A Lepage-type test for the GSHD

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The Problem

The **generalised secant hyperbolic distribution** (GSHD) was introduced in 2002 in [1]. The GSHD:

- Is a location-scale family of symmetric unimodal distributions of various tails;
- Includes the Cauchy and the uniform distributions as its limiting heavy-tail and light-tail cases; and
- is interesting in applications where the lack of normality is explained by the tail behaviour of the data distributions.

A member of the distribution is completely specified by the location, scale, and shape (tail) parameters.

We introduce a new location-scale rank test efficient for the GSHD. The new test is a family of Lepage-type tests, each of which combines the standardised location and scale rank statistics efficient under their alternative hypotheses for a specific distribution.

The two-sample linear rank procedures of location and scale alternatives efficient for the GSHD were introduced in 2006 in [2, 3]. Both the location and scale rank procedures are robust to distributional misspecifications. However, the scale procedures are extremely sensitive to the presence of outliers. Moreover, the location estimators are regular almost for the whole family, while the scale ratio rank estimator is only regular conditioned on no difference in location.

Location-scale rank test

We consider the two-sample location-scale problem with two independent samples of sizes m and n. Let $m+n=N$, let

$$c(i) = \begin{cases} \frac{1}{m}\sqrt{\frac{mn}{N}} & , \quad i < m, \\ -\frac{1}{n}\sqrt{\frac{mn}{N}} & , \quad i > m, \end{cases}$$

and let R denote the rank of an observation in the pooled data.

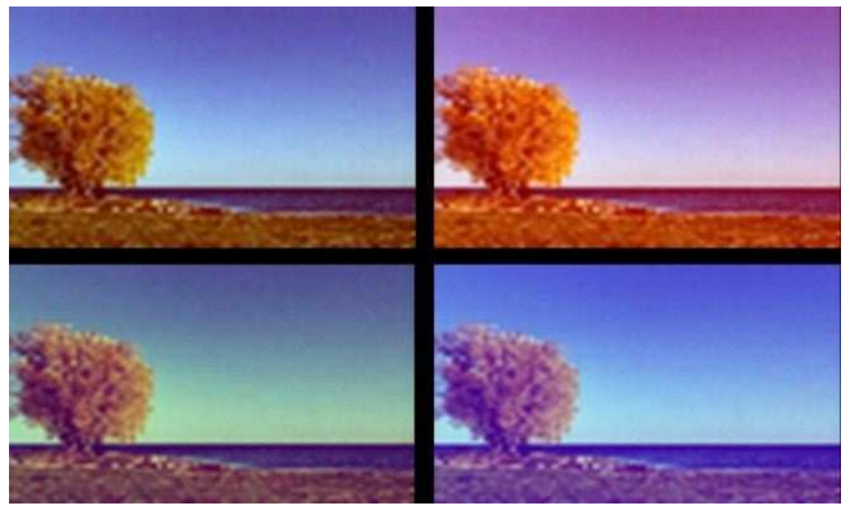
We have studied and compared the properties of the following location-scale rank tests:

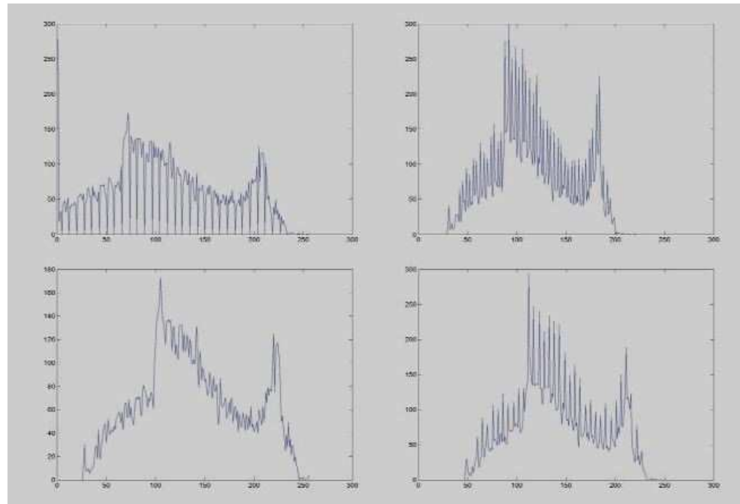
$$S(t=-\pi/2) = \frac{2\pi^2}{N^2}\left[\sum_{i=1}^{N}\sin\left(\pi\frac{i}{N}\right)\sum_{j=1}^{i}c(R_i)\right]^2 + \frac{8}{\pi^2}\left[\sum_{i=1}^{N}c(i)\ln\left(\tan\left(\frac{\pi}{2}\frac{R_i}{N+1}\right)\right)\cos\left(\pi\frac{R_i}{N+1}\right)\right]^2,$$

$$S(t=-\pi) = \frac{16\pi^2}{3N^2}\left[\left(\frac{2}{N}\sum_{i=[N/4]}^{3[N/4]}\cos\left(2\pi\frac{i}{N}\right)\sum_{j=1}^{i}c(R_i)\right)\right]^2 + \frac{8\pi^2}{N^2}\left[\sum_{i=1}^{N}\sin\left(2\pi\frac{i}{N}\right)\sum_{j=1}^{i}c(R_i)\right]^2,$$

