A Moment Matching Approach to the Valuation of a Volume Weighted Average Price Option

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In this paper we develop a method to find the price of an option whose payoff depends on a volume weighted average price (VWAP). It assumes that the stock follows a geometric Brownian motion and that the rate of trades evolves as a mean reverting process. The price is obtained by assuming that the VWAP at the final time has a lognormal distribution. The parameters of the approximating log normal distribution are obtained by matching the first two moments of the volume weighted average price with a log normal process. A price is then easily obtained when the market price of risk is a constant. The method is fast and can easily be extended to price exotics such as lookbacks, barriers, and digitals which have a VWAP component. We concentrate on the price for calls, prices for puts can be obtained in an analogous manner.

Keywords: Option Pricing; Moment Matching; Volume Weighted Average Price.

1. Introduction

A volume weighted average price (VWAP) is an average which gives more weight to periods of high trading than to periods of low trading in calculation of the average. Broker’s daily performance is frequently measured against a VWAP and it is become increasingly popular for institutional investors to place buy and sell orders at a VWAP. This paper addresses the problem of finding the price of an option where the payoff is

- The maximum of the difference between the final price of a stock and the VWAP and zero, i.e. \( \max(S(T) - VWAP(T), 0) \)
- The maximum of the difference between the VWAP and a fixed strike and zero, i.e. \( \max(VWAP(T) - K, 0) \).

This problem is certainly set in an incomplete market since there is no underlying with which to hedge the volume risk, and hence there is not a unique price. Any price which is obtained will include a market price of volume risk which must be determined from the market.

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The next section presents the definition of a VWAP together with several options of interest. Following this, the moment matching is done, then some numeric examples are given of the price, finally conclusions and further work is presented.

2. Definitions and Notation

2.1. The VWAP price

Denote the price of a stock at time $t$ as $S(t)$ and the number of trades of $S(t)$ per unit time as $U(t)$. Thus the turnover over the interval $t_1, t_2$ is $\int_{t_1}^{t_2} S(s)U(s)ds$ and the number of shares traded is $\int_{t_1}^{t_2} U(s)ds$. Assume also that $U(t) > 0$ for all $t$. The model used for of $S(t)$ and $U(t)$ will be specified later. The VWAP over the time interval $[0, t]$ is defined as

$$I(t) = \frac{Y(t)}{Z(t)} \quad \text{with} \quad I(0) = S(0)$$

(2.1)

where $Y(t) = \int_0^t S(s)U(s)ds$ and $Z(t) = \int_0^t U(s)ds$, for the continuous case. We assume that $\int_0^t U(s)ds \neq 0$, which is not unreasonable from a practical perspective. Note that this definition reduces to the average used for the Asian option when $U(t)$ is constant. This definition is easily extended to the more practical discrete case by discretized the time interval $[0, t]$ into the intervals $t_0 = 0; t_1, t_2, \ldots, t_N = t$ and forming the average as

$$I(t) = \frac{\sum_{i=0}^N S_i U_i}{\sum_{i=0}^N U_i}, \quad \text{with} \quad I(0) = S(0),$$

(2.2)

where $U_i = U(t_i)$ and $S_i = S(t_i)$. Likewise we assume $\sum_{i=0}^N U_i \neq 0$. For the remainder of this work we deal with the continuous case.

To give an idea of what the VWAP price looks like, Figure 1 shows a plot containing the stock price, the arithmetic average and the VWAP for several stocks. It is noticeable that there is little difference between an arithmetic average price and VWAP, so it seems reasonable to assume that the volume risk is not altering the distribution of the VWAP significantly from the arithmetic average in normal market conditions. We can also see that for small times, both the arithmetic average price and VWAP are quite sensitive to changes in the stock. This sensitivity to the stock price deteriorates as the averaging time increases.
2.2. Contracts of Interest

Here we are interested in European call options\(^a\) with a floating strike, which have a payoff at expiry \(T\) of

\[
V_{\text{floating}}(T) = (S_T - I(T))^+, \tag{2.3}
\]

and fixed strike with payoff at the final time of

\[
V_{\text{fixed}}(T) = (I(T) - K)^+. \tag{2.4}
\]

It is these contracts which we price.

3. The Models

We require models for both the stock and volume processes. We assume that the stock evolves as the familiar geometric Brownian motion

\[
dS(t) = \mu S(t)dt + \sigma S(t)dW_1(t), \tag{3.1}
\]

\(^a\)We use the notation \((\cdot)^+\) to denote \(\max(\cdot, 0)\)
with \( \mu \) being the drift of the stock, \( \sigma \) the variance and \( W_1 \) a Weiner process.

