8** (C)

<div align="center">

Part B, Mid-semester exam, 2010

</div>

1a. $3x(5x^2 - x + 2) = 2x(x^2 + 2x - 4) \Rightarrow 15x^3 - 3x^2 + 6x = 2x^3 + 4x^2 - 8x \Rightarrow 13x^3 - 7x^2 + 14x = 0$.
(Or $-13x^3 + 7x^2 - 14x = 0$.)

1b. $\dfrac{4(x-3)}{3} - 3 = 5$, $\dfrac{4x-12}{3} = 8$, $4x - 12 = 24$, $4x = 36$, $x = 9$.

2. $\dfrac{x}{2} + \dfrac{2x}{3} = -14$, $\dfrac{3x}{6} + \dfrac{4x}{6} = -14$, $\dfrac{7x}{6} = -14$, $7x = -84$, $x = -12$.

3. $|4x - 1| = 11$ so $4x - 1 = 11$ or $4x - 1 = -11$. Hence $4x = 12$ or $4x = -10$, so $x = 3$ or $x = -\dfrac{5}{2}$.

4. **(a)** $2(x - 1) \geq 3x - 2$, so $2x - 2 \geq 3x - 2$, so $0 \geq x$, or $x \leq 0$. **(b)** $(-\infty, 0]$.

(c) There is a shaded circle on the point $x = 0$, and the line has an arrow pointing towards the left.

5. (a) $\displaystyle\sum_{i=1}^{\infty}(3i+1)k$ (Any correct expression is OK.)   **(b)** $\displaystyle\sum_{i=1}^{3}(ix^2+2x+2i-3)$ (Any correct expression is OK.)

6. $x^{-2}y^3 \div y^2 \times x^{-2} + (x^2y)^3 \times (xy^2)^{-2} \div \dfrac{2}{x}$

$= \dfrac{x^{-2}y^3x^{-2}}{y^2} + x^6y^3 \times (x^{-2}y^{-4} \div \dfrac{2}{x}$

$= x^{-4}y + x^4y^{-1} \times \dfrac{x}{2}$

$= x^{-4}y + \dfrac{x^5y^{-1}}{2}$

(Or $= \dfrac{y}{x^4} + \dfrac{x^5}{2y}$)

7a. Let $s =$ sausages.
$3s - 2 = 25 \Rightarrow 3s = 27 \Rightarrow s = 9.$

7b. $x \times 8 + 25 = 185 \Rightarrow 8x = 160 \Rightarrow x = 20.$

7c. Let $x =$ the distance from home to UQ.
$\dfrac{1}{3}x + \dfrac{3}{4} \times \dfrac{2}{3}x + 2 = x$

$\dfrac{1}{3}x + \dfrac{1}{2}x + 2 = x$

$\dfrac{5}{6}x + 2 = x$

$2 = \dfrac{1}{6}x$

$x = 12\text{km}$

8. Removed.

9. LHS= $(2 \times -1 + 1) + (2 \times 0 + 1) + (2 \times 1 + 1) + (2 \times 2 + 1) = -1 + 1 + 3 + 5 = 8.$

RHS= $\dfrac{5}{x} \times \dfrac{x^2}{\sqrt{2}} \times \dfrac{1}{10x} \times 2\sqrt{2}x^3 = x^3.$ So $x^3 = 8 \Rightarrow x = 2.$

10. Let $x =$ number of customers.
$\dfrac{5}{8} \times 800 - 20 \times x + 22 = 9 \times x$

$\Rightarrow 500 - 20x + 22 = 9x$

$\Rightarrow 522 = 9x$

$\Rightarrow x = 18.$

14.4 Mid-semester exam, 2009

Part A

For each of the following 30 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. x is the first prime number.

2. $x = 3 - -3$.

3. $x = 1 + 2 \times -2$

4. $x = (-1)^2$

5. $x = 3 + 2 \div (4 \div 2) - 8 \times 2 \div 4$

6. $x = |\, 3 - 8 \,|$

7. $x = \sqrt{f} - f + \dfrac{f^2}{8}$, where $f = 4$

8. $x = |\, 4^0 \,|$

9. $y = 2x + 3$, where $y = -9$

10. $x - 2 = 2x - 7$

11. $xp - 4p = -p$

12. $2(x + 3) = 4$

13. $3x + 1 = -\dfrac{16}{2}$

14. $3x - 3 = -3 - 5x$

15. $x = \dfrac{\sqrt{3}}{2} \times 4\sqrt{3}$

16. The square root of x is 2

17. $x = (\sqrt{2})^2$

18. $x = |\, 2 - 6 \,| - |\, -2 - 6 \,| + |\, -1 \,|^3$

19. $x = \displaystyle\sum_{i=0}^{2} i$

20. $\displaystyle\sum_{j=-1}^{3} jx = 20$

21. $\displaystyle\sum_{i=-2}^{1} 3x = 12$

22. Removed.

23. $\sqrt{12} = x\sqrt{3}$

24. $16^{1/x} = 4$

25. $5^x = \dfrac{1}{25}$

26. $2^{-4} = 4 \times 2^x$

27. $-5 \div -5^x = -\dfrac{1}{5}$

28. $3^{x+4} = 6^{x+4}$

29. $3^x = x + 7$

30. $x^2 - 2 = -3 + 6^0$

For each of the following 7 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. The intervals $\ [-1, 3)$ and $(0, 7)\ $ overlap on the interval:

   (A) $[-1, 7)$ (B) $(0, 3)$ (C) $[0, 3)$ (D) $(-1, 3)$ (E) $[0, 3]$ (F) $[1, 3)$

2. Which one of the following statements is **true**?

   (A) $-4 \in \mathbb{R}$ but $-4 \notin \mathbb{Z}$. (B) $4 \in \mathbb{N}$ but $4 \notin \mathbb{Z}$. (C) $-4 \in \mathbb{Z}$ but $-4 \notin \mathbb{R}$.
   (D) $-4 \in \mathbb{Z}$ and $-4 \in \mathbb{N}$. (E) $4 \in \mathbb{Z}$ but $4 \notin \mathbb{N}$. (F) $4 \in \mathbb{Q}$ and $4 \in \mathbb{N}$.

3. $(x - 2)^2$ expands and simplifies to:

   (A) $x^2 - 4$ (B) $x^2 + 4$ (C) $x^2 - 4x - 4$ (D) $x^2 - 4x + 4$ (E) $2x + 4$ (F) $2x - 4$

4. $\dfrac{2}{x} + \dfrac{2}{x}$ equals:

   (A) $\dfrac{4}{x}$ (B) $\dfrac{4}{x^2}$ (C) $\dfrac{\sqrt{4}}{x}$ (D) $\dfrac{4}{2x}$ (E) $\dfrac{22}{2x}$ (F) $\dfrac{2}{x^2}$

5. $3a^2b + 3a^2b$ simplies to

   (A) $6a^2b$ (B) $6a^4b^2$ (C) $3a^4b^2$ (D) $9a^2b$ (E) $9a^4b^2$ (F) $6a^2b^2$

6. Which one of the following statements is **true**?

   (A) $\sqrt{a}(\sqrt{b} - \sqrt{c}) = \sqrt{ab} - ac$ (B) $\sqrt{x} + \sqrt{x} + \sqrt{x} = 3\sqrt{x}$ (C) $\sqrt{c} \times \sqrt{c} \times \sqrt{a} = \sqrt{ac}$
   (D) $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ (E) $(2x\sqrt{2})^2 = 8x$
   (F) None of statements (A), (B), (C), (D) or (E) is true.

7. Removed.

Part B

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{2}{3} - \dfrac{1}{3} \times \dfrac{1}{2} \div \left(\dfrac{1}{3}\right)^2$, leaving the answer as a fraction. Show all working.   (3 marks)

2. Solve for x: $\dfrac{6(x-2)}{2x} - 1 = 1$ (3 marks)

3. Solve $|-2x - 4| = 6$.   (2 marks)

4. **(a)** Find all x for which $3(x + 2) \le 4x - 3$, writing your answer in inequality
   form. (An example of inequality form is $x > \dots$).   (3 marks)

   (b) Write your answer to part (a) in interval format.   (1 mark)

   (c) Mark your answer to part (a) on the real line.   (1 mark)

5. **(a)** Write in summation (sigma) notation:

   $$4j + 8j + 12j + 16j$$ (2 marks)

   (b) Write in summation (sigma) notation:

   $$x^2 + 4x^3 + 9x^4 + 16x^5 + \dots$$ (3 marks)

6. Simplify $\sqrt{8} + \sqrt{2} + \sqrt{3}\,(\sqrt{6} + \sqrt{4} - \sqrt{12})$ (4 marks)

7. Simplify $(p^2 - q)(p - q^2) + p \times q - \dfrac{p^{-2}}{p} + q^{\frac{1}{2}} \times q^{\frac{5}{2}}$ (4 marks)

8. Removed.

9. Removed.

10. Simplify $\left(\dfrac{2}{x} + \dfrac{3}{x}\right) \times x^2 \div (10x) \times \sqrt{8}\,x^2$ (4 marks)

11. Solve for x: (5 marks)

$$\frac{\displaystyle\sum_{i=0}^{2} i^2 x}{\displaystyle\sum_{i=-1}^{1} (ix + 1)} = \sum_{i=-1}^{3} i(1 + x)$$

14.5 Solutions to mid-semester exam, 2009

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Part A, Mid-semester exam, 2009

</div>

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1	2	**2**	6	**3**	-3	**4**	1	**5**	0	**6**	5	**7**	0	**8**	1	**9**	-6
10	5	**11**	3	**12**	-1	**13**	-3	**14**	0	**15**	6	**16**	4	**17**	2	**18**	-3
19	3	**20**	4	**21**	1	**22**	$-$	**23**	2	**24**	2	**25**	-2	**26**	-6		
27	2	**28**	-4	**29**	2	**30**	0										

Working for some harder questions:

1. The first prime number is 2. It is the only even prime number as all other even numbers have 2 as a factor.

4. Anything squared is positive (or zero). No matter how many times you multiply 1 by itself, the answer is still 1.

8. Anything (except zero) to the power of 0 is 1.

15. $\sqrt{3} \times \sqrt{3} = 3$, so $x = 3 \times 4 \div 2$, so $x = 6$.

20. Expanding the sum gives $-x + 0x + 1x + +2x + 3x = 20$, so $5x = 20$, so $x = 4$.

21. Expanding the sum gives $3x + 3x + 3x + 3x = 12$, so $12x = 12$, so $x = 1$.

24. 4 is the square root of 16, and a number to the power of $\frac{1}{2}$ is the square root, so $x = 2$.

25. $5 = \sqrt{25}$, but the 25 is in the denominator, so $x = -2$.

26. $4 = 2^2$, so $2^{-4} = 2^2 + 2^x$, so $x = -6$.

28. It is very difficult to get two numbers raised to the same power equalling each other, unless that power is 0. $x + 4 = 0$, so $x = -4$.

Multichoice: 1 (B) **2** (F) **3** (D) **4** (A) **5** (A) **6** (B) **7** Removed.

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Part B, Mid-semester exam, 2009

</div>

1. $\dfrac{2}{3} - \dfrac{1}{3} \times \dfrac{1}{2} \div \left(\dfrac{1}{3}\right)^2 = \dfrac{2}{3} - \dfrac{1}{6} \div \dfrac{1}{9} = \dfrac{2}{3} - \dfrac{1}{6} \times \dfrac{9}{1} = \dfrac{2}{3} - \dfrac{9}{6} = \dfrac{4}{6} - \dfrac{9}{6} = -\dfrac{5}{6}$

2. $\dfrac{6(x-2)}{2x} - 1 = 1$, $\dfrac{6(x-2)}{2x} = 2$, $6(x-2) = 4x$, $6x - 12 = 4x$, $2x = 12$, $x = 6$

3. $|-2x - 4| = 6$ so $-2x - 4 = 6$ or $-2x - 4 = -6$. Hence $-2x = 10$ or $-2x = -2$, so $x = -5$ or $x = 1$.

4. **(a)** $3(x+2) \leq 4x - 3$, so $3x + 6 \leq 4x - 3$, so $-x \leq -9$, so $x \geq 9$. **(b)** $[9, \infty)$.

   (c) There is a shaded circle on the point $x = 9$, and the line has an arrow pointing towards the right.

5. **(a)** $\displaystyle\sum_{i=1}^{4} 4ij$ (Any correct expression is OK.)   **(b)** $\displaystyle\sum_{i=1}^{\infty} i^2 x^{i+1}$ (Any correct expression is OK.)

6. $\sqrt{8} + \sqrt{2} + \sqrt{3}\left(\sqrt{6} + \sqrt{4} - \sqrt{12}\right) = 2\sqrt{2} + \sqrt{2} + \sqrt{18} + \sqrt{12} - \sqrt{36}$
   $= 3\sqrt{2} + 3\sqrt{2} + 2\sqrt{3} - 6 = 6\sqrt{2} + 2\sqrt{3} - 6$

7. $(p^2 - q)(p - q^2) + p \times q - \dfrac{p^{-2}}{p} + q^{\frac{1}{2}} \times q^{\frac{5}{2}}$

$= p^3 - p^2 q^2 - qp + q^3 + pq - p^{-3} + q^3 = p^3 - p^2 q^2 + 2q^3 - p^{-3}$

8. Removed.

9. Removed.

10. $\left(\dfrac{2}{x} + \dfrac{3}{x}\right) \times x^2 \div (10x) \times \sqrt{8}\, x^2$

$= \dfrac{5}{x} \times x^2 \div (10x) \times 2\sqrt{2}\, x^2$

$= \dfrac{5x}{10x} \times 2\sqrt{2}x^2$

$= \dfrac{1}{2} \times 2\sqrt{2}x^2$

$= \sqrt{2}x^2$

11. $\dfrac{\displaystyle\sum_{i=0}^{2} i^2 x}{\displaystyle\sum_{i=-1}^{1}(ix+1)} = \sum_{i=-1}^{3} i(1+x)$

$\dfrac{0x + 1x + 4x}{-x + 1 + 0x + 1 + x + 1} = -1(1+x) + 0(1+x) + 1(1+x) + 2(1+x) + 3(1+x)$

$\dfrac{5x}{3} = 5 + 5x$

$5x = 15 + 15x$

$-10x = 15$

$x = -\dfrac{3}{2}$

14.6 Mid-semester exam, 2008

Part A

For each of the following 27 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. x is the second prime number.

2. $x = 5 - 4 \div 2 + 1^3 - 2 \times 4$.

3. $x = |-2 - 4| - |2 + 4|$

4. $x = \dfrac{f^2}{2} - \sqrt{f}$, where $f = 4$

5. $x = -|-2^0|$

6. $y = 3x - 4$, where $y = 8$

7. $x + 2 = 3x + 12$

8. $(2\sqrt{3})^2 \div x = -2$

9. $\displaystyle\sum_{i=-2}^{1} i = x$

10. $\displaystyle\sum_{i=1}^{3} (i^2 x - i) = 8$

11. $\displaystyle\sum_{i=0}^{2} 2x = 6$

12. $\dfrac{4^x}{4^{-1}} = 4^3$

13. The highest power of any variable in a linear equation is x

14. Removed.

15. $x = 3\sqrt{2} \times \sqrt{2}$

16. $\sqrt{-x} = \sqrt{2}$

17. $-2e - xe = -5e$

18. x is the gradient of a line parallel to $y = 2(3 - 2x)$

19. x is the gradient of a line perpendicular to $y = \dfrac{1}{5}x + 7$

20. $-3^x = -6^x$

21. $-2^{-x} = -\dfrac{1}{8}$

22. $5 = 25^{1/x}$

23. $(-1)^x + x = -1$

24. $x = -c^4 - c$, where $c = -1$

25. $x = \left(2\sqrt{2}\right)^{2/3}$

26. $x \le 0$ and the straight line distance between the points $(0, 2x)$ and $(0, 4x)$ is 6.

27. The point $(-2, -2)$ is a distance of $x\sqrt{2}$ from the origin.

For each of the following 9 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. The intervals $(-5, 1]$ and $[-1, 5)$ overlap on the interval:
   (A) $[-1, 1]$ (B) $[-1, 1)$ (C) $(-5, 0]$ (D) $(-5, 5)$ (E) $[1, 5)$ (F) None of the above.

2. The sum $\dfrac{3}{4} + \dfrac{2}{5}$ equals:
   (A) $\dfrac{5}{9}$ (B) $\dfrac{5}{20}$ (C) $\dfrac{6}{20}$ (D) $\dfrac{23}{20}$ (E) $\dfrac{1}{4}$ (F) $\dfrac{3}{5}$

3. Removed.

4. $(x + 3)^2$ expands and simplifies to:
   (A) $x^2 + 3x + 9$ (B) $x^2 + 9$ (C) $x^2 + 3x + 6$ (D) $x^2 + 3$ (E) $x^2 + 6x + 9$ (F) None of the above.

5. What is the the value of the gradient (m) and y-intercept (c) of the line $3y - 2x - 1 = 4x - 4$.

    (A) $m = -2$ and $c = 1$ (B) $m = -2$ and $c = -1$ (C) $m = 2$ and $c = 1$ (D) $m = 2$ and $c = -1$

    (E) $m = -2$ and $c = 4$ (F) None of the above.

6. The equation of the line passing through the points $(1, 0)$ and $(2, \pi^2)$ is:

    (A) $x = \pi^2$ (B) $y = 2x$ (C) $y = \pi$ (D) $y = \pi^2$ (E) $y = \pi^2 x - \pi^2$ (F) $y = \pi^2 x$

7. The standard form of equation for a vertical line is:

    (A) $y = mx + c$ (B) $y = c$ (C) $x = c$ (D) $x = my + c$ (E) $y = x$ (F) None of the above.

8. Removed.

9. Let $a, b \neq 0$. Which one of the following statements must be **false**?

    (A) $\dfrac{2}{a} \times \dfrac{1}{2b} = \dfrac{1}{a \times b}$ (B) $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b - a}{ab}$ (C) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{b}{a}$ (D) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = a^{-1}b$

    (E) $\dfrac{2}{a} + \dfrac{1}{b} = \dfrac{2a + b}{ab}$ (F) $\left(\dfrac{1}{a}\right) \div b = \dfrac{1}{ab}$

Part B

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{2}{3} - \dfrac{1}{4} \times \left(\dfrac{1}{2}\right)^3 \div \left(\dfrac{1}{4}\right)^2$, leaving the answer as a fraction. Show all working.   (3 marks)

2. Solve $|-2x + 6| = 8$.   (2 marks)

3. **(a)** Find all x for which $-5x + 4 \geq 2(-x - 1)$, writing your answer in inequality
   form. (An example of inequality form is $x > \ldots$).   (3 marks)

    (b) Write your answer to part (a) in interval format.   (1 mark)

    (c) Mark your answer to part (a) on the real line.   (1 mark)

4. **(a)** Write in summation (sigma) notation:

$$2i + 4i + 6i + 8i$$
(3 marks)

   (b) Write in summation (sigma) notation:

$$(x^3 - 1) + (x^4 - 2) + (x^5 - 3)$$
(3 marks)

5. Find x if $\quad 6x^6 + 3x - 6 = (3x^3 + 3)(2x^3 - 2) + 1$
(3 marks)

6. Simplify $\quad \dfrac{\sqrt{8}(\sqrt{2} - 2)(\sqrt{8} + 2)}{\sqrt{2}}$
(4 marks)

7. Simplify $\quad (ab)^2 \times a^{-3}b^{-2} \div (-a^{2/3}b^{1/3})^{-3}$
(3 marks)

8. Solve for x: $\quad \displaystyle\sum_{i=0}^{2}(3ix + 2) = \sum_{i=1}^{3}(ix + i)$
(5 marks)

9. Removed.

10. **(a)** Find the equation of the line passing through $(-4, 6)$, which is perpendicular to the line passing through the points $(4, -1)$ and $(6, -5)$.
(3 marks)

   (b) Find the distance between the points $(4, -1)$ and $(6, -5)$, expressing your answer in simplest form.
(2 marks)

   (c) Find the value of x if the distance between the points $(x, 1)$ and $(2, 5)$ is 5.
(4 marks)

14.7 Solutions to mid-semester exam, 2008

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

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Part A, Mid-semester exam, 2008

</div>

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1	3	**2**	-4	**3**	0	**4**	6	**5**	-1	**6**	4	**7**	-5	**8**	-6	**9**	-2
10	1	**11**	1	**12**	2	**13**	1	**14**	$-$	**15**	6	**16**	-2	**17**	3	**18**	-4
19	-5	**20**	0	**21**	3	**22**	2	**23**	-2	**24**	0	**25**	2	**26**	-3	**27**	2

Working for some harder questions:

1. The first prime number is 2 and the second is 3.

8. $(2\sqrt{3})^2 = 4 \times 3 = 12$. So $12 \div x = -2$, so $x = -6$.

10. Expanding the sum gives $x - 1 + 4x - 2 + 9x - 3 = 8$, so $14x = 14$, so $x = 1$.

11. Expanding the sum gives $2x + 2x + 2x = 6$, so $6x = 6$, so $x = 1$.

12. The power laws give $4^{x-(-1)} = 4^3$, so $4^{x+1} = 4^3$ so $x = 2$.

18. The original equation is $y = -4x + 6$, so any parallel line has the same gradient, so $x = -4$.

19. The products of the gradients of two perpendicular lines equals -1, so the new line must have gradient $x = -5$.

20. By thinking a bit, you'll see we must have $x = 0$, as $-3^0 = 1$ and $-6^0 = 1$.

21. The RHS equals -2^{-3}, so $-x = -3$ so $x = 3$.

22. $5 = \sqrt{25}$, so $5 = 25^{1/2}$, so $x = 2$.

23. This one is really a trial-and-error question.

24. BEDMAS says that when evaluating $-c^4$, you must first raise c to the power 4, then negate the answer. So $-c^4 = -1$, and $x = -1 - (-1) = 0$.

25. $(2\sqrt{2})^{2/3} = ((2\sqrt{2})^2)^{1/3} = 8^{1/3} = 2$, so $x = 2$.

26. The distance between $(0, 2x)$ and $(0, 4x)$ is $4x - 2x = 2x$, noting that all distances must be positive (so take the absolute value if it is negative). So $x = -3$, as we know $x \leq 0$.

Multichoice: 1 (A) **2** (D) **3** Removed. **4** (E) **5** (D) **6** (E) **7** (C) **8** Removed. **9** (E)

1. $\dfrac{2}{3} - \dfrac{1}{4} \times \left(\dfrac{1}{2}\right)^3 \div \left(\dfrac{1}{4}\right)^2 = \dfrac{2}{3} - \dfrac{1}{4} \times \dfrac{1}{8} \div \dfrac{1}{16} = \dfrac{2}{3} - \dfrac{1}{32} \times \dfrac{16}{1} = \dfrac{2}{3} - \dfrac{1}{2} = \dfrac{4}{6} - \dfrac{3}{6} = \dfrac{1}{6}$

2. $|-2x + 6| = 8$ so $-2x + 6 = 8$ or $-2x + 6 = -8$. Hence $-2x = 2$ or $-2x = -14$, so $x = -1$ or $x = 7$.

3. **(a)** $-5x + 4 \geq -2x - 2$, so $-3x \geq -6$, so $x \leq 2$.   **(b)** $(-\infty, 2]$.

   (c) There is a shaded circle on the point $x = 2$, and the line has an arrow pointing towards the left.

4. **(a)** $\displaystyle\sum_{j=1}^{4} 2ji$ (Any correct expression is OK.)   **(b)** $\displaystyle\sum_{i=1}^{3} (x^{i+2} - i)$ (Any correct is OK.)

5. Hence $6x^6 + 3x - 6 = 6x^6 - 6x^3 + 6x^3 - 6 + 1$, so $6x^6 + 3x - 6 = 6x^6 - 5$.

   Hence $3x - 6 = -6 + 1$, so $3x = 1$, so $x = 1/3$.

6. Note $\sqrt{8} = 2\sqrt{2}$, so $\dfrac{2\sqrt{2}\left(\sqrt{16} + 2\sqrt{2} - 2\sqrt{8} - 4\right)}{\sqrt{2}} = 2(4 + 2\sqrt{2} - 4\sqrt{2} - 4) = -4\sqrt{2}$.

7. $a^2b^2 \times a^{-3}b^{-2} \div (-a^{-2}b^{-1}) = a^{-1} \times -a^2b = -ab$.

8. LHS$= (3 \times 0x + 2) + (3 \times 1x + 2) + (3 \times 2x + 2) = 2 + 3x + 2 + 6x + 2 = 9x + 6$.

   RHS$= (1x + 1) + (2x + 2) + (3x + 3) = 6x + 6$.

   So $LHS = RHS$ means $9x + 6 = 6x + 6$, so $3x = 0$, so $x = 0$.

9. Removed.

10. **(a)** The gradient of the line through $(4, -1)$ and $(6, -5)$ is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - (-1)}{6 - 4} = \frac{-4}{2} = -2.$$

So the gradient of the perpendicular line is $1/2$ (as the product of the gradients equals -1.)
So the equation is $y = \frac{1}{2}x + c$, and $(-4, 6)$ is on the line, so $6 = -2 + c$, so $c = 8$, and the equation is

$$y = \frac{1}{2}x + 8.$$

(b) $d = \sqrt{(-5 - (-1))^2 + (6 - 4)^2} = \sqrt{(-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$.

(c) Squaring each side of the distance formula gives:

$$5^2 = (2 - x)^2 + (5 - 1)^2 \quad \text{so} \quad 25 = (2 - x)^2 + 16 \quad \text{so} \quad 9 = (2 - x)^2.$$

Hence $(2 - x) = 3$ or $(2 - x) = -3$, so $x = -1$ or $x = 5$.

14.8 Mid-semester exam, 2007

Part A

For each of the following 26 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. Removed (not relevant).

2. Removed (not relevant).

3. Find x if $x + y = -3$ and $y = -2$

4. $\dfrac{1}{2} - \dfrac{3}{4} = \dfrac{x}{8}$

5. $x = -\,|\,5^1\,|$

6. $y = 3x - 2$, where $y = -14$

7. $2x - 3 = 9 - x$

8. $x^3 + 3x = 0$

9. $x = \sqrt{|\,t\,|} - \dfrac{t}{2}$, where $t = -4$

10. $\dfrac{1}{x - 1} = -\dfrac{1}{3}$

11. $x = -c^4$, where $c = -1$

12. $5\pi + x\pi = 4\pi$

13. $3^5 = 3^{n+x} \times 3^{1-n}$

14. $\displaystyle\sum_{i=1}^{2} -2i = 2x$

15. $\displaystyle\sum_{i=0}^{1} (i^2 - ix) = 7$

16. $\displaystyle\sum_{i=0}^{1} x = -6$

17. $-5^{-x} = -5$

18. Removed (not relevant).

19. $(4\sqrt{3})^2 \div x = 8$

20. $(\sqrt{-x + 1})^2 = 5$

21. $(2t - 2) = x(1 - t) \qquad (t \neq 1)$

22. $|\,x\,| - 1 = x + 1$

23. Removed (not relevant).

24. $36^{1/x} = \dfrac{1}{6}$

25. $x + (-1)^x = 2$

26. Removed (not relevant).

For each of the following 9 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. The intervals $[0, 5]$ and $(-3, 2)$ overlap on the interval:

    (A) $(-3, 5]$ (B) $[-3, 2]$ (C) $[0, 2]$ (D) $[-3, 5]$ (E) $[1, 2)$ (F) $[0, 2)$

2. The inequality $x < -2$ **or** $x \geq 4$ can be written in interval form as:

    (A) $[4, \infty)$ (B) $(-\infty, -2)[4, \infty)$ (C) $[-2, 4]$ (D) $(-\infty, 4]$ (E) $(-2, -\infty)[4, \infty)$ (F) $(-\infty, -2)(4, \infty)$

3. Removed (not relevant).

4. Which one of the following statements is **true**?

    (A) $7 \in \mathbb{Z}$ but $7 \notin \mathbb{N}$.

    (B) $-7 \in \mathbb{Z}$ but $-7 \notin \mathbb{R}$.

    (C) $-7 \in \mathbb{R}$ but $-7 \notin \mathbb{Z}$.

    (D) $7 \in \mathbb{N}$ but $7 \notin \mathbb{Z}$.

    (E) $-7 \in \mathbb{Z}$ and $-7 \in \mathbb{N}$.

    (F) None of statements (A), (B), (C), (D) or (E) is true.

5. Removed (not relevant).

6. Which one of the following statements is **true**?

  (A) $\dfrac{2}{x} \times \dfrac{2}{y} = \dfrac{2}{x \times y}$

  (B) $\dfrac{2}{x} - \dfrac{2}{y} = \dfrac{2}{x - y}$

  (C) $\dfrac{2}{x} - \dfrac{2}{y} = \dfrac{0}{x - y}$

  (D) $\left(\dfrac{2}{x}\right) \div \left(\dfrac{1}{2y}\right) = \dfrac{y}{x}$

  (E) $\left(\dfrac{2}{x}\right) \div \left(\dfrac{2}{y}\right) = \dfrac{y}{x}$

  (F) $\dfrac{2}{x} + \dfrac{2}{y} = \dfrac{4}{x + y}$

7. Removed (not relevant).

8. The equation of the line passing through the points $(\pi^2, 0)$ and (π^2, π^2) is:

  (A) $x^2 = \pi$ (B) $x = \pi^2$ (C) $x = \pi^2 y$

  (D) $y^2 = \pi$ (E) $y = \pi^2$ (F) $y = \pi^2 x$

9. The equation of the line passing through the points $(1, \pi^2)$ and (π^2, π^4) is:

  (A) $x^2 = \pi$ (B) $x = \pi^2$ (C) $x = \pi^2 y$

  (D) $y^2 = \pi$ (E) $y = \pi^2$ (F) $y = \pi^2 x$

Part B

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{1}{2} + \dfrac{1}{3} \times \dfrac{1}{4} \div \left(\dfrac{1}{2}\right)^3$, leaving the answer as a fraction. Show all working.　(4 marks)

2. Solve　$|-3x - 4| = 1$　(2 marks)

3. **(a)** Find all x for which $-x - 1 < -3x - 3$, writing your answer in inequality form. (An example of inequality form is $x > \dots$).　(3 marks)

  (b) Write your answer to part (a) in interval format.　(1 mark)

  (c) Mark your answer to part (a) on the real line.　(1 mark)

4. Write in summation (sigma) notation:

  $$-3p^3 - 2p^3 - p^3 + p^3 + 2p^3$$　(2 marks)

5. Write in summation (sigma) notation:

  $$(2 - x) + (3 - x^2) + (4 - x^3) + (5 - x^4)$$　(3 marks)

6. **(a)** Let L be the line passing through the points $(-4, \sqrt{2})$ and $(0, -3\sqrt{2})$. Find the equation of L.　(4 marks)

  (b) Let L be the line from Part (a). Find the equation of the line perpendicular to L, passing through the point $(\sqrt{2}, 4)$.　(3 marks)

7. Solve for x: $\dfrac{3}{2x-3} = \dfrac{2}{5x-2}$ (3 marks)

8. Simplify $(3x^{-2} - y)(x^2 \times 2y)$ (3 marks)

9. Simplify $a^{-3}b^{-2} \div (a^{2/3}b^{1/3})^{-3}$ (3 marks)

10. Removed (not relevant).

11. Wally is at the point $(1, 4)$. First he walks in a straight line L parallel to the y-axis for a distance of 3 units. He then walks in a straight line M perpendicular to the y-axis for a distance of 3 units. Finally, he walks in a straight line N back to his starting point.

   (a) Find the equation of the line L. (1 mark)

   (b) Find the equation of the line M. (1 mark)

   (c) Find the equation of the line N. (3 marks)

   (d) What is the **total** distance he has walked? Express your answer as a surd in simplest form.

   (3 marks)

12. Solve for x: $\left(\displaystyle\sum_{i=0}^{1}(2ix - 3i) \right) \times \left(\displaystyle\sum_{i=1}^{2}(x + i) \right) = 27$ (6 marks)

14.9 Solutions to mid-semester exam, 2007

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

Part A, Mid-semester exam, 2007

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

3	-1	**4**	-2	**5**	-5	**6**	-4	**7**	4	**8**	0	**9**	4	**10**	-2	**11**	-1
12	-1	**13**	4	**14**	-3	**15**	-6	**16**	-3	**17**	-1	**18**	$-$	**19**	6	**20**	-4
21	-2	**22**	-1	**23**	$-$	**24**	-2	**25**	3	**26**	$-$						

Working for some harder questions:

9. Substitute -4 for t in the expression. So $\sqrt{|-4|} = \sqrt{4} = 2$, and $x = 2 - (-4/2) = 2 + 2 = 4$.

10. Multiplying each side by 3 and by $(x-1)$ gives $3 = -(x-1)$ so $3 = -x + 1$, which is easy to solve.

11. BEDMAS says that we raise c to the power 4 **before** we use the negative sign. Thus $-c^4 = -(-1^4) = -1$.

13. $3^5 = 3^{n+x} \times 3^{1-n} = 3^{n+x+1-n} = 3^{x+1}$. Hence $x = 4$.

16. Expanding the σ gives $x + x$. So $2x = -6$ and hence $x = -3$.

17. $-5 = -5^1$, so $-5^{-x} = -5^1$, so $-x = 1$, so $x = -1$.

19. $(4\sqrt{3})^2 = 4 \times 4 \times \sqrt{3} \times \sqrt{3} = 48$. Hence $x = 6$.

20. $(\sqrt{5})^2 = 5$, so $-x + 1 = 5$, so $x = -4$.

Multichoice.

1 (F) **2** (B) **4** (F) **6** (E) **8** (B) **9** (F)

Part B, Midsemester exam, 2007

1. $\dfrac{1}{2} + \dfrac{1}{3} \times \dfrac{1}{4} \div \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{2} + \dfrac{1}{3} \times \dfrac{1}{4} \div \dfrac{1}{8} = \dfrac{1}{2} + \dfrac{1}{12} \times \dfrac{8}{1} = \dfrac{1}{2} + \dfrac{8}{12} = \dfrac{6}{12} + \dfrac{8}{12} = \dfrac{14}{12} = \dfrac{7}{6}$

2. $|-3x - 4| = 1$ so $-3x - 4 = 1$ or $-3x - 4 = -1$. Hence $-3x = 5$ or $-3x = 3$, so $x = -5/3$ or $x = -1$

3. **(a)** $-x - 1 < -3x - 3$, so $2x - 1 < -3$, so $2x < -2$ so $x < -1$. **(b)** $(-\infty, -1)$. **(c)** There is a non-shaded circle on the point $x = -1$, and the line has an arrow pointing towards the left.

4. $\displaystyle\sum_{i=-3}^{2} ip^3$ (Any correct expression is OK.)   5. $\displaystyle\sum_{i=2}^{5}(i - x^{i-1})$ (Any correct is OK.)

6. **(a)** Points are $(-4, \sqrt{2})$ and $(0, -3\sqrt{2})$, so $m = \dfrac{-3\sqrt{2} - (\sqrt{2})}{0 - (-4)} = \dfrac{-4\sqrt{2}}{4} = -\sqrt{2}$.

   Hence $y = mx + c$ so $y = -\sqrt{2}x + c$, and $(0, -3\sqrt{2})$ is on the line, so $-3\sqrt{2} = -\sqrt{2} \times 0 + c$, so $c = -3\sqrt{2}$ and hence the equation of the line is $y = -\sqrt{2}x - 3\sqrt{2}$

   (b) Original gradient is $-\sqrt{2}$, so this gradient is $\dfrac{1}{\sqrt{2}}$. So $y = \dfrac{1}{\sqrt{2}}x + c$. Now $(\sqrt{2}, 4)$ is on line, so $4 = \dfrac{1}{\sqrt{2}} \times \sqrt{2} + c$, so $4 = 1 + c$, so $c = 3$ and equation is $y = \dfrac{1}{\sqrt{2}}x + 3$

7. Hence $3(5x - 2) = 2(2x - 3)$, so $15x - 6 = 4x - 6$, so $15x = 4x$, so $11x = 0$, so $x = 0$

8. There is a trick here; the second set of brackets contains a multiplication sign, not a plus sign. So the question is really $(3x^{-2} - y) \times 2x^2y$. This then simplifies to $6x^{2-2} - 2x^2y^2 = 6x^0y - 2x^2y^2 = 6y - 2x^2y^2$.

9. $a^{-3}b^{-2} \div (a^{2/3}b^{1/3})^{-3} = a^{-3}b^{-2} \div (a^{-2}b^{-1}) = a^{-3}b^{-2} \times a^2b^1 = a^{-3+2}b^{-2+1} = a^{-1}b^{-1}$

10. Removed (not relevant).

11. **(a)** He starts at $(1, 4)$, then moves on a vertical line to the point $(1, 7)$ (or $(1, 1)$ if you assume he moves in the negative direction; either is correct). The equation of this line is $x = 1$.

   (b) He then moves on a horizontal line to the point $(4, 7)$ (or $(-2, 7)$, or $(-2, 1)$ or $(4, 1)$, depending on whether he moves in the positive or negative directions). The equation of his line is $y = 7$ (or $y = 1$).

   (c) For this working we'll assume in Part (b) he was at $(4, 7)$, but any of the other points is correct and has similar working.

   Two points are $(1, 4)$ and $(4, 7)$. Then $m = \dfrac{7 - 4}{4 - 1} = \dfrac{3}{3} = 1$, so equation is $y = x + c$. But $(1, 4)$ is on line, so $4 = 1 + c$, so $c = 3$ and equation is $y = x + 3$.
   (The alternate answer for other points is $y = -x + 5$.)

   (d) He has walked around a right-angled triangle with sides of lengths 3, 3, and unknown hypotenuse. The hypotenuse length is $\sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$, so he has walked $3 + 3 + 3\sqrt{2} = 6 + 3\sqrt{2}$.

12. First expand the sums, so $\displaystyle\sum_{i=0}^{1}(2ix - 3i) = 0 + 2x - 3 = 2x - 3$, and $\displaystyle\sum_{i=1}^{2}(x + i) = x + 1 + x + 2 = 2x + 3$.

   So the question becomes

   $(2x - 3)(2x + 3) = 27$, so $4x^2 - 9 = 27$, so $4x^2 = 36$, so $x^2 = 9$, so $x = 3$ and $x = -3$.

14.10 Mid-semester exam, 2006

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Part A, Mid-semester exam, 2006

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For each of the following 26 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. Removed (not relevant).