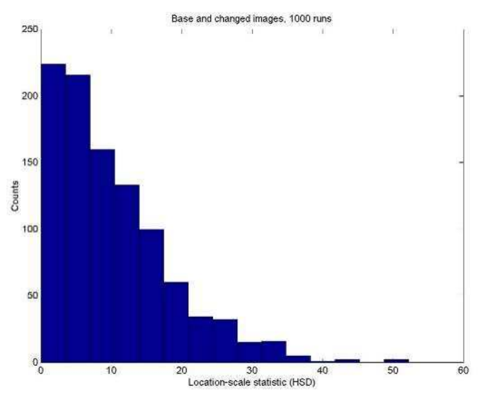
$$S_{CM} = \frac{2\pi^2}{N^2}\left[\sum_{i=1}^{N}\sin\left(\pi\frac{i}{N}\right)\sum_{j=1}^{i}c(R_i)\right]^2 + \frac{8\pi^2}{N^2}\left[\sum_{i=1}^{N}\sin\left(2\pi\frac{i}{N}\right)\sum_{j=1}^{i}c(R_i)\right]^2.$$

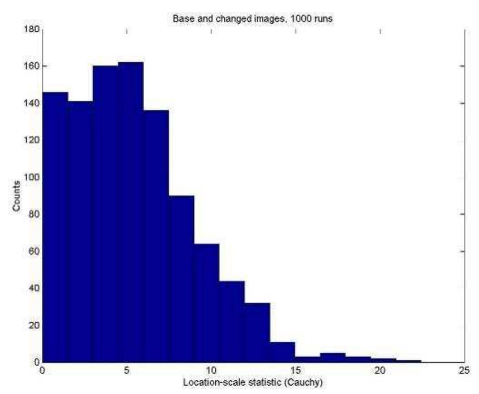
Numerical example

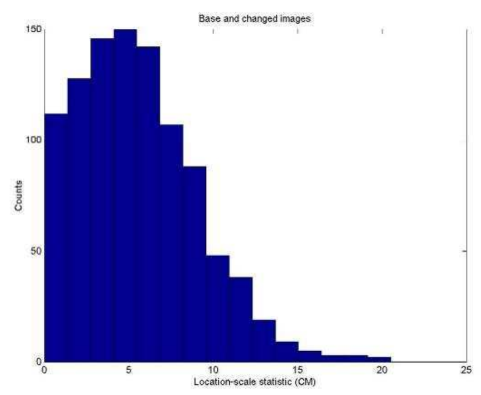




In this example, all images are 360×360 pixels in size and contain the same scene. The only difference in the images is the distribution of the intensity of the red channel. We want to compare the intensity of the red channel of these images by selecting small samples of pixels from the base (top left) and location-scale changed (bottom right) images. The exact distributions of the rank statistics for comparison of these images basing on 20 pixels from each of the images are shown below, for $S(t=-\pi/2)$, $S(t=-\pi)$ and S_{CM} correspondingly.







Although all three tests work well, S_{CM} is the best option. The test based on $S(t=-\pi/2, HSD)$ performs almost as well. The test based on $S(t=-\pi, Cauchy)$ loses its power due to the presence of the second mode.

Conclusions

The new location-scale test is fairly robust to distributional misspecification, as long as the order of tail is correct. However, the test is sensitive to the presence of outliers under the scale-only alternative. To overcome this lack of robustness in practice, we suggest applying the location-scale test built on the first two components of the Cramer-von Mises statistic, S_{CM}, to heavy-tailed distributions, and truncating the scale score function at certain points for normal-like and light-tailed distributions.

References

1. Vaughan D.C., The generalized secant hyperbolic distribution and its properties. *Comm. Statist. Theory Methods*, 31(2):219-238, 2002.
2. Kravchuk, O.Y., R-estimator of location of the GSHD. *Comm. Statist. Simulation Computation*, 35(1):1-18, 2006.
3. Kravchuk, O.Y. Two-sample scale rank procedures optimal for the GSHD, *Proceed. of ICSMRF*, Hawaii, January, 2006.