A model for volume is not quite as straight forward to decide on. We use a mean reverting process. We reason that the volume traded is driven by information. During ‘normal periods’ the flow of information results in volume being at a level around \( U_{\text{mean}} \). When new information is received the market reacts and the trading volume increases then some time later the volume settles back down to \( U_{\text{mean}} \). Having decided that a mean reverting process is to be used for the volume intensity, we choose a log normal mean reverting process. The analysis which follows is also applicable for both Ornstein Uhlenbeck (OU) and Cox Ingersoll Ross (CIR) type mean reverting processes, in fact any process which has enough moments available to calculate the expectation and variance of the VWAP. We denote the volume intensity by \( U(t) \) and write it as

\[
dU(t) = \alpha(U_{\text{mean}} - U(t))dt + \beta U(t)dW_2(t)
\]  

(3.2)

with \( \alpha \) being the speed of mean reversion, \( U_{\text{mean}} \) the long term average of the volume process, \( \beta \) the volatility of the volume process and \( W_2 \) a Weiner process. This is similar to [4] who use this model to model information flow which they proxy by number of trades. Assume that there is no correlation between the two Weiner processes \( W_1 \) and \( W_2 \). If this restriction is lifted then a modified system of differential equations which describe the expectation and variance of the VWAP is obtained. We also assume that \( \mu, \sigma, \alpha, U_{\text{mean}} \) and \( \beta \) are positive bounded constants. A later model might improve the model for \( U(t) \) by introducing jumps into the model.

4. The Approximation

A partial differential equation approach to the valuation of (2.3) and (2.4), requires four state variables as well as time which leads to a numerically intractable solution, and further the boundary conditions of the partial differential equation are very difficult to formulate. Monte Carlo can also be used to price (2.3) and (2.4) but has the draw back that it is slow.

To obtain a numerically tractable problem, we approximate the distribution of the VWAP by a log normal process \( \tilde{S} \). This is a common approximation used in the valuation of a Asian option. This has the advantage that it leads to a analytically tractable problem and at the very least gives a good approximation to the price of (2.3) and (2.4). In making this approximation we have simplified the problem of modeling the VWAP from \( I = I(t, S, U, Y, Z) \) to \( I = I(t, \tilde{S}) \). The process \( \tilde{S}(t) \) is given by

\[
d\tilde{S}(t) = \tilde{\mu}\tilde{S}(t)dt + \tilde{\sigma}\tilde{S}(t)d\tilde{W}(t),
\]  

(4.1)

with \( \tilde{\mu} \) being the drift of the process, \( \tilde{\sigma} \) the variance and \( \tilde{W}(t) \) a Weiner process. It is the parameters \( \tilde{\mu} \) and \( \tilde{\sigma} \) which we must find. Once the parameters \( \tilde{\mu} \) and \( \tilde{\sigma} \) have been obtained, we can derive standard partial differential equations to solve for the option
price since the options (2.3) and (2.4) are now reduced to $V_{\text{fixed}} = V_{\text{fixed}}(t, \tilde{S})$ and $V_{\text{floating}} = V_{\text{floating}}(t, S, \tilde{S})$.

5. Matching

The concept is simple, we want to match the first two moments of the VWAP given by (2.1) to the log normal process (4.1). This means that both the expectation and variance of the VWAP are required. To find these we use the following approximations

\[ \mathbb{E}\left( \frac{Y}{Z} \right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y, Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} \text{Var}(Z) \]  
\[ \text{Var}\left( \frac{Y}{Z} \right) \approx \left( \frac{E(Y)}{E(Z)} \right)^2 \left( \frac{\text{Var}(Y)}{(E(Y))^2} + \frac{\text{Var}(Z)}{(E(Z))^2} - 2 \frac{\text{Cov}(Y, Z)}{E(Y)E(Z)} \right), \]

see [5][p181]. The proof is easy and is based on a truncated Taylor’s series expansion around $(Y_{\text{mean}}, Z_{\text{mean}})$. The expectation and variance of (4.1) are well known and are

\[ \mathbb{E}(\tilde{S}(t)) = S(t)e^{\tilde{\mu}t} \quad \text{and} \quad \text{Var}(\tilde{S}(t)) = S(t)^2 e^{2\tilde{\mu}t}(e^{2\tilde{\sigma}^2t} - 1), \]

see for example [3][p95].