2. Removed (not relevant).

3. $x = t - \sqrt{\dfrac{t}{2}}$, where $t = 8$

4. $\dfrac{1}{2} + \dfrac{1}{4} = \dfrac{x}{8}$

5. $x = |-5^0|$

6. $\dfrac{1}{x+2} = \dfrac{2}{8}$

7. $3|x| = |-x|$

8. $x = -a^2 - a$, where $a = -1$

9. $4\pi - x\pi = 6\pi$

10. $y = 4x + 1$, where $y = -11$

11. $-4 + x = 3x$

12. $\sqrt{-10x} = \sqrt{10}$

13. $x + 2 = 6 - x$

14. Removed (not relevant).

15. $(2\sqrt{5})^2 \div x = 5$

16. $(\sqrt{-x})^2 = 6$

17. $x^6 = -x^4$

18. $2^{-x} = 8$

19. $16^{1/x} = 4$

20. $x \times (-1)^x = 3$

21. $3^4 = 3^{n+x} \times 3^{1-n}$

22. $\displaystyle\sum_{i=1}^{4} 2i = x + 14$

23. $\displaystyle\sum_{i=0}^{1} (i^2 - ix) = -5$

24. $\displaystyle\sum_{i=0}^{2} x = -6$

25. $(4t - 4) = x(1 - t)$ $\quad (t \neq 1)$

26. Removed (not relevant).

For each of the following 9 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. The intervals $\quad[-2, 2]$ and $(0, 3)\quad$ overlap on the interval:

     (a) $[-2, 3]$ **(b)** $[-2, 3)$ **(c)** $(0, 2]$ **(d)** $(0, 2)$ **(e)** $[0, 2]$ **(f)** $(-2, 3)$

2. The inequality $\quad x \geq 4$ **or** $x < -2\quad$ can be written in interval form as:

     (a) $[4, \infty)$ **(b)** $[-2, 4]$ **(c)** $(-2, -\infty)[4, \infty)$ **(d)** $(-\infty, -2][4, \infty)$ **(e)** $(-\infty, -2)[4, \infty)$ **(f)** $(-\infty, 4]$

3. Removed (not relevant).

4. Which one of the following statements is false?

     (a) $\dfrac{1}{a} \times \dfrac{1}{b} = \dfrac{1}{a \times b}$
     (b) $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b - a}{ab}$
     (c) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{b}{a}$

     (d) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = (ab)^{-1}$
     (e) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a + b}{ab}$
     (f) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = a^{-1}b$

5. Which one of the following statements is true?

     (a) $3 \in \mathbb{Z}$ but $3 \notin \mathbb{N}$ **(b)** $-3 \in \mathbb{Z}$ but $-3 \notin \mathbb{N}$ **(c)** $-3 \in \mathbb{N}$ but $-3 \notin \mathbb{Z}$
     (d) $3 \in \mathbb{N}$ but $3 \notin \mathbb{Z}$ **(e)** $-3 \in \mathbb{Z}$ and $-3 \in \mathbb{N}$ **(f)** None of statements (a), (b), (c), (d) or (e) is true.

6. The equation of the line passing through the points $(0, \pi^2)$ and (π^2, π^2) is:

     (a) $x = \pi$. **(b)** $x = \pi^2$. **(c)** $y = x \times \pi^2$. **(d)** $y = \pi$. **(e)** $y = \pi^2$. **(f)** $y = \pi^2 x$.

7. Removed (not relevant).

8. Removed (not relevant).

9. Which of the following statements is false?

   (a) $(2\sqrt{2x})^2 = 8x$.
   (b) $\sqrt{x} + \sqrt{x} + \sqrt{x} = \sqrt{9x}$.
   (c) $\sqrt{a} \times \sqrt{b} \times \sqrt{c} = \sqrt{abc}$.
   (d) $\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{a+b+c}$.
   (f) None of statements (a), (b), (c), or (d) is false.

Part B, Midsemester exam, 2006

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{1}{3} \times \dfrac{1}{2} + \dfrac{1}{5} \div \left(\dfrac{1}{2}\right)^2$, leaving the answer as a fraction. Show all working. (3 marks)

2. (a) Find all x for which $-2x + 4 \geq x + 1$, writing your answer in inequality
   form. (An example of inequality form is $x > \dots$). (3 marks)

   (b) Write your answer to part (a) in interval format. (1 mark)

   (c) Mark your answer to part (a) on the real line. (1 mark)

3. Write in summation (sigma) notation:

$$11x + 10x + 9x + 8x + 7x + 6x + 5x + 4x + 3x + 2x + x$$
(2 marks)

4. Write in summation (sigma) notation:

$$(4x + 4) + (6x + 9) + (8x + 16) + (10x + 25) + (12x + 36)$$
(3 marks)

5. Find the equation of the line passing through the points $(1, -2\sqrt{5})$ and $(2, -\sqrt{5})$.

   (4 marks)

6. Solve for x: $\dfrac{1}{4x + 4} = \dfrac{1}{5x + 6}$. (2 marks)

7. Removed (not relevant).

8. Solve $|-x + 2| = 4$. (2 marks)

9. Let L be the line $y = x + 4$, and let M be the line perpendicular to L passing through the point $(-1, 3)$. Find the distance between the y-intercept of L and the x-intercept of M. Express your answer as a surd in simplest form. (6 marks)

10. Simplify $-(xy^2) \div (x^{-1}y^{-2}) \times (x^{1/2}y^2)^{-2}$. (3 marks)

11. Find x if $(5x + 5)(2x + 1) = 10x^2 + x + 5$. (3 marks)

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12. Removed (not relevant).

13. It can be shown that for any $n \geq 0$, $\quad \sum_{i=1}^{n} \frac{1}{2^i} = 1 - 2^{-n}$.

    (a) Show this is true for $n = 2$ and $n = 3$. (3 marks)

    (b) Find $\quad 1 - \sum_{i=1}^{100} \frac{1}{2^i}$. (3 marks)

$$\sum_{i=1}^{n} \frac{1}{2^i} \qquad 95 \qquad .$$

14.11 Solutions to mid-semester exam, 2006

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

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Part A, Mid-semester exam, 2006

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Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

3. 6 **4.** 6 **5.** 1 **6.** 2 **7.** 0 **8.** 0 **9.** -2 **10.** -3 **11.** -2 **12.** -1
13. 2 **14.** $-$ **15.** 4 **16.** -6 **17.** 0 **18.** -3 **19.** 2 **20.** -3 **21.** 3 **22.** 6
23. 6 **24.** -2 **25.** -4 **26.** $-$

Working:

3. Substitute 8 in for t in the expression. So $\sqrt{t/2} = \sqrt{8/2} = \sqrt{4} = 2$, and $x = 8 - 2 = 6$.

7. $\mid -x \mid = \mid x \mid$. Hence we have $3 \mid x \mid = \mid x \mid$, so $x = 0$.

8. BEDMAS says that we do the squaring **before** we make a negative. Then substitute -1 in for a in the expression. So $-a^2 + a = -(-1^2) - (-1) = -1 + 1 = 0$.

16. We cannot take the square root of a negative number, so $-x$ must be positive, so x must be negative. If we take the square root of both sides we get $\sqrt{-x} = \sqrt{6}$. Hence $x = -6$.

20. -1^x either equals 1 (if x is even), or -1 (if x is odd). Hence we have $x \times 1 = 3$ (if x is even), or $x \times -1 = 3$ (if x is odd). Clearly, x must be 3 or -3. In either case, x is odd, so we must have $x \times -1 = 3$, so $x = -3$.

21. $3^4 = 3^{n+x} \times 3^{1-n} = 3^{n+x+1-n} = 3^{x+1}$. Hence $x = 3$.

Multichoice.

1. Answer is **(c)**.

2. Answer is **(e)**.

4. Answer is **(d)**. All other statements are true, but look at this one. LHS is $\dfrac{1}{a} \div \dfrac{1}{b} = \dfrac{1}{a} \times \dfrac{b}{1} = \dfrac{b}{a}$, whereas RHS is $(ab)^{-1} = \dfrac{1}{ab}$.

5. Answer is **(b)**. -3 **is** an integer, but it is **not** a natural number.

6. In each case, the y-coordinate is the same, so the line is horizontal, with equation $y = \pi^2$. Hence answer is **(e)**.

9. Answer is **(d)**. You can check that the others are all true using simple calculations.

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Part B, Midsemester exam, 2006

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1. $\dfrac{1}{3} \times \dfrac{1}{2} + \dfrac{1}{5} \div \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{3} \times \dfrac{1}{2} + \dfrac{1}{5} \div \dfrac{1}{4} = \dfrac{1}{6} + \dfrac{1}{5} \times \dfrac{4}{1} = \dfrac{1}{6} + \dfrac{4}{5} = \dfrac{5}{30} + \dfrac{24}{30} = \dfrac{29}{30}$.

2. **(a)** $-2x + 4 \geq x + 1$, so $-3x \geq -3$, so $x \leq 1$. **(b)** $(-\infty, 1]$. **(c)**
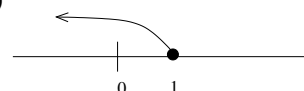
3. $\displaystyle\sum_{i=1}^{11} ix.$ (Any correct expression is ok.)

4. $\displaystyle\sum_{i=2}^{6}(2ix + i^2).$ (Any correct expression is ok.)

5. Let $(x_1, y_1) = (1, -2\sqrt{5})$ and $(x_2, y_2) = (2, -\sqrt{5})$, so $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-\sqrt{5} - (-2\sqrt{5})}{2 - 1} = \dfrac{\sqrt{5}}{1} = \sqrt{5}.$

Hence $y = mx + c$ so $y = \sqrt{5}x + c$, and $(1, -2\sqrt{5})$ is on the line, so $-2\sqrt{5} = \sqrt{5} \times 1 + c$, so $-2\sqrt{5} - \sqrt{5} = c$, so $c = -3\sqrt{5}$ and hence the equation of the line is $y = \sqrt{5}x - 3\sqrt{5}.$

6. $\dfrac{1}{4x + 4} = \dfrac{1}{5x + 6}$ so $4x + 4 = 5x + 6$ so $4x - 5x = 6 - 4$, so $-x = 2$, so $x = -2.$

8. $|-x + 2| = 4$ so $-x + 2 = 4$ or $-x + 2 = -4$. Hence $-x = 2$ or $-x = -6,$ so $x = -2$ or $x = 6.$

9. The line L has gradient 1, so the line M (which is perpendicular to L) has gradient -1. M passes through the point $(-1, 3)$ and so we get $y = -1x + c$, so $3 = -1 \times -1 + c$, so $3 = 1 + c$, so $c = 2$. Hence M is the line $y = -x + 2.$
From the equation for L, its y-intercept is $(0, 4)$. The x-intercept of M is where the y-coordinate is 0, so $0 = -x + 2$, so $x = 2$. Hence the x-intercept is $(2, 0)$.
We want to find the distance between $(x_1, y_1) = (0, 4)$ and $(x_2, y_2) = (2, 0)$. So $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2 - 0)^2 + (0 - 4)^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}.$

10. $-(xy^2) \div (x^{-1}y^{-2}) \times (x^{1/2}y^2)^{-2} = -xy^2 \times (x^1 y^2) \times (x^{-1}y^{-4}) = -x^{(1+1-1)}y^{(2+2-4)} = -x^1 y^0 = -x.$

11. $(5x + 5)(2x + 1) = 10x^2 + x + 5$, so $10x^2 + 5x + 10x + 5 = 10x^2 + x + 5$, so $15x + 5 = x + 5$, so $15x = x$, so $14x = 0$, so $x = 0.$

12. Removed (not relevant).

13. **(a)** When $n = 2$, on the LHS we get $\dfrac{1}{2^1} + \dfrac{1}{2^2} = \dfrac{2}{4} + \dfrac{1}{4} = \dfrac{3}{4}$. On the RHS we get $1 - 2^{-2} = 1 - \dfrac{1}{2^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4}.$ So LHS=RHS, so it is true for $n = 2.$

When $n = 3$, on the LHS we get $\dfrac{1}{2^1} + \dfrac{1}{2^2} + \dfrac{1}{2^3} = \dfrac{4}{8} + \dfrac{2}{8} + \dfrac{1}{8} = \dfrac{7}{8}.$ On the RHS we get $1 - 2^{-3} = 1 - \dfrac{1}{2^3} = 1 - \dfrac{1}{8} = \dfrac{7}{8}.$ So LHS=RHS, so it is true for $n = 3.$

(b) To find $1 - \displaystyle\sum_{i=1}^{100} \dfrac{1}{2^i}$, first let's find $\displaystyle\sum_{i=1}^{100} \dfrac{1}{2^i}.$ Substitute $n = 100$ into the RHS of the equation, giving

$\displaystyle\sum_{i=1}^{100} \dfrac{1}{2^i} = 1 - 2^{-100} = 1 - \dfrac{1}{2^{100}}.$ Hence $1 - \displaystyle\sum_{i=1}^{100} \dfrac{1}{2^i} = 1 - (1 - \dfrac{1}{2^{100}}) = \dfrac{1}{2^{100}}.$

14.12 Mid-semester exam, 2005

For each of the following 32 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. Removed (not relevant).

2. Removed (not relevant).

3. $x = -|6|$

4. $2^{-x} = 4$

5. $x = f(-1)$ where $f(a) = (-a)^2 - a$

6. $x = f(-1)$ where $f(a) = -a^2 - a$

7. $x = f(8)$ where $f(t) = t - \sqrt{\dfrac{t}{2}}$

8. $\dfrac{x}{3} - \dfrac{2}{3} = \dfrac{2x}{3}$

9. $\pi^x \div \pi^{-2} = \dfrac{1}{\pi^4}$

10. $\dfrac{1}{x} + 1 = 0$

11. $\displaystyle\sum_{i=x}^{3} i = 0$

12. Removed (not relevant).

13. $x = \displaystyle\sum_{i=0}^{2} i^2$

14. $\displaystyle\sum_{i=x-1}^{x} i = -5$

15. $8^{2/x} = 4$

16. $\sqrt{24} = -x\sqrt{6}$

17. $\sqrt{8x} = 4\sqrt{3}$

18. $2^x \div 2^{-x} = 2^{-8}$

19. $y = 3x - 7$ where $y = 2$

20. $\dfrac{4^{x+1}}{2^{x+1}} = \dfrac{1}{8}$

21. $f(t) = \sqrt{t} + 4$ has range $[x, \infty)$

22. $f(t) = \sqrt{3t} + x$ has range $[-3, \infty)$

23. $f(y) = \dfrac{1}{|y|}$ has domain $(-\infty, \infty) \setminus \{x\}$

24. $f(t) = \sqrt{t + x}$ has domain $[-3, \infty)$

25. $3^{-x} = 3^{n+1} \times 3^{1-n}$

26. $2|x| = |-x|$

27. $x \times -1^x = 5$

28. $\dfrac{1}{x} = 1^{-x}$

29. $(t - 1) = x(1 - t)$ $\quad (t \neq 1)$

30. x is gradient of a line parallel to $y(t) = 3 - 2t$.

31. x is gradient of a line perpendicular to $y(t) = \dfrac{t}{3} + 2$.

32. x is the y-intercept on the graph of $y(t) = 3t^2 - 4t$.

For each of the following 6 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. Let $f(x) = -2|x|$. What is the **range** of $f(x)$?

   (a) $[0, \infty)$.   **(b)** $[2, \infty)$.   **(c)** $(-\infty, 0) \cup (0, \infty)$.   **(d)** $(-\infty, -2]$.   **(e)** $(-\infty, 0]$.   **(f)** (∞, ∞).

2. Let $f(x) = \dfrac{1}{\sqrt{x - 1}}$. What is the **domain** of $f(x)$?

   (a) $(-\infty, 0) \cup (0, \infty)$.   **(b)** $[1, \infty)$.   **(c)** $(-\infty, \infty)$.   **(d)** $[-1, \infty)$.   **(e)** $[0, \infty)$.   **(f)** $(1, \infty)$.

3. Which one of the following statements is false?

   (a) $\dfrac{2}{a} \times \dfrac{1}{2b} = \dfrac{1}{a \times b}$ **(b)** $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b - a}{ab}$ **(c)** $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{b}{a}$

   (d) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = a^{-1}b$ **(e)** $\dfrac{2}{a} + \dfrac{1}{b} = \dfrac{2a + b}{ab}$ **(f)** $\left(\dfrac{1}{a}\right) \div b = \dfrac{1}{ab}$

4. Which one of the following statements is true?

   (a) $\mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{N}$. **(b)** $\mathbb{Z} \subseteq \mathbb{N} \subseteq \mathbb{R}$. **(c)** $\mathbb{R} \subseteq \mathbb{N} \subseteq \mathbb{Z}$.
   (d) $\mathbb{R} \subseteq \mathbb{Z} \subseteq \mathbb{N}$. **(e)** $\mathbb{N} \subseteq \mathbb{R} \subseteq \mathbb{Z}$. **(f)** None of statements (a), (b), (c), (d) or (e) is true.

5. Removed (not relevant).

6. Removed (not relevant).

The following seven equations have their graphs drawn below, in random order, with eight extra graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to O.

(7 marks)

(1) $y = |-x|$. **(2)** $y = -x^2 + 4$. **(3)** $y = \sqrt{31}$. **(4)** $x + (1.7)^4 = -0.05$.

(5) $y - 2x - 4 = 0$. **(6)** $4(y - 1) = -2(x + 2)$. **(7)** $\dfrac{y}{-3} = \dfrac{x}{2} + \dfrac{5}{11}$.

Graph A   Graph B   Graph C   Graph D   Graph E   Graph F   Graph G   Graph H

Graph I   Graph J   Graph K   Graph L   Graph M   Graph N   Graph O
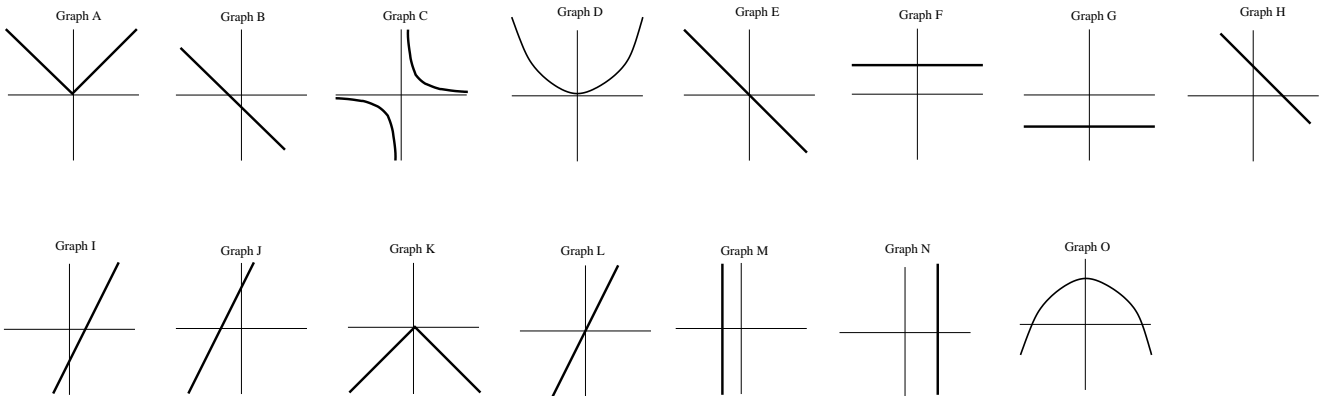
Part B, Midsemester exam, 2005

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{1}{4} + \dfrac{1}{2} \div \left(\dfrac{1}{2}\right)^2 \times \dfrac{1}{3}$, leaving the answer as a fraction. Show all working. (3 marks)

2. Solve for x: $(x - \sqrt{2})(x + \sqrt{2}) = 2(x^2 - 1)$. (2 marks)

3. **(a)** Find all x for which $-3x + 2 \leq -x - 4$, writing your answer in inequality
   form. (An example of inequality form is $x > \dots$). (3 marks)

   (b) Write your answer to part (a) in interval format. (1 mark)

   (c) Mark your answer to part (a) on the real line. (1 mark)

4. Solve $|-2x + 3| = 7$. (2 marks)

5. **(a)** Write in summation (sigma) notation:

   $$(2x - 4)^2 + (3x - 6)^2 + (4x - 8)^2 + (5x - 10)^2 + (6x - 12)^2$$ (2 marks)

(b) Write in summation (sigma) notation:

$$10x + 9x + 8x + 7x + 6x + 5x + 4x + 3x + 2x$$

(2 marks)

6. Removed (not relevant).

7. If $f(x) = x^2 + 1$, evaluate and simplify $\dfrac{f(x+h) - f(x)}{h}$. (4 marks)

8. Find the distance between the points $(\sqrt{3}, \sqrt{2})$ and $(-\sqrt{3}, -\sqrt{2})$, expressing your answer in simplest form. (3 marks)

9. Find the equation of the line passing through the points $(0, \sqrt{5})$ and $(2, -3\sqrt{5})$.

(4 marks)

10. Solve the following system of two simultaneous equations:

$$6x - 3y = 3$$
$$-2x + 5y = -1.$$

(4 marks)

11. Solve the following system of two simultaneous equations:

$$4x + 2y = 6$$
$$-6x - 3y = 9.$$

(3 marks)

12. Removed (not relevant).

14.13   Solutions to mid-semester exam, 2005

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

<div align="center">Part A, Mid-semester exam, 2005</div>

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1. -1	**2.** 4	**3.** -6	**4.** -2	**5.** 2	**6.** 0	**7.** 6	**8.** -2	**9.** -6	**10.** -1	
11. -3	**13.** 5	**14.** -2	**15.** 3	**16.** -2	**17.** 6	**18.** -4	**19.** 3	**20.** -4	**21.** 4	
22. -3	**23.** 0	**24.** 3	**25.** -2	**26.** 0	**27.** -5	**28.** 1	**29.** -1	**30.** -2	**31.** -3	
32. 0										

Working:

4. Note that $4 = 2^2$, so we have $2^{-x} = 2^2$, so $x = -2$.

5. Substitute -1 in for a in the expression. So $(--1)^2 + 1 = 1 + 1 = 2$.

6. BEDMAS says that we do the squaring **before** we make a negative. Then substitute -1 in for a in the expression. So $-a^2 + a = -(-1^2) - (-1) = -1 + 1 = 0$.

7. Substitute 8 in for t in the expression. So $\sqrt{t/2} = \sqrt{8/2} = \sqrt{4} = 2$, and $x = f(8) = 8 - 2 = 6$.

8. The denominators are all 3, so multiplying each side by 3 gives $x - 2 = 2x$, so $x = -2$.

9. We have $\pi^x \div \pi^{-2} = \pi^{-4}$, so $\pi^{x+2} = \pi^{-4}$, so $x + 2 = -4$, so $x = -6$.

10. $1/x = -1$, so multiplying each side by x we get $x = -1$.

14. We are summing the values of i as i goes from from $(x-1)$ to x. Hence $(x-1) + x = -5$, so $2x - 1 = -5$, so $2x = -4$ so $x = -2$.

15. $8^{2/x} = 4$ so $(2^3)^{(2/x)} = 2^2$ so looking at the powers, $3 \times (2/x) = 2$ so $6/x = 2$, so $x = 3$.

16. $\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$. Hence $x = -2$.

17. $4\sqrt{3} = \sqrt{16 \times 3} = \sqrt{48}$. Hence $8x = 48$ so $x = 6$.

18. $2^x \div 2^{-x} = 2^{x+x} = 2^{2x}$. Hence $2x = -8$ so $x = -4$.

19. $2 = 3x - 7$, so $3x = 9$ so $x = 3$.

20. Note that $\dfrac{4^{x+1}}{2^{x+1}} = \left(\dfrac{4}{2}\right)^{x+1} = 2^{x+1}$. Also, $1/8 = 2^{-3}$. Hence $2^{x+1} = 2^{-3}$, so $x + 1 = -3$, so $x = -4$.

21. Note that $\sqrt{t}$ has range $[0, \infty)$, so when we add 4 to this, the range must be $[4, \infty)$, so $x = 4$.

22. Note that $\sqrt{3t}$ has range $[0, \infty)$, so to have range $[-3, \infty)$ we must subtract 3 from this, so $x = -3$.

23. This function exists at every input value except we cannot divide by 0. Hence the domain of f is everything except the point 0, so $x = 0$.

24. This function only exists when we are taking the square root of a positive number or 0. For the domain to be $[-3, \infty)$, we must have the situation that when x is added to each number in the domain, we get a positive number or 0. Hence $x = 3$.

25. $3^{n+1} \times 3^{1-n} = 3^{n+1+1-n} = 3^2$. Hence $x = -2$.

26. $|-x| = |x|$. Hence we have $2 |x| = |x|$, so $x = 0$.

27. -1^x either equals 1 (if x is even), or -1 (if x is odd). Hence we have $x \times 1 = 5$ (if x is even), or $x \times -1 = 5$ (if x is odd). Clearly, x must be 5 or -5. In either case, x is odd, so we must have $x \times -1 = 5$, so $x = -5$.

28. We have $x^{-1} = 1^{-x}$, and trial and error gives $x = 1$.

30. The gradient is the coefficient of t, which is $x = -2$.

31. The gradient of the given line is the coefficient of t, which is $1/3$. Hence, as our line is perpendicular, we need $x \times 1/3 = -1$, so $x = -3$.

32. The y-intercept is always the value of the constant term. Here the constant term is 0, so we have $x = 0$.

Multichoice.

1. Range is all possible values of $f(x)$. Now $|x|$ gives all values from 0 to ∞, including 0, so when we multiply it by -2, we get all values from $-\infty$ to 0. Hence answer is **(e)**.

2. The domain represents every point at which it is possible to evaluate the function. This function does not work when $x - 1 < 0$ (as then we would have $\sqrt{}$ of a negative number), or when $\sqrt{x-1} = 0$ (as then we would be dividing by 0). Hence domain is everything else, which is 1 to ∞, but not including 1. Hence answer is **(f)**.

3. Answer is **(e)**. All other statements are true, but look at this one. LHS is $\dfrac{2}{a} + \dfrac{1}{b} = \dfrac{2b}{ab} + \dfrac{a}{ab} = \dfrac{2b+a}{ab}$, which does not equal the RHS.

4. Answer is **(f)**; this is easy.

Matching graphs with equations. In each case we give the letter of the corresponding graph.

1. A **2.** O **3.** F **4.** M **5.** J **6.** E **7.** B

Part B, Midsemester exam, 2005

1. $\dfrac{1}{4} + \dfrac{1}{2} \div \left(\dfrac{1}{2}\right)^2 \times \dfrac{1}{3} = \dfrac{1}{4} + \dfrac{1}{2} \div \dfrac{1}{4} \times \dfrac{1}{3} = \dfrac{1}{4} + \dfrac{1}{2} \times \dfrac{4}{1} \times \dfrac{1}{3} = \dfrac{1}{4} + \dfrac{4}{6} = \dfrac{3}{12} + \dfrac{8}{12} = \dfrac{11}{12}$.

2. $(x - \sqrt{2})(x + \sqrt{2}) = 2(x^2 - 1)$, so $x^2 - x\sqrt{2} + x\sqrt{2} - \sqrt{2} \times \sqrt{2} = 2x^2 - 2$, so $x^2 - 2 = 2x^2 - 2$, so $x^2 = 2x^2$, so $x^2 = 0$, so $x = 0$.

3. **(a)** $-3x + 2 \le -x - 4$, so $-2x \le -6$, so $-x \le -3$, so $x \ge 3$. **(b)** $[3, \infty)$. **(c)**
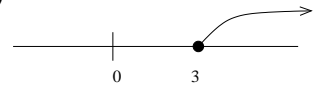
4. $|-2x + 3| = 7$ so $-2x + 3 = 7$ or $-2x + 3 = -7$. Hence $-2x = 4$ or $-2x = -10$ so $x = -2$ or $x = 5$.

5. **(a)** $\displaystyle\sum_{i=2}^{6}(ix - 2i)^2$. (Any correct expression is ok.) **(b)** $\displaystyle\sum_{i=2}^{10} ix$. (Any correct expression is ok.)

6. **(a)**

(b) $\{-1, 0, 1\} \setminus (\{0, 1, 2\} \cup \{0, 3, 6\}) = \{-1, 0, 1\} \setminus \{0, 1, 2, 3, 6\} = \{-1\}$.

(c) From the diagram, equals $\{0, 1\} \setminus \{0\} = \{1\}$.

(d) Look at the expression. The answer is those elements that are in C, but not B, then not A, then not C. Ignoring the middle part of this, we have those elements in C but not C. Hence the answer must be the emptyset, $\emptyset$.

(e) We have $C \cup \emptyset = C$, $A \cup \emptyset = A$ and $B \cup \emptyset = B$, so the answer is $C \cap A \cap B = \{0\}$.

7. We have $f(x) = x^2 + 1$, so $f(x + h) = (x + h)^2 + 1 = (x + h)(x + h) + 1 = x^2 + 2xh + h^2 + 1$.

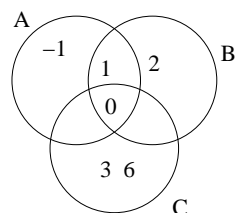So $\dfrac{f(x + h) - f(x)}{h} = \dfrac{(x^2 + 2xh + h^2 + 1) - (x^2 + 1)}{h} = \dfrac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \dfrac{2xh + h^2}{h} = 2x + h$.

8. Let $(x_1, y_1) = (\sqrt{3}, \sqrt{2})$ and $(x_2, y_2) = (-\sqrt{3}, -\sqrt{2})$.

Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(\sqrt{3} + \sqrt{3})^2 + (\sqrt{2} + \sqrt{2})^2} =$

$\sqrt{(2\sqrt{3})^2 + (2\sqrt{2})^2} = \sqrt{4 \times 3 + 4 \times 2} = \sqrt{12 + 8} = \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

9. Let $(x_1, y_1) = (0, \sqrt{5})$ and $(x_2, y_2) = (2, -3\sqrt{5})$, so $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3\sqrt{5} - \sqrt{5}}{2 - 0} = \dfrac{-4\sqrt{5}}{2} = -2\sqrt{5}$.

Hence $y = mx + c$ so $y = -2\sqrt{5}x + c$, and $(0, \sqrt{5})$ is on the line, so $\sqrt{5} = -2\sqrt{5} \times 0 + c$, so $c = \sqrt{5}$ and hence the equation of the line is $y = -2\sqrt{5}x + \sqrt{5}$.

10. To solve: $6x - 3y = 3$ (1) $-2x + 5y = -1$ (2)

Multiply both sides of (2) by 3, giving $-6x + 15y = -3$ and then add each side of this to each side of (1), giving $0x + 12y = 0$, so $12y = 0$, so $y = 0$.

Now substitute $y = 0$ into (1), giving $6x = 3$, so $x = 1/2$. Hence the solution is $(x, y) = (1/2, 0)$.

Now we should always check the answer. Substitute $(x, y) = (1/2, 0)$ into Equations (1) and (2), and see that both equations are satisfied, so the answer **must** be correct.

11. To solve: $4x + 2y = 6$ (1) $-6x - 3y = 9$ (2)

Multiply both sides of (1) by 3, giving $12x + 6y = 18$,and multiply both sides of (2) by 2, giving $-12x - 6y = 18$. Add each side of these two equations together, giving $0x + 0y = 36$, so $0 = 36$. This is impossible, so there is no solution to these equations.

14.14 Mid-semester exam, 2004

For each of the following 23 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. removed (not relevant)

2. removed (not relevant)

3. $x = |-3^0|$

4. $(\sqrt{-x})^2 = 5$

5. $x = f(2)$ where $f(a) = (-a)^2 + a$.

6. $x = f(2)$ where $f(a) = -a^2 + a$.

7. $x = f(-1)$ where $f(t) = \dfrac{t^2}{1/2}$

8. $\dfrac{x}{3} + \dfrac{2}{3} = 0$.

9. $\pi^3 \div \pi^{-2} = \dfrac{1}{\pi^x}$

10. $x + 1 = \dfrac{8}{2}$

11. $\displaystyle\sum_{i=1}^{x} 2i = 12$

12. Removed (not relevant).

13. $x = \displaystyle\sum_{i=-2}^{2} i$

14. $4^x \div 2^5 = 2^{-1}$

15. $\sqrt{18} = x\sqrt{2}$

16. $2^{x+1} = 4^{x+1}$

17. $f(t) = \sqrt{t+4}$ has range $[x, \infty)$

18. $f(y) = \dfrac{1}{|y+4|}$ has domain $(-\infty, \infty) \setminus \{x\}$

19. $x \times 1^x = -2$

20. $-2 + x = 3x$

21. $3^x = 3^{n+1} \div 3^{n-1}$

22. $4^6 = 4^{-x} \div 4^x$

23. $x^4 = -x^5$

For each of the following 10 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. What is the value of the gradient (m) and y-intercept (c) of the line $3y + 6x = 12$.

   (a) $m = 2$ and $c = 12$.   (b) $m = 6$ and $c = 12$.   (c) $m = -2$ and $c = 4$.
   (d) $m = -2$ and $c = -4$.   (e) $m = 2$ and $c = 4$.   (f) $m = 2$ and $c = -4$.

2. Let $f(x) = |x| - 1$. What is the **range** of $f(x)$?
   (a) $[0, \infty)$.   (b) $(-1, \infty)$.   (c) $(-\infty, 0) \cup (0, \infty)$.   (d) $[-1, \infty)$.   (e) $(-\infty, \infty)$.   (f) $(1, \infty)$.

3. Let $f(x) = \dfrac{\sqrt{x}}{|x|}$. What is the **domain** of $f(x)$?
   (a) $(-\infty, -1) \cup (-1, \infty)$.   (b) $[1, \infty)$.   (c) $(-\infty, 0) \cup (0, \infty)$.   (d) $(-\infty, \infty)$.   (e) $[0, \infty)$.   (f) $(0, \infty)$.

4. Which one of the following statements is false?

   (a) $\dfrac{1}{a} \times \dfrac{1}{b} = \dfrac{1}{a \times b}$ (b) $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b-a}{ab}$ (c) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{b}{a}$
   (d) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = (ab)^{-1}$ (e) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab}$ (f) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = a^{-1}b$

5. Which one of the following statements is true?
   (a) $3 \in \mathbb{Z}$ but $3 \notin \mathbb{N}$ (b) $-3 \in \mathbb{Z}$ but $-3 \notin \mathbb{N}$ (c) $-3 \in \mathbb{N}$ but $-3 \notin \mathbb{Z}$
   (d) $3 \in \mathbb{N}$ but $3 \notin \mathbb{Z}$ (e) $-3 \in \mathbb{Z}$ and $-3 \in \mathbb{N}$ (f) None of statements (a), (b), (c), (d) or (e) is true.

6. Let A and B be sets. Which one of the following statements must be true?

    (a) $A \subseteq A \cap B$.   **(b)** $B \subseteq A \cap B$. **(c)** $A \cup B \subseteq A$.

    (d) $A \cup B \subseteq A \cap B$. **(e)** $A \cup \emptyset \subseteq B$.   **(f)** None of statements (a), (b), (c), (d) or (e) must be true.

7. Removed (not relevant).

8. Removed (not relevant).

9. Which one of the following statements is true?

    (a) $(\sqrt{4x})^2 = 2x$.   **(b)** $2\sqrt{x} + 2\sqrt{x} = \sqrt{4x}$. **(c)** $\sqrt{a} + (\sqrt{b} \times \sqrt{c}) = \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{c}$.

    (d) $\sqrt{a} - \sqrt{b} = \sqrt{a - b}$. **(e)** $(\sqrt{4x})^2 = 4x$.   **(f)** None of statements (a), (b), (c), (d) or (e) is true.

10. The equation of the line passing through the points $(0, \pi^2)$ and (π^2, π^2) is:

    (a) $x = \pi$. **(b)** $x = \pi^2$. **(c)** $y = x \times \pi^2$. **(d)** $y = \pi$. **(e)** $y = \pi^2$. **(f)** $y = \pi^2 x$.

The following seven equations have their graphs drawn below, in random order, with eight extra graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to O.

        (1) $-2(y - x) = 4$.   **(2)** $y + 5 = 0$.   **(3)** removed (not relevant)   **(4)** $x - (1.7)^{27} = 14$.

(7 marks)   **(5)** $y = \mid x \mid \times (-1)^3$. **(6)** $y = \dfrac{-1}{x}$.   **(7)** $\dfrac{y}{\sqrt{7}} = x$.
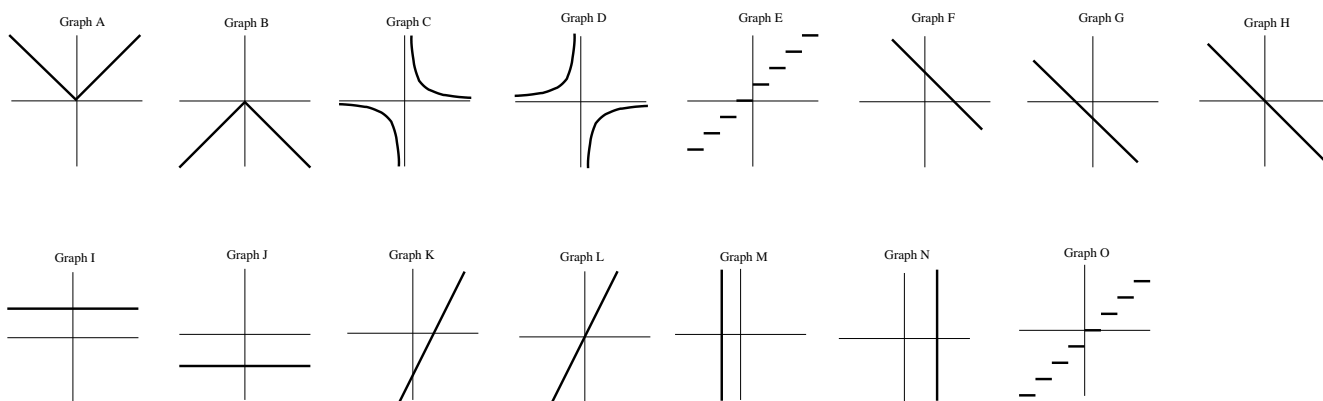
Part B, Midsemester exam, 2004

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{1}{2} \times \dfrac{1}{4} + \dfrac{1}{2} \div \dfrac{1}{3}$, leaving the answer as a fraction. Show all working.   (2 marks)

2. Solve for x: $\dfrac{1}{3x - 4} = \dfrac{1}{4x - 3}$.   (2 marks)

3. Show that: $\dfrac{1}{\sqrt{2}} \left(\dfrac{\sqrt{18}}{3} + \dfrac{\sqrt{8}}{3} \right) = \left(\dfrac{3^2 + 4^2}{2^3 + 1^3} \right)^{\frac{1}{2}}$.   (4 marks)

4. Simplify $\dfrac{a^{-1}b^2}{a^{-3}b^{-2}} \times \dfrac{1}{(ab^2)^2} \div a^{-1}$.   (4 marks)

5. Removed (not relevant). (3 marks)

6. (a) Find all x for which $-3x + 7 > x + 3$, writing your answer in inequality form. (An example of inequality form is $x > \ldots$). (2 marks)

   (b) Write your answer to part (a) in interval format. (1 mark)

   (c) Mark your answer to part (a) on the real line. (1 mark)

7. Solve $|-x - 4| = 6$. (2 marks)

8. Deleted. (3 marks)

9. Write in summation (sigma) notation $\quad 3x_1 + 6x_2 + 9x_3 + 12x_4 + \ldots + 30x_{10}$. (3 marks)

10. If $f(x) = 2x$, evaluate and simplify $\dfrac{f(x + h) \times f(x - h)}{f(2)}$. (4 marks)

11. Let $A = \{x \mid x \in \mathbb{Z}, \ x^2 \leq 4\}$, $B = \{1, 2, 3, 4\}$ and $C = \{1, 0, -1, -3\}$.

    (a) Explain why $A = \{-2, -1, 0, 1, 2\}$. (1 mark)

    (b) Mark the sets A, B and C on a Venn diagram, with the elements of each set written on the diagram. (2 marks)

    (c) Write down the set $(B \cap C) \setminus A$. (1 mark)

    (d) Write down the set $(A \cup B) \cap C$. (1 mark)

    (e) Write down the set $(B \cup \emptyset) \cup (A \cap \emptyset) \cup (C \cap \emptyset)$. (1 mark)

12. Find the distance between the points $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{8}, \sqrt{8})$, expressing your answer in simplest form. (2 marks)

13. Find the equation of the line parallel to $2(y + 101) = x + y + 2$ and passing through $(-3, -4)$. (3 marks)

14. For $i \in \mathbb{N}$, let $R_i = \displaystyle\sum_{j=1}^{i} j$. Evaluate $\displaystyle\sum_{k=1}^{3} \dfrac{1}{R_k}$. (4 marks)

15. Solve the following system of two simultaneous equations:

$$2x - 2y = 6$$
$$3x = -5 - 4y.$$

(4 marks)

14.15 Solutions to mid-semester exam, 2004

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

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Part A, Mid-semester exam, 2004

</div>

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1. 6   **2.** -3 **3.** 1   **4.** -5 **5.** 6   **6.** -2 **7.** 2   **8.** -2 **9.** -5 **10.** 3
11. 3   **13.** 0   **14.** 2   **15.** 3   **16.** -1 **17.** 0   **18.** -4 **19.** -2 **20.** -1 **21.** 2
22. -3 **23.** -1 or 0

Working:

4. We cannot take the square root of a negative number, so $-x$ must be positive, so x must be negative. If we take the square root of both sides we get $\sqrt{-x} = \sqrt{5}$. Hence $x = -5$.

