3.1 Sigma Notation

Sigma notation is a shorthand way of writing long sums. The letter Σ is a Greek capital sigma standing for sum. We often want to expand sigma notation into a long sum, or reduce a long sum into sigma notation. Sometimes we need to solve an equation that uses sigma notation. This can look tricky, but is actually easy to solve. Let's look at the following equation:

$$\sum_{i=3}^{6} ix = 36$$

The first step is to expand the left-hand side; that is, the sum. For the moment forget about the right-hand side.

$$\sum_{i=3}^{6} ix = 3x + 4x + 5x + 6x$$

Next try to simplify or factorise the expanded sum.

$$3x + 4x + 5x + 6x = 18x$$

Then match up the right-hand side of the original equation with the simplifed left-hand side.

$$18x = 36$$

Solve this like you would a normal equation.

$$\frac{18x}{18} = \frac{36}{18}$$

$$x = 2$$

Practice Question 1

$$\sum_{i=2}^{5} 2ix = 84$$

Another use of sigma notation is to reduce a sum into an expression that involves Σ. Sometimes this is easy; $2+3+4+5$ is clearly $\sum_{i=2}^{5} i$. Other times you can't see the answer straight away. To reduce a sum we need to find the pattern between the terms. In the previous example each term had an increase of 1. Let's look at a harder sum:

$$\frac{x}{4} + \frac{x}{9} + \frac{x}{16} + \frac{x}{25}$$

What changes from term to term? Is there a pattern? In this case it is the denominator that is changing. By looking at the numbers closely we can see that they are all square numbers ($2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$). Looking at the base of each of the square numbers we see that they are increasing by 1 each term. An increase of 1, suggests that this might be the right place for i, as i increases by 1 each term. So putting all those ideas together, we want the denominator to be the square of i. Since the numerator stays the same in each term, it can remain as x in our sigma expression.

$$\sum_{\cdots}^{\cdots} \frac{x}{i^2}$$

Next we have to work out our upper and lower bounds. What is the first value in our sum? It was $\frac{x}{4}$. What value of i gives this as the first term? $\frac{x}{i^2} = \frac{x}{4}$ when $i = 2$, so our lower bound is 2. Repeat the same process for the upper bound. $\frac{x}{i^2} = \frac{x}{25}$ when $i = 5$, so our upper bound is 5. Finally write your upper and lower bound into the expression, and we have successfully reduced the sum.

$$\frac{x}{4} + \frac{x}{9} + \frac{x}{16} + \frac{x}{25} = \sum_{i=2}^{5} \frac{x}{i^2}$$

Practice Question 2

$$9y + 12y + 15y + 18y$$

Discussion Questions

Work through these problems with the person next to you or in a small group.

1. $\sum_{i=3}^{7} \frac{3i}{5x} = 5$

2. A tree has 1 apple on day 1. At the end of day 2 it has grown another 2 apples, thus it has a total of 3 apples. At the end of day 3 it has a total of 7 apples. On day 4 it has 15 apples, on day 5 it has 31 apples and so on.
   Write a formula for the number of apples on the tree using sigma notation and hence solve for the number of apples on the tree at the end of day 8.

3. Reduce this sum using sigma notation:

$$1 - x^2 + 2 - 2x^2 + 3 - 3x^2 + 4 - 4x^2 + 5 - 5x^2$$