The idea of the method is to obtain the variance and expectation of $\frac{Y}{Z}$ by the approximations in (5.1) and (5.2) and substitute these into (5.3) and (5.4) to find $\tilde{\mu}$ and $\tilde{\sigma}$ at any time.

In order to evaluated (5.1) and (5.2) a large number of expectations are required. The method for finding the expectations is long and tedious but relatively straightforward, so only several of the expectations will be found to demonstrate the technique. To start with, we find $\mathbb{E}(S(t))$ which we already know. To find this we write down the stochastic differential equation for $S(t)$

\[ dS(t) = \mu S(t)dt + \sigma S(t)dW_1(t) \]

which is really short hand for

\[ S(t) - S(0) = \int_0^t \mu S(s)ds + \int_0^t \sigma S(s)dW_1(s). \]

Taking expectation of (5.6) and using the property that the expectation of an Ito integral is 0 we obtain

\[ \mathbb{E}(S(t) - S(0)) = \mathbb{E}(\int_0^t \mu S(s)ds + \int_0^t \sigma S(s)dW_1(s)) \]
\[ = \mathbb{E}(\int_0^t \mu S(s)ds). \]
Then moving the expectation inside the integral we have

$$
E(S(t) - S(0)) = \int_0^t \mu E(S(s)) ds,
$$

(5.9)

and finally differentiating with respect to time we obtain

$$
\frac{dE(S(t))}{dt} = \mu E(S(t)).
$$

(5.10)

This is a simple differential equation, which can be solved together with the initial condition that $E(S(0)) = S(0)$. Solving this gives us the expected result that $E(S(t)) = S(0)e^{\mu t}$ matching (5.3).

To further illustrate the method we do one more as an example, and then simply list the system of 19 differential equations in the Appendix. This system of equations can either be solved numerically, or a symbolic computer package such as Maple can be used. We need the expectation of $Y^2(t)$, i.e. $E(Y^2(t))$. To obtain this we appeal to Ito’s Lemma and form

$$
d(Y^2(t)) = 2Y(t)dY(t)
$$

(5.12)

and

$$
d(Y^2(t)) = 2Y(t)U(t)S(t)dt.
$$

(5.13)

Repeating the arguments above, we need to solve the equation

$$
\frac{dE(Y^2(t))}{dt} = 2E(Y(t)U(t)S(t))
$$

(5.14)

with the initial condition $E(Y^2(0)) = Y^2(0)$. To solve (5.14), $E(Y(t)U(t)S(t))$ is required, so we repeat the above procedure for $E(Y(t)U(t)S(t))$ and the remaining expectations required. The full system of differential equations is given in the Appendix.

Solving this system of differential equations means we can find the expectation and variance of the VWAP at any time by substituting the necessary expectations found from the system of ordinary differential equations into (5.1) and (5.2). The solution of this type of ordinary differential equation system is well known and is dominated by the positive eigenvalues. Fortunately we can find the eigenvalues (with the help of Maple) of the ordinary differential equation system, they are listed in Table 1.

It is interesting to note that $U_{mean}$ does not appear in the eigenvalues. From the eigenvalues in Table 1 we can see there are many positive eigenvalues and that they are all real values. The fact that there are many positive values is to be expected since many of the expectations which the ordinary differential equations represent are increasing functions with time, i.e. the expectation of the stock is a monotonically increasing function in time. To obtain a numeric solution the values of these eigenvalues should not be too big. $\mu$ and $\sigma$ are small values, both typically between...
0 and 0.5 so these parameters do not pose a problem. The quantity $-2\alpha + \beta^2$ occurs throughout the eigenvalues a number of times, this quantity should be small, or negative, so that numeric solutions can be obtained, i.e. we ideally like to have $-2\alpha + \beta^2 \leq 0$ otherwise the solution will blow up too fast. If this occurs, then the parameters should be rescaled to enforce this condition.

Now for any given time we can easily match the expectation and variance obtained from the differential equations and approximations given by (5.1) and (5.2) to the log normal distribution of $\tilde{S}(t)$ and obtain $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ since rewriting (5.3) and (5.4) we have

\[
\tilde{\mu}(t) = \frac{1}{t} \log \frac{E(\tilde{S}(t))}{\tilde{S}(0)} \quad (5.15)
\]

\[
\tilde{\sigma}(t) = \sqrt{\frac{1}{t} \log \frac{Var(\tilde{S}(t)) + (E(\tilde{S}(t)))^2}{(E(\tilde{S}(t)))^2}} \quad (5.16)
\]

So now for a given time $T$ we can now find $\tilde{\mu}$ and $\tilde{\sigma}$ which matches the final distribution of the VWAP to a lognormal distribution, these are given by $\tilde{\mu}(T)$ and $\tilde{\sigma}(T)$.