5. Substitute 2 in for a in the expression. So $(-2)^2 + 2 = 4 + 2 = 6$.

6. BEDMAS says that we do the squaring **before** we make a negative. Then substitute 2 in for a in the expression. So $-a^2 + a = -(2^2) + 2 = -4 + 2 = -2$.

7. Substitute -1 in for t in the expression. So $t^2 = (-1)^2 = 1$ and $\dfrac{1}{1/2} = 2$.

8. So $\dfrac{x}{3} = \dfrac{-2}{3}$ so $x = -2$.

9. $\pi^3 \div \pi^{-2} = \pi^3 \times \pi^2 = \pi^{3+2} = \pi^5 = \dfrac{1}{\pi^{-5}}$

10. So $2(x + 1) = 8$, so $2x + 2 = 8$, so $2x = 6$ so $x = 3$.

14. $4^x = (2^2)^x = 2^{2x}$, so $2^{2x} \div 2^5 = 2^{2x-5}$ so $x = 2$.

16. RHS $= 4^{x+1} = (2^2)^{x+1} = 2^{2\times(x+1)} = 2^{2x+2}$. So $2^{x+1} = 2^{2x+2}$. The only time that this can be true is if both powers are equal, so $x + 1 = 2x + 2$, so $x = -1$.

17. We know that $\sqrt{x}$ is always positive, and has range $[0, \infty)$. Hence $x = 0$.

18. The domain represents every point at which it is possible to evaluate the function. This function works everywhere, **except** where we would have something divided by 0. This occurs when we substitute in -4. Hence the domain is everything, except $x = -4$.

19. $1^x = 1$ for every possible value of x. Hence $x \times 1 = -2$, so $x = -2$.

21. $3^{n+1} \div 3^{n-1} = \dfrac{3^{n-1} \times 3^2}{3^{n-1}} = 3^2$, so $x = 2$.

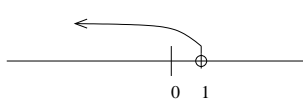22. $4^{-x} \div 4^x = 4^{-x-x} = 4^{-2x}$, so $x = -3$.

Multichoice.

1. $3y + 6x = 12$, so $3y = -6x + 12$, so $y = -2x + 4$. Hence $m = -2$ and $c = 4$, so answer is **(c)**.

2. Range is all possible values of $f(x)$. Now $|x|$ gives all values from 0 to ∞, including 0, so when we subtract 1, we get all values from -1 to ∞, including -1. Hence answer is **(d)**.

3. The domain represents every point at which it is possible to evaluate the function. This function does not work when $x < 0$ (as then we would have $\sqrt{\ }$ of a negative number), or when $x = 0$ (as then we would be dividing by 0). Hence domain is everything else, which is 0 to ∞, but not including 0. Hence answer is **(f)**.

4. Answer is **(d)**. All other statements are true, but look at this one. LHS is $\dfrac{1}{a} \div \dfrac{1}{b} = \dfrac{1}{a} \times \dfrac{b}{1} = \dfrac{b}{a}$, whereas RHS is $(ab)^{-1} = \dfrac{1}{ab}$.

5. Answer is **(b)**. -3 **is** an integer, but it is **not** a natural number.

6. Answer is **(f)**. Simple examples show that none of the others has to be true. For example, try the sets $A = \{1\}$ and $B = \{2\}$.

7. Removed (not relevant).

8. Removed (not relevant).

9. Answer is **(e)**. Can't be (a), as LHS equals $4x$, not $2x$. Can't be (b), as LHS equals $4\sqrt{x}$. Can't be (c), as LHS cannot be simplified. Can't be (d), as we know from class this is not true. But (e) is right, as LHS equals $4x$.

10. In each case, the y-coordinate is the same, so the line is horizontal, with equation $y = \pi^2$. Hence answer is **(e)**.

Matching graphs with equations. In each case we give the letter of the corresponding graph.

1. K   2. J   3. O   4. N   5. B   6. D   7. L

Part B, Midsemester exam, 2004

1. $\dfrac{1}{2} \times \dfrac{1}{4} + \dfrac{1}{2} \div \dfrac{1}{3} = \dfrac{1}{8} + \dfrac{3}{2} = \dfrac{1}{8} + \dfrac{12}{8} = \dfrac{13}{8}.$

2. $\dfrac{1}{3x-4} = \dfrac{1}{4x-3}$ so $4x - 3 = 3x - 4$ so $x = -1$.

3. LHS $= \dfrac{1}{\sqrt{2}}\left(\dfrac{3\sqrt{2}}{3} + \dfrac{2\sqrt{2}}{3}\right) = \dfrac{1}{\sqrt{2}}\left(\dfrac{5\sqrt{2}}{3}\right) = \dfrac{5\sqrt{2}}{3\sqrt{2}} = \dfrac{5}{3}.$ RHS $= \left(\dfrac{9+16}{8+1}\right)^{1/2} = \left(\dfrac{25}{9}\right)^{1/2} = \dfrac{5}{3}.$

4. LHS $= \dfrac{a^{-1}b^2}{a^{-3}b^{-2}a^2b^4} \times a^1 = \dfrac{a^{-1}b^2a^1}{a^{-1}b^2} = \dfrac{b^2}{a^{-1}b^2} = \dfrac{1}{a^{-1}} = a.$

5. Removed (not relevant).

6. **(a)** $-3x + 7 > x + 3$ so $-4x > -4$ so $x < 1$.   **(b)** $(-\infty, 1)$ **(c)**

7. $|-x - 4| = 6$ so $-x - 4 = 6$ or $-x - 4 = -6$.

   Hence $-x = 10$ or $-x = -2$ so $x = -10$ or $x = 2$.

8. Deleted.

9. $\displaystyle\sum_{i=1}^{10} 3ix_i.$ (But any correct expression is ok.)

10. Now $f(x + h) = 2(x + h)$, $f(x - h) = 2(x - h)$ and $f(2) = 2 \times 2 = 4$. Hence $\dfrac{f(x+h) \times f(x-h)}{f(2)} =$

    $\dfrac{2(x+h) \times 2(x-h)}{4} = \dfrac{4(x+h)(x-h)}{4} = (x+h)(x-h) = x^2 + xh - xh - h^2 = x^2 - h^2.$

11. (a) Those are the integers whose square is less than or equal to 4.

(b)
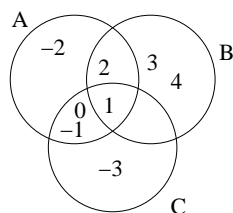
(c) The emptyset, $\emptyset$.   **(d)** $\{0, -1, 1\}$ **(e)** B, which is $\{1, 2, 3, 4\}$.

12. Let $(x_1, y_1) = (-\sqrt{2}, -\sqrt{2})$ and $(x_2, y_2) = (\sqrt{8}, \sqrt{8})$.

Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-\sqrt{2} - \sqrt{8})^2 + (-\sqrt{2} - \sqrt{8})^2} =$

$\sqrt{(-\sqrt{2} - 2\sqrt{2})^2 + (-\sqrt{2} - 2\sqrt{2})^2} = \sqrt{(-3\sqrt{2})^2 + (-3\sqrt{2})^2} = \sqrt{9 \times 2 + 9 \times 2} = \sqrt{18 + 18} = \sqrt{36} = 6.$

13. $2(y + 101) = x + y + 2$ so $y = x - 200$, so $m = 1$. As lines are parallel, new equation is $y = 1x + c$. Now $(-3, -4)$ is on the line so $-4 = -3 + c$ so $c = -1$ and equation is $y = x - 1$.

14. To evaluate the sum, we need R_1, R_2 and R_3. Now $R_1 = \displaystyle\sum_{j=1}^{1} j = 1$, $R_2 = \displaystyle\sum_{j=1}^{2} j = 1 + 2 = 3$ and

$R_3 = \displaystyle\sum_{j=1}^{3} j = 1 + 2 + 3 = 6.$

Hence $\displaystyle\sum_{k=1}^{3} \frac{1}{R_k} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} = \frac{6}{6} + \frac{2}{6} + \frac{1}{6} = \frac{9}{6} = 1.5.$

15. To solve: $2x - 2y = 6$ (1) $3x = -5 - 4y$ (2)

From (1), $x - y = 3$ so $x = y + 3$. Substitute this into (2), giving $3(y + 3) = -5 - 4y$, so $3y + 9 = -5 - 4y$, so $7y = -14$ and hence $y = -2$.

Substitute $y = -2$ into (1) gives us $2x + 4 = 6$ so $2x = 1$ so $x = 1$. Hence the solution is $(x, y) = (1, -2)$.

Now we should always check the answer. Substitute $(x, y) = (1, -2)$ into Equations (1) and (2), and see that both equations are satisifed, so the answer **must** be correct.

14.16 Mid-semester exam, 2003

For each of the following 20 questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks. (Hint: in each case, x is an integer between -6 and 6 inclusive.)

1. removed (not relevant)

2. removed (not relevant)

3. $x = |-\sqrt{25}|$

4. $(2\sqrt{5})^2 \div x = -5$.

5. $x = f(-2)$ where $f(a) = (-a)^2 + a$.

6. $x = f(-2)$ where $f(a) = -a^2 + a$.

7. $x = f(4)$ where $f(t) = \dfrac{t}{\sqrt{t}} - t$

8. $\dfrac{x}{3} + 2 = 0$.

9. $\pi^3 \times \pi^{-2} = \dfrac{1}{\pi^x}$

10. $\dfrac{1}{x+1} = \dfrac{2}{8}$

11. $\displaystyle\sum_{i=0}^{3} x = -4$

12. Removed (not relevant).

13. Removed (not relevant).

14. $2^x \times 4 = 2^{-3}$

15. $\sqrt{-10x} = 2\sqrt{10}$

16. $2^x = 3^x$

17. $f(t) = -\sqrt{t} - x$ has range $(-\infty, -4]$

18. $f(y) = \dfrac{1}{2y+6}$ has domain $(-\infty, \infty) \setminus \{x\}$

19. $x \times -1^x = 3$

20. $\dfrac{1}{2} + \dfrac{1}{3} = \dfrac{2.5}{x}$

For each of the following 13 multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. What is the the value of the gradient (m) and y-intercept (c) of the line $4y - 8x - 12 = 0$.

   (a) $m = -2$ and $c = 3$.   **(b)** $m = -2$ and $c = -3$.   **(c)** $m = 2$ and $c = 3$.
   (d) $m = 2$ and $c = -3$.   **(e)** $m = 8$ and $c = 12$.   **(f)** $m = -8$ and $c = -12$.

2. Let $f(x) = x + 1$. What is the **range** of $f(x)$?

   (a) $[0, \infty)$.   **(b)** $[1, \infty)$.   **(c)** $(-\infty, 0) \cup (0, \infty)$.   **(d)** $[-1, \infty)$.   **(e)** $(-\infty, \infty)$.   **(f)** $(1, \infty)$.

3. Let $f(x) = \dfrac{\sqrt{x}}{x+1}$. What is the **domain** of $f(x)$?

   (a) $(-\infty, -1) \cup (-1, \infty)$.   **(b)** $[1, \infty)$.   **(c)** $(-\infty, 0) \cup (0, \infty)$.   **(d)** $(-\infty, \infty)$.   **(e)** $[0, \infty)$.   **(f)** $(0, \infty)$.

4. Which one of the following statements is true?

   (a) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{a+b}$ **(b)** $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a-b}$ **(c)** $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{a}{b}$
   (d) $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{1}{ab}$ **(e)** $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{ab}{a+b}$ **(f)** $\left(\dfrac{1}{a}\right) \div \left(\dfrac{1}{b}\right) = \dfrac{b}{a}$

5. Which one of the following statements is true?

   (a) $\mathbb{Z} \cap \mathbb{N} = \mathbb{Z}$.   **(b)** $\mathbb{Z} \cap \mathbb{R} = \emptyset$.   **(c)** $\mathbb{Z} \cup \mathbb{N} = \mathbb{R}$.
   (d) $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$.   **(e)** $\mathbb{N} \setminus \mathbb{Z} = \emptyset$.   **(f)** None of those statements is true.

6. Let A and B be sets. Which one of the following statements must be true?

   (a) $A \cup B \subseteq A$.   **(b)** $A \cup B \subseteq A \cap B$.   **(c)** $A \cap B \subseteq A \cup B$.
   (d) $A \cap B \subseteq A \setminus B$.   **(e)** $(A \cup \emptyset) \cup (B \cup \emptyset) \subseteq A$.   **(f)** None of those statements must be true.

7. Removed (not relevant).

8. Removed (not relevant)

9. Which one of the following statements is true?
   (a) $(\sqrt{3x})^2 = 3x^2$.
   (b) $2\sqrt{x} + 2\sqrt{x} = \sqrt{8x}$.
   (c) $\sqrt{a} \times (\sqrt{b} \times \sqrt{c}) = \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{c}$.
   (d) $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$.
   (e) None of statements (a), (b), (c) or (d) is true.

10. The equation of the line passing through the points $(2, 3)$ and $(2, 6)$ is:
    (a) $x = 2$.   (b) $y = 2$.   (c) $y = 3$.   (d) $y = 6$.   (e) $y = 2x + 1$.   (f) $y = 3x$.

11. The line passing through the points $(0, 1)$ and $(1, 4)$ has gradient m and y-intercept c, where:
    (a) $m = -1$ and $c = -3$.   (b) $m = -3$ and $c = -1$.   (c) $m = 1$ and $c = 3$.
    (d) $m = 3$ and $c = 1$.   (e) $m = 1$ and $c = 4$.   (f) $m = -1$ and $c = -4$.

12. Let the point (a, e) be on the line $y_1 = mx + c$, let the point (a, f) be on the line $y_2 = 2mx + c$ and let the point (a, g) be on the line $y_3 = mx + 2c$. Which one of the following statements must be true:
    (a) $f = 2e$.   (b) $g = 2e$.   (c) $f = e + g$.   (d) $f + g = e$.   (e) $f + g = 2e$.   (f) $f + g = 3e$.

13. If $x = \dfrac{2^{n+2}}{2^{n-1}}$, which of the following is true:
    (a) $x = 2^{-2}$.   (b) $x = 2^{-1}$.   (c) $x = 2^0$.   (d) $x = 2^1$.   (e) $x = 2^2$.   (f) $x = 2^3$.

The following seven equations have their graphs drawn in the Midsemester 2004 exam paper, in random order, with eight extra graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation.

(7 marks)

Equation 1 is $-\sqrt{7} \times y = x$.

Equation 2 is $x + \sqrt{6.9} = 5$.

Equation 3 is $y = |-x|$.

Equation 4 is $y - 2x + 3 = -\pi$.

Equation 5 is $y = (-1)^7 \pi$.

Equation 6 is $y = \dfrac{\pi}{x}$.

Equation 7 is $4(y + x) = 2$.

Part B, Midsemester exam, 2003

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $\dfrac{1}{2} - \dfrac{1}{3} \times \left(\dfrac{1}{2} \div 2 \div \dfrac{1}{4}\right)$, leaving the answer as a fraction. Show all working.   (2 marks)

2. Solve for x: $\dfrac{(-2x - 1)}{(x - 1)} = 1$.   (3 marks)

3. Removed (not relevant).

4. Simplify $-(xy^2) \div (x^{-1}y^{-2}) \times (xy)^{-2} + x(x^{1/2}y^{-1})^{-2}$.   (4 marks)

5. Removed (not relevant).   (3 marks)

6. (a) Find all x for which $-2x - 2 < x + 4$, writing your answer in inequality form. (An example of inequality form is $x > \dots$). (2 marks)

   (b) Write your answer to part (a) in interval format. (1 mark)

   (c) Mark your answer to part (a) on the real line. (1 mark)

7. Solve $|2x + 3| = 5$. (3 marks)

8. Write in summation (sigma) notation $-2x - 3x - 4x - 5x - 6x - \dots - 99x - 100x$. (3 marks)

9. Deleted. (3 marks)

10. If $f(x) = 2x + 1$, evaluate and simplify $\dfrac{f(x + h) - f(x - h)}{h}$. (3 marks)

11. Let $A = \{x \mid x \in \mathbb{Z}, \ x^2 = 1 \text{ or } x^2 = 4\}$, $B = \{0, 1, 2, 3, 4\}$ and $C = \{1, 0, -1, -3\}$.

    (a) Explain why $A = \{-2, -1, 1, 2\}$. (1 mark)

    (b) Mark the sets A, B and C on a Venn diagram, with the elements of each set written on the diagram. (2 marks)

    (c) Write down the set $B \setminus (A \cap C)$. (1 mark)

    (d) Write down the set $(A \cap B) \cup C$. (1 mark)

    (e) Write down the set $(B \setminus \emptyset) \cap (A \setminus \emptyset) \cap (C \setminus \emptyset)$. (1 mark)

12. Find the distance between the points $(0, 0)$ and $(\sqrt{12}, \sqrt{12})$, expressing your answer as a surd in simplest form. (2 marks)

13. Find the equation of the line perpendicular to $y - 3x + 4 = 0$ and passing through the point $(6, -1)$. (3 marks)

14. Let $x_i = 2i$ for $i \in \{0, 1, 2, 3\}$. Evaluate $\displaystyle\sum_{i=0}^{3} \frac{(x_i - 3)^2}{4}$. (4 marks)

15. Solve the following system of two simultaneous equations:

$$8x - 4y = 3$$
$$3x + 2y = 2.$$

(4 marks)

14.17 Solutions to mid-semester exam, 2003

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

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Part A, Mid-semester exam, 2003

</div>

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1. 0 **2.** -3 **3.** 5 **4.** -4 **5.** 2 **6.** -6 **7.** -2 **8.** -6 **9.** -1 **10.** 3
11. -1 **14.** -5 **15.** -4 **16.** 0 **17.** 4 **18.** -3 **19.** -3 **20.** 3

Working:

5. Substitute -2 in for a in the expression. So $(2)^2 - 2 = 4 - 2 = 2$.

6. BEDMAS says that we do the squaring **before** we make a negative. Then substitute -2 in for a in the expression. So $-a^2 + a = -(-2^2) + -2 = -4 - 2 = -6$.

7. Substitute 4 in for t in the expression. So $\dfrac{t}{\sqrt{t}} - t = 2 - 4 = -2$.

8. So $\dfrac{x}{3} = -2$, so $x = -6$.

9. $\pi^3 \times \pi^{-2} = \pi^{3-2} = \pi^1 = \dfrac{1}{\pi^{-1}}$

10. So $2(x+1) = 8$, so $2x + 2 = 8$, so $2x = 6$ so $x = 3$.

14. $2^x \times 4 = 2^x \times 2^2 = 2^{x+2} = 2^{-3}$ so $x + 2 = -3$ so $x = -5$.

16. $2^x = 3^x$. The only time that this can be true is if both powers are equal to 0.

17. We know that $\sqrt{t}$ is always positive, and that $-\sqrt{t}$ has range $(-\infty, 0]$. Hence $x = 4$.

18. The domain represents every point at which it is possible to evaluate the function. This function works everywhere, **except** where we would have something divided by 0. This occurs when we substitute -3. Hence the domain is everything, except $x = -3$.

19. $-1^x = 1$ for every possible even value of x and -1 for every odd value of x. As x is obviously 3 or -3 and these are both odd then we have $x \times -1 = 3$, so $x = -3$.

20. $\dfrac{1}{2} + \dfrac{1}{3} = \dfrac{3}{6} + \dfrac{2}{6} = \dfrac{5}{6} = \dfrac{2.5}{3}$, so $x = 3$.

Multichoice.
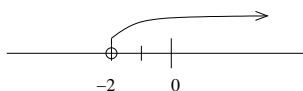
1. $4y - 8x - 12 = 0$, so $4y = 8x + 12$, so $y = 2x + 3$. Hence $m = 2$ and $c = 3$, so answer is **(c)**.

2. Range is all possible values of $f(x)$. Now $x + 1$ gives all values from $-\infty$ to ∞. Hence answer is **(e)**.

3. The domain represents every point at which it is possible to evaluate the function. This function does not work when $x < 0$ (as then we would have $\sqrt{}$ of a negative number), or when $x = -1$ (as then we would be dividing by 0). Hence domain is everything else, which is 0 to ∞. Hence answer is **(e)**.

4. Answer is **(f)**. All other statements are false, but look at this one. LHS is $\dfrac{1}{a} \div \dfrac{1}{b} = \dfrac{1}{a} \times \dfrac{b}{1} = \dfrac{b}{a} = RHS$.

5. Answer is **(e)**. Every natural number is also an integer.

6. Answer is **(c)**. Simple examples show that none of the others has to be true. For example, try the sets $A = \{1\}$ and $B = \{2\}$.

7. Removed (not relevant).

8. Removed (not relevant).

9. Answer is **(f)**. Can't be (a), as LHS equals $3x$, not $3x^2$. Can't be (b), as LHS equals $4\sqrt{x}$. Can't be (c), as LHS equals $\sqrt{a}\sqrt{b}\sqrt{c}$. Can't be (d), as we know from class this is not true.

10. In each case, the x-coordinate is the same, so the line is horizontal, with equation $x = 2$. Hence answer is **(a)**.

11. Let $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (1, 4)$. Using the formula from class the slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4 - 1}{1 - 0} = 3$. Then $y = 3x + c$ and substituting the point $(0, 1)$ in gives $y = 3x + 1$. Hence the answer is **(d)**.

12. Substitute the coordinates of each point into the equation of the line passing through that point. The point (a, e) gives us the equation $e = ma + c$. The point (a, f) gives the equation $f = 2ma + c$ and the point (a, g) gives $g = ma + 2c$. Then substitute the expressions for e, f and g into each of the answers, and the only correct one is **(f)**.

13. $\dfrac{2^{n+2}}{2^{n-1}} = 2^{n+2} \times 2^{-n+1} = 2^{n+2-n+1} = 2^3$. Hence the answer is **(f)**.

Matching graphs with equations. In each case we give the letter of the corresponding graph.

1. H 2. N 3. A 4. K 5. J 6. C 7. F

Part B, Midsemester exam, 2003

1. $\dfrac{1}{2} - \dfrac{1}{3} \times \left(\dfrac{1}{2} \div 2 \div \dfrac{1}{4} \right) = \dfrac{3}{6} - \dfrac{2}{6} \times \left(\dfrac{1}{2} \times \dfrac{1}{2} \times 4 \right) = \dfrac{3}{6} - \dfrac{2}{6} \times \dfrac{4}{4} = \dfrac{3}{6} - \dfrac{2}{6} = \dfrac{1}{6}$.

2. $\dfrac{(-2x - 1)}{x - 1} = 1$, so $-2x - 1 = x - 1$, so $-3x = 0$, so $x = 0$.

3. Removed (not relevant).

4. $-(xy^2) \div (x^{-1}y^{-2}) \times (xy)^{-2} + x(x^{1 \div 2}y^{-1})^{-2} = -xy^2 \times (x^1 y^2) \times (x^{-2}y^{-2}) + x(x^{-1}y^2) = -x^{(1+1-2)}y^{(2+2-2)} + x^{(1-1)}y^2 = -y^2 + y^2 = 0$.

5. Removed (not relevant).

6. **(a)** $-2x - 2 < x + 4$ so $-3x < 6$ so $x > -2$. **(b)** $(-2, \infty)$ **(c)**

7. $|2x + 3| = 5$ so $2x + 3 = 5$ or $2x + 3 = -5$.

   Hence $2x = 2$ or $2x = -8$ so $x = 1$ or $x = -4$.

8. $\displaystyle\sum_{i=2}^{100} -ix$.   (But any correct expression is ok.)

9. Deleted.

10. Now $f(x + h) = 2(x + h) + 1 = 2x + 2h + 1$ and $f(x - h) = 2(x - h) + 1 = 2x - 2h + 1$. Hence $\dfrac{f(x + h) - f(x - h)}{h} = \dfrac{2x + 2h + 1 - (2x - 2h + 1)}{h} = \dfrac{4h}{h} = 4$.

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11. (a) Those are the integers whose square is equal to 1 or 4.

(b)
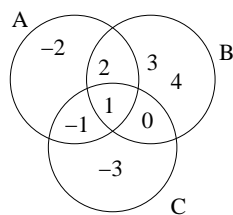
(c) $\{0, 2, 3, 4\}$.   **(d)** $\{-3, -1, 0, 1, 2\}$ **(e)** $\{1\}$.

12. Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (\sqrt{12}, \sqrt{12})$.

Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(0 - \sqrt{12})^2 + (0 - \sqrt{12})^2} =$

$\sqrt{(-\sqrt{12})^2 + (-\sqrt{12})^2} = \sqrt{12 + 12} = \sqrt{24} = 2\sqrt{6}$.

13. $y - 3x + 4 = 0$ so $y = 3x - 4$, so $m = 3$. As lines are perpendicular, new equation is $y = -\dfrac{1}{3}x + c$. Now $(6, -1)$ is on the line so $-1 = -\dfrac{1}{3} \times 6 + c$ so $-1 = -2 + c$ so $c = 1$ and equation is $y = -\dfrac{1}{3}x + 1$.

14. To evaluate the sum, we need x_0, x_1, x_2 and x_3. Now $x_0 = 2 \times 0 = 0$, $x_1 = 2 \times 1 = 2$, $x_2 = 2 \times 2 = 4$ and $x_3 = 2 \times 3 = 6$.

Hence $\displaystyle\sum_{i=0}^{3} \frac{(x_i - 3)^2}{4} = \frac{(x_0 - 3)^2}{4} + \frac{(x_1 - 3)^2}{4} + \frac{(x_2 - 3)^2}{4} + \frac{(x_3 - 3)^2}{4} = \frac{(0 - 3)^2}{4} + \frac{(2 - 3)^2}{4} + \frac{(4 - 3)^2}{4} +$

$\dfrac{(6 - 3)^2}{4} = \dfrac{9}{4} + \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{9}{4} = \dfrac{9 + 1 + 1 + 9}{4} = \dfrac{20}{4} = 5$.

15. To solve: $8x - 4y = 3$ (1) $3x + 2y = 2$ (2)

Multiply (2) by 2, so $6x + 4y = 4$. Add this to (1), giving $8x - 4y + 6x + 4y = 3 + 4 \Rightarrow 14x = 7 \Rightarrow x = \dfrac{1}{2}$.

Substitute $x = \dfrac{1}{2}$ into (1) gives us $8 \times \dfrac{1}{2} - 4y = 3$ so $-4y = -4 + 3$ so $y = \dfrac{1}{4}$. Hence the solution is $(x, y) = (\dfrac{1}{2}, \dfrac{1}{4})$.

Now we should always check the answer. Substitute $(x, y) = (\dfrac{1}{2}, \dfrac{1}{4})$ into Equations (1) and (2), and see that both equations are satisfied, so the answer **must** be correct.

14.18 Mid-semester exam, 2002

<div align="center">

Part A, Mid-semester exam, 2002

</div>

For each of the following thirteen questions, find the correct value of x. There is no need to show any working. Each correct answer is worth 1 mark. (Hint: in each case, x is an integer between -4 and 4 inclusive.)

1. removed (not relevant)

2. removed (not relevant)

3. $x = -|\sqrt{4}|$

4. $x = f(2)$ where $f(t) = 2t - 2t^2$

5. $(\{0, 1, 2, 4\} \cap \{1, 4, -1\}) \setminus (\{1, 2, x\}) = \emptyset$

6. $2x + 3 = 6 - x$

7. $4x + 5 = 6x + 5$

8. $x = f(0)$ where $f(t) = \dfrac{1}{t-1} + \dfrac{1}{t+1}$

9. $\displaystyle\sum_{i=1}^{x} i = 10$

10. Removed (not relevant).

11. $|\sqrt{32}| \times x = -8\sqrt{2}$

12. $2^x = \dfrac{1}{4}$

13. $8^{2/x} = 4$

For each of the following ten multichoice questions, find the correct answer. There is no need to show any working. Each correct answer is worth 1 mark.

1. What is the the value of the gradient (m) and y-intercept (c) of the line $3y - 2x - 1 = 4x - 4$.

   (a) $m = -2$ and $c = 1$.   **(b)** $m = -2$ and $c = -1$.   **(c)** $m = 2$ and $c = 1$.
   (d) $m = 2$ and $c = -1$.   **(e)** $m = -2$ and $c = 4$.

2. Let $f(x) = |x| + 1$. What is the **range** of $f(x)$?

   (a) $[0, \infty)$.   **(b)** $[1, \infty)$.   **(c)** $(-\infty, \infty)$.   **(d)** $[-1, \infty)$.   **(e)** $(-\infty, 0) \cup (0, \infty)$.

3. Let $f(x) = \dfrac{1}{\sqrt{x}}$. What is the **domain** of $f(x)$?

   (a) $(0, \infty)$.   **(b)** $(-\infty, \infty)$.   **(c)** $[0, \infty)$.   **(d)** $(-\infty, 0) \cup (0, \infty)$.   **(e)** $[1, \infty)$.

4. Which of the following statements is true?

   (a) $2 \in \mathbb{Z}$ but $2 \notin \mathbb{N}$.   **(b)** $-2 \in \mathbb{Z}$ but $-2 \notin \mathbb{N}$.   **(c)** $-2 \in \mathbb{N}$ but $-2 \notin \mathbb{Z}$.
   (d) $2 \in \mathbb{N}$ but $2 \notin \mathbb{Z}$.   **(e)** $-2 \in \mathbb{Z}$ and $-2 \in \mathbb{N}$.

5. Let A and B be sets. Which of the following statements is true?

   (a) $A \cap B = (A \cup B) \setminus B$.   **(b)** $A \cap B = A \setminus (A \setminus B)$.   **(c)** $A \cap B = (A \setminus B) \cup \emptyset$.
   (d) Both (a) and (b) are true.   **(e)** Both (b) and (c) are true.

6. Which of the following statements is true?

   (a) $\mathbb{R} \subseteq \mathbb{N} \subseteq \mathbb{Z}$.   **(b)** $\mathbb{Z} \subseteq \mathbb{N} \subseteq \mathbb{R}$.   **(c)** $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$.
   (d) $\mathbb{R} \subseteq \mathbb{Z} \subseteq \mathbb{N}$.   **(e)** None of statements (a), (b), (c) or (d) is true.

7. Removed (not relevant).

8. Removed (not relevant).

9. Which of the following statements is false?

   (a) $(2\sqrt{2x})^2 = 8x$.   **(b)** $\sqrt{x} + \sqrt{x} + \sqrt{x} = \sqrt{9x}$.
   (c) $\sqrt{a} \times \sqrt{b} \times \sqrt{c} = \sqrt{abc}$.   **(d)** $\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{a+b+c}$.

10. The equation of the line passing through the points $(0,0)$ and $(2,4)$ is:

    (a) $x = 2$.   **(b)** $y = 4$.   **(c)** $y = x$.   **(d)** $y = 2x$.   **(e)** $y = 2x + 4$.

The following seven equations have their graphs drawn below, in random order, with nine extra graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to P.

(7 marks)

Equation 1 is $y = \sqrt{x}$.   Equation 2 is $y = \sqrt{7} \times x$.

Equation 3 is $-x + \sqrt{5} = 6.7$.   Equation 4 is $y = -\mid x \mid$.

Equation 5 is $y = -\mid x \mid + 2 + \mid -x \mid$.   Equation 6 is $\pi y = -2\pi x - 3\pi$.

Equation 7 is $y = \dfrac{1}{-x}$.
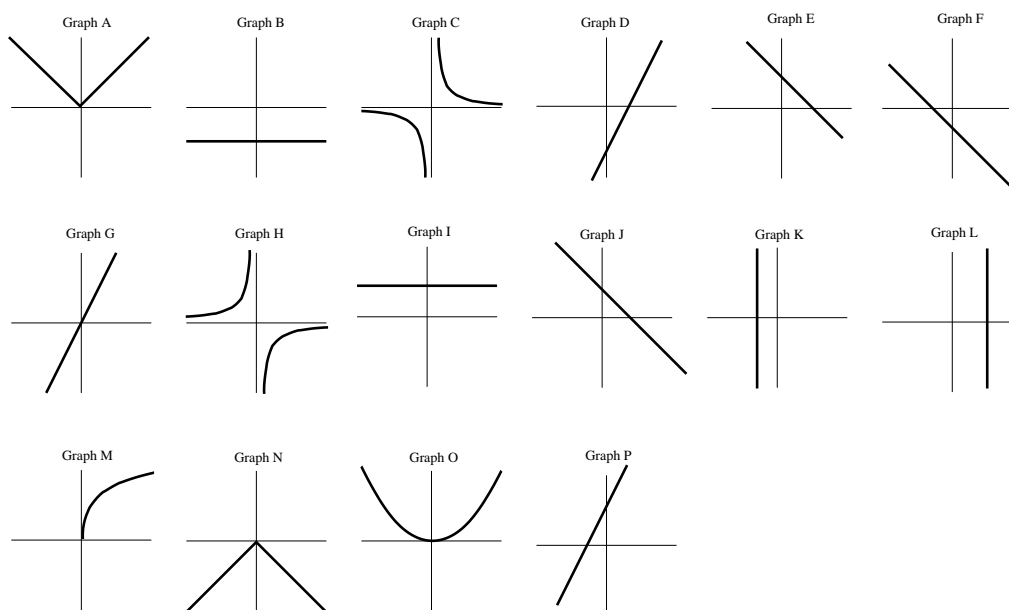
Part B, Midsemester exam, 2002

Each of the following questions carries the stated number of marks. Write your answers in the space provided. Part marks will be awarded for correct working.

1. Evaluate $(\dfrac{1}{2} + \dfrac{1}{4} \div \dfrac{1}{2}) \div \dfrac{1}{3}$, leaving the answer as a fraction. Show all working.   (2 marks)

2. Solve for x: $\dfrac{5}{2x + 4} = (2\sqrt{10})^2 - 8 \times (2 + 3)$.   (4 marks)

3. Expand and simplify $\dfrac{(\sqrt{6} - \sqrt{2})(\sqrt{3} + 1)}{\sqrt{2}}$.   (3 marks)

4. Simplify $a(-ab)^3 \div (ab^2) \div b^{-2} \times a^{-3}$.   (4 marks)

5. Simplify $\dfrac{3^n - 3^{n-1}}{3^{n+1} - 3^{n-1}}$.   (3 marks)

6. (a) Find all x for which $-2(2x+4) \geq 2x - 2$, writing your answer in inequality form. (An example of inequality form is $x > \ldots$). (3 marks)

(b) Write your answer to part (a) in interval format. (1 mark)

(c) Mark your answer to part (a) on the real line. (1 mark)

7. Removed (not relevant). (5 marks)

8. Write in summation (sigma) notation $12i + 10i + 8i + 6i + 4i + 2i - 2i - 4i - 6i$. (4 marks)

9. Deleted. (3 marks)

10. If $f(x) = x^2$, evaluate and simplify $\dfrac{f(x+h) - f(x)}{h}$. (3 marks)

11. Let $A = \{-4, 0, 1, -1\}$, $B = \{-4, -3, 0, 4\}$ and $C = \{3, 1, -4, 4\}$.

   (a) Mark the sets A, B and C on a Venn diagram, with the elements of each set written on the diagram. (3 marks)

   (b) Write down the set $A \setminus (C \cup B)$. (1 mark)

   (c) Write down the set $(A \cap B) \setminus (A \cap B \cap C)$. (1 mark)

   (d) Write down the set $(A \cap C) \cap (A \cup B)$. (1 mark)

   (e) Write down the set $((B \cap \emptyset) \cup (A \cup \emptyset)) \setminus (B \setminus \emptyset)$. (1 mark)

12. Find the distance between the points $(-1, -3)$ and $(-6, -8)$, expressing your answer as a surd in simplest form. (2 marks)

13. Find the equation of the line perpendicular to $y - x = -(x+1)$ and passing through the point $(-4, -2)$. (3 marks)

14. Do the points $(2, -1)$, $(-3, 9)$ and $(0, 2)$ all lie on the same line? Explain your answer carefully, showing working. (4 marks)

15. Solve the following system of two simultaneous equations:
$$2x + 4y = -(x + 2y) + 9$$
$$-2x - 5y = -3.$$
(4 marks)

16. Solve the following system of two simultaneous equations:
$$2a + b = 2(a - b) + 3b$$
$$3a + b = -3(b - a) + 4 + 4b.$$
(4 marks)

14.19   Solutions to mid-semester exam, 2002

In some cases we use the shorthand notation LHS to mean the left-hand side of the given expression, and RHS to mean the right-hand side.

Part A, Mid-semester exam, 2002

Single number answer: In each case we give the value for x. After the answers we show the working for some of the harder questions.

1. 3 **2.** -2 **3.** -2 **4.** -4 **5.** 4 **6.** 1 **7.** 0
8. 0 **9.** 4 **11.** -2 **12.** -2 **13.** 3

Working:

4. Substitute 2 in for t in the expression. So $2 \times 2 - 2 \times 2^2 = 4 - 8 = -4$.

5. $(\{0, 1, 2, 4\} \cap \{1, 4, -1\}) \setminus (\{1, 2, x\}) = (\{1, 4\}) \setminus (\{1, 2, x\}) = \emptyset$. Thus $x = 4$.

8. Substitute 0 in for t in the expression. So $\dfrac{1}{t-1} + \dfrac{1}{t+1} = \dfrac{1}{0-1} + \dfrac{1}{0+1} = -1 + 1 = 0$.

11. So $| \sqrt{32} | \times x = | \sqrt{4 \times 4 \times 2} | \times x = | 4\sqrt{2} | \times x = 4\sqrt{2} \times x = -8\sqrt{2}$ so $x = -2$.

12. $2^x = \dfrac{1}{4}$ so $2^x = \dfrac{1}{2^2}$ so $2^x = 2^{-2}$ so $x = -2$.

13. $8^{2/x} = 4$ so $2^{3(2/x)} = 2^2$ so looking at the powers, $3 \left(\dfrac{2}{x} \right) = 2$ so $\dfrac{6}{x} = 2$. Hence $x = 3$.

Multichoice.

1. $3y - 2x - 1 = 4x - 4$, so $3y = 6x - 3$, so $y = 2x - 1$. Hence $m = 2$ and $c = -1$, so answer is **(d).**

2. Range is all possible values of $f(x)$. Now $|x|$ gives all values from 0 to ∞, so adding 1 gives all values from 1 to ∞. Hence answer is **(b).**

3. The domain represents every point at which it is possible to evaluate the function. This function does not work when $x < 0$ (as then we would have $\sqrt{}$ of a negative number), or when $x = 0$ (as then we would be dividing by 0). Hence domain is everything else, which is 0 to ∞ (not including 0). Hence answer is **(a).**

4. Answer is **(b).** -2 is an integer, but it is **not** a natural number.

5. Answer is **(b).** Simple examples show that none of the others have to be true. For example, try the sets $A = \{1\}$ and $B = \{2\}$.

