8 Functions

A *function* specifies a rule by which an **input** is converted to a **unique output**. Functions are usually written as

$$f(x) = expression$$

where x represents the input to the function and *expression* gives the output of the function. Often the output is called y, so we can plot the graph on a set of axes. The titles $f(x)$ and y are the same, the use of $f(x)$ is that it tells what the input value is. Take the function $f(x) = 2x$ this means take any value x and multiply it by 2. So, $f(1) = 2 \times 1$, $f(3) = 2 \times 3$, $f(14) = 2 \times 14$, $f(a) = 2 \times a$. Even $f(\square) = 2 \times \square$. Whatever is in the brackets after the name of the function (in the case above, f) is what we replace x with in the formula. Of course we don't have to use f and x, $g(a) = 2a$ is the same function with different names for the variables. It has the same inputs and outputs as $f(x) = 2x$. Try not to get caught up in thinking about x; it's just a way of telling us where to put the value we want in. It's like an empty box waiting for us to fill it.

Practice Question 1

Given $f(a) = 2a + 7$, find $f(2)$, $f(-4)$, $f(\triangle)$ and $f(b + c)$.

The **domain** of a function is the set of all possible x values that can be used as inputs, and the **range** of a function is the set of all possible y values that arise as outputs. The domain and range are often written in *interval format.*

When trying to work out what the domain and range of a function are, ask yourself some questions. For domain, ask yourself what possible values can I put into this function so that I get a number out. For range, ask yourself are there any numbers that I can't get as a y value?
Look at the function $f(x) = x^2$. I can put any number in because I know that I can square any real number. Therefore my domain is all numbers from negative infinity to positive infinity, $(-\infty, \infty)$. When I square a number I know that I get a positive number so my output can not contain negative numbers. I check to see if I can get zero, yes $0^2 = 0$. There is no limit to how big my output can get, so my range must be all positive numbers including zero, $[0, \infty)$.

Key points to remember for domain and range:

- You can't take the square root of a negative number.
- You can't divide by zero, so you can't have a value of x that makes the denominator 0.
- By convention the square root of every number is always positive.
- The absolute value of every number is always positive.
- Every number squared is always 0 or positive.

Practice Question 2

Find the domain of $f(x) = \dfrac{2}{x - 3}$

Practice Question 3

Find the range of $f(x) = \sqrt{x + 3}$

Composition of functions is where instead of substituting a number for x into the function, we substitute another function. Remember x is just holding a place for us; it's like an empty box telling us where to substitute the value between the brackets at the start. For example, given the functions $f(x) = 2x^2 + 4$ and $g(x) = 4x$, find $f(g(x))$. That means wherever there is an x in the function $f(x)$ put $g(x)$. Let's find $f(g(1))$. The value of $g(1)$ is 4. Substituting this in as the input for f, we get $f(4) = 40$. Therefore $f(g(1)) = 40$.

Practice Question 4
Let $f(x) = 2x^2 + 4$ and $g(x) = 4x$. Find $f(g(2))$ and $f(g(-1))$.

Often we want to express the composition of functions as a function itself using the variable x. Here we actually need to place the function $g(x)$ where we see x in f to find a formula for $f(g(x))$. Look at the equations below and see how $g(x)$ is substituted in and then the function simplified. Once again, let $f(x) = 2x^2 + 4$ and $g(x) = 4x$.

$$\begin{aligned} f(x) &= 2x^2 + 4 \\ f(g(x)) &= 2(g(x))^2 + 4 \\ &= 2(4x)^2 + 4 \\ &= 2(16x^2) + 4 \\ &= 36x^2 + 4 \end{aligned}$$

Note that in general, $f(g(x))$ is **not** the same as $g(f(x))$.

Practice Question 5
Let $g(x) = \sqrt{x}$ and $h(x) = 4x - 2$. Find $g(h(x))$ and $h(g(x))$.

Discussion Questions
Work through these problems with the person next to you or in a small group.

1. Find the domain of the following functions:
   (a) $f(x) = |x|$ (b) $g(x) = \sqrt{-x}$ (c) $h(x) = \sqrt{-x+4}$ (d) $k(x) = \dfrac{1}{x+5}$
2. Find the range of the following functions:
   (a) $f(x) = |x| - 7$ (b) $g(x) = -x^2$ (c) $h(x) = \sqrt{-x} - 5$ (d) $k(x) = 9$
3. Let $f(x) = x^2 + 2x + 5$, $g(x) = \sqrt{x-5} + 3$ and $h(x) = \dfrac{1}{x+4}$.
   Find the following compositions of functions:
   (a) $f(g(x))$ (b) $g(f(x))$ (c) $h(f(x))$ (d) $h(g(x))$ (e) $g(h(x))$