Which means we now have the parameters $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ for the process $\tilde{S}(t)$ for any time.

### 6. Pricing

In order to find a price for the fixed strike, we construct a portfolio consisting of two options $V(t, \tilde{S})$ and $V_1(t, \tilde{S})$, and then from standard no arbitrage arguments we obtain

\[
\frac{\partial V}{\partial t} + \frac{1}{2} (\tilde{\sigma}(T) \tilde{S})^2 \frac{\partial^2 V}{\partial S^2} + (\tilde{\mu}(T) \tilde{S} - \lambda(t, \tilde{S}) \tilde{\sigma}(T) \tilde{S}) \frac{\partial V}{\partial S} - r\tilde{V} = 0, \quad (6.1)
\]

where $\lambda(t, \tilde{S})$ is the market price of $\tilde{S}$ risk, and final condition $V(T, \tilde{S}) = (\tilde{S}(T) - K)^+$. An analytic solution to (6.1) exists when the market price of risk is constant,
this solution is almost the same as the Black Scholes equation.

In the floating strike case we again form a portfolio of two options \( V = V(t, S, \tilde{S}) \) and \( V_1(t, S, \tilde{S}) \), and by no arbitrage arguments again we obtain

\[
\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma(t)S)^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma(t)S \tilde{S}(T) \frac{\partial^2 V}{\partial S \partial \tilde{S}} + \frac{1}{2}(\tilde{\sigma}(t)\tilde{S})^2 \frac{\partial^2 V}{\partial \tilde{S}^2} + rS \frac{\partial V}{\partial S} + (\tilde{\mu}(T)\tilde{S} - \lambda(t, \tilde{S})\tilde{S}(T)) \frac{\partial V}{\partial \tilde{S}} - rV = 0
\]

(6.2)

with final condition \( V(T, \tilde{S}) = (S(T) - \tilde{S}(T))^+ \). See [2] or [7] for details on constructing these partial differential equations.

We can see that the drift of \( \tilde{S}(t) \) is present in both of these equations unlike the Black Scholes equation which does not contain the drift of the stock. This is a consequence of the problem being set in an incomplete market with no underlying in \( \tilde{S}(t) \) with which to hedge with, this also leads to the market price of risk \( \lambda(t, \tilde{S}) \) being present in these equations.

We know all the values in (6.1) and (6.2) except for the market price of risk \( \lambda(t, \tilde{S}) \). In reality the trader must look to the market to obtain \( \lambda(t, \tilde{S}) \). We can however place some constraints on the market price of risk. To begin, we can see that the VWAP price over the interval \([0, T]\) is bounded above by the maximum of the stock price over the interval, \( S_{\text{max}} = \max_{0 \leq t \leq T} S_t \), and below by the minimum, \( S_{\text{min}} = \min_{0 \leq t \leq T} S_t \), over the time interval \([0, T]\), i.e.

\[
S_{\text{min}} \leq \frac{Y(T)}{Z(T)} \leq S_{\text{max}}.
\]

(6.3)

Using the bounds given by (6.3) upper and lower bounds for the floating strike contract at the final time are

\[
(S(T) - S_{\text{max}})^+ \leq (S(T) - I(T))^+ \leq (S(T) - S_{\text{min}})^+
\]

(6.4)

and

\[
(S_{\text{min}} - K)^+ \leq (I(T) - K)^+ \leq (S_{\text{max}} - K)^+,
\]

(6.5)

see [6]. All of these bounds have analytic expressions. This means that for a given market price of risk we can solve (6.2) and (6.1), if the solution obtained is not between those given in (6.4) and (6.5) then the market price of risk used is not valid. Depending on the form of the market price of risk there may be no analytic solution. In such cases we will have to evaluate (6.1) and (6.2) via a numerical scheme such as finite differences or use Monte Carlo methods.

7. Numeric Example

As an example, we demonstrate the method on the system

\[
dS(t) = 0.1S(t)dt + 0.15S(t)dW_1(t)
\]

\[
dU(t) = 100(110 - U(t))dt + 2U(t)dW_2(t)
\]
over the time interval \([0, 0.5]\) with \(S_0 = 110\) and \(U_0 = 80\).