6. Answer is **(c).**

7. Removed (not relevant).

8. Removed (not relevant).

9. Answer is **(d).** You can check that the others are all true using simple calculations.

10. Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2, 4)$. Using the formula from class, the slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4 - 0}{2 - 0} = 2$. Then $y = 2x + c$ and substituting the point $(0, 0)$ in gives $y = 2x$. Hence the answer is **(d).**

Matching graphs with equations.   In each case we give the letter of the corresponding graph.

1. M   2. G   3. K   4. N   5. I   6. F   7. H

1. $\left(\dfrac{1}{2}+\dfrac{1}{4}\div\dfrac{1}{2}\right)\div\dfrac{1}{3}=\left(\dfrac{1}{2}+\dfrac{1}{4}\times 2\right)\times 3=\left(\dfrac{1}{2}+\dfrac{1}{2}\right)\times 3=3.$

2. $\dfrac{5}{2x+4}=(2\sqrt{10})^2-8\times 5$ so $\dfrac{5}{2x+4}=(40)-40$ so $\dfrac{5}{2x+4}=0$ so $5=0$, which is impossible. Hence there is no solution.

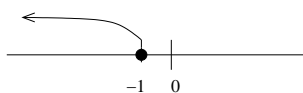3. $\dfrac{(\sqrt{6}-\sqrt{2})(\sqrt{3}+1)}{\sqrt{2}}=\dfrac{(\sqrt{6}\times\sqrt{3})+(1\times\sqrt{6})-(\sqrt{2}\times\sqrt{3})-(\sqrt{2}\times 1)}{\sqrt{2}}=\dfrac{3\sqrt{2}+\sqrt{6}-\sqrt{6}-\sqrt{2}}{\sqrt{2}}=\dfrac{2\sqrt{2}}{\sqrt{2}}=$
2.

4. $a(-ab)^3\div(ab^2)\div(b)^{-2}\times a^{-3}=a(-a^3b^3)\times a^{-1}b^{-2}\times b^2\times a^{-3}=-a^{1+3-1-3}b^{3-2+2}=-b^3.$

5. $\dfrac{3^n-3^{n-1}}{3^{n+1}-3^{n-1}}=\dfrac{3^{n-1}(3-1)}{3^{n-1}(3^2-1)}=\dfrac{3-1}{9-1}=\dfrac{2}{8}=\dfrac{1}{4}.$

6. **(a)** $-2(2x+4)\geq 2x-2$ so $-4x-8\geq 2x-2$ so $-6x\geq 6$ so $x\leq -1$.   **(b)** $(-\infty,-1]$
   (c)

7. Removed (not relevant).

8. $\displaystyle\sum_{j=-3}^{6}2ji.$ (But any correct expression is ok.)

9. Deleted.

10. Now $f(x+h)=(x+h)^2=x^2+h^2+2xh$. Hence $\dfrac{f(x+h)-f(x)}{h}=\dfrac{x^2+h^2+2xh-x^2}{h}=\dfrac{h^2+2xh}{h}=h+2x.$

11. **(a)**

    (b) $\{-1\}$ **(c)** $\{0\}$ **(d)** $\{-4,1\}$ **(e)** $\{-1,1\}$

12. Let $(x_1,y_1)=(-1,-3)$ and $(x_2,y_2)=(-6,-8)$.
    Then $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(-1+6)^2+(-3+8)^2}=\sqrt{(5)^2+(5)^2}=\sqrt{25+25}=\sqrt{50}=5\sqrt{2}.$

13. $y-x=-(x+1)$ so $y=-1$, so the slope is zero and line is horizontal. As lines are perpendicular, new line must be vertical. Hence, as $(-4,-2)$ is on line, equation must be $x=-4$.

14. Assume that $(2,-1)$ and $(-3,9)$ lie on the same line. Then from class we know that $m=\dfrac{10}{-5}=-2$. So the equation of the line is $y=2x+c$, and substituting in the point $(2,-1)$ we get $-1=-2\times 2+c$ so $3=c$, so the equation is $y=-2x+3$. If all the points are on the same line then $(0,2)$ must satisfy this equation. But it doesn't: when $x=0$ we would have $y=3$, instead of $y=2$. Hence the points $(2,-1)$, $(-3,9)$ and $(0,2)$ **do not** all lie on the same line. Note: You can start with any two of the three points and reach the same conclusion.

15. To solve: $2x + 4y = -(x + 2y) + 9$ (1) $-2x - 5y = -3$ (2)

From (1), $2x + 4y = -x - 2y + 9$ so $3x = -6y + 9$ so $x = -2y + 3$. Substitute this into (2), giving $-2(-2y + 3) - 5y = -3$, so $4y - 6 - 5y = -3$, so $-y = 3$, hence $y = -3$.

Substitute $y = -3$ into (1) gives us $2x + 4 \times -3 = -(x + 2 \times -3) + 9$ so $2x - 12 = -x + 6 + 9$ so $3x = 27$, hence $x = 9$. Hence the solution is $(x, y) = (9, -3)$.

Now we should always check the answer. Substitute $(x, y) = (9, -3)$ into Equations (1) and (2), and see that both equations are satisifed, so the answer **must** be correct.

16. To solve: $2a + b = 2(a - b) + 3b$ (1) $3a + b = -3(b - a) + 4 + 4b$ (2)

From (1), $2a + b = 2a - 2b + 3b$ we get that $b = b$ (which is obvious and not very helpful).

From (2), $3a + b = -3b + 3a + 4 + 4b$ so $3a + b = 3a + b + 4$ so we get that $0 = 4$, which is impossible. Hence the two simultaneous equations (1) and (2) have no solution.

14.20   Final exam, June 2010

1. Answer each of the following questions. Each part is worth 1 mark.   (5 marks)

   (a) Let $a = 1, b = 2, c = 3$ and $d = 4$. Evaluate $(ab)^c + \dfrac{dc}{b}$.

   (b) Evaluate $\dfrac{1}{5} + \dfrac{1}{3} \div \dfrac{3}{4}$ (show all working).

   (c) Find $\log_2 8$.

   (d) Find e^{2x} if $x = 0$.

   (e) Write down any two **linear** functions.

2.   (a) Solve for x where $\dfrac{(3x^2)^3 \cdot x^{-2}}{(3x)^2} = 2^4 \times \dfrac{3}{2}$.   (4 marks)

   (b) Let $x = \sqrt{3} - 2$ and $y = \sqrt{3} + 4$. Evaluate $2x - 3xy$, leaving your answer as a simplified surd.

                                                                                       (4 marks)

3. Let $f(x) = -x^2 + 2$ and let $g(x) = \sqrt{|x| - 1}$.

   (a) Determine $f(3)$.   (1 mark)

   (b) Determine $g(-5)$.   (1 mark)

   (c) Find the domain of $f(x)$.   (1 mark)

   (d) Find the range of $f(x)$.   (1 mark)

   (e) Find the domain of $g(x)$.   (2 marks)

   (f) Find the range of $g(x)$.   (1 mark)

   (g) Determine $f(g(x))$. Simplify your answer.   (2 marks)

4.   (a) A cellar contains only bottles of port, sherry, claret and brandy. Of these, $\frac{1}{5}$ of the bottles are port and $\frac{1}{5}$ are claret. The cellar also contains 25 bottles of sherry and 35 bottles of brandy. How many bottles of port are there in the cellar? Show all working.   (2 marks)

   (b) A ladder is placed up against a wall at an angle of $60°$ to the ground. If the base of the ladder is $3m$ away from the wall, how high up the wall does the ladder reach? (Hint: $\cos 60° = 0.5, \sin 60° \approx 0.866, \tan 60° \approx 1.732$.)   (3 marks)

5. Solve each of the following equations.

   (a) $3x + 4 = 5x - 6$ (1 mark)
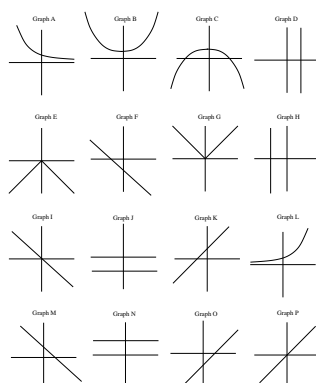
   (b) $t(4t - 2)(2t + 8) = 0$ (2 marks)

   (c) $x^2 = 25$ (2 marks)

6. Sketch the graphs of the following equations, showing all $x-$ and $y-$intercepts.

   (a) $y = 3x - 2$ (3 marks)

   (b) $y = x^2 - 4x - 5$ (5 marks)

7. Below are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to P.   (8 marks)

$$(1)\ x = 2 \qquad (2)\ y = \mid 7x \mid \qquad (3)\ y = e^{-2x} \quad (4)\ 3y = 2x + 6$$
$$(5)\ -y = x^2 - 4 \quad (6)\ y - 4x - 2 = -5 \quad (7)\ y = e^{0.3x} \quad (8)\ y = x^2 + \pi$$

Graph A Graph B Graph C Graph D

Graph E Graph F Graph G Graph H

Graph I Graph J Graph K Graph L

Graph M Graph N Graph O Graph P

8. Sarah goes for a walk from $(1, 0)$ to $(4, 3)$. Simon also goes for a walk, from $(2, 0)$ to $(2, 6)$.

    (a)   **(i)** Find the equation of the line Sarah walks.   (2 marks)

       (ii) Find the equation of the line Simon walks.   (2 marks)

      (iii) Find the coordinates of the point where their paths intersect.   (3 marks)

    (b) Recall that Sarah walks from $(1, 0)$ to $(4, 3)$ What distance has she walked? Write your answer as a simplified surd.   (2 marks)

    (c) Bill is walking along the line $2x + 3y = 6$, while Ben is walking along the line $-4x - 6y = 6$. Will their paths cross? If so, where?   (3 marks)

9. I have \$2000 to invest for 5 years. Bank A pays interest at 11% per annum compounded monthly for 5 years. Bank B has a special introductory offer of interest at 12% compounded monthly for the first year, followed by their standard offer of interest at 10% compounded continuously for the next 4 years. Bank C pays interest at 13% per annum compounded half-yearly for 5 years. Which is the best investment strategy? Show all working. (Ignore all fees and taxes.)

    (Hint: $(1.12)^{12} \approx 3.896$, $(1.11)^5 \approx 1.685$, $(1.01)^{12} \approx 1.127$, $(1.13)^5 \approx 1.842$

              $(1.009)^{60} \approx 1.712$, $(1.009)^5 \approx 1.046$, $e^{0.5} \approx 1.649$ $(1.065)^5 \approx 1.37$

              $e^{0.4} \approx 1.492$ $(1.065)^{10} \approx 1.877.$)

                                                                (7 marks)

10. In each of Parts (a), (b) and (c) find y', **simplifying where possible**.

    (a) $y = \dfrac{3}{2}x^4 - 5x + 2e^{3x} - \pi$ (1 mark)

    (b) $y = \dfrac{4n - n^3 + 2}{3n^4}$ (5 marks)

    (c) $y = (2x^3 - 1)^{-3} - 7\sin x + \ln x$ (4 marks)

11. Find each of the following integrals.

    (a) $\displaystyle\int -5\,dx$ (1 mark)

    (b) $\displaystyle\int \left(2x + e^{4x} + x^{-1}\right)\,dx$ (4 marks)

    (c) $\displaystyle\int \left(\sum_{i=1}^{3} 2ix\right)\,dx$ (2 marks)

12. A rocket takes off vertically at time $t = 0$, with *displacement* $S(t) = at^2 - bt$, where a and b are constants.

    (a) When $t = 0$, the rocket has *velocity* $v(0) = 7$. When $t = 1$, the rocket has *velocity* $v(1) = 13$. Find the values of a and b and an expression for velocity at any time, t.   (3 marks)

    (b) At what time does the value of the *velocity* of the rocket equal four times that of the *acceleration*?   (2 marks)

    (c) How far does the rocket travel between $t = 1$ and $t = 3$?   (2 marks)

13.   **(a)** Let $y = x^2$. Find the equation of the tangent line to the curve at the point $(3, 9)$.   (4 marks)

    (b) The *normal* to a curve at a particular point is the straight line that passes through the point in question on the curve and is *perpendicular* to the tangent to the curve at that point. Find the equation of the normal to the curve $y = x^2$ at the point $(3, 9)$.   (4 marks)

14. Let $f(x) = (x - 1)e^x + 2$. Find and classify all the critical points of y. (Your answer should include the x and y-coordinates of all critical points.)   (7 marks)

14.21 Solutions to final exam, June 2010

1. **(a)** $(1 \times 2)^3 + \dfrac{4 \times 3}{2} = 8 + 6 = 14.$

   (b) $\dfrac{1}{5} + \dfrac{1}{3} \div \dfrac{3}{4} = \dfrac{1}{5} + \dfrac{1}{3} \times \dfrac{4}{3} = \dfrac{1}{5} + \dfrac{4}{9} = \dfrac{29}{45}.$

   (c) $8 = 2^3$, so $\log_2 8 = 3.$

   (d) $e^0 = 1.$

   (e) Any two linear functions. For example, $y = 2, x = -2, y = 6x - 4.$

2a. $\dfrac{27x^6 \cdot x^{-2}}{9x^2} = 16 \times \dfrac{3}{2}$
   $\Rightarrow 3x^2 = 24 \Rightarrow x^2 = 8 \Rightarrow x = 2\sqrt{2}.$

2b. $2(\sqrt{3} - 2) - 3(\sqrt{3} - 2)(\sqrt{3} + 4)$
   $= 2\sqrt{3} - 4 - 3(3 + 4\sqrt{3} - 2\sqrt{3} - 8)$
   $= 2\sqrt{3} - 4 - 9 - 6\sqrt{3} - 24$
   $= -4\sqrt{3} - 37$

3. Let $f(x) = -x^2 + 2$ and let $g(x) = \sqrt{|x| - 1}.$

   (a) $f(3) = -3^2 + 2 = -7.$

   (b) $g(4) = \sqrt{5 - 1} = 2.$

   (c) Domain of $f(x)$ is $(-\infty, \infty)$ or $\mathbb{R}.$

   (d) Range of $f(x)$ is $(-\infty, 2)$ or $x \le 2.$

   (e) Domain of $g(x)$ is $(-\infty, -1] \cup [1, \infty).$

   (f) Range of $g(x)$ is $(0, \infty)$ or $x \ge 0.$

   (g) $f(g(x)) = f(\sqrt{|x| - 1}) = -(\sqrt{|x| - 1})^2 + 2 = -|x| + 3.$

4a. $\frac{1}{5}x + \frac{1}{5}x + 25 + 35 = x$
   $\frac{2}{5}x + 60 = x$
   $\frac{3}{5}x = 60$
   $x = 100$

4b. Draw a diagram. Let x be the height the ladder reaches. $\tan \theta = \dfrac{opp}{adj}$, so $\tan 60° = \dfrac{x}{3} \Rightarrow x = 3 \tan 60° \approx 5.2m.$

5a. $10 = 2x \Rightarrow x = 5.$

5b. Here we have three things multiplied together equaling zero, so either the first or second or third (or all three) things have to equal zero. $t = 0, 4t - 2 = 0 \Rightarrow t = \frac{1}{2}$ or $2t + 8 = 0 \Rightarrow x = -4.$

5c. $x = \pm 5.$

6a. Rising line to the right, $y-$intercept -2, $x-$intercept $\frac{2}{3}.$

6b. Happy parabola (starting high and finishing high), $y-$intercept -5, $x-$intercepts $-1, 5.$

7. (1) Graph D (2) Graph G (3) Graph A (4) Graph K
   (5) Graph C (6) Graph O (7) Graph L (8) Graph B

8. **(a)i)** $m = \dfrac{3-0}{4-1} = 1$, so $y = x + c$. Sub in $x = 1, y = 0$ in and c therefore equals -1. So the equation is $y = x - 1$.

(a)ii) $m = \dfrac{6-0}{2-2}$ which is not defined. Looking at the two points we can see that the line is vertical, so of the form $x = c$. The $x-$ coordinate of both points is 2, so the equation is $x = 2$.

(a)ii) The two lines have to equal each other when they intersect, and the second line is $x = 2$ so we can substitute this into $y = x - 1$. So the point of intersection is $(2, 1)$.

(b) Let $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (4, 3)$. Then the distance d is given by $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(1-4)^2 + (0-3)^2} = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$.

(c) Multiplying Bill's line by 2 gives $4x + 6y = 12$. Add this to Ben's line and we get $0 = 18$ which doesn't make sense. Hence the lines are parallel and their paths will not cross.

9. Bank A $\quad F = 2000 \times (1 + \frac{0.11}{12})^{60} \approx \$3457.83$

   Bank B $\quad$ 1st year $F = 2000 \times (1 + \frac{0.12}{12})^{12} \approx \$2253.65$

   Years 2-5 $F \approx 2253.65 \times (e^{0.1 \times 4}) \approx \$3362.05$

   Bank C $\quad F = 2000 \times (1 + \frac{0.13}{2})^{10} \approx \$3754.27$

   Therefore Bank C is better. (These figures will vary depending on whether you use the calculators values or the ones supplied in the question.)

10. **(a)** $y' = 6x^3 - 5 + 6e^{3x}$

    (b) As we have a fraction, let's use quotient rule with $u = 4n - n^3 + 2$ and $v = 3n^4$. Then $u' = 4 - 3n^2$ and $v' = 12n^3$. So

    $$y' = \frac{u'v - uv'}{v^2} = \frac{(4 - 3n^2) \times 3n^4 - (4n - n^3 + 2) \times 12n^3}{(3n^4)^2} = \frac{12n^4 - 9n^6 - 48n^4 + 12n^6 - 24n^3}{9n^8} = \frac{6x^2}{9n^8}$$

    (c) Let $u = (2x^3 - 1)$, so $u' = 6x^2$, and $y = -3u^{-4} - 7\sin x + \ln x$. Using the chain rule we get $y' = -3u^{-4} \times 6x^2 - 7\cos x + \dfrac{1}{x} = -18x^2(2x^3 - 1)^{-4} - 7\cos x + \dfrac{1}{x}$.

11. **(a)** $\displaystyle\int -5\,dx = -5x + C$

    (b) $\displaystyle\int (2x + e^{4x} + x^{-1}) = x^2 + \frac{1}{4}e^{4x} + \ln x + C$

    (c) $\displaystyle\int (2x + 4x + 6x)\,dx = \int 12x\,dx = 6x^2 + C$

12. **(a)** $v(t) = S'(t) = 2at - b$. Now $v(0) = 7$ so $b = -7$, and $v(1) = 13$, so $2a + 7 = 13$ so $a = 3$. So $v(t) = 3t - 7$.

    (b) $a(t) = v'(t) = 3$. So $3t - 7 = 4 \times 3$, so $t = \dfrac{19}{3}$.

    (c) $S(3) = 3 \times 3^2 + 7 \times 3 = 48$, $S(1) = 3 \times 1^2 + 7 \times 1 = 10$, so the rocket travels $38m$ between $t = 1$ and $t = 3$.

13a. $y = x^2$, so $y' = 2x$. The derivative is the slope, so at the point $(3, 9)$ the slope is $2 \times 3 = 6$. Therefore the equation of the tangent line at $(3, 9)$ is $y = 6x + C$. We can substitute $x = 3$ and $y = 9$ in, giving $9 = 6 \times 3 + C$, so $C = -9$. Hence the answer is $y = 6x - 9$.

13b. As the normal to the curve is perpendicular to the tangent at that point, the gradient of the normal is $-\dfrac{1}{6}$.

Therefore the equation of the normal at $(3, 9)$ is $y = -\dfrac{1}{6}x + C$. We can substitute $x = 3$ and $y = 9$ in, giving $9 = -\dfrac{1}{6} \times 3 + C$, so $C = \dfrac{19}{2}$. Hence the answer is $y = -\dfrac{1}{6}x + \dfrac{19}{2}$.

14. $f(x) = (x - 1)e^x + 2$. We can use the product rule to differentiate $f(x)$.

Let $u = x - 1$, so $u' = 1$, and let $v = e^x$, so $v' = e^x$.

So $f'(x) = u'v + uv'$
$= 1 \times e^x + (x - 1) \times e^x$
$= e^x + xe^x - e^x$
$= xe^x$

At a critical point we have $f'(x) = 0$. So $xe^x = 0$, so $x = 0$ or $e^x = 0$. The latter is not possible, so $x = 0$. When $x = 0$, $y = 1$.

$f''(x) = e^x + xe^x$, and $f''(0)$ is positive, so $(0, 1)$ is a local minimum.

14.22 Final exam, June 2009

1. Answer each of the following questions. Each part is worth 1 mark. (6 marks)

   (a) Find all x for which $2x - 1 > x + 2$

   (b) Evaluate $\dfrac{1}{3} - \dfrac{2}{3} \div \dfrac{2}{6}$

   (c) Find $\log_8 64$

   (d) What is the value of the gradient, m, and y-intercept, c, of the line $3y - 2x - 1 = 4x - 4$?

   (e) Which are the following are linear functions? Circle your answers.
   $$y = 3x \qquad x = -\pi \qquad y = \frac{1}{x} \qquad y = x^2$$
   $$y^2 = x^2 + 4 \qquad x^2 + xy = yx + x^2 - y \qquad x = 3y - 4$$

2. Find all x for which $|2x + 5| - 4 = 11$ (2 marks)

3. Let $f(x) = 3 + \sqrt{2 - x}$ and let $g(x) = 5 - x^2$.

(a) Determine $f(-2)$. (1 mark)

(b) Determine $g(4)$. (1 mark)

(c) Find the domain of $f(x)$. (1 mark)

(d) Find the range of $f(x)$. (1 mark)

(e) Find the domain of $g(x)$. (1 mark)

(f) Find the range of $g(x)$. (1 mark)

(g) Determine $f(g(x))$. (2 marks)

4. Solve for x where $\dfrac{(2x^6)^2}{x^5 x^6} = 2^4$. (4 marks)

5. Show that $x = \sqrt{3}$ is a solution to the equation
$$x^2 + 2^3 - \frac{\sqrt{12}}{2} - 18 \div 3 + \frac{x^4}{x} - 5x^0 - \sqrt{3} \times 2 = 0.$$ (4 marks)

6. Write the following in summation (sigma) notation:
$-3x + 1 - 5x + 2 - 7x + 3 - 9x + 4$ (2 marks)

7. (a) Find the equation of the line passing through the points $(-2, -3)$ and $(4, 9)$. (3 marks)

   (b) Find the equation of the line parallel to $y = -3x + 1$ and passing through the point $(6, -15)$.
   (3 marks)

8. (a) Find the straight line distance between the points $(4, -2)$ and $(1, -3)$, expressing your answer as a surd in simplest form. (3 marks)

   (b) Find the equation of the line perpendicular to $2x - y = 3 + 2x$ and passing through the point $(6, -15)$.
   (3 marks)

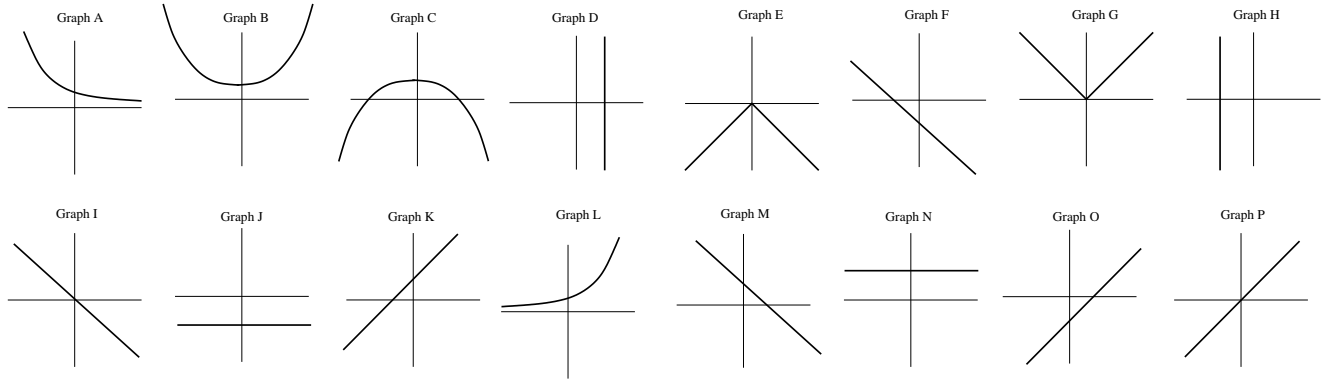9. Solve each of the following equations.

   (a) $\displaystyle\sum_{0}^{1}(ix + x) = 6x + 12$ (2 marks)

   (b) $2c(3c - 6) = 0$ (2 marks)

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(c) $y = x^2 - 4$. Sketch the graph, showing all $x-$ and $y-$ intercepts. (4 marks)

10. Below are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to P. (8 marks)

(1) $y = \frac{-x^2}{2} + \pi$ (2) $y = e^{-\frac{1}{2}x}$ (3) $y + x - 3 = 0$ (4) $x + 1 = 4$

(5) $-2y = \mid x \mid$ (6) $y = e$ (7) $y + 2 = x + 2$ (8) $-y + 3^2 = x + 9$

| Graph A | Graph B | Graph C | Graph D | Graph E | Graph F | Graph G | Graph H |
| Graph I | Graph J | Graph K | Graph L | Graph M | Graph N | Graph O | Graph P |

11. **(a)** Solve the following system of two simultaneous equations.
$$\begin{aligned} x - 4y &= -4 \\ -3x + 12y &= 4 \end{aligned}$$
(3 marks)

(b) Solve the following system of two simultaneous equations.
$$\begin{aligned} \frac{4x}{3} - y &= 2 \\ 2x + \frac{y}{2} &= -1 \end{aligned}$$
(3 marks)

12. I have \$500 to invest for 5 years. Bank A pays 11% interest per annum, compounding monthly for 5 years. Bank B has a special introductory offer of interest at 12% interest per annum, compounding monthly for the first year, followed by their standard offer of interest at 10% compounding continuously for the next 4 years. Which is the better investment strategy and by how much? (Ignore all fees and taxes.)

(Hint: $(1.12)^{12} \approx 3.896$, $(1.11)^5 \approx 1.685$, $1.01^{12} \approx 1.127$.

$(1.009)^{60} \approx 1.712$, $(1.009)^5 \approx 1.046$, $e^{0.5} \approx 1.649$. (6 marks)

$e^{0.4} \approx 1.492$.   )

13. In each of Parts (a), (b) and (c) find y', **simplifying where possible**.

(a) $y = -2x^4 + 7x + 3e^{2x} - 8$ (2 marks)

(b) $y = \dfrac{2x^3}{x^3 + 1}$ (4 marks)

(c) $y = -2(x^3 + 6)^{-2} + 7 \sin x$ (4 marks)

14. Find each of the following integrals.

(a) $\displaystyle\int 4 \, dx$ (1 mark)

(b) $\displaystyle\int \left(x - e^{-2x} + \frac{2}{x} + 3 \sin x \right) dx$ (4 marks)

15. A rocket takes off vertically at time $t = 0$ with **acceleration** $a(t) = 2t + 6$. When $t = 1$ the rocket has **velocity** $v(1) = 9$ and when $t = 3$ the rocket has **displacement** $S(3) = 40$.

    (a) Find an expression for the **velocity** of the rocket at any time t.   (3 marks)

    (b) Find an expression for the **displacement** of the rocket at any time t.   (3 marks)

    (c) At what time does the **velocity** of the rocket equal three times its acceleration?   (3 marks)

16. Let $f(x) = x^3 - 2x$. Find the equation of the tangent line to $f(x)$ at the point $(2, 4)$.   (5 marks)

17. Let $f(x) = (x - 1)^2(x + 2)$. Find and classify all the critical points of $f(x)$. (Your answer should include the x and y-coordinates of all critical points.)   (10 marks)

14.23   Solutions to final exam, June 2009

1. **(a)** $2x - 1 > x + 2$, so $x > 3$.

   (b) $\dfrac{1}{3} - \dfrac{2}{3} \div \dfrac{2}{6} = \dfrac{1}{3} - \dfrac{2}{3} \times \dfrac{6}{2} = \dfrac{1}{3} - 2 = -\dfrac{5}{3}$.

   (c) $64 = 8^2$, so $\log_8 64 = 2$.

   (d) $3y - 2x - 1 = 4x - 4$, so $3y = 6x - 3$, so $y = 2x - 1$. So gradient, $m = 2$ and y-intercept, $c = -1$.

   (e) $y = 3x, x = -\pi, x^2 + xy = yx + x^2 - y, x = 3y - 4$

2. $|2x + 5| = 15$, so $2x + 5 = 15$ or $2x + 5 = -15$. Hence $x = 5$ or $x = -10$.

3. Let $f(x) = 3 + \sqrt{2 - x}$ and let $g(x) = 5 - x^2$.

(a) $f(-2) = 3 + \sqrt{4} = 5$.

(b) $g(4) = 5 - 4^2 = -11$.

(c) Domain of $f(x)$ is $(-\infty, 2]$ or $x \leq 2$.

(d) Range of $f(x)$ is $[3, \infty)$ or $x \geq 3$.

(e) Domain of $g(x)$ is $(-\infty, \infty)$ or $\mathbb{R}$.

(f) Range of $g(x)$ is $(-\infty, 5]$ or $x \leq 5$.

(g) $f(g(x)) = f(5 - x^2) = 3 + \sqrt{2 - (5 - x)^2} = 3 + \sqrt{x^2 - 3}$.

4. Solve for x where $\dfrac{(2x^6)^2}{x^5 x^6} = 2^4$.

   $\dfrac{4x^{12}}{x^{11}} = 16$, so $4x = 16$, so $x = 4$.

5. Show that $x = \sqrt{3}$ is a solution to the equation
   $$x^2 + 2^3 - \frac{\sqrt{12}}{2} - 18 \div 3 + \frac{x^4}{x} - 5x^0 - \sqrt{3} \times 2 = 0.$$
   LHS$= (\sqrt{3})^2 + 8 - \dfrac{2\sqrt{3}}{2} - 6 + x^3 - 5 - 2\sqrt{3}$
   $= 3 + 8 - \sqrt{3} - 6 + 3\sqrt{3} - 5 - 2\sqrt{3}$
   $= 0 =$RHS

6. $-3x + 1 - 5x + 2 - 7x + 3 - 9x + 4 = \displaystyle\sum_{i=1}^{4}((-2i - 1)x + i)$ Any correct answer is fine.

7. **(a)** Let $(x_1, y_1) = (-2, -3)$ and $(x_2, y_2) = (4, 9)$. Then $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{9 - (-3)}{4 - (-2)} = \dfrac{12}{6} = 2$. Hence equation is $y = 2x + c$. Now $(4, 9)$ is on the line, so $9 = 2 \times 4 + c$, so $c = 1$. Hence the answer is $y = 2x + 1$.

   (b) $m = -3$, so $y = -3x + c$ is the equation. Now $(6, 15)$ is on the line, so $-15 = -3 \times 6 + c$, so $c = 3$. Hence the answer is $y = -3x + 3$.

8. **(a)** Let $(x_1, y_1) = (4, -2)$ and $(x_2, y_2) = (1, -3)$. Then the distance d is given by
   $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(4 - 1)^2 + (-2 - (-3))^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$.

   (b) The original equation simplifies to $y = -3$, which means that the perpendicular line has to be of the form $x = c$. As $(6, 15)$ is on the line, the answer is $x = 6$.

9. Solve each of the following equations.

   (a) $\sum_{0}^{1}(ix + x) = 6x + 12$

   LHS$= x + x + x = 3x$. So $3x = 6x + 12$, so $-3x = 12$, so $x = -4$

   (b) $2c(3c - 6) = 0$

   Here we have two things multiplied together equaling zero, so either the first or second (or both) things have to equal zero.

   So $2c = 0$, so $c = 0$, or $3c - 6 = 0$, so $c = 2$. We have two answers: $c = 0$ or $c = 2$.

   (c) $y = x^2 - 4$. We could use the quadratic formula to solve $0 = x^2 - 4$ (with $a = 1, b = 0, c = -4$) to find the x-intercepts. There is a quicker way though.

   $x^2 - 4 = 0$ is the same as $x^2 = 4$, which means that $x = 2$ or -2. You can easily check these answers by using the quadratic formula. The graph is a happy quadratic, with a y-intercept of -4, and crossing the x-axis at -2 and 2. The graph is symmetrical about the y-axis, meaning that $(0, -4)$ is the turning point.

10.
   (1) Graph C   (2) Graph A   (3) Graph M   (4) Graph D
   (5) Graph E   (6) Graph N   (7) Graph P   (8) Graph I

11. (a) Multiplying the first equation by 3 gives $3x - 12y = -12$. Adding this to the second equation gives $0 = -8$. This is clearly false, so there is no solution to this pair of equations. The lines are parallel.

    (b) Multiplying the first equation by 3 and the second equation by -2 gives $4x - 3y = 6$ and $-4x - y = 2$. Adding these gives $-4y = 8$, so $y = -2$ and hence $x = 0$. Hence the solution is $(0, -2)$, which is checked by substituting into both equations.

12. Bank A $F = 500 \times (1 + \frac{0.11}{12})^{60} \approx \$856$

    Bank B   1st year $F = 500 \times (1 + \frac{0.12}{12})^{12} \approx \$563.50$

    Years 2-5 $F \approx 563.50 \times (1 + e^{0.1 \times 4}) \approx \$840.74$

    Therefore Bank A is better by approximately \$16.74. (These figures will vary depending on whether you use the calculators values or the ones supplied in the question.)

13. (a) $y' = -8x^3 + 7 + 6e^{2x}$

    (b) As we have a fraction, let's use quotient rule with $u = 2x^3$ and $v = x^3 + 1$. Then $u' = 6x^2$ and $v' = 3x^2$. So

    $$y' = \frac{u'v - uv'}{v^2} = \frac{6x^2 \times (x^3 + 1) - 2x^3 \times 3x^2}{(x^3 + 1)^2} = \frac{6x^5 + 6x^2 - 6x^5}{(x^3 + 1)^2} = \frac{6x^2}{(x^3 + 1)^2}$$

    (c) Let $u = (x^3 + 6)$, so $u' = 3x^2$, and $y = -2u^{-2} + 7\sin x$. Using the chain rule we get $y' = 4u^{-3} \times 3x^2 + 7\cos x = 12x^2(x^3 + 6)^{-3} + 7\cos x$.

14. (a) $\int 4\,dx = 4x + C$

    (b) $\int (x - e^{-2x} + \frac{2}{x} + 3\sin x)\,dx = \frac{x^2}{2} + \frac{1}{2}e^{-2x} + 2\ln x - 3\cos x + C$.

15. (a) $v = \int a\,dt = \int (2t + 6)\,dt = t^2 + 6t + C$. Now $v(1) = 9$, so $1^2 + 6 + C = 9$, so $7 + C = 9$, so $C = 2$. Hence $v(t) = t^2 + 6t + 2$.

(b) $S = \int v\, dt = \int (t^2 + 6t + 2)\, dt = \dfrac{t^3}{3} + 3t^2 + 2t + C$. Now $S(3) = 40$, so $\dfrac{3^3}{3} + 3 \times 3^2 + 2 \times 3 + C = 40$, so $C = -2$. Hence $S(t) = \dfrac{t^3}{3} + 3t^2 + 2t - 2$.

(c) $v(t) = 3a(t)$, so $t^2 + 6t + 2 = 3(2t + 6)$, so $t^2 + 6t + 2 = 6t + 18$, so $t^2 = 16$, so $t = \pm 4$. However, we can't have negative time, so $t = 4$.

16. $f(x) = x^3 - 2x$, so $f'(x) = 3x^2 - 2$. The derivative is the slope, so at the point $(2, 4)$ the slope is $3 \times 2^2 - 2 = 10$. Therefore the equation of the tangent line at $(2, 4)$ is $y = 10x + C$. We can substitute $x = 2$ and $y = 4$ in, giving $4 = 10 \times 2 + C$, so $C = 16$. Hence the answer is $y = 10x - 16$.

17. $f(x) = (x - 1)^2(x + 2)$. We can use the product rule to differentiate $f(x)$.

Let $u = (x - 1)^2$, so $u' = 2(x - 1)$ (using the chain rule), and let $v = x + 2$, so $v' = 1$.

So $f'(x) = u'v + uv'$
$= 2(x - 1) \times (x + 2) + (x - 1)^2 \times 1$.
$= 2(x^2 + x - 2) + x^2 - 2x + 1$
$= 2x^2 + 2x - 4 + x^2 - 2x + 1$
$= 3x^2 - 3$

At a critical point we have $f' = 0$. So $3x^2 - 3 = 0$, so $3x^2 = 3$, so $x^2 = 1$. This means $x = \pm 1$. When $x = -1$, $y = 4$, and when $x = 1$, $y = 0$.

$f''(x) = 6x$, and $f''(-1)$ is negative, so $(-1, 4)$ is a local maximum. $f''(1)$ is positive, so $(1, 0)$ is a local minimum.

14.24 Final exam, June 2008

1. Answer each of the following questions. Each part is worth 1 mark. (4 marks)

   (a) Find all x for which $\quad -x - 2 > 2$

   (b) Find all x for which $\quad |3x| = \dfrac{1}{4}$

   (c) Simplify $\ 3^{n+2} \div 3^n$

   (d) Find $\quad \log_3 3^{n+2}$

2. Answer each of the following questions, in each case expressing your answer in **interval form**.

   (a) Find the domain of $f(x) = \dfrac{1}{x^2 - 1}$ (2 marks)

   (b) Find the domain of $f(x) = -4\sqrt{-x}$ (2 marks)

   (c) Find the range of $f(x) = -|2x|$ (2 marks)

3. Find R if $1 = \displaystyle\sum_{i=1}^{3} \dfrac{(i+1)R}{2i}$ (5 marks)

4. Write the following in summation (sigma) notation:
   $(3x + 1) + (5x + 4) + (7x + 9) + (9x + 16) + (11x + 25) + (13x + 36)$ (3 marks)

5. (a) Find the equation of the line passing through the points $(0, 2)$ and $(2, 0)$. (3 marks)

   (b) Find the straight line distance between the points $(0, 2)$ and $(2, 0)$, expressing your answer as a surd in simplest form. (2 marks)

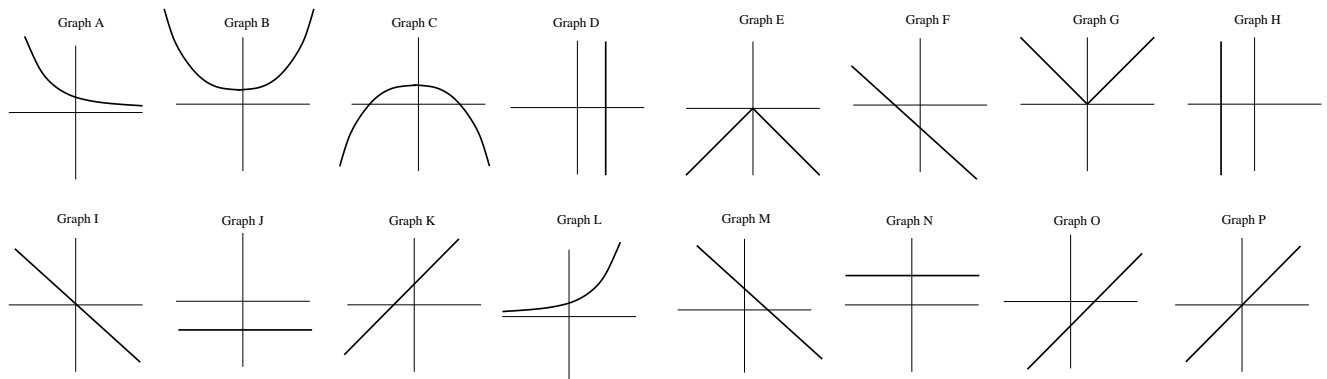6. Solve each of the following equations.