### 7.1. Finding \(\tilde{\mu}(t)\) and \(\tilde{\sigma}(t)\)

To price the option using this method, we only need the values of \(\tilde{\mu}(T)\) and \(\tilde{\sigma}(T)\) at the final time, i.e. \(\tilde{\mu}(T)\) and \(\tilde{\sigma}(T)\). But it is interesting to see how \(\tilde{\mu}(t)\) and \(\tilde{\sigma}(t)\) evolve over time.

First the ordinary differential equations are solved for each time over the interval \([0, 0.5]\), and at each time we use the approximation to the mean and variance from (5.1) and (5.2) to obtain the values of \(\tilde{\mu}(t)\) and \(\tilde{\sigma}(t)\). To benchmark our results to, we simulate the VWAP. To do this we discretise the time from the start to the end of the contract into \(N\) partitions, i.e. \(t_0, t_1, \ldots, t_N\) and compute the value of \(S(t)\) and \(U(t)\) at each time \(t_i\). The exact solution of \(S(t)\) is used to evolve \(S(t)\) and Milstein’s Scheme is used to evolve \(U(t)\),

\[
S_{i+1} = S_i \exp((\mu - \frac{1}{2}\sigma^2)dt + \sigma \sqrt{dt} Z_{i+1}^S)
\]

and

\[
U_{i+1} = U_i + \alpha(U_{\text{mean}} - U_i)dt + \beta U_i \sqrt{dt} Z_{i+1}^U + \frac{1}{2} \gamma^2 U_i dt ((Z_{i+1}^U)^2 - 1),
\]

where \(Z_{i+1}^S\) and \(Z_{i+1}^U\) are independent normal random numbers with mean 0 and variance 1. Then the sums

\[
Y_j = \sum_{i=0}^{N} S_i U_i
\]

and

\[
Z_j = \sum_{i=0}^{N} U_i
\]

are computed, so for each simulation \(j\) we find the VWAP

\[
I_j = \frac{Y_j}{Z_j}
\]

We repeat this \(M\) times. Finally we approximate the expectation of \(\frac{Y}{Z}\) by

\[
\mathbb{E}\left(\frac{Y}{Z}\right) \approx \frac{1}{M} \sum_{j=1}^{M} I_j \quad (7.1)
\]

and variance by

\[
\text{Var}\left(\frac{Y}{Z}\right) \approx \frac{1}{M} \sum_{j=1}^{M} I_j^2 - \left(\frac{1}{M} \sum_{j=1}^{M} I_j\right)^2 \quad (7.2)
\]

We then use these values of the expectation and variance to find \(\tilde{\mu}(t)\) and \(\tilde{\sigma}(t)\) for the simulation which matches the log normal distribution using (5.15) and (5.16).
The results obtained for $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ from both the simulation and the ordinary differential equations are shown in Figure 2. We can see in Figure 2 that $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ obtained from the moment matching approach qualitatively matches that obtained from simulation and is especially good for times longer than 0.2. The approximation to $\hat{\mu}(t)$ and $\hat{\sigma}(t)$ for small time is not as good as larger time.

Finally we plot the probability density function of the VWAP distribution at the final time obtained from simulation in Figure 3. We also plot the corresponding lognormal probability density function obtained by solving the ordinary differential equations and also from matching the mean and variance of the simulation to the log normal distribution. We can see that the log normal approximation to the VWAP matches the empirical probability density function well and that there is little difference between the lognormal fit obtained from the ordinary differential equations and simulation. It was noticed that as $\sigma$ increased, the approximating probability density function deteriorated which is expected since the log normal
approximation to the arithmetic average used in the pricing of the Asian option works well only for small $\sigma$.

![Empirical verse fitted probability density function at the final time for the VWAP. Note how well the value obtained for $\bar{\mu}(t)$ and $\bar{\sigma}(t)$ by simulation matches those obtained from solving the system of ordinary differential equations. $2 \times 10^6$ simulations and time split into $10^4$ intervals. ($dS = 0.15 dt + 0.15 S dW_1, dU = 110(100 - U) dt + 2U dV_2, S(0) = 110, U(0) = 80$).]