   (a) $\dfrac{6x - 9}{3} = 2x + 4$ (2 marks)

   (b) $(x - 4)(x + 4) = 9$ (2 marks)

   (c) $x^2 + 5x + 6 = 0$ (4 marks)

7. Below are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to P. (8 marks)

   (1) $3y + 4x = 0$ (2) $y = e^{-x-4}$ (3) $y = 0.0001x - \pi^2$ (4) $y - e = e^2 - e$
   (5) $|-4| = x$ (6) $-y = 2 \mid x \mid$ (7) $y - 3 = 2 - x$ (8) $y = -2x^2 + 2^2$

Graph A   Graph B   Graph C   Graph D   Graph E   Graph F   Graph G   Graph H

Graph I   Graph J   Graph K   Graph L   Graph M   Graph N   Graph O   Graph P

8. Let $f(x) = \dfrac{(\sqrt{x} + x)(x - 2\sqrt{x})}{x}$

    (a) Expand and fully simplify $f(x)$.   (3 marks)

    (b) Find $f(9) - f(3)$, simplifying your answer.   (2 marks)

9. Let $f(x) = e^x \times (e^{2x})^2 \div e^{3x}$

    (a) Fully simplify $f(x)$ (3 marks)

    (b) Find $\dfrac{f(0) - f(1)}{1 - e^2}$, simplifying your answer.   (3 marks)

10.   **(a)** Solve the following system of two simultaneous equations.
$$\begin{aligned} 8x - y &= 2 \\ -4x + 3y &= 4 \end{aligned}$$
    (3 marks)

    (b) Solve the following system of two simultaneous equations.
$$\begin{aligned} x^2 + y &= 3 \\ x^2 - 2y &= -3 \end{aligned}$$
    (3 marks)

11. I need \$1000 in 5 years' time. I have two alternative places in which I can invest. Bank A pays 12% interest per annum, compounding continuously, and Bank B pays 12% interest per annum, compounding monthly. For each bank, find the amount I would need to invest in that bank now in order to achieve my savings target? (Ignore all fees and taxes.)

    (Hint: $12^5 \approx 248,832$, $1.12^5 \approx 1.762$, $1.12^{60} \approx 897.597$.

          $1.01^5 \approx 1.051$, $1.01^{12} \approx 1.127$, $1.01^{60} \approx 1.817$.

          $e^{0.12} \approx 1.128$, $e^5 \approx 148.413$, $e^{0.6} \approx 1.822$.)

    (6 marks)

12. In each of Parts (a), (b) and (c) find y', **simplifying where possible**.

    (a) $y = -2x^{-3} - 4x + e^x + 5$ (1 mark)

    (b) $y = \dfrac{1 - e^x + x}{e^x}$ (5 marks)

    (c) $y = \sin(2x^2 + 5) + x^2$ (3 marks)

13. Let $y = \dfrac{x^3}{3} - 2x^2 + 3x$.

    (a) Find the x and y-coordinates of all critical points y.   (5 marks)

    (b) Classify each critical point as a local maximum or minimum.   (3 marks)

14. Find each of the following integrals.

(a) $\displaystyle\int (x^{-1} + 9\sin x)\, dx$ (2 marks)

(b) $\displaystyle\int_0^1 (\sqrt{x} + e^x)\, dx$ (5 marks)

(c) $\displaystyle\int_{-1}^1 (x + x^2 + x^3 + x^4)\, dx$. (Show all working.) (5 marks)

15. A ball is thrown vertically into the air at time $t = 0$ with **acceleration** $a(t) = -10$.

(a) When $t = 2$ the ball has a **velocity** $v = 0$. Find an expression for the **velocity** of the ball at any time t. (2 marks)

(b) When $t = 1$ the ball has a **displacement** $S = 22$. Find an expression for the **displacement** of the ball at any time t. (2 marks)

(c) How far does the ball travel between $t = 1$ and $t = 2$? (2 marks)

16. Shown below are three copies of the graph $y = \sin x$, for $x \in [-2\pi, 2\pi]$. In each case, sketch your graph on the set of axes.

(a) Sketch a graph of $y = 2\sin x$, for $x \in [-2\pi, 2\pi]$. (2 marks)

(b) Sketch a graph of $y = \sin 2x$, for $x \in [-2\pi, 2\pi]$. (2 marks)

(c) Sketch a graph of $y = 1 + \sin x$, for $x \in [-2\pi, 2\pi]$. (3 marks)

17. The following diagram shows the graph of the *derivative* $f'(x)$ of a function $f(x)$ on the interval $x \in [-0.5, 6.5]$.

(a) How many critical points does the *function* $f(x)$ have on this interval? (2 marks)

(b) Write down the (approximate) x-coordinate(s) of the point(s) at which the *function* $f(x)$ has a local maximum? (2 marks)

(c) Write down, in interval format, the (approximate) interval(s) on which the *function* $f(x)$ is increasing? (2 marks)

(d) Write down the (approximate) x-coordinate(s) of the point(s) at which the *function* $f(x)$ has its steepest negative slope? (2 marks)

(e) Write down the (approximate) x-coordinate(s) of the point(s) at which the *second derivative* $f''(x)$ is equal to zero? (2 marks)

14.25 Solutions to final exam, June 2008

1. **(a)** $-x - 2 > 2$, so $-x > 4$, so $x < -4$.

    (b) $|3x| = \dfrac{1}{4}$, so $|x| = \dfrac{1}{12}$. Hence $x = -\dfrac{1}{12}$ or $x = \dfrac{1}{12}$.

    (c) $3^{n+2} \div 3^n = 3^{n+2-n} = 3^2 = 9$

    (d) $\log_3 3^{n+2} = n + 2$

2. **(a)** We cannot divide by 0, so $x^2 - 1$ cannot equal 0. This means x^2 cannot equal 1, so x cannot equal ± 1. Any x other than this two is able to be put into the function. In interval form this is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

    (b) Recalling that we cannot take the square root of a negative number, we must have $-x \geq 0$, so $x \leq 0$. In interval form this is $(-\infty, 0]$.

    (c) The range is the set of all possible output values. We know that $|2x|$ has range $[0, \infty)$, and when we take the negative of of this we get the range, $(-\infty, 0]$.

3. Expanding the RHS gives $\dfrac{(1+1)R}{2 \times 1} + \dfrac{(2+1)R}{2 \times 2} + \dfrac{(3+1)R}{2 \times 3}$

    $\dfrac{2R}{2} + \dfrac{3R}{4} + \dfrac{4R}{6}$

    Using 12 as a common denominator, we have RHS$= \dfrac{12R}{12} + \dfrac{9R}{12} + \dfrac{8R}{12} = \dfrac{29R}{12}$

    So $1 = \dfrac{29R}{12}$, so $R = \dfrac{12}{29}$.

4. $(3x + 1) + (5x + 4) + (7x + 9) + (9x + 16) + (11x + 25) + (13x + 36) = \displaystyle\sum_{i=1}^{6} ((2i + 1)x + i^2)$

5. **(a)** Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (2, 0)$. Then $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0 - 2)}{2 - 0)} = \dfrac{-2}{2} = -1$. Hence equation is $y = -x + c$. Now $(2, 0)$ is on the line, so $0 = -1 \times 2 + c$, so $c = 2$.
    Hence the answer is $y = -x + 2$.

    (b) LLet $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (2, 0)$. Then the distance d is given by
    $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(0 - 2)^2 + (2 - 0)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$.

6. **(a)** $\dfrac{6x - 9}{3} = 2x + 4$, so $6x - 9 = 6x + 12$, so $-9 = 12$ which doesn't make sense. Hence there is no solution.

    (b) $(x - 4)(x + 4) = 9$, so $x^2 - 16 = 9$, so $x^2 = 25$, so $x = \pm 5$.

    (c) $x^2 + 5x + 6 = 0$. We can use the quadratic formula (with $a = 1, b = 5, c = 6$) or factorise. Let's factorise.
    $x^2 + 5x + 6 = (x + 2)(x + 3) = 0$, so either $x + 2 = 0$ or $x + 3 = 0$. So are answers are $x = -2$ or $x = -3$. Check by using the quadratic formula.

7. (1) Graph I   (2) Graph A   (3) Graph O   (4) Graph N
    (5) Graph D   (6) Graph E   (7) Graph M   (8) Graph C

8. Let $f(x) = \dfrac{(\sqrt{x} + x)(x - 2\sqrt{x})}{x}$

    (a) $f(x) = \dfrac{x\sqrt{x} - 2\sqrt{x}\sqrt{x} + x^2 - 2x\sqrt{x}}{x} = \dfrac{x^2 - x\sqrt{x} - 2x}{x} = x - \sqrt{x} - 2.$

    (b) $f(9) - f(3) = 9 - \sqrt{9} - 2 - (3 - \sqrt{3} - 2) = 9 - 3 - 2 - 3 + \sqrt{3} + 2 = 3 + \sqrt{3}$

9. Let $f(x) = e^x \times (e^{2x})^2 \div e^{3x}$

    (a) $f(x) = e^x \times e^{4x} \div e^{3x} = e^{2x}$

    (b) $\dfrac{f(0) - f(1)}{1 - e^2} = \dfrac{e^0 - e^2}{1 - e^2} = \dfrac{1 - e^2}{1 - e^2} = 1$

10.   **(a)** Multiplying the second equation by 2 gives $-8x + 6y = 8$. Adding this to the first equation gives $5y = 10$, so $y = 2$ and hence $x = \frac{1}{2}$. Hence the solution is $(\frac{1}{2}, 2)$, which is checked by substituting into both equations.

    (b) Multiplying the first equation by -1 gives $-x^2 - y = -3$. Adding this to the second equation gives $-3y = -6$, so $y = 2$ and hence $x = \pm 1$. Hence the solutions are $(-1, 2)$ and $(1, 2)$, which are checked by substituting into both equations.

11. Let A be the amount I need to invest in Bank A. Then $B \times e^{0.12 \times 5} = 1000$, so $B \times 1.822 \approx 1000$, so $B \approx 1000/1.822 \approx \$548.85$.

    Let B be the amount I need to invest in Bank B. Then $A \times (1 + 0.01)^{5 \times 12} = 1000$, so $A \times 1.817 \approx 1000$, so $A \approx 1000 \div 1.817 \approx \$550.36$.

12.   **(a)** $y' = 6x^{-4} - 4 + e^x$

    (b) We have a fraction so let's use the quotient rule with $u = 1 - e^x + x$ and $v = e^x$. Then $u' = -e^x + 1$ and $v' = e^x$. So
$$y' = \frac{u'v - uv'}{v^2} = \frac{(-e^x + 1) \times e^x - (1 - e^x + x) \times e^x}{(e^x)^2} = \frac{e^{2x} + e^x - e^x + e^{2x} - xe^x}{e^{2x}} = -\frac{xe^x}{e^{2x}} = -\frac{x}{e^x}$$

    (c) Let $u = 2x^2 + 5$, so $u' = 4x$, and the chain rule gives that $y' = 4x \times \cos(2x^2 + 5) + 2x = 4x\cos(2x^2 + 5) + 2x$.

13. $y = \dfrac{x^3}{3} - 2x^2 + 3x.$

    (a) First, $y' = x^2 - 4x + 3$. At a critical point we have $y' = 0$, so $x^2 - 4x + 3 = 0$. This is a quadratic equation, with coefficients $a = 1$, $b = -4$ and $c = 3$. Use the quadratic formula (or factoring) to solve this, giving $x = 1$ or $x = 3$.

    Hence critical points occur when $x = 1$ and $x = 3$. We need to find the y-coordinates of the critical points. When $x = 1$, $y = \frac{4}{3}$. When $x = 3$, $y = 0$.

    (b) To classify these critical points, note that $y'' = 2x - 4$. When $x = 1$, y'' is negative, so there is a local maximum. When $x = 3$, y'' is positive, so there is a local minimum.

    Hence at $(1, \frac{4}{3})$ there is a local maximum and at $(3, 0)$ there is a local minimum.

14.   **(a)** $\displaystyle\int (x^{-1} + 9\sin x)\, dx = \int \left(\frac{1}{x} + 9\sin x\right) dx = \ln x - 9\cos x + C$

    (b) $\displaystyle\int_0^1 (\sqrt{x} + e^x)\, dx = \int_0^1 (x^{\frac{1}{2}} + e^x)\, dx = \left[\frac{2}{3}x^{\frac{3}{2}} + e^x\right]_0^1 = \frac{2}{3} + e - (0 + 1) = e + \frac{1}{3}$

    (c) $\displaystyle\int_{-1}^1 (x + x^2 + x^3 + x^4)\, dx = \left[\frac{x^2}{2} + \frac{x^3}{3} + +\frac{x^4}{4} + \frac{x^5}{5}\right]_{-1}^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right)$
    $= \dfrac{2}{3} + \dfrac{2}{5} = \dfrac{10}{15} + \dfrac{6}{15} = \dfrac{16}{15}$

15. **(a)** $v = \int a\,dt = \int -10\,dt = -10t + C$. Now $v(2) = 0$, so $-10 \times 2 + C = 0$, so $-20 + C = 0$, so $C = 20$. Hence $v(t) = -10t + 20$.

(b) $S = \int v\,dt = \int -10t + 20\,dt = -5t^2 + 20t + C$. Now $S(1) = 22$, so $-5 \times 1^2 + 20 \times 1 + C = 22$, so $C = 7$. Hence $S(t) = -5t^2 + 20t + 7$.

(c) At $t = 2$ its displacement is $-5 \times 2^2 + 20 \times 2 + 7 = 27$. At $t = 1$ its displacement is 22 (this is given in the question). Hence it travels a distance of $27 - 22 = 5$.

16. **(a)** The graph is stretched vertically by a factor of 2.

(b) The graph is squashed horizontally by a factor of 2, so the graph repeats twice over the domain.

(c) Every point on the graph is moved up by 1 unit.

17. **(a)** There are four critical points, at $x = 0, 2, 4, 6$. This is when the derivative equals zero.

(b) There are two local maxima, at $x = 2, 6$. This is when the derivative goes from positive to zero to negative.

(c) The function is increasing when the derivative is positive, so on the intervals $(0, 2)$ and $(4, 6)$.

(d) The steepest negative slope is when the derivative has the smallest negative value, so at $x = 3$.

(e) The second derivative equals zero when the derivative has zero slope, so at $x = 1, 3, 5$.

14.26 Final exam, June 2007

1. Answer each of the following questions. Each part is worth 1 mark. (5 marks)

   (a) Find all x for which $\quad 3 \leq -3 - 2x$

   (b) Find all x for which $\quad |x + 1| = 2$

   (c) Evaluate $\quad \dfrac{1}{2} + \dfrac{3}{5} \div \dfrac{2}{5}\quad$ (show all working)

   (d) Find $\quad \log_3\left(\dfrac{1}{9}\right)$

   (e) Explain briefly why $\quad \log_2 8 = 3$

2. Answer each of the following questions, in each case expressing your answer in **interval form**.

   (a) Find the domain of $f(x) = \sqrt{x + 2}$ (2 marks)

   (b) Find the domain of $f(x) = \dfrac{1}{e^x}$ (2 marks)

   (c) Find the range of $f(x) = \dfrac{x^2}{2}$ (2 marks)

3. **(a)** Find the equation of the line passing through the points $(-2, -1)$ and $(-4, -5)$. (3 marks)

   (b) Find the straight line distance between the points $(-2, -1)$ and $(-4, -5)$, expressing your answer as a surd in simplest form. (3 marks)

4. Solve each of the following equations.

   (a) $\quad 2 - \dfrac{4}{x} = -1$ (2 marks)

   (b) $\quad x^2 - 4x + 5 = 0$ (2 marks)

   (c) $\quad 4x(x + 1)(x^2 + 2x - 3) = 0$ (4 marks)

5. Below are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph, by writing the le tter of the corresponding graph next to each equation. Note that the graphs are labelled A to P. (8 marks)

   (1) $y = \sqrt{23}$ (2) $y = e^{-x}$ (3) $4x = -5y$ (4) $x = 2\pi + \pi^2 - 4$
   (5) $y = -\,|-8x\,|$ (6) $3^2 + 2x + 3y = 3^7$ (7) $-2y + 5x - 4 = 2$ (8) $y = -10x^2 + 1$
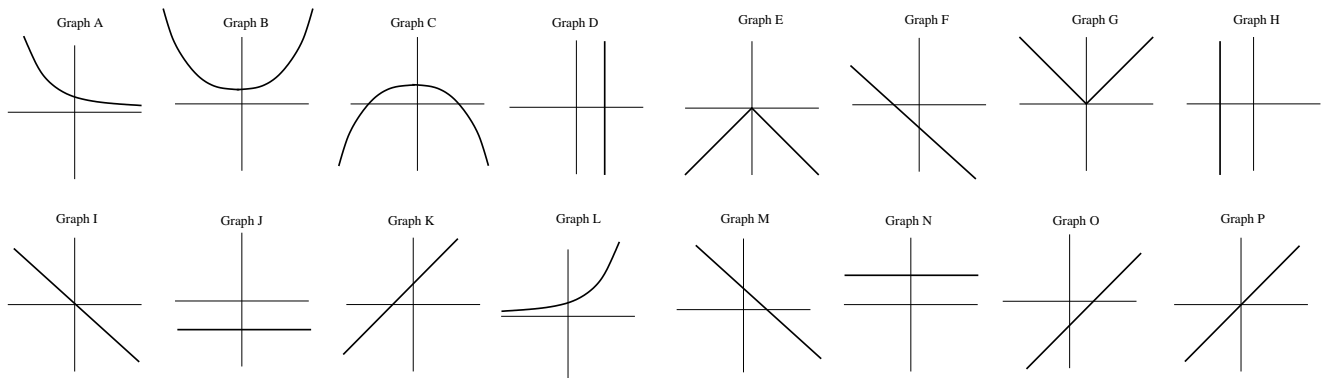
6. Let $f(x) = (2x - \sqrt{8})(x + \sqrt{8})$

   (a) Expand and simplify $f(x)$.   (3 marks)

   (b) Find $f(2\sqrt{2})$, simplifying your answer.   (2 marks)

7.   **(a)** Solve the following system of two simultaneous equations.
$$3x + 2y = -1$$
$$2x + 3y = -4$$
                                                                         (3 marks)

   (b) Solve the following system of two simultaneous equations.
$$-3x + 4y = 6$$
$$\frac{3x}{2} - 2y = 7$$
                                                                         (3 marks)

8. Simplify the following, **without using a calculator**:

$$e^2 \div e^3 \times \frac{1}{e} - e(e + e^{-3})$$
                                                                          (4 marks)

9. Define the following functions:
$$f(x) = x + 1, \ g(x) = x - 1, \ h(x) = 2x, \ n(x) = x^2 \text{ and } p(x) = 2x + 1.$$

Use composition of functions and one or more of f, g, h, n and p to define a new function $y(x) = x^2 - 2x + 1$.

                                                                          (3 marks)

10. I need \$1000 in 5 years' time. I have two alternative places in which I can invest. Bank A pays 12% interest per annum, compounding monthly, and Bank B pays 12% interest per annum, compounding continuously. For each bank, find the amount I would need to invest in that bank now in order to achieve my savings target? (Ignore all fees and taxes.)

   (Hint: $12^5 \approx 248,832$, $1.12^5 \approx 1.762$, $1.12^{60} \approx 897.597$.

             $1.01^5 \approx 1.051$, $1.01^{12} \approx 1.127$, $1.01^{60} \approx 1.817$.

             $e^{0.12} \approx 1.128$, $e^5 \approx 148.413$, $e^{0.6} \approx 1.822$.)

                                                                          (6 marks)

11.   **(a)** Write the following in summation (sigma) notation:

$$(y_2 - x_1) + (y_3 - x_2) + (y_4 - x_3) + (y_5 - x_4) + (y_6 - x_5) + \ldots$$
                                                                          (3 marks)

   (b) Solve for x: $\displaystyle\sum_{i=0}^{4} 2ix = \sum_{i=-1}^{3} ix$ (4 marks)

12. In each of Parts (a), (b) and (c) find y', **simplifying where possible**.

   (a) $y = 2x^3 - 5x + 2e^x + 10$ (1 mark)

   (b) $y = \dfrac{x^2 + 3}{2x^2}$ (4 marks)

   (c) $y = 3(2x^2 + 5)^{-1}$ (4 marks)

13. Let $y = -2x^3 + 3x^2 - 3$. Find and classify all critical points of y. (Your answer should include the x and y-coordinates of all critical points.)   (7 marks)

continued over...

14. Find each of the following integrals.

(a) $\int 0 \, dx$ (1 mark)

(b) $\int \left(e^{4x} + x^2 - \dfrac{4}{x} \right) dx$ (3 marks)

(c) $\int_0^1 (e^x - e^{-x} + 2) \, dx$ (4 marks)

15. A rocket takes off vertically at time $t = 0$ with **acceleration** $a(t) = 6t$. When $t = 1$ the rocket has **velocity** $v(1) = 5$ and **displacement** $S(1) = 6$.

(a) Find an expression for the **velocity** of the rocket at any time t.   (2 marks)

(b) Find an expression for the **displacement** of the rocket at any time t.

                                                                                                                                   (2 marks)

(c) How far does the rocket travel between $t = 1$ and $t = 2$?   (2 marks)

16. Let $f(x) = xe^{-x}$

(a) Show that $f'(x) = e^{-x}(1 - x)$ (3 marks)

(b) Find the x-coordinates of the point(s) at which the **value** of $f(x)$ is equal to the **derivative** of $f(x)$. (There is no need to find the y-coordinates.)

                                                                                                                                   (2 marks)

(c) Find $f''(x)$ (3 marks)

17. Let $f(x) = 2x - \sin(2x - \pi)$

(a) Find $f'(x)$ (3 marks)

(b) $f(x)$ has two critical points on the interval $x \in [0, 2\pi]$. Find the x-coordinates of these points.

(Hint: $\sin x = 0$ for $x = 0, \pi, 2\pi,$ and $\cos x = 0$ for $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\sin x = 1$ for $x = \dfrac{\pi}{2}, \dfrac{5\pi}{2},$ and $\cos x = 1$ for $x = 0, 2\pi$

                                                                                                                                   (6 marks)

18. In each case, roughly sketch the graph of the function $f(x)$ with the given properties. In each case, mark all given points.

(a) $f(x)$ is a polynomial of degree 4, with $f(-3) = f(-1) = f(1) = f(3) = 0$ and $f(0) = -2$.

                                                                                                                                   (3 marks)

(b) $f'(x) = -0.3$ for all $x \in (-\infty, \infty)$ and $f(4) = 0$ (2 marks)

(c) $f(-2) = 2$ $f(0) = 1$ $f'(-2) = 0$

    $f'(x) < 0$ whenever $x > -2$ $f'(x) > 0$ whenever $x < -2$

                                                                                                                                   (4 marks)

14.27 Solutions to final exam, June 2007

1. **(a)** $-3 \le -3 - 2x$, so $6 \le -2x$, so $-2x \ge 6$, so $x \le -3$.

   (b) $|x + 1| = 2$, so $x + 1 = 2$ or $x + 1 = -2$. Hence $x = 1$ or $x = -3$.

   (c) $\dfrac{1}{2} + \dfrac{3}{5} \div \dfrac{2}{5} = \dfrac{1}{2} + \dfrac{3}{5} \times \dfrac{5}{2} = \dfrac{1}{2} + \dfrac{3}{2} = \dfrac{4}{2} = 2.$

   (d) $9 = 3^{-2}$, so $\log_3 \left(\dfrac{1}{9} \right) = -2.$

   (e) $8 = 2^3$, so $\log_2 8 = 3.$

2. Answer each of the following questions, in each case expressing your answer in **interval form**.

   (a) Recalling that we cannot take the square root of a negative number, we must have $x + 2 \ge 0$, so $x \ge -2$. In interval form this is $[-2, \infty)$.

   (b) We cannot divide by 0, but we know that e^x is **never** equal to 0. Hence the domain is $(-\infty, \infty)$.

   (c) The range is the set of all possible output values. We know that x^2 has range $[0, \infty)$, and when we divide this by 2 we get the same range, $[0, \infty)$.

3. **(a)** Let $(x_1, y_1) = (-2, -1)$ and $(x_2, y_2) = (-4, -5)$. Then $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-5 - (-1)}{-4 - (-2)} = \dfrac{-4}{-2} = 2$. Hence equation is $y = 2x + c$. Now $(-2, -1)$ is on the line, so $-1 = 2 \times -2 + c$, so $c = 3$.
   Hence the answer is $y = 2x + 3$.

   (b) Let $(x_1, y_1) = (-2, -1)$ and $(x_2, y_2) = (-4, -5)$. Then the distance d is given by
   $$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-2 - (-4))^2 + (-1 - (-5))^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}.$$

4. Solve each of the following equations.

   (a) $2 - \dfrac{4}{x} = -1$ so $3 = \dfrac{4}{x}$, so $3x = 4$, so $x = \dfrac{4}{3}$.

   (b) $x^2 - 4x + 5 = 0$ is a quadratic equation, with $a = 1$, $b = -4$ and $c = 5$. Then the solution is given by:
   $$x = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2} = \frac{4 \pm \sqrt{-4}}{2}.$$
   We cannot take the square root of a negative number, so there are no solutions.

   (c) If we have an equation in which things multiply together to give 0, the only way this can happen is if one of the things equals 0. Hence we must have $4x = 0$, or $x + 1 = 0$, or $x^2 + 2x - 3 = 0$. The first two are easy to solve: $x = 0$ or $x = -1$. The third one is a quadratic equation which can be solved by factoring or using the quadratic formula with $a = 1$, $b = 2$ and $c = -3$. Using the quadratic formula we get:
   $$x = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times -3}}{2} = \frac{-2 \pm \sqrt{16}}{2} = \frac{-2 \pm 4}{2}, \text{ so } x = 1 \text{ or } x = -3.$$
   Hence the solutions are $x = 0$, $x = -1$, $x = 1$ nd $x = -3$.

5.
   (1) Graph N (2) Graph A (3) Graph I (4) Graph D
   (5) Graph E (6) Graph M (7) Graph O (8) Graph C

6. **(a)** $f(x) = 2x^2 + 2x \times \sqrt{8} - x \times \sqrt{8} - 8 = 2x^2 + x\sqrt{8} - 8.$

   (b) $f(2\sqrt{2}) = 2 \times (2\sqrt{2})^2 + 2\sqrt{2} \times \sqrt{8} - 8 = 16 + 2\sqrt{2} \times 2\sqrt{2} - 8 = 16 + 8 - 8 = 16.$

7. **(a)** Multiplying the first equation by -3 and the second equation by 2 gives $-9x - 6y = 3$ and $4x + 6y = -8$. Adding these gives $-5x = -5$, so $x = 1$ and hence $y = -2$. Hence the solution is $(1, -2)$, which is checked by substituting into both equations.

(b) Multiplying the second equation by 2 gives $3x - 4y = 14$. Adding this to the second equation gives $0 = 20$. This is clearly false, so there is no solution to this pair of equations. (That is, the lines are parallel.)

8. $e^2 \div e^3 \times \dfrac{1}{e} - e(e + e^{-3}) = e^{2-3} \times e^{-1} - e(e + e^{-3}) = e^{-1-1} - e \times e - e \times e^{-3} = e^{-2} - e^2 - e^{-2} = -e^2$.

9. This is one of the hardest questions on this paper. We need to get the function y, which includes an x^2 term. This gives us a hint: the original function $n(x) = x^2$ might help. Then $n(f(x)) = (x + 1)^2 = x^2 + 2x + 1$, which is not correct. Next, $n(g(x)) = (x - 1)^2 = x^2 - 2x + 1$, which is correct. Hence the answer is $n(g(x))$.

10. Let A be the amount I need to invest in Bank A. Then $A \times (1 + 0.01)^{5 \times 12} = 1000$, so $A \times 1.817 = 1000$, so $A = 1000 \div 1.817 = \$550.36$.

Let B be the amount I need to invest in Bank B. Then $B \times e^{0.12 \times 5} = 1000$, so $B \times 1.822 = 1000$, so $B = 1000/1.822 = \$548.85$.

11. (a) $\displaystyle\sum_{i=1}^{\infty} (y_{i+1} - x_i)$.

(Note that there are many possible answers. If yours expands to the correct expression then you will receive full marks.)

(b) To solve this we expand the left-hand side, then the right-hand side, and make them equal.

The left-hand side expands to: $2 \times 0 \times x + 2 \times 1 \times x + 2 \times 2x + 2 \times 3x + 2 \times 4x = 0x + 2x + 4x + 6x + 8x = 20x$.

The right-hand side expands to: $-x + 0x + x + 2x + 3x = 5x$.

Equating the two sides gives $20x = 5x$, so $20x - 5x = 0$, so $15x = 0$, so $x = 0$.

12. (a) $y' = 6x^2 - 5 + 2e^x$

(b) There are two ways to do this.

The first is to use the quotient rule with $u = x^2 + 3$ and $v = 2x^2$. Then $u' = 2x$ and $v' = 4x$. So

$$y' = \frac{u'v - uv'}{v^2} = \frac{2x \times 2x^2 - (x^2 + 3) \times 4x}{(2x)^2} = \frac{4x^3 - 4x^3 - 12x}{4x^4} = \frac{-12x}{4x^4} = -\frac{3}{x^3}$$

There is an easier way if you notice that the original equation can be rewritten as $y = \dfrac{x^2}{2x^2} + \dfrac{3}{2x^2} = \dfrac{1}{2} + \dfrac{3}{2}x^{-2}$. Then you can differentiate this directly, giving $y' = -2 \times \dfrac{3}{2}x^{-3} = \dfrac{-3}{x^3}$.

(c) Let $u = 2x^2 + 5$, so $u' = 4x$, and the chain rule gives that $y' = 3 \times 4x \times -1 \times (2x^2 + 5)^{-2} = -12x(2x^2 + 5)^{-2}$.

13. First, $y' = -6x^2 + 6x$. At a critical point we have $y' = 0$, so $-6x^2 + 6x = 0$. We can cancel through by 6, giving $-x^2 + x = 0$. This is a quadratic equation, with coefficients $a = -1$, $b = 1$ and $c = 0$. Use the quadratic formula (or factoring) to solve this, giving $x = 0$ or $x = 1$.

Hence critical points occur when $x = 0$ and $x = 1$. To classify these, note that $y'' = -12x + 6$. When $x = 0$, y'' is positive, so there is a local minimum. When $x = 1$, y'' is negative, so there is a local maximum.

Finally, we need to find the y-coordinates of the critical points. When $x = 0$, $y(0) = -3$. When $x = 1$, $y(1) = -2 \times 1^3 + 3 \times 1^2 - 3 = -2 + 3 - 3 = -2$.

Hence at $(0, -3)$ there is a local minimum and at $(1, -2)$ there is a local maximum.

14. **(a)** $\displaystyle\int 0\,dx = C$

(b) $\displaystyle\int \left(e^{4x} + x^2 - \frac{4}{x}\right)\,dx = \frac{1}{4}e^{4x} + \frac{x^3}{3} - 4\ln x + C.$

(c) $\displaystyle\int_0^1 (e^x - e^{-x} + 2)\,dx = \left[e^x + e^{-x} + 2x\right]_0^1 = (e^1 + e^{-1} + 2 \times 1) - (e^0 + e^{-0} + 2 \times 0) = e + e^{-1} + 2 - 1 - 1 = e + e^{-1}.$

15. **(a)** $v = \displaystyle\int a\,dt = \int 6t\,dt = 3t^2 + C.$ Now $v(1) = 5$, so $3 \times 1^2 + C = 5$, so $3 + C = 5$, so $C = 2$. Hence $v(t) = 3t^2 + 2.$

(b) $S = \displaystyle\int v\,dt = \int 3t^2 + 2\,dt = t^3 + 2t + C.$ Now $S(1) = 6$, so $1^3 + 2 \times 1 + C = 6$, so $C = 3$. Hence $S(t) = t^3 + 2t + 3.$

(c) At $t = 2$ its displacement is $2^3 + 2 \times 2 + 3 = 15$. At $t = 1$ its displacement is 6 (this is given in the question). Hence it travels a distance of $15 - 6 = 9$.

16. **(a)** Let $u = x$ and $v = e^{-x}$, so $u' = 1$ and $v' = -e^{-x}$. Then the product rule says that $f' = u'v + uv' = 1 \times e^{-x} + x \times -e^{-x}$, so $f'(x) = e^{-x}(1 - x)$.

(b) This happens when $f'(x) = f(x)$, so $e^{-x}(1 - x) = xe^{-x}$. Now we can divide each side by e^{-x}, as it never equals 0. Hence $(1 - x) = x$, so $1 = 2x$, so $x = 1/2$.

(c) We need to differentiate $f'(x) = e^{-x}(1 - x)$. Let $u = e^{-x}$ and $v = (1 - x)$, so $u' = -e^{-x}$ and $v' = -1$. Then the product rule says $f'' = u'v + uv' = -e^{-x} \times (1 - x) + e^{-x} \times -1 = e^{-x}(-1 + x - 1) = e^{-x}(x - 2)$.

17. **(a)** First, let $g(x) = \sin(2x - \pi)$. We need to use the chain rule to differentiate $g(x)$. Let $u = 2x - \pi$, so $u' = 2$ and the chain rule says $g' = 2\cos(2x - \pi)$.
Then we can use this to find the derivative of $f(x)$, giving $f'(x) = 2 - 2\cos(2x - \pi)$.

(b) At a critical point we have $f' = 0$. So $2 - 2\cos(2x - \pi) = 0$, so $2\cos(2x - \pi) = 2$, so $\cos(2x - \pi) = 1$. Now, the question tells us that $\cos x = 1$ when $x = 0$ or $x = 2\pi$.
So $\cos(2x - \pi) = 1$ when $2x - \pi = 0$ or $2x - \pi = 2\pi$. Hence $2x = \pi$ or $2x = 3\pi$, so $x = \dfrac{\pi}{2}$ or $x = \dfrac{3\pi}{2}$.

18. The answers to Parts (a), (b) and (c) are shown in order from left to right.
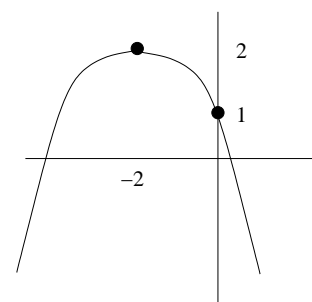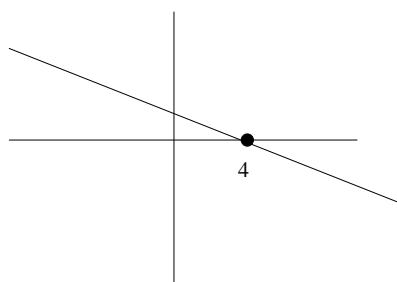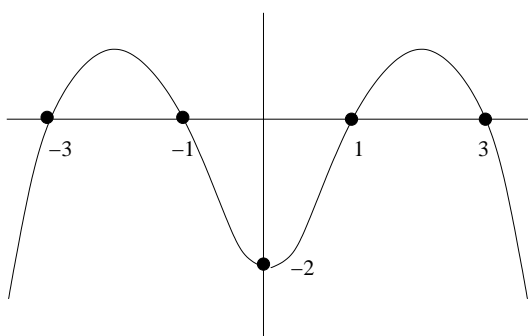
14.28 Final exam, June 2006

1. Answer each of the following questions. Each part is worth 1 mark.

   (a) Find all x for which $-x - 3 < 1$.

   (b) If $f(x) = -x^2 - x$, find $f(2)$.

   (c) If $f(x) = -x^2 - x$, find $f(-1)$.

   (d) Find all values of x for which $(3x - 1)(4x + 3) = 0$.

   (e) Explain briefly why $\log_{10} 100 = 2$.

   (f) Find $\log_{16} 4$.

   (g) Explain briefly why $\ln 1 = 0$.

   (h) Find the derivative of $f(x) = 3x^2 - 6x + e^x$.

2. Below are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to P.   (8 marks)

   (1) $y = \pi^2 + 2\pi + 1$ (2) $y = |-2x|$ (3) $y = -x^2 + 4$ (4) deleted
   (5) $x = \sqrt{69}$ (6) $-y + x + 2 = 0$ (7) $4y = -3x$ (8) $13x = -5y - 4^8$
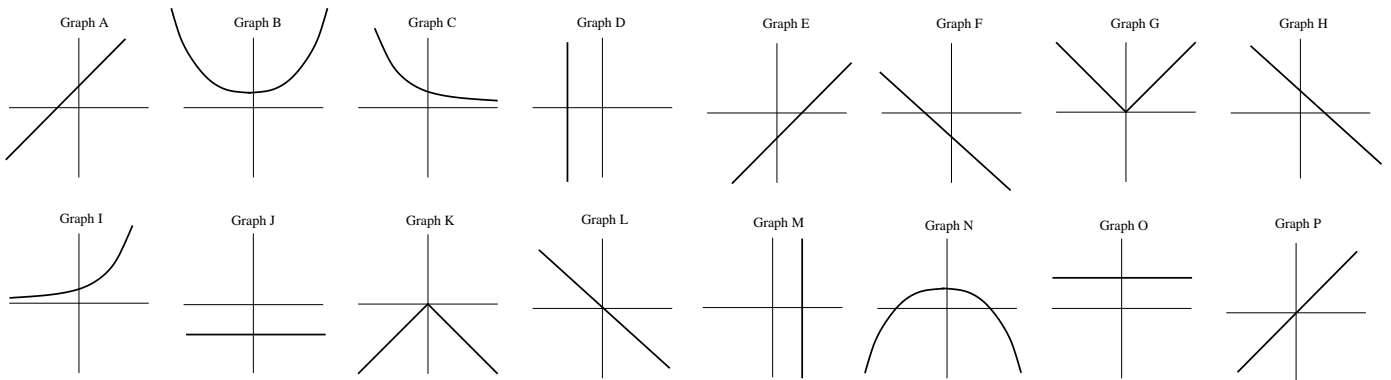
3. Find the equation of the line passing through the points $(-1, -1)$ and $(-2, -7)$.   (3 marks)

4. (a) Find the **domain** of $f(x) = \dfrac{1}{\sqrt{x} - 2}$. Write your answer in interval format.

                                                                                  (2 marks)

   (b) Find the **range** of $g(x) = |x| - 2$. Write your answer in interval format.

                                                                                  (2 marks)

5. Let $f(x) = (x - \sqrt{12})(x + \sqrt{3})$.

   (a) Expand and simplify $f(x)$.   (4 marks)

   (b) Find $f(\sqrt{3})$, expressing your answer as a surd in simplest form.

                                                                                  (3 marks)

6. (a) Solve the following system of two simultaneous equations.
$$
\begin{aligned}
2x + 5y &= 6 \\
3x + 4y &= 9
\end{aligned}
$$
                                                                                  (4 marks)

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(b) Solve the following system of two simultaneous equations.

$$x + \frac{y}{2} = 2$$
$$-2x - y = -4$$

(3 marks)

(c) Find the x and y coordinates of **two** points that satisfy the equations in Part (b). (2 marks)

7. Simplify the following, **without using a calculator**: (3 marks)

$$e^2\left(e^3 + e^{-2}\right) - \frac{e}{(e^{-3})^2}.$$

8. Let $f(x) = x^2 - 4$, and let $g(x) = 2(x - 1)$. Find and simplify $f(g(x))$. (4 marks)

9. I have \$1000 to invest for 5 years. Bank A pays interest at 12% per annum, compounding annually for 2 years, then compounding continuously for the next 3 years. Bank B pays interest at 12% per annum, compounding monthly for 5 years. Which is the better investment strategy, and by how much? (Ignore all fees and taxes.)

(Hint: $1.12^2 \approx 1.254$, $1.12^5 \approx 1.762$, $1.12^{60} \approx 897.597$.

$1.01^5 \approx 1.051$, $1.01^{36} \approx 1.431$, $1.01^{60} \approx 1.817$.

$e^{0.12} \approx 1.128$, $e^{0.36} \approx 1.433$, $e^{0.6} \approx 1.822$.   )

(6 marks)

10. I toss a fair coin 6 times in a row. In each of Parts (a) to (c), find the probability that:

(a) All 6 tosses come up Heads? (2 marks)

(b) The first 5 tosses are Heads or the last 5 tosses are Heads (or both)? (2 marks)

(c) All 6 tosses are Heads given that the first 4 tosses are Heads? (2 marks)

11. Find R if $\dfrac{1+R}{R} = \displaystyle\sum_{i=1}^{3} \dfrac{1}{2i-1}$.   (Leave your answer as a fraction; show all working.)

(5 marks)

12. Let $f(x) = x^2 - 5$.

(a) Find $f(-2)$. (1 mark)

(b) Solve $f(x) = 20$. (2 marks)

(c) Solve $|f(x)| = 4$. (4 marks)

13. In each of Parts (a), (b) and (c) find y', **simplifying where possible**.

(a) $y = 2(e^{2x} + 1)^9$ (3 marks)

(b) $y = \dfrac{x^2 - x + 1}{x + 1}$ (4 marks)

(c) $y = \sqrt{e^x}$. (3 marks)

14. Let $y = 2x^3 + 3x^2 - 12x + 6$. Find and classify all critical points of y. (Your answer should include the x and y-coordinates of all critical points.) (7 marks)

15. Find each of the following integrals.

(a) $\displaystyle\int (6x - 2)\, dx$ (1 mark)

(b) $\displaystyle\int \left(\frac{2}{x} + e^{-x}\right) dx$ (2 marks)

(c) $\displaystyle\int_1^4 3\sqrt{x}\, dx$ (4 marks)

16. A rocket takes off vertically at time $t = 0$ with acceleration $a(t) = pt + 2$, where p is a constant. When $t = 0$ the rocket has **velocity** $v(0) = 0$ and **displacement** $S(0) = 0$. When $t = 1$ the rocket has **displacement** $S(1) = 5$.

   (a) Find the value of p.

   (7 marks)

   (b) How far does the rocket travel between $t = 1$ and $t = 2$? (2 marks)

17. Let $f(x) = (x^2 + 2x - 1)e^{-x}$. Find the x-coordinates of the point(s) at which the value of $f(x)$ is equal to the derivative of $f(x)$. (There is no need to find the y-coordinates.)

   (6 marks)

18. The following diagram shows the graph of the **derivative** $f'(x)$ of a function $f(x)$ on the interval $x \in [-3.5, 3.5]$.
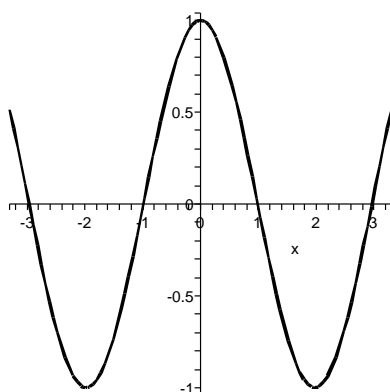
   (a) How many critical points does the function $f(x)$ have on this interval?

   (2 marks)

   (b) Write down the (approximate) x-coordinate(s) of the point(s) at which the function $f(x)$ has a local maximum.

   (2 marks)

   (c) Write down, in interval format, the (approximate) interval(s) on which the function $f(x)$ is decreasing.

   (2 marks)

   (d) Write down the (approximate) x-coordinate(s) of the point(s) at which the function $f(x)$ has its steepest negative slope.

   (2 marks)

   (e) Write down the (approximate) x-coordinate(s) of the point(s) at which the **second derivative** $f''(x)$ is equal to 0.

   (3 marks)

14.29   Solutions to final exam, June 2006

1. **(a)** $-x - 3 < 1$, so $-x < 4$, so $x > -4$.

    (b) $f(2) = -(2^2) - 2 = -6$.

    (c) $f(-1) = -(-1^2) + 1 = 0$.

    (d) $(3x - 1) = 0$ or $(4x + 3) = 0$, so $x = 1/3$ or $x = -3/4$.

    (e) $100 = 10^2$, so 2 is the power to which we raise 10 in order to get 100. Hence the log (to base 10) of 100 is 2.

    (f) $4 = 16^{1/2}$, so $\log_{16} 4 = 1/2$.

    (g) Anything to the power 0 equals 1, so $e^0 = 1$, so $\ln 1 = 0$.

    (h) $f'(x) = 6x - 6 + e^x$.

2. (1) Graph O   (2) Graph G   (3) Graph N   (4) Deleted
    (5) Graph M   (6) Graph A   (7) Graph L   (8) Graph F

3. Let $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (-2, -7)$. Then $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-7 - (-1)}{-2 - (-1)} = \dfrac{-6}{-1} = 6$. Hence equation is $y = 6x + c$. Now $(-1, -1)$ is on the line, so $-1 = 6 \times -1 + c$, so $c = 5$.

    Hence the answer is $y = 6x + 5$.

4. **(a)** We cannot find the square root of a negative number, and cannot divide by 0. Hence we must have $x \geq 0$ and $\sqrt{x} - 2 \neq 0$. Hence $x \geq 0$ and $x \neq 4$. In interval format, this is $[0, 4)(4, \infty)$.

    (b) We know that $|x| \geq 0$. Hence the range is $[-2, \infty)$.

5. **(a)** $f(x) = x^2 + \sqrt{3}x - 2\sqrt{3}x - \sqrt{36} = x^2 - \sqrt{3}x - 6$.

    (b) $f(\sqrt{3}) = 3 - 3 - 6 = -6$.

6. **(a)** Multiplying the first equation by -3 and the second equation by 2 gives $-6x - 15y = -18$ and $6x + 8y = 18$. Adding these gives $-7y = 0$, so $y = 0$ and hence $x = 3$. Hence the solution is $(3, 0)$, which is checked by substituting into both equations.

    (b) Multiplying the first equation by -2 gives $-2x - y = -4$, which is identical to the second equation. Hence there is an infinite number of solutions.

    (c) Any point on the first line will also be on the second line, so we can find any two such points that we want. For example, substitute $x = 0$ into the first equation, giving $y = 4$, so $(0, 4)$ is one such point. Then substitute $y = 0$ into the equation giving $x = 2$, so $(2, 0)$ is a second point.

7. $e^2 \left(e^3 + e^{-2} \right) - \dfrac{e}{(e^{-3})^2} = e^{3+2} + e^{-2+2} - \dfrac{e}{e^{-6}} = e^5 + e^0 - e^7 = e^5 + 1 - e^7$.

8. $f(g(x)) = (2(x - 1))^2 - 4 = 4(x^2 - 2x + 1) - 4 = 4x^2 - 8x$.

9. For Bank A, after two years the balance is $1000(1 + 0.12)^2 = 1254$. Then this amount is invested at continuous compouding for three more years, so the final balance is $1254e^{0.12 \times 3} = 1254e^{0.36} = 1797.40$.

    For Bank B, after 5 years the balance is $1000(1 + 0.12/12)^{5 \times 12} = 1000(1.01)^{60} = 1817$.

    Hence Bank B is the better strategy, by $1817 - 1797.40 = \$19.60$.

10. I toss a fair coin 6 times in a row. In each of Parts (a) to (c), find the probability that:

    (a) Prob equals $(1/2)^6 = 1/64$.

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(b) Prob(first 5 are heads) equals $(1/2)^5 = 1/32$.

Prob(last 5 are heads) equals $1/32$.

By the principle of inclusion/exclusion, Prob(first 5 heads OR last 5 heads) equals $1/32+1/32-1/64 = 3/64$.

(c) All tosses are independent, so if first 4 are heads, Prob(all heads) = Prob(last two are heads) equals $(1/2)^2 = 1/4$.

11. $RHS = \sum_{i=1}^{3} \dfrac{1}{2i-1} = \dfrac{1}{2-1} + \dfrac{1}{4-1} + \dfrac{1}{6-1} = \dfrac{1}{1} + \dfrac{1}{3} + \dfrac{1}{5} = \dfrac{15}{15} + \dfrac{5}{15} + \dfrac{3}{15} = \dfrac{23}{15}$.

Hence $\dfrac{1+R}{R} = \dfrac{23}{15}$, so $15(1+R) = 23R$, so $15 + 15R = 23R$, so $15 = 8R$, so $R = \dfrac{15}{8}$.

12. Let $f(x) = x^2 - 5$.

(a) $f(-2) = 4 - 5 = -1$.

(b) $f(x) = 20$, so $x^2 - 5 = 20$, so $x^2 = 25$, so $x = 5$ or $x = -5$.

(c) $|f(x)| = 4$, so $|x^2 - 5| = 4$, so $x^2 - 5 = 4$ or $x^2 - 5 = -4$. Solve each of these separately.
If $x^2 - 5 = 4$ then $x^2 = 9$ so $x = 3$ or $x = -3$. If $x^2 - 5 = -4$ then $x^2 = 1$, so $x = 1$ or $x = -1$.
Hece the answer is $x = -3$, $x = -1$, $x = 1$ or $x = 3$.

13. **(a)** $y' = 2 \times 2e^{2x} \times 9 \times (e^{2x} + 1)^8 = 36e^{2x}(e^{2x} + 1)^8$.

(b) Let $u = x^2 - x + 1$ and $v = x + 1$, so $u' = 2x - 1$ and $v' = 1$. By the quotient rule, $y' = \dfrac{u'v - uv'}{v^2} = \dfrac{(2x-1)(x+1) - (x^2 - x + 1)}{(x+1)^2} = \dfrac{2x^2 + 2x - x - 1 - x^2 + x - 1}{(x+1)^2} = \dfrac{x^2 + 2x - 2}{(x+1)^2}$.

(c) There is a trick here. Note that the function can be rewritten as $y = (e^x)^{1/2} = e^{0.5x}$. Then $y' = 0.5e^{0.5x}$.

14. First, $y' = 6x^2 + 6x - 12$. At a critical point we have $y' = 0$, so $6x^2 + 6x - 12 = 0$. We can cancel through by 6, giving $x^2 + x - 2 = 0$. This is a quadratic equation, with coefficients $a = 1$, $b = 1$ and $c = -2$. Use the quadratic formula (or factoring) to solve this, giving $x = 1$ or $x = -2$.

From above, critical points occur at $x = 1$ and $x = -2$. To classify these, note that $y'' = 12x + 6$. When $x = 1$, y'' is positive, so there is a local minimum. When $x = -2$, y'' is negative, so there is a local maximum.

Finally, we need to find the y-coordinates of the critical points. When $x = 1$, $y(1) = 2 + 3 - 12 + 6 = -1$, so $(1, -1)$ is a local minimum. When $x = -2$, $y(-2) = 2 \times (-2)^3 + 3 \times (-2)^2 - 12 \times (-2) + 6 = 26$, so there is a local maximum at $(-2, 26)$.

15. **(a)** $\displaystyle\int (6x - 2)\, dx = 3x^2 - 2x + C$

(b) $\displaystyle\int \left(\dfrac{2}{x} + e^{-x} \right) dx = 2\ln x - e^{-x} + C$

(c) $\displaystyle\int_1^4 3\sqrt{x}\, dx = \left[3 \times 2/3 \times x^{3/2} \right]_1^4 = \left[2\sqrt{x^3} \right]_1^4 = 2\sqrt{4^3} - 2\sqrt{1^3} = 2 \times 8 - 2 \times 1 = 14$.

16. A rocket takes off vertically at time $t = 0$ with acceleration $a(t) = pt + 2$, where p is a constant. When $t = 0$ the rocket has **velocity** $v(0) = 0$ and **displacement** $S(0) = 0$. When $t = 1$ the rocket has **displacement** $S(1) = 5$.

(a) $v = \displaystyle\int a\, dt = \int pt + 2\, dt = \dfrac{pt^2}{2} + 2t + C$. Now $v(0) = 0$, so $C = 0$. Hence $v(t) = \dfrac{pt^2}{2} + 2t$.

$S = \displaystyle\int v\, dt = \int \dfrac{pt^2}{2} + 2t\, dt = \dfrac{pt^3}{6} + t^2 + D$. Now $S(0) = 0$, so $D = 0$, and hence $S(t) = \dfrac{pt^3}{6} + t^2$.

Finally, $S(1) = 5$, so $\dfrac{p \times 1^3}{6} + 1^2 = 5$, so $\dfrac{p}{6} = 4$, so $p = 24$.

(b) Because $p = 24$, we have $S(t) = \dfrac{pt^3}{6} + t^2 = \dfrac{24t^3}{6} + t^2 = 4t^3 + t^2$.

At $t = 2$, $S(2) = 4 \times 2^3 + 2^2 = 36$. At $t = 1$, we are given that $S(1) = 5$. Hence between $t = 1$ and $t = 2$ the rocket travels $36 - 5 = 31$.

17. First we need to use the product rule to find the derivative.

Let $u(x) = x^2 + 2x - 1$ and $v(x) = e^{-x}$, so $u' = 2x + 2$ and $v' = -e^{-x}$.

Then $y' = u'v + uv' = (2x + 2)e^{-x} + (x^2 + 2x - 1) \times -e^{-x} = e^{-x}(2x + 2 - x^2 - 2x + 1) = e^{-x}(-x^2 + 3)$.

Now, when the value of $f(x)$ equals the slope of $f(x)$, we have $(x^2 + 2x - 1)e^{-x} = e^{-x}(-x^2 + 3)$.

Because e^{-x} never equals 0 we can cancel e^{-x}, giving $(x^2 + 2x - 1) = (-x^2 + 3)$, so $2x^2 + 2x - 4 = 0$, so $x^2 + x - 2 = 0$. This is a quadratic equation, which can be solved (using the quadratic formula or factoring) to give $x = 1$ or $x = -2$.

Hence when $x = 1$ and $x = -2$, the value of $f(x)$ equals the slope of $f(x)$.

18. **(a)** The function has a critical point when the derivative equals zero, which happens 4 times in the diagram.

(b) At a local maximum, the value of the derivative should be positive, then zero, then negative. This happens at $x = -3$ and $x = 1$.

(c) The function is decreasing when the derivative has a negative value. This happens on the intervals $[-3, -1]$ and $[1, 3]$.

(d) This happens when the derivative has its largest negative values, which occurs when $x = -2$ and $x = 2$.

(e) The second derivative equals zero when the derivative has a local maximum or minimum. This happens when $x = -2$, $x = 0$ and $x = 2$.

14.30   Final exam, June 2005

1. Find R if $R = \sum_{i=-2}^{1} \dfrac{\mid i \mid}{2i+1}$.   (3 marks)

2.   **(a)** Find all x such that $-3x - 4 \geq -2(x - 3)$.   (3 marks)

   (b) Write your answer to Part (a) in interval format.   (1 mark)

   (c) Mark your answer to Part (a) on the real line.   (1 mark)

3. For each of Parts (a) to (f), let
$$
\begin{aligned}
A &= \{x \mid x \in \mathbb{N}, x \leq 10 \text{ and } x \text{ is even}\}, \\
B &= \{x \mid x \in \mathbb{N}, x \leq 10 \text{ and } x \text{ is divisible by } 3\}, \text{ and} \\
C &= \{x \mid x \in \mathbb{N} \text{ and } 3 \leq x \leq 6\}.
\end{aligned}
$$

   (a) Write down the sets A, B and C, by listing their elements.   (2 marks)

   (b) Mark the elements of the sets A, B and C on a Venn diagram.   (2 marks)

   (c) Write down the elements of the set $(A \cap B) \cup C$.   (2 marks)

   (d) Write down the elements of the set $(A \setminus B) \setminus (C \cap A)$.   (2 marks)

   (e) Let x be a natural number between 1 and 10 (inclusive) chosen at random. In each of Parts (i), (ii) and (iii), find the probability that:

   (i) $x \in A \cup B$.   (2 marks)

   (ii) x is both even and divisible by 3.   (2 marks)

   (iii) $x \in C$ given that $x \in A$.   (2 marks)

   (f) Let x and y each be a natural number between 1 and 10 (inclusive) chosen independently at random. In each of Parts (i) and (ii), find the probability that:

   (i) both x and y are divisible by 3.   (2 marks)

   (ii) $x \in A$ or $y \in C$ (or both).   (2 marks)

4. Solve the following system of two simultaneous equations.
$$
\begin{aligned}
3x - 4y &= 5 \\
5x - 3y &= 1.
\end{aligned}
$$
     (4 marks)

5. Find the equation of the line passing through the points $(1, 1)$ and $(-2, 7)$.   (3 marks)

6. Let $f(x) = (x - \sqrt{8})(x + \sqrt{2})$.

   (a) Expand and simplify $f(x)$.   (3 marks)

   (b) Find $f(\sqrt{6})$, expressing your answer as a surd in simplest form.   (2 marks)

7. Shown below are two copies of the graph of $y = \sin x$, for $x \in [-2\pi, 2\pi]$. In each of Parts (a) and (b), sketch your graph on the set of axes. **Note: to save space, the diagram is only shown for Part (a).**

   (a) Sketch a graph of $y = \dfrac{1}{2}\sin\left(\dfrac{1}{2}x\right)$ for $x \in [-2\pi, 2\pi]$.   (2 marks)
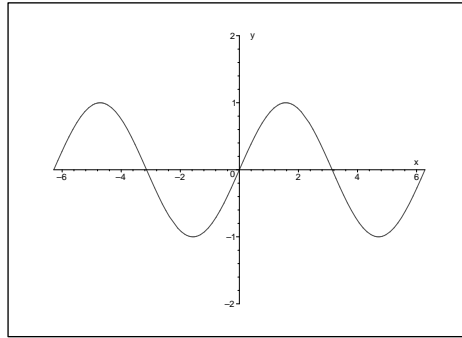
(b) Sketch a graph of $y = 1\sin 2x$ for $x \in [-2\pi, 2\pi]$. (2 marks)

8. Deleted.

9. The following diagram shows the graph of a **function** $f(x)$ on the interval $x \in [-2, 3]$. Parts (a) to (d) contain questions about the **derivative** $f'(x)$ on this interval.
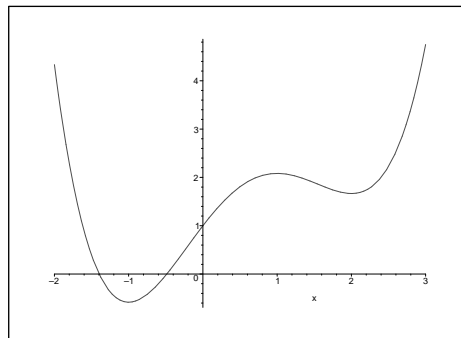
(a) Find the (approximate) x-coordinate(s) of the point(s) at which $f'(x) = 0$? (2 marks)

(b) Write down the (approximate) interval(s) on which f' is negative. (2 marks)

(c) The derivative f' has a critical point between $x = 1$ and $x = 2$. Is this critical point a local maximum or local minimum (of f')? Briefly explain your answer. (2 marks)

(d) Rewrite these values in increasing order: $f'(3), f'(1), f'(0), f'(-2)$. (2 marks)

10. In each of Parts (a) to (c), find $\dfrac{dy}{dx}$, simplifying where possible. (2 marks, 3 marks, 4 marks)

(a) $y = -2x^4 + 6x + 1$ **(b)** $y = \cos(\sqrt{x})$ **(c)** $y = x \ln x - x$

11. In each of Parts (a) and (b), find $\dfrac{dy}{dx}$, simplifying where possible. (2 marks, 3 marks)

(a) $y = 3(x^2 + 1)^4$ **(b)** $y = \dfrac{x^2 - 1}{x^2 + 1}$

12. Here are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph. (8 marks)

(1) $y = x^2 + 1$ (2) $y = e^{2x}$ (3) $0.001y - 0.004x = 0.002$ (4) $y = -2 \mid x \mid$

(5) $x - y = 0$ (6) $y = \pi^2$ (7) $x = \sqrt{27}$ (8) $y = 3^{-x}$
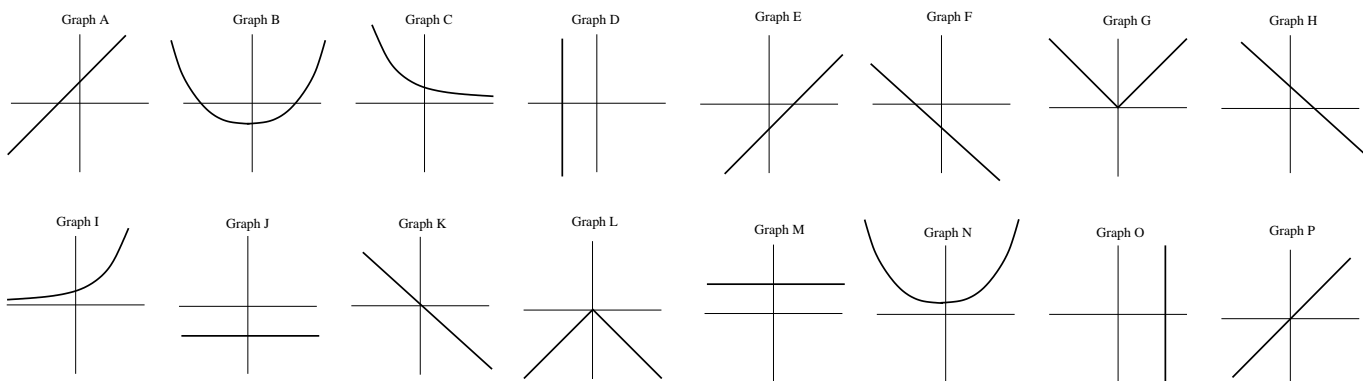
13. A rocket takes off vertically at time $t = 0$ with acceleration $a(t) = 6t + 2$. When $t = 1$ the rocket is travelling with **velocity** $v(1) = 7$ and has **displacement** $S(1) = 7$.

   (a) Find an expression for the **velocity** of the rocket at any time t.   (3 marks)

   (b) How far does the rocket travel between $t = 1$ and $t = 2$?   (4 marks)

   (c) At what time(s) does the **velocity** of the rocket equal 10?   (3 marks)

14. Find each of the following integrals. (1 mark, 2 marks, 3 marks, 4 marks)

   (a) $\displaystyle\int (4x + 3)\, dx$ **(b)** $\displaystyle\int \left(\frac{1}{x} + 2e^{-2x}\right) dx$ **(c)** $\displaystyle\int_0^1 \left(e^x + e^{-x}\right) dx$ **(d)** $\displaystyle\int_{-1}^1 (x + x^2 + x^3)\, dx$

15. For each of Parts (a) to (d), assume that an initial amount of \$1000 is invested in an account earning 10% per annum (with no taxes or fees). (It may help to note that $e^{0.3} \approx 1.35$, $1.1^3 \approx 1.33$ and $1.05^6 \approx 1.34$.) All four parts are worth 1 mark each.

   (a) What will be the balance F in the account after 3 years, if interest compounds annually?
   (b) What will be the balance F in the account after 3 years, if interest compounds each six months?

   (c) What will be the balance F in the account after 3 years, if interest compounds continuously?

   (d) Briefly explain why your answer to Part (c) should be larger than your answer to Part (a)?

16. For each of Parts (a) to (d), let $y = (x^2 - 2x + 1)e^x$.

   (a) Show that $y' = (x^2 - 1)e^x$.   (4 marks)

   (b) Find all values of x for which $y' = 0$.   (2 marks)

      Recall that $y = (x^2 - 2x + 1)e^x$ and $y' = (x^2 - 1)e^x$.

   (c) Find y''.   (3 marks)

   (d) Hence (or any way that you like), find and classify all critical points of y.   (3 marks)

17. Find the point(s) at which the circle with centre $(0, 0)$ and radius $\sqrt{5}$ intersects the curve $y = x^2 + 1$.

       (6 marks)

14.31   Solutions to final exam, June 2005

1. $R = \sum_{i=-2}^{1} \dfrac{|i|}{2i+1}$, so $R = \dfrac{|-2|}{2 \times -2 + 1} + \dfrac{|-1|}{2 \times -1 + 1} + \dfrac{|0|}{2 \times 0 + 1} + \dfrac{|1|}{2 \times 1 + 1}$, so $R = \dfrac{2}{-3} + \dfrac{1}{-1} + \dfrac{0}{1} + \dfrac{1}{3}$, so
$R = \dfrac{-1}{3} - 1$, so $R = \dfrac{-1}{3} - \dfrac{3}{3}$, so $R = \dfrac{-4}{3}$.

2. **(a)** $-3x - 4 \geq -2(x-3)$, so $-3x - 4 \geq -2x + 6$, so $-x \geq 10$, so $x \leq -10$.
   (b) $(-\infty, -10]$.   **(c)**   The answer has a shaded circle on -10, and an arrow pointing left.

3. **(a)** $A = \{2, 4, 6, 8, 10\}$, $B = \{3, 6, 9\}$ and $C = \{3, 4, 5, 6\}$. **(b)**   **(c)** $\{3, 4, 5, 6\}$
   (d) $\{2, 8, 10\}$ **(e)(i)** $\dfrac{7}{10}$ **(ii)** $\dfrac{1}{10}$ **(iii)** $= \dfrac{P(A \cap C)}{P(A)} = \dfrac{2/10}{5/10} = \dfrac{2}{5}$.
   (f)(i) $\dfrac{3}{10} \times \dfrac{3}{10} = \dfrac{9}{100}$ **(ii)** $\dfrac{5}{10} + \dfrac{4}{10} - \dfrac{5}{10} \times \dfrac{4}{10} = \dfrac{9}{10} - \dfrac{20}{100} = \dfrac{90}{100} - \dfrac{20}{100} = \dfrac{70}{100} = \dfrac{7}{10}$.

4. To solve: $3x - 4y = 5$ (1) $5x - 3y = 1$ (2)

   Multiplying (1) by 5 gives $15x - 20y = 25$ and multiplying (2) by -3 gives $-15x + 9y = -3$. Adding these together gives $-11y = 22$, so $y = -2$.

   Substitute $y = -2$ into (1) gives us $3x - 4 \times -2 = 5$, so $3x + 8 = 5$ so $3x = -3$, so $x = -1$.

   Hence the solution is $(x, y) = (-1, -2)$.

   Now we should always check the answer. Substitute $(x, y) = (-1, -2)$ into Equations (1) and (2), and see that both equations are satisfied, so the answer **must** be correct.

5. Let $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (-2, 7)$. From the notes $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-1}{-2-1} = \dfrac{6}{-3} = -2$. So the equation is $y = -2x + c$. Substitute in the point $(1, 1)$, so $1 = -2 \times 1 + c$, so $1 = -2 + c$, so $c = 3$. The equation is $y = -2x + 3$.

6. **(a)** $f(x) = (x - \sqrt{8})(x + \sqrt{2})$, so $f(x) = x \times x + x \times \sqrt{2} - \sqrt{8} \times x - \sqrt{8} \times \sqrt{2}$, so $f(x) = x^2 + \sqrt{2}x - \sqrt{8}x - \sqrt{8 \times 2}$, so $f(x) = x^2 + \sqrt{2}x - \sqrt{2 \times 2 \times 2}x - \sqrt{16}$, so $f(x) = x^2 + \sqrt{2}x - 2\sqrt{2}x - 4$, so $f(x) = x^2 - \sqrt{2}x - 4$.
   (b) $f(\sqrt{6}) = (\sqrt{6})^2 - \sqrt{2} \times \sqrt{6} - 4$, so $f(\sqrt{6}) = 6 - \sqrt{2 \times 6} - 4$, so $f(\sqrt{6}) = 2 - \sqrt{2 \times 2 \times 3}$, so $f(\sqrt{6}) = 2 - 2\sqrt{3}$.

7. The dashed graph on the left is $y = \dfrac{1}{2} \sin\left(\dfrac{1}{2}x\right)$. On the right is $y = 1 \sin 2x$.

8. Deleted.

9. **(a)** The points at which $f'(x) = 0$ are the critical points of $f(x)$, so the x-coordinates are $x = -1$, $x = 1$ and $x = 2$.

   (b) The intervals where the slope $(f'(x))$ is negative are the intervals on which f is decreasing, so $(-2, -1) \cup (1, 2)$.

   (c) As f is decreasing between $x = 1$ and $x = 2$, the critical point of f' between $x = 1$ and $x = 2$ must be a local minimum.

   (d) The slope at $x = 3$ and $x = 0$ is positive, with the slope at $x = 3$ being the steeper of the two. The slope at $x = 1$ is approximately zero and the slope at $x = -2$ is negative. In increasing order: $f'(-2), f'(1), f'(0)$ and $f'(3)$.

10. **(a)** $\dfrac{dy}{dx} = -8x^3 + 6$.

   (b) Using the chain rule from the notes, $\dfrac{dy}{dx} = -\sin(\sqrt{x}) \times \dfrac{1}{2}x^{-1/2} = -\dfrac{sin(\sqrt{x})}{2\sqrt{x}}$.

(c) Firstly, we need to use the product rule on $x \ln x$. Let $g = x \ln x$, $u = x$ and $v = \ln x$, so $u' = 1$ and $v' = \dfrac{1}{x}$. So $g' = 1 \times \ln x + \dfrac{1}{x} \times x = \ln x + 1$. Thus $\dfrac{dy}{dx} = \ln x + 1 - 1 = \ln x$.

11. **(a)** Using the chain rule from the notes, $\dfrac{dy}{dx} = 12(x^2 + 1)^3 \times 2x = 24x(x^2 + 1)^3$.

(b) Using the quotient rule, let $u = x^2 - 1$ and $v = x^2 + 1$. Now $u' = 2x$ and $v' = 2x$. So $\dfrac{dy}{dx} =$
$$\dfrac{u' \times v - v' \times u}{v^2} = \dfrac{2x \times (x^2 + 1) - 2x \times (x^2 - 1)}{(x^2 + 1)^2} = \dfrac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} = \dfrac{4x}{(x^2 + 1)^2}.$$

12. **1.** N **2.** I **3.** A **4.** L **5.** P **6.** M **7.** O **8.** C

13. **(a)** $v(t) = \int a(t)dt = \int 6t + 2 \, dt = 3t^2 + 2t + C$. We know that $v(1) = 7$, so $7 = 3(1)^2 + 2 \times 1 + C$, so $7 = 3 + 2 + C$, so $C = 2$. Hence $v(t) = 3t^2 + 2t + 2$.

(b) $S(t) = \int v(t)dt = \int 3t^2 + 2t + 2 \, dt = t^3 + t^2 + 2t + C$. We know that $S(1) = 7$, so $7 = (1)^3 + (1)^2 + 2 \times 1 + C$, so $7 = 1 + 1 + 2 + C$ so $C = 3$, so $S(t) = t^3 + t^2 + 2t + 3$. We know that $S(1) = 7$. Now $S(2) = (2)^3 + (2)^2 + 2 \times 2 + 3 = 8 + 4 + 4 + 3 = 19$. Thus between $t = 1$ and $t = 2$, the rocket travelled $S(2) - S(1) = 19 - 7 = 12$.

(c) $v(t) = 3t^2 + 2t + 2$, so we need to solve $10 = 3t^2 + 2t + 2$. Rearranging, we get $0 = 3t^2 + 2t - 8$, which we can slove using the quadratic formula. $t = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 3 \times -8}}{2 \times 3} = \dfrac{-2 \pm \sqrt{4 + 96}}{6} = \dfrac{-2 \pm \sqrt{100}}{6} = \dfrac{-2 \pm 10}{6}$, so $t = \dfrac{-2 + 10}{6}$ or $t = \dfrac{-2 - 10}{6}$, so $t = \dfrac{8}{6}$ or $t = \dfrac{-12}{6}$, so $t = \dfrac{4}{3}$ or $t = -2$. But t can't be negative, so $t = \dfrac{4}{3}$.

14. **(a)** $= 2x^2 + 3x + C$. **(b)** $= \ln x - \dfrac{1}{2}(2e^{-2x}) + C = \ln x - e^{-2x} + C$.

(c) $= [e^x - e^{-x}]_0^1 = (e^1 - e^{-1}) - (e^0 - e^0) = (e - e^{-1}) - (1 - 1) = e - \dfrac{1}{e}$.

(d) $= \left[\dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \dfrac{1}{4}x^4 \right]_{-1}^1 = \left(\dfrac{1}{2}(1)^2 + \dfrac{1}{3}(1)^3 + \dfrac{1}{4}(1)^4 \right) - \left(\dfrac{1}{2}(-1)^2 + \dfrac{1}{3}(-1)^3 + \dfrac{1}{4}(-1)^4 \right)$

$= \left(\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} \right) - \left(\dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} \right) = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{2}{3}$.

15. **(a)** $F = 1000(1 + 0.1)^3 = 1000(1.1)^3 = 1000 \times 1.33 = \$1330$.

(b) $F = 1000 \left(1 + \dfrac{0.1}{2} \right)^{3 \times 2} = 1000(1.05)^6 = 1000 \times 1.34 = \$1340$.

(c) $F = 1000e^{3(0.1)} = 1000e^{0.3} = 1000 \times 1.35 = \$1350$.

(d) Because with interest compounding continuously, you are earning interest on your interest.

16. **(a)** Using the product rule from the notes, let $u = x^2 - 2x + 1$ and $v = e^x$. Now $u' = 2x - 2$ and $v' = e^x$. So $\dfrac{dy}{dx} = u' \times v + v' \times u = (2x - 2)(e^x) + (e^x)(x^2 - 2x + 1) = e^x(2x - 2 + x^2 - 2x + 1) = (x^2 - 1)e^x$.

(b) If $\dfrac{dy}{dx} = 0$, then $(x^2 - 1)e^x = 0$, so either $e^x = 0$ (which is impossible) or else $(x^2 - 1) = 0$, so $x^2 = 1$, so $x = 1$ or $x = -1$.

(c) Using the product rule from the notes, let $u = x^2 - 1$ and $v = e^x$. Now $u' = 2x$ and $v' = e^x$. So $\dfrac{d^2y}{dx^2} = u' \times v + v' \times u = 2xe^x + e^x(x^2 - 1) = e^x(2x + x^2 - 1)$.

(d) From the notes we know that a critical point is a local maximum if $y''(x) < 0$ and a local minimum if $y''(x) > 0$. We found y'' in Part **(c)**. Hence $y''(1) = e^1(2 \times 1 + 1^2 - 1) = e^1(2)$, which is positive, so at

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$x = 1$ there is a local minimum. Next, $y''(-1) = e^{-1}(2 \times -1 + (-1)^2 - 1) = e^{-1}(-2)$, which is negative, so at $x = -1$ there is a local maximum. The y-coordinates are $y(1) = (1^2 - 2 \times 1 + 1)e^1 = 0e^1 = 0$ and $y(-1) = ((-1)^2 - 2 \times -1 + 1)e^{-1} = 4e^{-1} = \dfrac{4}{e}$. So there is a local minimum at $(1, 0)$ and a local maximum at $(-1, \dfrac{4}{e})$.

17. The equation of a circle is $(x-a)^2 + (y-b)^2 = r^2$. Using centre $(0, 0)$ and radius $\sqrt{5}$, we get $(x-0)^2 + (y-0)^2 = (\sqrt{5})^2$, so $x^2 + y^2 = 5$. To solve: $\qquad y = x^2 + 1 \qquad (1) \qquad x^2 + y^2 = 5 \qquad (2)$

Rearrange (2) to get $x^2 = 5 - y^2$ and substitute into (1). Then $y = 5 - y^2 + 1$, so $0 = -y^2 - y + 6$, which we can solve using the quadratic formula. $y = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4 \times -1 \times 6}}{2 \times -1} = \dfrac{1 \pm \sqrt{1 + 24}}{-2} = \dfrac{1 \pm \sqrt{25}}{-2} = \dfrac{1 \pm 5}{-2}$. so $y = \dfrac{1 + 5}{-2}$ or $\dfrac{1 - 5}{-2}$, so $y = -3$ or $y = 2$.

Firstly, substituting $y = -3$ into (1) gives us $-3 = x^2 + 1$, so $x^2 = -4$, which is impossible, so $y = -3$ is not a solution.

Secondly, substituting $y = 2$ into (1) gives us $2 = x^2 + 1$, so $x^2 = 1$ so $x = 1$ or $x = -1$. Hence the solutions are $(x, y) = (1, 2)$ and $(x, y) = (-1, 2)$.

Now we should always check the answer. Substitute $(x, y) = (1, 2)$ and then $(x, y) = (-1, 2)$ into Equations (1) and (2), and see that both equations are satisfied, so both answers **must** be correct.

14.32   Final exam, Dec 2004

1. Find R if $\quad -6 = \sum_{i=0}^{2} \left((-2)^i \times i \times R \right).$ (4 marks)

2. Removed (not relevant). (5 marks)

3. Let A, B and C be sets. In each of Parts (a), (b), (c) and (d), you are given an expression involving these sets, and a Venn diagram. On each Venn diagram, **shade the region** which corresponds to the expression. If the expression equals the empty set, state that the answer is the empty set. **Note: to save space, the diagram is only shown for Part (a).**

(a) $(A \cup B) \cup (A \cup C) \cup (B \cup C).$ (2 marks)
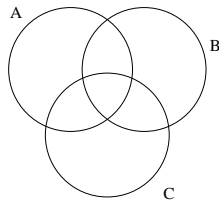
(b) $(A \setminus B) \cap (B \setminus A).$ (2 marks)

(c) $(A \setminus C) \setminus B.$ (2 marks)

(d) $A \setminus (C \setminus B).$ (2 marks)

4. Simplify $\quad \dfrac{2^{n+1} - 2^n + 2^{n-1}}{2^n - 2^{n-1} + 2^{n-2}}.$ (3 marks)

5. Solve the following system of two simultaneous equations.
$$6x + 4y = 1$$
$$-3x - 2y = 5.$$
(3 marks)

6. Find the straight line distance between the points $(4, 2)$ and $(1, 5)$, expressing your answer as a surd in simplest form. (2 marks)

7. I roll *two fair, six-sided* dice, one coloured green and the other coloured blue. Let g denote the number on the green die and b the number on the blue die. What is the probability that:

(a) $g = 3$ and $b = 4$? (2 marks)

(b) $g < b$? (2 marks)

(c) g is odd? (2 marks)

(d) g and b are both prime numbers?   (1 is not prime) (2 marks)

8. Find the equation of the line passing through the points $(-\sqrt{2}, 2\sqrt{6})$ and $(2\sqrt{2}, -\sqrt{6})$, expressing the gradient and y-intercept as surds in simplest form. (3 marks)

9. (a) Find the derivative of $f(x) = \dfrac{2}{x}.$ (1 mark)

(b) Hence, or any other way that you like, explain why $f(x)$ has no critical points. (2 marks)

(c) Show that the gradient of $f(x)$ at the point $(-2, -1)$ is $-\dfrac{1}{2}$. Find the line which is tangential to the curve at the point $(-2, -1)$. (3 marks)

10. In each of Parts (a) to (f), find $\dfrac{dy}{dx}$, simplifying where possible. (1 mark, 1 mark, 2 marks, 3 marks, 3 marks, 4 marks).

   (a) $y = 3x^3 - 4x^2 + 2$ **(b)** $y = \dfrac{1}{\sqrt{2}}$ **(c)** $y = 3e^{3x} + 3e^2$

   (d) $y = (3x - 1)^{10}$ **(e)** $y = \dfrac{\ln x}{x}$ **(f)** $y = \sqrt{(\sqrt{x} + 4)}$

11. Shown below are two copies of the graph of $y = \sin x$, for $x \in [-2\pi, 2\pi]$. In each of Parts (a) and (b), sketch your graph on the set of axes. **Note: to save space, the diagram is only shown for Part (a).**

   (a) Sketch a graph of $y = -\sin x$ for $x \in [-2\pi, 2\pi]$.   (2 marks)
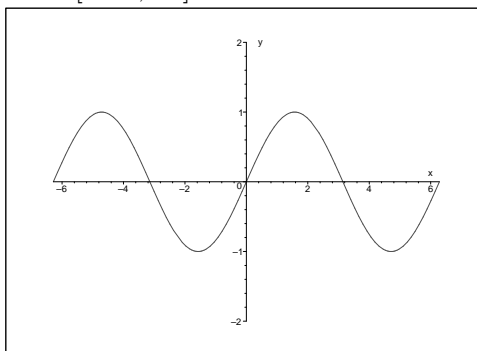
   (b) Sketch a graph of $y = \dfrac{1}{2} \sin 2x$ for $x \in [-2\pi, 2\pi]$.   (2 marks)

12. The following diagram contains the graph of a function and the graph of the derivative of the function. The graphs are labelled A and B. Clearly identify which graph represents the function, and which represents the derivative. Explain your answer. (Most marks will be assigned to your explanation.)
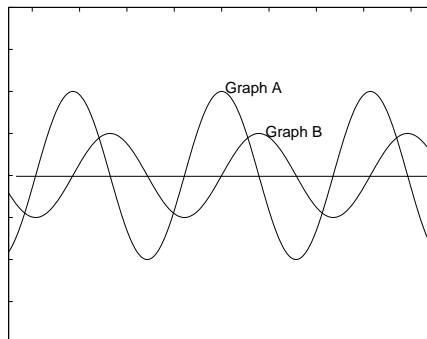
                                                      (3 marks)

13. Here are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph.   (8 marks)

   (1) $y = -(-x)^2$ (2) $y = x + 2^{10}$ (3) $x + y = 0$ (4) removed (not relevant)

   (5) $y = \pi^x$ (6) $y = |\sqrt{2}x|$ (7) $y = 4(x - 3) - 4x$ (8) $x + x + x = -y - y - y - 3$

Graph A   Graph B   Graph C   Graph D   Graph E   Graph F

Graph G   Graph H   Graph I   Graph J   Graph K   Graph L

Graph M   Graph N   Graph O   Graph P
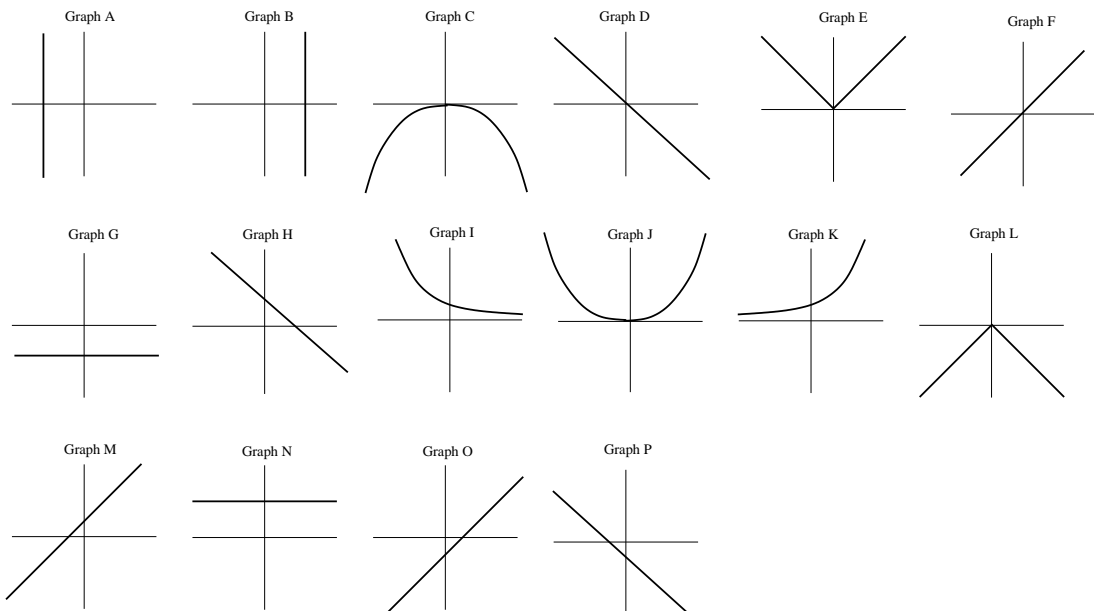
14. A rocket takes off vertically at time $t = 0$, with **displacement** $S(t) = pt^2 - qt$, where p and q are constants.

(a) When $t = 0$, the rocket has **velocity** $v(0) = 5$. When $t = 1$, the rocket has velocity $v(1) = 12$. Find the values of p and q and an expression for the **velocity** of the rocket at any time t. (3 marks)

(b) At what time is the **velocity** of the rocket exactly equal to its **acceleration**? (2 marks)

(c) How far does the rocket travel between $t = 1$ and $t = 2$? (2 marks)

15. Find each of the following integrals. (1 mark, 2 marks, 2 marks, 2 marks)

(a) $\displaystyle\int (x^3 + 2)\, dx$ **(b)** $\displaystyle\int (\cos x + x)\, dx$ **(c)** $\displaystyle\int_{-1}^{0} e^{2x}\, dx$ **(d)** $\displaystyle\int_{1}^{2} (x^{-1} + 1)\, dx$

16. **(a)** Consider the following isoceles, right-angled triangle, with two sides of length 1 unit.
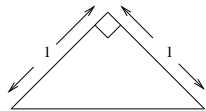
Using Pythagoras' formula $h^2 = a^2 + b^2$, determine the length of the hypotenuse. (2 marks)

(b) Hence find $\cos(\pi/4)$ and $\sin(\pi/4)$. (2 marks)

17. For each of Parts (a) to (d), let $y = e^x \cos x$.

(a) Show that $\dfrac{dy}{dx} = e^x(\cos x - \sin x)$. (3 marks)

(b) Using the result from Question 16(b), show that $x = \pi/4$ is a solution to $\dfrac{dy}{dx} = 0$. (2 marks)

(c) Find $\dfrac{d^2y}{dx^2}$. (4 marks)

(d) Hence (or any other way that you like), determine whether $x = \pi/4$ is a maximum or minimum. (3 marks)

18. Find all roots of the following polynomial. $z^6 - 9z^3 + 8 = 0$ (5 marks)

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14.33   Solutions to final exam, Dec 2004

1. $-6 = \sum_{i=0}^{2}((-2)^i \times i \times R)$ so $-6 = ((-2)^0 \times 0 \times R) + ((-2)^1 \times 1 \times R) + ((-2)^2 \times 2 \times R)$ so $-6 = (0) + (-2R) + (4 \times 2R)$
so $-6 = -2R + 8R$ so $-6 = 6R$, so $R = -1$.

2. Removed (not relevant).
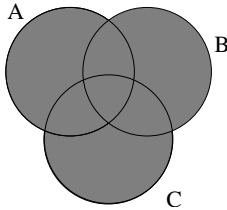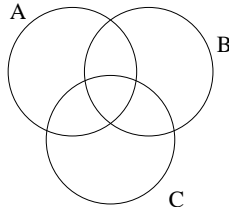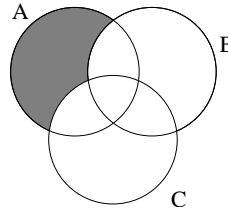
3. **(a)**   **(b)**   **(c)**   **(d)**
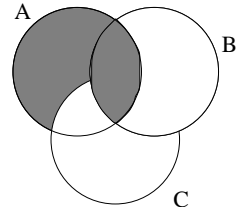
4. Now $\dfrac{2^{n+1} - 2^n + 2^{n-1}}{2^n - 2^{n-1} + 2^{n-2}} = \dfrac{2^{n-1} \times 2^2 - 2^{n-1} \times 2^1 + 2^{n-1}}{2^{n-2} \times 2^2 - 2^{n-2} \times 2^1 + 2^{n-2}} = \dfrac{2^{n-1}(2^2 - 2^1 + 1)}{2^{n-2}(2^2 - 2^1 + 1)} = \dfrac{2^{n-1}}{2^{n-2}} = \dfrac{2^{n-2} \times 2^1}{2^{n-2}} = 2.$

5. To solve: $6x + 4y = 1$ (1) $-3x - 2y = 5$ (2)

Multiply (2) by 2, giving $-6x - 4y = 10$. Adding this to (1) gives $6x + 4y - 6x - 4y = 1 + 10$, so $0 = 11$. This is impossible, so there is no solution.

6. Let $(x_1, y_1) = (4, 2)$ and $(x_2, y_2) = (1, 5)$.
Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(4-1)^2 + (2-5)^2} = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}.$

7. The table below contains all 36 possible outcomes when two dice are thrown. In each case, the first number is the value shown on the green die, and the second number is the value shown on the blue die.

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 2	3 3	3 4	3 5	3 6
4 1	4 2	4 3	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

(a) $g = 3$ and $b = 4$ happens in 1 case: Prob$(g = 3, b = 4) = \dfrac{1}{36}$.

(b) $g < b$ in 15 cases: Prob$(g < b) = \dfrac{15}{36} = \dfrac{5}{12}$.

(c) g is odd in 18 cases: Prob$(g$ is odd$) = \dfrac{18}{36} = \dfrac{1}{2}$.

(d) g and b are both prime numbers in 9 cases: Prob$(g$ and b both prime$) = \dfrac{9}{36} = \dfrac{1}{4}$.

8. Let $(x_1, y_1) = (-\sqrt{2}, 2\sqrt{6})$ and $(x_2, y_2) = (2\sqrt{2}, -\sqrt{6})$. From the notes $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-\sqrt{6} - 2\sqrt{6}}{2\sqrt{2} - (-\sqrt{2})} = \dfrac{-3\sqrt{6}}{3\sqrt{2}} = \dfrac{-\sqrt{6}}{\sqrt{2}} = -\dfrac{\sqrt{3}\sqrt{2}}{\sqrt{2}} = -\sqrt{3}$. So the equation is $y = -\sqrt{3}x + c$.

Substitute in the point $(-\sqrt{2}, 2\sqrt{6})$, so $2\sqrt{6} = -\sqrt{3} \times -\sqrt{2} + c$, so $2\sqrt{6} = \sqrt{6} + c$, so $c = \sqrt{6}$. The equation is $y = -\sqrt{3}x + \sqrt{6}$.

9. **(a)** $f'(x) = \dfrac{-2}{x^2}$.

(b) The critical points of $f(x)$ are where $f'(x) = 0$. Thus a critical point is where $\dfrac{-2}{x^2} = 0$, so $-2 = 0$ (which is impossible), hence there are no critical points.

(c) From **(a)** we know that the tangent line has gradient $m = \dfrac{-2}{x^2}$. So at the point $(-2, -1)$, $m = \dfrac{-2}{(-2)^2} = \dfrac{-2}{4} = \dfrac{-1}{2}$, so the equation is $y = \dfrac{-1}{2}x + c$. Substitute in $(-2, -1)$, giving $-1 = -\dfrac{1}{2} \times -2 + c$, so $-1 = 1 + c$, so $c = -2$. Thus the equation is $y = -\dfrac{1}{2}x - 2$.

10. (a) $\dfrac{dy}{dx} = 9x^2 - 8x$.   **(b)** $\dfrac{dy}{dx} = 0$.   **(c)** $\dfrac{dy}{dx} = 9e^{3x}$.

(d) Using the chain rule, $\dfrac{dy}{dx} = 10(3x - 1)^9 \times 3 = 30(3x - 1)^9$.

(e) Using the quotient rule, let $u = \ln x$ and $v = x$. Now $u' = \dfrac{1}{x}$ and $v' = 1$.

So $\dfrac{dy}{dx} = \dfrac{u' \times v - v' \times u}{v^2} = \dfrac{\dfrac{1}{x} \times x - 1 \times \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.

(f) $y = \sqrt{(\sqrt{x} + 4)} = (\sqrt{x} + 4)^{1/2} = (x^{1/2} + 4)^{1/2}$.

Now using the chain rule, $\dfrac{dy}{dx} = \dfrac{1}{2}(x^{1/2} + 4)^{-1/2} \times \dfrac{1}{2}x^{-1/2} = \dfrac{1}{4}x^{-1/2}(x^{1/2} + 4)^{-1/2} =$

$\dfrac{1}{4x^{1/2}(x^{1/2} + 4)^{1/2}} = \dfrac{1}{4\sqrt{x}\sqrt{\sqrt{x} + 4}} = \dfrac{1}{4\sqrt{x(\sqrt{x} + 4)}}$.

11. The dashed graph on the left is $y = -\sin x$. On the right is $y = \dfrac{1}{2}\sin 2x$.
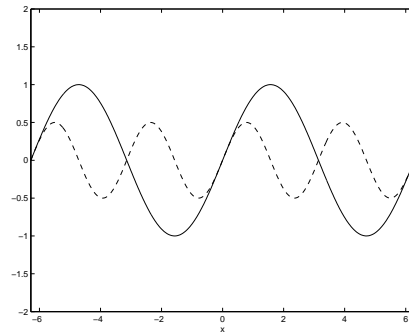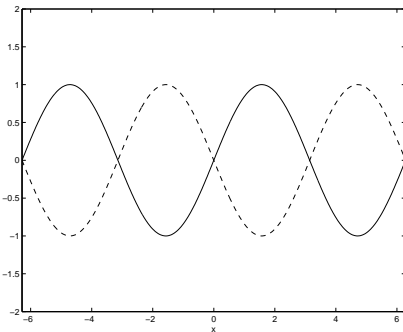
12. The **function** is Graph **B**. The **derivative** is Graph **A**.
Explanation: Suppose that Graph A is the function (and Graph B its derivative). The increasing sections of Graph A (between $y = 0$ and the local maxima) should have a positive derivative (gradient) but the corresponding section of Graph B is actually negative. This means that Graph B is definitely not the derivative of Graph A. Hence Graph B must be the function.

13. **1.** C   **2.** M   **3.** D   **4.** A   **5.** K   **6.** E   **7.** G   **8.** P

14. (a) $v(t) = \dfrac{dS}{dt} = 2pt - q$. We know that $v(0) = 5$, so $5 = 2 \times p \times 0 - q$, so $q = -5$, so $v(t) = 2pt + 5$.

We also know that $v(1) = 12$, so $12 = 2 \times p \times 1 + 5$, so $12 = 2p + 5$, so $7 = 2p$, so $p = \dfrac{7}{2}$. Hence $v(t) = 2 \times \dfrac{7}{2}t + 5 = 7t + 5$.

(b) We know that $a(t) = \dfrac{dv}{dt} = 7$. So for velocity to equal acceleration, we need $v(t) = a(t)$, so $7t + 5 = 7$, so $7t = 2$, so $t = \dfrac{2}{7}$.

(c) $S(t) = \dfrac{7}{2}t^2 + 5t$, so $S(1) = \dfrac{7}{2} \times 1^2 + 5 \times 1 = \dfrac{7}{2} + 5 = 8.5$. Now $S(2) = \dfrac{7}{2} \times 2^2 + 5 \times 2 = 14 + 10 = 24$. Thus between $t = 1$ and $t = 2$, the rocket travelled $S(2) - S(1) = 24 - 8.5 = 15.5$.

15. (a) $= \dfrac{x^4}{4} + 2x + C.$ **(b)** $= \sin x + \dfrac{x^2}{2} + C.$

(c) $= [\dfrac{1}{2}e^{2x}]^0_{-1} = (\dfrac{1}{2}e^{2 \times 0}) - (\dfrac{1}{2}e^{2 \times -1}) = \dfrac{1}{2}e^0 - \dfrac{1}{2}e^{-2} = \dfrac{1}{2} - \dfrac{1}{2}e^{-2} = \dfrac{1}{2}(1 - e^{-2}) = \dfrac{1}{2}(1 - \dfrac{1}{e^2}).$

(d) $\int_1^2 (x^{-1} + 1)dx = \int_1^2 (\dfrac{1}{x} + 1)dx = [\ln x + x]_1^2 = (\ln 2 + 2) - (\ln 1 + 1) = \ln 2 + 2 - \ln 1 - 1 =$
$\ln 2 + 2 - 0 - 1 = \ln 2 + 1.$

16. **(a)** $h^2 = a^2 + b^2$, so $h^2 = 1^2 + 1^2$, so $h^2 = 2$, so $h = \pm\sqrt{2}$. But you can't get a negative length, so $h = \sqrt{2}$.

(b) Now we have the triangle below (same as in the notes).
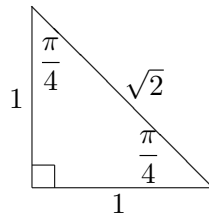
Now from the notes we know that $\cos\theta = \dfrac{adjacent}{hypotenuse}$ and $\sin\theta = \dfrac{opposite}{hypotenuse}$. So looking at the above triangle, we get $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ and $\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$.

17. **(a)** Using the product rule from the notes, let $u = e^x$ and $v = \cos x$. Now $u' = e^x$ and $v' = -\sin x$.
So $\dfrac{dy}{dx} = u' \times v + v' \times u = e^x(\cos x) + (-\sin x)e^x = e^x(\cos x - \sin x).$

(b) If $\dfrac{dy}{dx} = 0$, then $e^x(\cos x - \sin x) = 0$, so either $e^x = 0$ (which is impossible) or else $\cos x - \sin x = 0$.
Now from **Q16(b)** we know that $\cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4}$, so $\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{4} = 0$. Hence $x = \dfrac{\pi}{4}$ is a solution.

(c) Using the product rule from the notes, let $u = e^x$ and $v = \cos x - \sin x$. Now $u' = e^x$ and $v' = -\sin x - \cos x$. So $\dfrac{d^2y}{dx^2} = u' \times v + v' \times u = e^x(\cos x - \sin x) + (-\sin x - \cos x)e^x = e^x\cos x - e^x\sin x - e^x\sin x - e^x\cos x = -2e^x\sin x.$

(d) From the notes we know that a critical point is a maximum if $y''(x) < 0$ and a local minimum if $y''(x) > 0$. We found y'' in part **(c)**. So $y''(\dfrac{\pi}{4}) = -2e^{\pi/4}\sin\dfrac{\pi}{4} = -2e^{\pi/4} \times \dfrac{1}{\sqrt{2}} = -\sqrt{2}e^{\pi/4}$ which is negative. Hence $x = \dfrac{\pi}{4}$ is a local maximum.
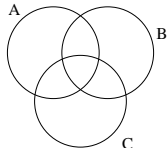
18. Let $x = z^3$, so now the equation reads $x^2 - 9x + 8 = 0$. We can use the quadratic formula from the notes to solve. So $x = \dfrac{9 \pm \sqrt{(-9)^2 - 4 \times 1 \times 8}}{2 \times 1} = \dfrac{9 \pm \sqrt{81 - 32}}{2} = \dfrac{9 \pm \sqrt{49}}{2} = \dfrac{9 \pm 7}{2}$, so $x = \dfrac{9 + 7}{2}$ or $x = \dfrac{9 - 7}{2}$, so $x = 8$ or $x = 1$. Recalling $x = z^3$, we must have $z^3 = 8$ or $z^3 = 1$, so $z = 2$ or $z = 1$. Hence the roots of the polynomial are $z = 1$ and $z = 2$.

14.34 Final exam, June 2004

1. Find R if $\quad -6 = \sum_{i=1}^{3} \left((-1)^i \times i \times R^{i-1} \right)$. (4 marks)

2. Removed (not relevant). (5 marks)

3. Let A, B and C be sets. In each of Parts (a), (b), (c) and (d), you are given an expression involving these sets, and a Venn diagram. On each Venn diagram, **shade the region** which corresponds to the expression. If the expression equals the empty set, state that the answer is the empty set. **Note: to save space, the diagram is only shown for Part (a).**

   (a) $(A \cap B) \cup (A \cap C)$. (2 marks)

   (b) $(A \setminus B) \cap (C \setminus B)$. (2 marks)

   (c) $(A \setminus B) \cup ((B \cap C) \setminus A)$. (2 marks)

   (d) $((\emptyset \setminus B) \cup (B \setminus \emptyset)) \cap (B \cup (B \setminus B))$. (2 marks)

4. Let $f(x) = 2^x$. Simplify $\quad \dfrac{f(n+2) - f(n)}{f(n-1)}$. (3 marks)

5. Solve the following system of two simultaneous equations.
$$6x + 4y = 1$$
$$-3x - 8y = 4.$$
(3 marks)

6. Find the straight line distance between the points $(3\sqrt{3}, -2)$ and $(\sqrt{3}, 2)$, expressing your answer as a surd in simplest form. (2 marks)

7. Removed (not relevant).

8. Find the equation of the line passing through the points $(-2, -5)$ and $(-4, -1)$. (3 marks)

9. In Parts (a), (b) and (c) of this question, use the function $f(x) = ax^2 + bx + c$, where a, b and c are constants.

   (a) Deleted.

   (b) Hence, or any other way that you like, show that $f(x) = ax^2 + bx + c$ has a critical point at $x = \dfrac{-b}{2a}$. (There is no need to classify the critical point as a local maximum or local minimum.) (2 marks)

   (c) Suppose that the point $(1, 2a + b)$ lies on the curve $y = ax^2 + bx + c$. Find the line which is tangential to the curve at the point $(1, 2a + b)$. (3 marks)

10. In each of Parts (a) to (g), find $\dfrac{dy}{dx}$, simplifying where possible. (1 mark, 1 mark, 2 marks, 3 marks, 3 marks, 3 marks)

(a) $y = 3x^2 - 2x - 4$ **(b)** $y = \dfrac{-3}{x}$ **(c)** $y = 3e^{3x} + 3e^2$ **(d)** $y = \sin(4x)$

(e) $y = (4x + 2)^5$ **(f)** $y = \dfrac{\ln x}{x}$ **(g)** $y = \sqrt{(x^2 + 4)}$

11. Shown below are two copies of the graph of $y = \sin x$, for $x \in [-2\pi, 2\pi]$. In each of Parts (a) and (b), sketch your graph on the set of axes. **Note: to save space, the diagram is only shown for Part (a).**

   (a) Sketch a graph of $y = -2\sin x$ for $x \in [-2\pi, 2\pi]$. (2 marks)
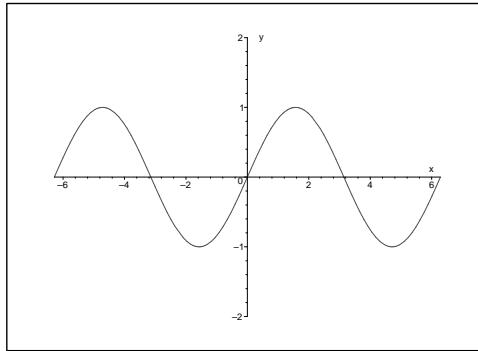
   (b) Sketch a graph of $y = \dfrac{1}{2}\sin 2x$ for $x \in [-2\pi, 2\pi]$. (2 marks)

12. The following diagram shows the graph of the **derivative** f' of a certain function f on the interval $x \in [-4.1, 1.2]$. Parts (a) to (d) contain questions about the **function** f on this interval.
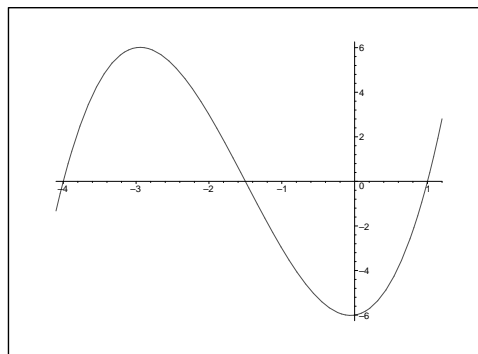
   (a) What are the x-coordinate(s) of all critical points of f? (1.5 marks)

   (b) Write down the interval(s) on which f is increasing. (1.5 marks)

   (c) What are the x-coordinate(s) at which f has a local maximum? (1.5 marks)

   (d) What are the x-coordinate(s) at which f attains its steepest positive slope? (1.5 marks)

13. Here are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph. (8 marks)

   (1) $y = -x^2$ (2) $y = 147x + 682$ (3) $x + y = 0$ (4) $x = -\,|-4\,|$

   (5) $y = 2^{-x}$ (6) $y = |-2x|$ (7) $y + 4x = 4(x + 1)$ (8) $(-2)^9(x + 1) = (-3)^{10}y$

Graph A   Graph B   Graph C   Graph D   Graph E   Graph F

Graph G   Graph H   Graph I   Graph J   Graph K   Graph L

Graph M   Graph N   Graph O   Graph P
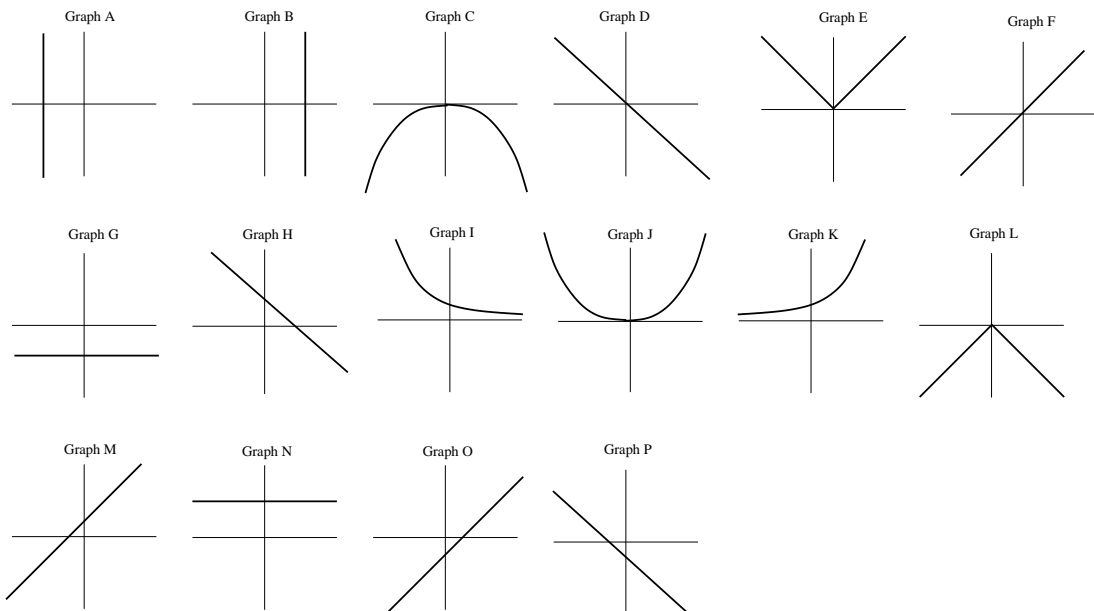
14. A rocket takes off vertically at time $t = 0$, with **velocity** $v(t) = pt + 4$, where p is a constant.

   (a) When $t = 0$, the rocket has **displacement** $S(0) = 2$. When $t = 1$, the rocket has displacement $S(1) = 10$. Find the value of p and an expression for the **displacement** of the rocket at any time t.   (3 marks)

   (b) At what time is the **velocity** of the rocket exactly equal to its **acceleration**?   (2 marks)

   (c) How far does the rocket travel between $t = 1$ and $t = 2$?   (2 marks)

15. Find each of the following integrals. (1 mark, 2 marks, 2 marks, 2 marks, 2 marks)

   (a) $\displaystyle\int (-6x^2 + 2x)\, dx$ **(b)** $\displaystyle\int \left(\frac{-1}{x} + e^{2x} \right)\, dx$ **(c)** $\displaystyle\int (2\sin x + 4)\, dx$ **(d)** $\displaystyle\int_0^1 e^{-x}\, dx$ **(e)** $\displaystyle\int_1^2 (x^{-2} + 1)\, dx$

16. For each of Parts (a) to (d), let $y = x^2 e^{-2x}$.

   (a) Show $\dfrac{dy}{dx} = e^{-2x}(2x - 2x^2)$.   (3 marks)

   (b) Find all values of x for which $\dfrac{dy}{dx} = 0$.   (2 marks)

   (c) Recall that $\dfrac{dy}{dx} = e^{-2x}(2x - 2x^2)$. Using the product rule, find $\dfrac{d^2y}{dx^2}$.   (3 marks)

   (d) Hence (or any other way that you like), classify all critical points of y.   (3 marks)

17. Solve the following system of two simultaneous equations.
$$y = 3x^2 + 2x - 3$$
$$y = 2x^2 - 2x + 9.$$
    (5 marks)

14.35 Solutions to final exam, June 2004

1. $-6 = \sum_{i=1}^{3}((-1)^i \times i \times R^{i-1})$ so $-6 = ((-1)^1 \times 1 \times R^{1-1}) + ((-1)^2 \times 2 \times R^{2-1}) + ((-1)^3 \times 3 \times R^{3-1})$ so
   $-6 = (-R^0) + (2R^1) + (-3R^2)$, so $-6 = -1 + 2R - 3R^2$ so $0 = -3R^2 + 2R + 5$.
   Using the quadratic formula from the notes, $R = \dfrac{-2 \pm \sqrt{2^2 - 4 \times -3 \times 5}}{2 \times -3} = \dfrac{-2 \pm \sqrt{4 + 60}}{-6} = \dfrac{-2 \pm \sqrt{64}}{-6} = $
   $\dfrac{-2 \pm 8}{-6}$, so $R = \dfrac{-2 + 8}{-6}$ or $R = \dfrac{-2 - 8}{-6}$, so $R = \dfrac{6}{-6}$ or $R = \dfrac{-10}{-6}$, so $R = -1$ or $R = \dfrac{5}{3}$.

2. Removed (not relevant).
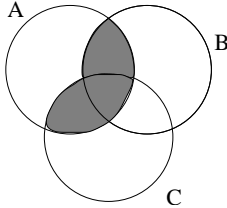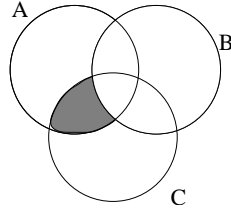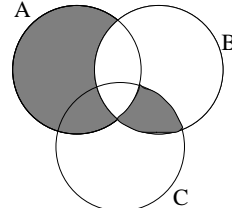
3. **(a)**   **(b)**   **(c)**   **(d)**
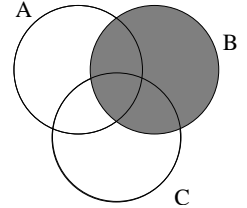
4. Now $f(n + 2) = 2^{n+2}$ and $f(n - 1) = 2^{n-1}$. Hence $\dfrac{f(n+2) - f(n)}{f(n-1)} = \dfrac{2^{n+2} - 2^n}{2^{n-1}} = \dfrac{2^n \times 2^2 - 2^n}{2^n \times 2^{-1}} = $
   $\dfrac{2^n(2^2 - 1)}{2^n \times 2^{-1}} = \dfrac{(2^2 - 1)}{2^{-1}} = \dfrac{4 - 1}{2^{-1}} = \dfrac{3}{2^{-1}} = 3 \times 2^1 = 6$.

5. To solve: $6x + 4y = 1$ (1) $-3x - 8y = 4$ (2)

   Multiplying (1) by 2 gives $12x + 8y = 2$. Adding this to (2) gives $12x + 8y - 3x - 8y = 4 + 2$, so $9x = 6$, so
   $x = \dfrac{2}{3}$.
   Substitute $x = \dfrac{2}{3}$ into (1) gives us $6 \times \dfrac{2}{3} + 4y = 1$, so $4 + 4y = 1$ so $4y = -3$, so $y = -\dfrac{3}{4}$.
   Hence the solution is $(x, y) = (\dfrac{2}{3}, -\dfrac{3}{4})$.

   Now we should always check the answer. Substitute $(x, y) = (\dfrac{2}{3}, -\dfrac{3}{4})$ into Equations (1) and (2), and see
   that both equations are satisfied, so the answer **must** be correct.

6. Let $(x_1, y_1) = (3\sqrt{3}, -2)$ and $(x_2, y_2) = (\sqrt{3}, 2)$. Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = $
   $\sqrt{(3\sqrt{3} - \sqrt{3})^2 + (-2 - 2)^2} = \sqrt{(2\sqrt{3})^2 + (-4)^2} = \sqrt{12 + 16} = \sqrt{28} = 2\sqrt{7}$.

7. Removed (not relevant).

8. Let $(x_1, y_1) = (-2, -5)$ and $(x_2, y_2) = (-4, -1)$. From the notes $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1 - (-5)}{-4 - (-2)} = \dfrac{-1 + 5}{-4 + 2} = $
   $\dfrac{4}{-2} = -2$. So the equation is $y = -2x + c$. Substitute in the point $(-2, -5)$, so $-5 = -2 \times -2 + c$, so
   $-5 = 4 + c$, so $c = -9$. The equation is $y = -2x - 9$.

9. **(a)** Deleted.

   (b) The critical points of $f(x)$ are where $f'(x) = 0$. Thus a critical point is where $2ax + b = 0$, so $2ax = -b$,
   so $x = -\dfrac{b}{2a}$.

   (c) From **(a)** we know that the curve $y = ax^2 + bx + c$ has slope $2ax + b$ at any point. So at the point
   $(1, 2a + b)$, $m = 2a \times 1 + b = 2a + b$, so the equation of the tangent line is $y = (2a + b)x + c$. Substitute
   in the point $(1, 2a + b)$, giving $2a + b = (2a + b) \times 1 + c$, so $c = 0$. Thus the equation is $y = (2a + b)x$.

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10. (a) $\dfrac{dy}{dx} = 6x - 2$.　　**(b)** $y = -\dfrac{3}{x} = -3x^{-1}$. Now $\dfrac{dy}{dx} = 3x^{-2} = \dfrac{3}{x^2}$.　　**(c)** $\dfrac{dy}{dx} = 9e^{3x}$.

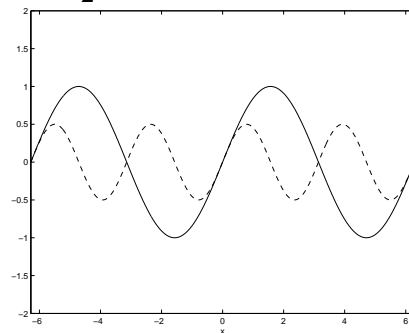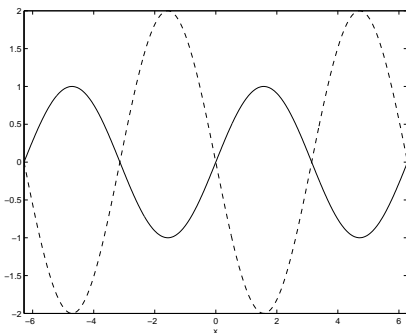**(d)** Using the chain rule from the notes, $\dfrac{dy}{dx} = \cos(4x) \times 4 = 4\cos(4x)$.

(e) Using the chain rule, $\dfrac{dy}{dx} = 5(4x+2)^4 \times 4 = 20(4x+2)^4$.

(f) Using the quotient rule, let $u = \ln x$ and $v = x$. Now $u' = \dfrac{1}{x}$ and $v' = 1$.

So $\dfrac{dy}{dx} = \dfrac{u' \times v - v' \times u}{v^2} = \dfrac{\frac{1}{x} \times x - 1 \times \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.

(g) $y = \sqrt{(x^2+4)} = (x^2+4)^{1/2}$. Now using the chain rule, $\dfrac{dy}{dx} = \dfrac{1}{2}(x^2+4)^{-1/2} \times 2x = x(x^2+4)^{-1/2} = \dfrac{x}{(x^2+4)^{1/2}} = \dfrac{x}{\sqrt{x^2+4}}$.

11. The dashed graph on the left is $y = -2\sin x$. On the right is $y = \dfrac{1}{2}\sin 2x$.

12. (a) The critical points of $f(x)$ are where $f'(x) = 0$, so the x-coordinates are $x = -4$, $x = -1.5$ and $x = 1$.

(b) The intervals on which f is increasing is where the slope ($f'(x)$) is positive, so $(-4, -1.5) \cup (1, 1.2]$.

(c) A local maximum has positive slope before it and negative slope after it, so there is a local maximum at $x = -1.5$.

(d) The steepest positive slope (greatest $f'(x)$ value) is at $x = -3$.

13. 1. C　**2.** M　**3.** D　**4.** A　**5.** I　**6.** E　**7.** N　**8.** P

14. (a) $S(t) = \int v(t)dt = \int pt + 4dt = \dfrac{1}{2}pt^2 + 4t + C$. We know that $S(0) = 2$, so $2 = \dfrac{1}{2}p \times 0^2 + 4 \times 0 + C$,

so $C = 2$, so $S(t) = \dfrac{1}{2}pt^2 + 4t + 2$. We also know that $S(1) = 10$, so $10 = \dfrac{1}{2}p \times 1^2 + 4 \times 1 + 2$, so

$10 = \dfrac{1}{2}p + 4 + 2$, so $4 = \dfrac{1}{2}p$, so $p = 8$. Hence $S(t) = \dfrac{1}{2} \times 8t^2 + 4t + 2 = 4t^2 + 4t + 2$.

(b) $v(t) = 8t + 4$. We know that $a(t) = \dfrac{dv}{dt} = 8$. So for velocity to equal acceleration, we need $v(t) = a(t)$,

so $8t + 4 = 8$, so $8t = 4$, so $t = \dfrac{1}{2}$.

(c) We know that $S(1) = 10$. Now $S(2) = 4 \times 2^2 + 4 \times 2 + 2 = 16 + 8 + 2 = 26$. Thus between $t = 1$ and $t = 2$, the rocket travelled $S(2) - S(1) = 26 - 10 = 16$.

15. (a) $= -2x^3 + x^2 + C$.　　**(b)** $= -\ln x + \dfrac{1}{2}e^{2x} + C$.　　**(c)** $= -2\cos x + 4x + C$.

(d) $= [-e^{-x}]_0^1 = (-e^{-1}) - (-e^0) = -e^{-1} + 1 = -\dfrac{1}{e} + 1$.

(e) $= [-x^{-1} + x]_1^2 = [-\dfrac{1}{x} + x]_1^2 = (-\dfrac{1}{2} + 2) - (-\dfrac{1}{1} + 1) = -\dfrac{1}{2} + 2 - (-1 + 1) = -\dfrac{1}{2} + 2 = \dfrac{3}{2}$.

16. (a) Using the product rule from the notes, let $u = x^2$ and $v = e^{-2x}$. Now $u' = 2x$ and $v' = -2e^{-2x}$.

So $\dfrac{dy}{dx} = u' \times v + v' \times u = 2x(e^{-2x}) + (-2e^{-2x})x^2 = 2xe^{-2x} - 2x^2e^{-2x} = e^{-2x}(2x - 2x^2)$.

(b) If $\dfrac{dy}{dx} = 0$, then $e^{-2x}(2x - 2x^2) = 0$, so either $e^{-2x} = 0$ (which is impossible) or else $2x - 2x^2 = 0$, so $2x(1 - x) = 0$, so $x = 0$ or $x = 1$.

(c) Using the product rule from the notes, let $u = e^{-2x}$ and $v = 2x - 2x^2$. Now $u' = -2e^{-2x}$ and $v' = 2 - 4x$. So $\dfrac{d^2y}{dx^2} = u' \times v + v' \times u = -2e^{-2x}(2x - 2x^2) + (2 - 4x)e^{-2x} = $
$-4xe^{-2x} + 4x^2e^{-2x} + 2e^{-2x} - 4xe^{-2x} = 4x^2e^{-2x} - 8xe^{-2x} + 2e^{-2x} = e^{-2x}(4x^2 - 8x + 2)$.

(d) From the notes we know that a critical point is a local maximum if $y''(x) < 0$ and a local minimum if $y''(x) > 0$. We found y'' in Part **(c)**. Hence $y''(0) = e^0(4 \times 0^2 - 8 \times 0 + 2) = 1(2) = 2$, which is positive, so at $x = 0$ there is a local minimum. Next, $y''(1) = e^{(-2 \times 1)}(4 \times 1^2 - 8 \times 1 + 2) = e^{-2}(4 - 8 + 2) = -2e^{-2} =$, which is negative, so at $x = 1$ there is a local maximum. The y-coordinates are $y(0) = 0^2e^0 = 0$ and $y(1) = 1^2e^{(-2 \times 1)} = e^{-2} = \dfrac{1}{e^2}$. So there is a local minimum at $(0, 0)$ and a local maximum at $(1, \dfrac{1}{e^2})$.

17. To solve: $y = 3x^2 + 2x - 3$ (1) $y = 2x^2 - 2x + 9$ (2)

Let (1) = (2) because $y = y$, so $3x^2 + 2x - 3 = 2x^2 - 2x + 9$, so $x^2 + 4x - 12 = 0$.

Using the quadratic formula, $x = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 1 \times -12}}{2 \times 1} = \dfrac{-4 \pm \sqrt{16 + 48}}{2} = \dfrac{-4 \pm \sqrt{64}}{2} = \dfrac{-4 \pm 8}{2}$,

so $x = \dfrac{-4 + 8}{2}$ or $\dfrac{-4 - 8}{2}$, so $x = 2$ or $x = -6$, so there are two solutions.

Firstly, substituting $x = 2$ into (1) gives us $y = 3 \times 2^2 + 2 \times 2 - 3$, so $y = 12 + 4 - 3$ so $y = 13$. Hence one solution is $(x, y) = (2, 13)$.

Secondly, substituting $x = -6$ into (1) gives us $y = 3 \times (-6)^2 + 2 \times -6 - 3$, so $y = 108 - 12 - 3$ so $y = 93$. Hence another solution is $(x, y) = (-6, 93)$

Now we should always check the answer. Substitute $(x, y) = (2, 13)$ and then $(x, y) = (-6, 93)$ into Equations (1) and (2), and see that both equations are satisfied, so both answers **must** be correct.

14.36   Final exam, 2003

1. Find R if $1 = \sum_{i=1}^{3} \frac{(i+1)R}{2i}.$ (4 marks)

2. Removed (not relevant).   (5 marks)

3. Let A, B and C be sets. In each of Parts (a), (b), (c) and (d), you are given an expression involving these sets. In each case, **shade the region** on a Venn diagram which corresponds to the expression. If the expression equals the empty set, state that the answer is the empty set (2 marks each part).

   (a) $A \cap (B \cup C)$ **(b)** $(A \setminus B) \setminus C$ **(c)** $(A \cap B \cap C) \setminus C$ **(d)** $B \setminus (A \cap C)$

4. Simplify $\dfrac{(x^{-2})^3 (x^2 y)}{x^{-3} y^3} \div (x^{-2} y^{-2}).$ (3 marks)

5. Solve the following system of two simultaneous equations.
$$3x + 2y = \sqrt{2} \quad \text{and} \quad -5x - 7y = 2\sqrt{2}.$$ (4 marks)

6. Find the straight line distance between the points $(0, -\sqrt{7})$ and $(\sqrt{11}, 0)$, expressing your answer as a surd in simplest form.   (2 marks)

7. I roll *two fair, six-sided* dice, one coloured green and the other blue. Let t be the total obtained by adding the scores on both dice. If $t = 9$, I continue to roll again until t does not equal 9. Let g denote the number shown on the green die, and b denote the number shown on the blue die (so we know that $g + b \neq 9$).

   (a) Using a table of all possible outcomes (or any other way you like), show that the sample space has 32 possible outcomes.   (2 marks)

   (b) What is the probability that $t \geq 8$?   (2 marks)

   (c) What is the probability that t is odd?   (2 marks)

   (d) What is the probability that $g = b - 1$?   (2 marks)

8. Find the equation of the line passing through the points $(1, 4)$ and $(3, 14)$.   (3 marks)

9. Deleted.

10. **(a)** Show that the slope of the curve $y = x + 4\sqrt{x}$ at the point $(4, 12)$ is 2.
     (3 marks)

     (b) Hence, or otherwise, find the equation of the line which is tangential to the curve $y = x + 4\sqrt{x}$ at the point $(4, 12)$.   (2 marks)

11. (4 marks)   If $y = \dfrac{e^x}{x^2}$, find and simplify $\dfrac{dy}{dx}$.   **12.** (4 marks)   If $y = x^4 \ln x$, find and simplify $\dfrac{dy}{dx}$.

13. Let $f(x) = \dfrac{x^3}{3} - 2x^2 + 3x - 1$.

   (a) Find the coordinates of all critical points of $f(x)$.   (4 marks)

   (b) Classify each critical point as a local maximum or minimum.   (3 marks)

14. The following two sets of axes each contain a graph of $y = \sin x$, for $x \in [-2\pi, 2\pi]$. On the first set of axes, sketch a graph of $y = 2 \sin x$. **Note: to save space, the diagram is shown in the final exam paper for June 2004.**   (2 marks)

   Also sketch a graph of $y = \dfrac{1}{2} \sin \dfrac{x}{2}$.   (2 marks)

15. The following diagram contains the graph of a function and the graph of the derivative of the function. The graphs are labelled A and B. Clearly identify which graph represents the function, and which represents the derivative. Explain your answer. (Most marks will be assigned to your explanation.)
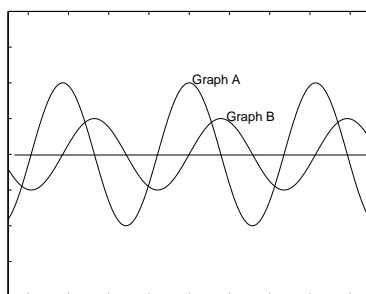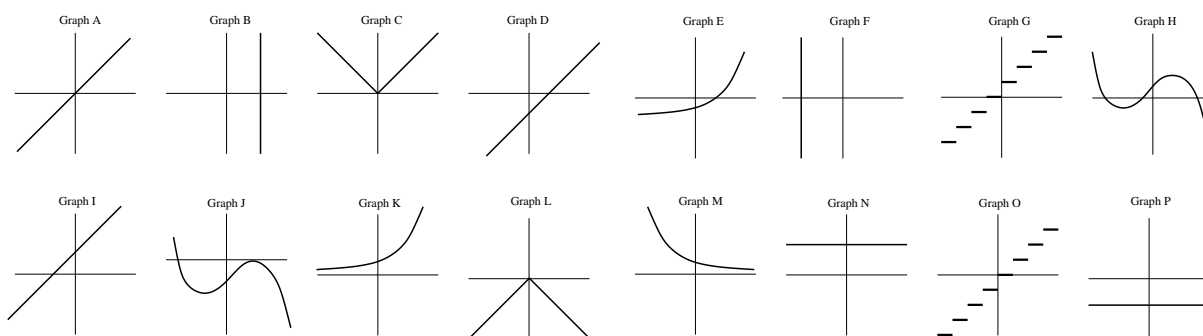
(3 marks)

16. Below are eight equations, with their graphs drawn below the equations, in random order, with eight extra (unused) graphs included. Match each equation with its graph, by writing the letter of the corresponding graph next to each equation. Note that the graphs are labelled A to P. (8 marks)

(1) removed (not relevant) (2) $y = \dfrac{3x}{2} + (-1)^4$. (3) $y = -(x+2)(x+4)(x-4)$. (4) $y = \pi^{10}$.

(5) $y = e^{-2x}$. (6) $y = |-x|$. (7) $y + 4 = 4x + 4$. (8) $-4x = -2$.

17. A ball is thrown vertically into the air at time $t = 0$, with **acceleration** $a(t) = -10$.

(i) When $t = 3$, the ball has **velocity** $v = 0$. Find an expression for the **velocity** of the ball at any time t. (3 marks)

(ii) When $t = 1$, the ball has **displacement** $S = 27$. Find an expression for the **displacement** of the ball at any time t. (3 marks)

(iii) At what time(s) does the ball have **displacement** $S = 2$? (3 marks)

18. (5 marks) Evaluate $\displaystyle\int_{-1}^{1} (x + x^2 + x^3 + x^4)\, dx$. **19.** (3 marks) Find $\displaystyle\int \left(-\cos x + \dfrac{2}{x} + e^{-x} \right) dx$.

20. Let $y = (1 - e^x)^{10}$.

(a) Using the chain rule (or otherwise), find $\dfrac{dy}{dx}$. (4 marks)

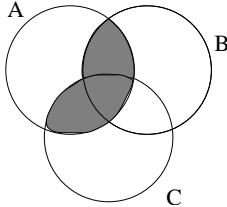(b) Find all values of x for which $\dfrac{dy}{dx} = 0$. (2 marks)

21. Solve for x: $\displaystyle\sum_{i=1}^{2} \left(\sum_{j=i+1}^{3} \left(\dfrac{ix^2}{4} + jx + i + j \right) \right) = 0$. (Hint: first expand the summation signs and then solve the resulting quadratic equation.) (6 marks)
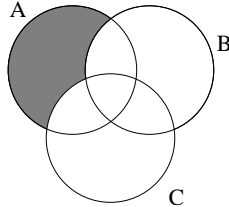
14.37 Solutions to final exam, 2003

1. $1 = \sum\limits_{i=1}^{3} \dfrac{(i+1)R}{2i}$ so $1 = \dfrac{(1+1)R}{2 \times 1} + \dfrac{(2+1)R}{2 \times 2} + \dfrac{(3+1)R}{2 \times 3}$ so $1 = \dfrac{2R}{2} + \dfrac{3R}{4} + \dfrac{4R}{6}$ so $1 = R + \dfrac{3R}{4} + \dfrac{2R}{3}$ so $1 = \dfrac{12R + 9R + 8R}{12}$ so $12 = 29R$. Hence $R = \dfrac{12}{29}$
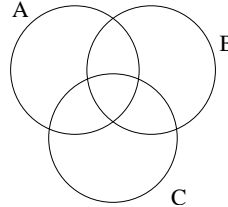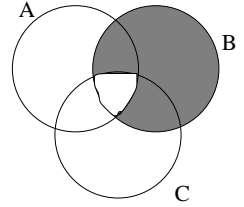
2. Removed (not relevant).

3. (a)   **(b)**   **(c)**   **(d)**

4. $\dfrac{(x^{-2})^3(x^2y)}{x^{-3}y^3} \div (x^{-2}y^{-2}) = \dfrac{x^{-6}x^2y}{x^{-3}y^3} \times x^2y^2 = \dfrac{x^{(-6+2+2)}y^{(1+2)}}{x^{-3}y^3} = \dfrac{x^{-2}y^3}{x^{-3}y^3} = \dfrac{x^{-2}}{x^{-3}} = x^{-2} \times x^3 = x^{-2+3} = x.$

5. To solve: $3x + 2y = \sqrt{2}$ (1) $-5x - 7y = 2\sqrt{2}$ (2)

From (1), $\dfrac{3}{2}x + y = \dfrac{\sqrt{2}}{2}$ so $y = -\dfrac{3}{2}x + \dfrac{\sqrt{2}}{2}$. Substitute this into (2), giving $-5x - 7(-\dfrac{3}{2}x + \dfrac{\sqrt{2}}{2}) = 2\sqrt{2}$, so $-5x + \dfrac{21}{2}x - \dfrac{7\sqrt{2}}{2} = 2\sqrt{2}$, so $\dfrac{11}{2}x = \dfrac{11\sqrt{2}}{2}$ and hence $x = \sqrt{2}$.

Substitute $x = \sqrt{2}$ into (1) gives us $3 \times \sqrt{2} + 2y = \sqrt{2}$, so $2y = -2\sqrt{2}$ so $y = -\sqrt{2}$. Hence the solution is $(x, y) = (\sqrt{2}, -\sqrt{2})$.

Now we should always check the answer. Substitute $(x, y) = (\sqrt{2}, -\sqrt{2})$ into Equations (1) and (2), and see that both equations are satisfied, so the answer **must** be correct.

6. Let $(x_1, y_1) = (0, -\sqrt{7})$ and $(x_2, y_2) = (\sqrt{11}, 0)$.

Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(0 - \sqrt{11})^2 + (-\sqrt{7} - 0)^2} = \sqrt{(-\sqrt{11})^2 + (-\sqrt{7})^2} = \sqrt{11 + 7} = \sqrt{18} = 3\sqrt{2}$.

7. (a) The table below contains all 32 possible outcomes when the two dice are thrown and the total doesn't equal 9. The numbers along the top are the values shown on the blue die and the numbers along the side are the values shown on the green die. The entries in the table are the totals.

t	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	–
4	5	6	7	8	–	10
5	6	7	8	–	10	11
6	7	8	–	10	11	12

(b) $t \geq 8$ happens in 11 cases: Prob$(t \geq 8) = \dfrac{11}{32}$.   **(c)** t is odd in 14 cases: Prob$(t$ is odd$) = \dfrac{14}{32} = \dfrac{7}{16}$.

(d) $g = b - 1$ happens in 4 cases: Prob$(g = b - 1) = \dfrac{4}{32} = \dfrac{1}{8}$.

8. Let $(x_1, y_1) = (1, 4)$ and $(x_2, y_2) = (3, 14)$. From the notes $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{14 - 4}{3 - 1} = \dfrac{10}{2} = 5$. So the equation is $y = 5x + c$. Substitute in the point $(1, 4)$, so $4 = 5 \times 1 + c$, so $c = -1$. The equation is $y = 5x - 1$.

9. Deleted.

10. **(a)** $y = x + 4\sqrt{x} = x + 4x^{1/2}$. Using the chain rule from the notes, $\dfrac{dy}{dx} = 1 + 4 \times \dfrac{1}{2}x^{-1/2} = 1 + \dfrac{2}{x^{1/2}} = 1 + \dfrac{2}{\sqrt{x}}$.

   To find the slope at $(4, 12)$ substitute $x = 4$, so $\dfrac{dy}{dx} = 1 + \dfrac{2}{\sqrt{4}} = 1 + 1 = 2$.

   (b) From **(a)** we know that $m = 2$ at $(4, 12)$, so the equation is $y = 2x + c$. Substitute in $(4, 12)$, giving $12 = 2 \times 4 + c$, so $c = 4$. The equation is $y = 2x + 4$.

11. Using the quotient rule from the notes, let $u = e^x$ and $v = x^2$. Now $u' = e^x$ and $v' = 2x$.
    So $\dfrac{dy}{dx} = \dfrac{u' \times v - v' \times u}{v^2} = \dfrac{e^x \times x^2 - e^x \times 2x}{(x^2)^2} = \dfrac{xe^x(x-2)}{x^4} = \dfrac{e^x(x-2)}{x^3}$.

12. Using the product rule from the notes, let $u = x^4$ and $v = \ln x$. Now $u' = 4x^3$ and $v' = \dfrac{1}{x}$.
    So $\dfrac{dy}{dx} = u' \times v + v' \times u = 4x^3 \times \ln x + \dfrac{1}{x} \times x^4 = 4x^3 \ln x + x^3 = x^3(4\ln x + 1)$.

13. **(a)** Now $f'(x) = \dfrac{3x^2}{3} - 4x + 3 = x^2 - 4x + 3$. The critical points are where $f'(x) = 0$, so $x^2 - 4x + 3 = 0$. This can be solved using the quadratic formula from the notes: $x = \dfrac{4 \pm \sqrt{16 - 4 \times 1 \times 3}}{2 \times 1} = \dfrac{4 \pm \sqrt{16 - 12}}{2} = \dfrac{4 \pm \sqrt{4}}{2} = \dfrac{4 \pm 2}{2}$, so $x = \dfrac{4+2}{2}$ or $x = \dfrac{4-2}{2}$, so $x = 3$ or $x = 1$. Substitute $x = 3$ and $x = 1$ back into $f(x)$ to find the y-coordinates: $f(1) = \dfrac{1^3}{3} - 2(1)^2 + 3 \times 1 - 1 = \dfrac{1}{3} - 2 + 3 - 1 = \dfrac{1}{3}$ and $f(3) = \dfrac{3^3}{3} - 2(3)^2 + 3 \times 3 - 1 = \dfrac{27}{3} - 2 \times 9 + 9 - 1 = 9 - 18 + 9 - 1 = -1$. So the critical points are at $(1, \dfrac{1}{3})$ and $(3, -1)$.

    (b) From the notes we know that a critical point is a local maximum if $f''(x) < 0$ and a local minimum if $f''(x) > 0$. Now $f''(x) = 2x - 4$, so $f''(1) = 2 \times 1 - 4 = -2$, so at $(1, \dfrac{1}{3})$ there is a local maximum. $f''(3) = 2 \times 3 - 4 = 2$, so at $(3, -1)$ there is a local minimum.

14. The dashed graph on the left is $2\sin x$. On the right is $\dfrac{1}{2}\sin\dfrac{1}{2}x$.
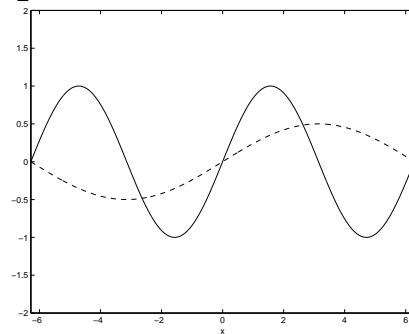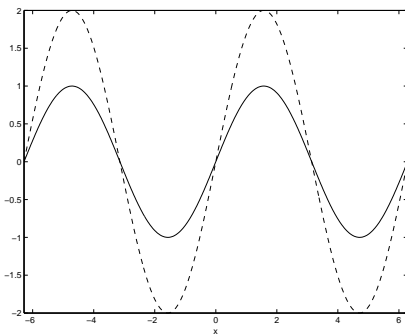
15. The function is Graph B. The derivative is Graph A.
    Explanation: Suppose that Graph A is the function (and Graph B is its derivative). The increasing sections of Graph A (between the zero and the local maximums) must have a positive derivative (gradient) but the corresponding section of Graph B is actually negative. This means that Graph B is definitely not the derivative of Graph A. Hence Graph A **must** be the derivative of Graph B.

16. **1.** G **2.** I **3.** H **4.** N **5.** M **6.** C **7.** A **8.** B

17. **(i)** $v(t) = \int a(t) = -10t + C$. We also know that $v(3) = 0$, so $0 = -10 \times 3 + C$, so $C = 30$. Hence $v(t) = -10t + 30$.

(ii) $S(t) = \int v(t) = -5t^2 + 30t + C$. We also know that $S(1) = 27$, so $27 = -5 \times 1^2 + 30 \times 1 + C$, so $27 = -5 + 30 + C$, so $C = 2$. Hence $S(t) = -5t^2 + 30t + 2$.

(iii) To find at what time the ball has displacement $S = 2$, we must solve $2 = S(t) = -5t^2 + 30t + 2$, so $0 = -5t^2 + 30t$, so $0 = t(-5t + 30)$. Thus the ball has displacement $S = 2$ when $t = 0$ and when $(-5t + 30) = 0$. Thus the ball has displacement $S = 2$ when $t = 0$ and $t = 6$.

18. $\int_{-1}^{1}(x+x^2+x^3+x^4)dx = [\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}]_{-1}^{1} = [\frac{1^2}{2} + \frac{1^3}{3} + \frac{1^4}{4} + \frac{1^5}{5}] - [\frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4} + \frac{(-1)^5}{5}] =$
$(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) - (\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{2}{3} + \frac{2}{5} = \frac{10}{15} + \frac{6}{15} = \frac{16}{15}$.

19. $\int(-\cos x + \frac{2}{x} + e^{-x})dx = -\sin x + 2\ln x - e^{-x} + C$.

20. (a) Using the chain rule from the notes, $\frac{dy}{dx} = 10(1 - e^x)^9 \times -e^x = -10e^x(1 - e^x)^9$.

(b) If $\frac{dy}{dx} = 0$, then $-10e^x(1 - e^x)^9 = 0$, so either $e^x = 0$ (which is impossible) or else $(1 - e^x) = 0$, so $1 = e^x$, so $x = 0$.

21. $\sum_{i=1}^{2}(\sum_{j=i+1}^{3}(\frac{ix^2}{4} + jx + i + j)) = 0$, so $(\sum_{j=2}^{3}(\frac{x^2}{4} + jx + 1 + j)) + (\sum_{j=3}^{3}(\frac{2x^2}{4} + jx + 2 + j)) = 0$.
Then $((\frac{x^2}{4} + 2x + 1 + 2) + (\frac{x^2}{4} + 3x + 1 + 3)) + (\frac{2x^2}{4} + 3x + 2 + 3) = 0$, so $\frac{2x^2}{4} + 5x + 7 + \frac{2x^2}{4} + 3x +$
$5 = 0$, so $x^2 + 8x + 12 = 0$. Using the quadratic formula from the notes, $x = \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times 12}}{2 \times 1} =$
$\frac{-8 \pm \sqrt{64 - 48}}{2} = \frac{-8 \pm \sqrt{16}}{2} = \frac{-8 \pm 4}{2}$, so $x = \frac{-8 + 4}{2}$ or $x = \frac{-8 - 4}{2}$, so $x = -2$ or $x = -6$.

14.38 Final exam, 2001

1. Find R if $\quad \dfrac{2}{R} = \displaystyle\sum_{i=-1}^{2} \dfrac{1}{\mid 2i \mid -1}.$ (4 marks)

2. Removed (not relevant). (5 marks)

3. Let $A = \{a, b, d, e\}$, $B = \{a, d, f, g\}$ and $C = \{a, c, e, f\}$. For parts (b), (c), (d) and (e), either write the elements of the answer set, or (if appropriate) state that the answer is the empty set. Each part is worth 2 marks.

   (a) Mark the three sets A, B and C on a Venn diagram, with the elements of each set written in appropriate places on the diagram.

   (b) Write the set $(A \cap C) \setminus B$.   (c) Write the set $B \setminus (B \cap B)$.

   (d) Write the set $(A \cup B) \cap (C \setminus A)$.   (e) Write the set $(A \setminus \emptyset) \cap (B \setminus \emptyset) \cap (C \setminus \emptyset)$.

4. Simplify $\quad \dfrac{2^{n+1} - 2^n}{2^n}.$ (2 marks)

5. Solve the following system of two simultaneous equations.
$$3y - x = 6$$
$$2y + 3x = 15.$$
(4 marks)

6. Find the straight line distance between the points $(4, 2)$ and $(1, 5)$, expressing your answer as a surd in simplest form. (2 marks)

7. I roll *two fair, six-sided* dice, one coloured green and the other blue. Let g denote the number shown on the green die, and b denote the number shown on the blue die. What is the probability that:

   (a) $g + b = 7$? (2 marks)

   (b) g is odd and b is even? (2 marks)

   (c) $g < b$? (2 marks)

8. Deleted.

9. (a) Show that the slope of the curve $y = \sqrt{x}$ at the point $(4, 2)$ is $\dfrac{1}{4}$. (3 marks)

   (b) Hence, or otherwise, find the equation of the line which is tangential to the curve $y = \sqrt{x}$ at the point $(4, 2)$. (2 marks)

10. (4 marks) If $y = \sin(e^{2x})$, find $\dfrac{dy}{dx}$.   11. (4 marks) If $y = \dfrac{(2x + 1)}{(2x - 1)}$, find $\dfrac{dy}{dx}$.

12. Let $f(x) = -x^3 + 9x^2 - 24x - 3$.

   (a) Find the coordinates of all critical points of $f(x)$. (5 marks)

   (b) Classify each critical point as a local maximum or minimum. (3 marks)

13. The following two sets of axes each contain a graph of $y = \sin x$, for $x \in [-2\pi, 2\pi]$. On the first set of axes, sketch a graph of $y = 2 \sin x$. **Note: to save space, the diagram is shown in the final exam paper for June 2004.** (2 marks)

   Also sketch a graph of $y = \dfrac{1}{2} \sin 2x$. (2 marks)

14. The following diagram contains the graph of a function and the graph of the derivative of the function. Clearly identify which graph represents the function, and which represents the derivative. Explain your answer. (Most marks will be assigned to your explanation.) (3 marks)
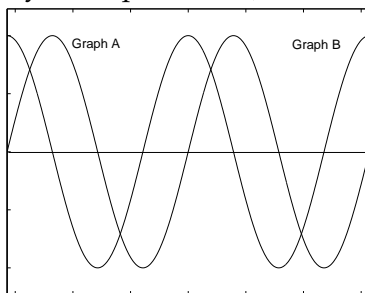
15. Below are eight equations, with their graphs drawn below the equations. Match each equation with its graph. The graphs are labelled A to P. (8 marks)

(1) $y^2 + 2y + 2 = (y+1)^2 - x$. (2) $y = -(x-1)(x-3)$. (3) $3x - 3 + 2y = 0$. (4) $y = \sqrt{\dfrac{1}{16}}$.

(5) $y = -e^{-x}$. (6) $y = -\mid x \mid$. (7) $x = (-1)^4 y$. (8) $x - \sqrt{5} = -2$.
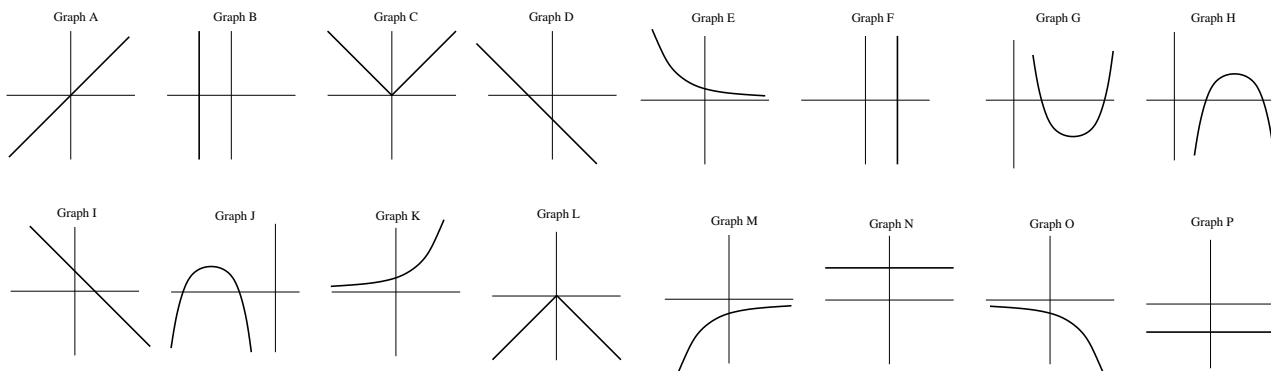
16. A rocket takes off vertically at time $t = 0$, with velocity $v(t) = 6t$ metres/second.

(i) Find an expression for the displacement of the rocket at any time t. Assume the rocket launching pad is 6 metres **below** ground level, so the displacement at $t = 0$ is -6. (4 marks)

(ii) At what time t is the rocket 42 metres above ground (so it has travelled 48 metres in total). (3 marks)

17. (3 marks) Evaluate $\displaystyle\int_{-1}^{1} (4x^3 + e^x + 1)\, dx$. **18.** (4 marks) Find $\displaystyle\int \left(\frac{2}{x} - \sin x + e^{2x} \right) dx$.

19. For parts (a), (b), (c) and (d), let $y = e^{-x}\left(-x^2 + x - 1\right)$.

(a) Using the product rule (or otherwise), show that $\dfrac{dy}{dx} = e^{-x}(x^2 - 3x + 2)$. (4 marks)

(b) Find all values of x for which $\dfrac{dy}{dx} = 0$. (3 marks)
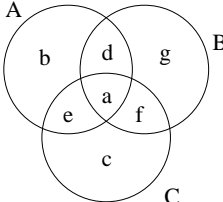
(c) Find the second derivative of y. (3 marks)

(d) Hence, or otherwise, find and classify all local maxima and minima of the function y. (4 marks)

14.39 Solutions to final exam, 2001

1. $\dfrac{2}{R} = \displaystyle\sum_{i=-1}^{2} \dfrac{1}{|2i|-1}$ so $\dfrac{2}{R} = (\dfrac{1}{|2 \times -1|-1}) + (\dfrac{1}{|2 \times 0|-1}) + (\dfrac{1}{|2 \times 1|-1}) + (\dfrac{1}{|2 \times 2|-1})$ so $\dfrac{2}{R} = \dfrac{1}{|-2|-1} +$

$\dfrac{1}{|0|-1} + \dfrac{1}{|2|-1} + \dfrac{1}{|4|-1}$ so $\dfrac{2}{R} = \dfrac{1}{2-1} + \dfrac{1}{0-1} + \dfrac{1}{2-1} + \dfrac{1}{4-1}$ so $\dfrac{2}{R} = \dfrac{1}{1} + \dfrac{1}{-1} + \dfrac{1}{1} + \dfrac{1}{3}$ so $\dfrac{2}{R} = 1-1+1+\dfrac{1}{3}$

so $\dfrac{2}{R} = \dfrac{4}{3}$ so $6 = 4R$ so $\dfrac{6}{4} = R$. Hence $R = \dfrac{3}{2}$.

2. Removed (not relevant).

3. **(a)**   **(b)** $\{e\}$ **(c)**   The emptyset, $\emptyset$.   **(d)** $\{f\}$ **(e)** $\{a\}$

4. $\dfrac{2^{n+1} - 2^n}{2^n} = \dfrac{2^n \times 2^1 - 2^n}{2^n} = \dfrac{2^n(2-1)}{2^n} = (2-1) = 1.$

5. To solve: $3y - x = 6$ (1) $2y + 3x = 15$ (2)

   From (1), $-x = 6-3y$ so $x = -6+3y$. Substitute this into (2), giving $2y+3(-6+3y) = 15$, so $2y-18+9y = 15$, so $11y = 33$ and hence $y = 3$.

   Substitute $y = 3$ into (1) gives us $3 \times 3 - x = 6$, so $9 - x = 6$ so $-x = 6 - 9$, so $-x = -3$ and hence $x = 3$. Hence the solution is $(x, y) = (3, 3)$.

   Now we should always check the answer. Substitute $(x, y) = (3, 3)$ into Equations (1) and (2), and see that both equations are satisfied, so the answer **must** be correct.

6. Let $(x_1, y_1) = (4, 2)$ and $(x_2, y_2) = (1, 5)$.
   Then $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(4-1)^2 + (2-5)^2} = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}.$

7. **(a)** The table below contains all 36 possible outcomes when two dice are thrown. In each case, the first number is the value shown on the green die, and the second number is the value shown on the blue die.

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 3	3 3	3 4	3 5	3 6
4 1	4 2	4 4	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

   $g + b = 7$ in 6 cases: Prob$(g + b = 7) = \dfrac{6}{36} = \dfrac{1}{6}.$

   (b) g is odd and b is even in 9 cases: Prob(g is odd and b is even)$= \dfrac{9}{36} = \dfrac{1}{4}.$

   (c) $g < b$ in 15 cases: Prob$(g < b) = \dfrac{15}{36} = \dfrac{5}{12}.$

8. Deleted.

9. **(a)** $y = \sqrt{x} = x^{1/2}$. Hence $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2x^{1/2}} = \dfrac{1}{2\sqrt{x}}.$

   So the slope at $(4, 2)$ is $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2 \times 2} = \dfrac{1}{4}.$

(b) From **(a)** we know that $m = \dfrac{1}{4}$ at $(4, 2)$, so the equation is $y = \dfrac{1}{4}x + c$. Substitute in $(4, 2)$, so $2 = 4 \times \dfrac{1}{4} + c$, so $c = 1$. The equation is $y = \dfrac{1}{4}x + 1$.

10. Using the chain rule from the notes, $\dfrac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x} = 2e^{2x}\cos(e^{2x})$.

11. Using the quotient rule from the notes, let $u = 2x + 1$ and $v = 2x - 1$. Now $u' = 2$ and $v' = 2$.
    So $\dfrac{dy}{dx} = \dfrac{u' \times v - v' \times u}{v^2} = \dfrac{2(2x-1) - 2(2x+1)}{(2x-1)^2} = \dfrac{4x - 2 - 4x - 2}{(2x-1)^2} = \dfrac{-4}{(2x-1)^2}$.

12. **(a)** Now $f'(x) = -3x^2 + 18x - 24$. The critical points are where $f'(x) = 0$, so $-3x^2 + 18x - 24 = 0$, so $-3(x^2 - 6x + 8) = 0$, so $x^2 - 6x + 8 = 0$. This can be solved using the quadratic formula from the notes. $x = \dfrac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 8}}{2 \times 1} = \dfrac{6 \pm \sqrt{36 - 32}}{2} = \dfrac{6 \pm \sqrt{4}}{2} = \dfrac{6 \pm 2}{2}$, so $x = \dfrac{6 + 2}{2}$ or $x = \dfrac{6 - 2}{2}$, so $x = 4$ or $x = 2$. Substitute $x = 4$ and $x = 2$ back into $f(x)$ to find the y-coordinates. $f(2) = -2^3 + 9(2)^2 - 24 \times 2 - 3 = -8 + 36 - 48 - 3 = -23$. $f(4) = -4^3 + 9(4)^2 - 24 \times 4 - 3 = -64 + 144 - 96 - 3 = -19$. So the critical points are at $(2, -23)$ and $(4, -19)$.

    (b) From the notes we know that a critical point is a local maximum if $f''(x) < 0$ and a local minimum if $f''(x) > 0$. Now $f''(x) = -6x + 18$, so $f''(2) = -6 \times 2 + 18 = -12 + 18 = 6$, so at $(2, -23)$ there is a local minimum. $f''(4) = -6 \times 4 + 18 = -6$, so at $(4, -19)$ there is a local maximum.

13. The dashed graph on the left is $y = 2\sin x$. On the right is $y = \dfrac{1}{2}\sin 2x$.
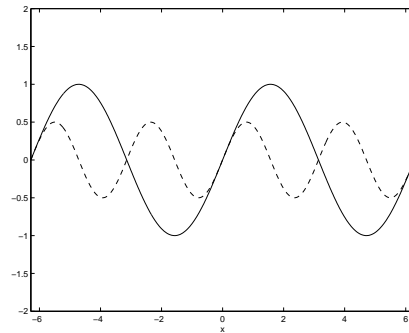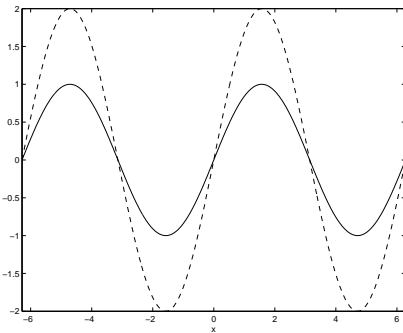
14. The function is Graph A. The derivative is Graph B.
    Explanation: Suppose that Graph B is the function (and Graph A its derivative). The increasing sections of Graph B (between the zero and the local maximums) should a positive derivative (gradient) but the corresponding section of Graph A is actually negative. This means that Graph A is definitely not the derivative of Graph B.

15. **1.** B **2.** H **3.** I **4.** N **5.** M **6.** L **7.** A **8.** F

16. **(i)** $S(t) = \int v(t) = 3t^2 + C$. We also know that $S(0) = -6$, so $-6 = 3 \times 0^2 + C$, so $C = -6$. Hence $S(t) = 3t^2 - 6$.

    (ii) To find at what time the rocket has displacement $S = 42$, we must solve $42 = S(t) = 3t^2 - 6$, so $48 = 3t^2$, so $16 = t^2$. Thus the rocket has displacement $S = 42$ when $t = -4$ (impossible) and when $t = 4$.

17. $\int_{-1}^{1}(4x^3 + e^x + 1)dx = [x^4 + e^x + x]_{-1}^{1} = [1^4 + e^1 + 1] - [(-1)^4 + e^{-1} - 1] = 1 + e + 1 - (1 + \dfrac{1}{e^1} - 1) = 1 + e + 1 - 1 - \dfrac{1}{e} + 1 = 2 + e - \dfrac{1}{e}$.

18. $\int(\dfrac{2}{x} - \sin x + e^{2x})dx = 2\ln x + \cos x + \dfrac{1}{2}e^{2x} + C$.

19. **(a)** Using the product rule from the notes, let $u = e^{-x}$ and $v = -x^2 + x - 1$. Now $u' = -e^{-x}$ and $v' = -2x + 1$.

So $\dfrac{dy}{dx} = u' \times v + v' \times u = -e^{-x}(-x^2 + x - 1) + e^{-x}(-2x + 1) = e^{-x}x^2 - e^{-x}x + e^{-x} - 2xe^{-x} + e^{-x} = x^2 e^{-x} - 3xe^{-x} + 2e^{-x} = e^{-x}(x^2 - 3x + 2)$.

(b) If $\dfrac{dy}{dx} = 0$, then $e^{-x}(x^2 - 3x + 2) = 0$, so either $e^{-x} = 0$ (which is impossible) or else $(x^2 - 3x + 2) = 0$.

Using the quadratic formula from the notes, $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 2}}{2 \times 1} = \dfrac{3 \pm \sqrt{9 - 8}}{2} = \dfrac{3 \pm \sqrt{1}}{2} = \dfrac{3 \pm 1}{2}$, so $x = \dfrac{3 + 1}{2}$ or $x = \dfrac{3 - 1}{2}$, so $x = 2$ or $x = 1$.

(c) Note that the second derivative of y is the same as the first derivative of $\dfrac{dy}{dx}$. Using the product rule from the notes, let $u = e^{-x}$ and $v = x^2 - 3x + 2$. Now $u' = -e^{-x}$ and $v' = 2x - 3$.

So $\dfrac{d^2y}{dx^2} = u' \times v + v' \times u = -e^{-x}(x^2 - 3x + 2) + e^{-x}(2x - 3) = -x^2 e^{-x} + 3xe^{-x} - 2e^{-x} + 2xe^{-x} - 3e^{-x} = -x^2 e^{-x} + 5xe^{-x} - 5e^{-x} = e^{-x}(-x^2 + 5x - 5)$.

(d) From the notes we know that a critical point is a local maximum if $\dfrac{d^2y}{dx^2} < 0$ and a local minimum if $\dfrac{d^2y}{dx^2} > 0$. Now $y''(1) = e^{-1}(-(1)^2 + 5 \times 1 - 5) = e^{-1}(-1 + 5 - 5) = e^{-1} \times -1 = -e^{-1} = \dfrac{-1}{e}$. The y-coordinate for $x = 1$ is $y(1) = e^{-1}(-1^2 + 1 - 1) = -e^{-1} = \dfrac{-1}{e}$. $y''(2) = e^{-2}(-2^2 + 5 \times 2 - 5) = e^{-2}(-4 + 10 - 5) = e^{-2} = \dfrac{1}{e^2}$. The y-coordinate for $x = 2$ is $y(2) = e^{-2}(-2^2 + 2 - 1) = -3e^{-2} = \dfrac{-3}{e^2}$.

Hence at $(1, \dfrac{-1}{e})$ there is a local maximum and at $(2, \dfrac{-3}{e^2})$ there is a local minimum.