**Fig. 3.**

7.2. **Solving the Partial Differential Equation**

Now that $\bar{\mu}$ and $\bar{\sigma}$ have been obtained, we move onto solving the partial differential equations. If we assume that $\lambda$ is a constant then there is a closed form solution to the fixed strike option since the partial differential equation essentially becomes the Black Scholes equation. This solution is given by

$$V_{fixed}(0) = e^{-(r-\bar{\mu}(T)+\bar{\sigma}(T))^T(\lambda T)}S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where

$$d_1 = d_2 + \bar{\sigma}(T)\sqrt{T}$$

and

$$d_2 = \frac{\log(S(0)/K) + (\bar{\mu}(T) - \bar{\sigma}(T)\lambda - \frac{1}{2}\bar{\sigma}^2(T))T}{\bar{\sigma}(T)\sqrt{T}}$$

(7.3)
where \( \Phi(\cdot) \) is the cumulative normal distribution function, see [1]. Solutions to the floating strike option, and when the fixed strike has a constant market price of risk, must be solved numerically using Monte Carlo or the Finite Difference methods.

7.2.1. Fixed Strike

Using (7.3) a typical solution to the price is found in Figure 4. Note that the bounds of the VWAP option are those given by solving (6.5). We can see how the value of \( \lambda \) shifts the solution up and down relative to the Black Scholes call and the European fixed strike Asian call price.

8. Extensions

This technique can be easily incorporated into a variance reduction scheme as a control variate when Monte Carlo is used to value options which are functions of a VWAP.

The pricing of exotics such as lookbacks, barriers and digitals which are based on a VWAP can also be easily priced. The hedging however would possibly be difficult.
9. Conclusions and Further Work

We have demonstrated a fast method to approximate the price of an option which is a function of the VWAP for both fixed and floating strikes. The method can easily be extended to price exotics. The approximation works well for values of $\sigma$ up to about 0.2. Even when $\sigma$ is not in this range the method can be used as a control variate in a Monte Carlo scheme. A natural extension to this work is to match higher moments of the VWAP distribution to another process, perhaps to a shifted lognormal distribution. As well as this a better approximation to the variance and mean of the VWAP at the final time can be used.

This work is part of a larger work dealing with the pricing and hedging of options based on a VWAP. Further work includes a practical hedging strategy and more accurate pricing methods for these options.

10. Appendix

This appendix lists in full all the differential equations which need to be solved to get the expectations. Throughout it was assumed that the correlation between $W_1$ and $W_2$ is 0.

\[
\begin{align*}
\frac{d}{dt} E(S) &= \mu E(S), E(S_0) = S_0 \\
\frac{d}{dt} E(U) &= \alpha(U_{\text{mean}} - E(U)), E(U_0) = U_0 \\
\frac{d}{dt} E(Z) &= E(U), E(Z_0) = 0 \\
\frac{d}{dt} E(SU) &= (\mu - \alpha)E(SU) + \alpha U_{\text{mean}} E(S), E(S_0 U_0) = S_0 U_0 \\
\frac{d}{dt} E(Y) &= E(SU), E(Y_0) = 0 \\
\frac{d}{dt} E(Y^2) &= 2E(SU), E(Y^2_0) = 0 \\
\frac{d}{dt} E(Y SU) &= E(S^2 U^2) + \alpha U_{\text{mean}} E(Y SU) + (\mu - \alpha)E(Y SU), E(Y_0 S_0 U_0) = 0 \\
\frac{d}{dt} E(S^2 U^2) &= (2\mu - 2\alpha + \sigma^2 + \beta^2)E(S^2 U^2) + 2\alpha U_{\text{mean}} E(US^2), E(S^2_0 U^2_0) = S^2_0 U^2_0 \\
\frac{d}{dt} E(S^2 U) &= (2\mu - \alpha + \sigma^2)E(S^2 U) + \alpha U_{\text{mean}} E(S^2), E(S^2_0 U_0) = S^2_0 U_0 \\
\frac{d}{dt} E(S^2) &= (2\mu + \sigma^2)E(S^2), E(S^2_0) = S^2_0 \\
\frac{d}{dt} E(Y S) &= \mu E(S) + E(S^2 U), E(Y_0 S_0) = 0 \\
\frac{d}{dt} E(Y U) &= E(U^2 S) - \alpha E(Y U) + \alpha U_{\text{mean}} E(Y), E(Y_0 U_0) = 0 \\
\frac{d}{dt} E(Y Z) &= E(Y U) + E(Z SU), E(Y_0 Z_0) = 0 \\
\frac{d}{dt} E(U^2 S) &= 2\alpha U_{\text{mean}} E(US) + (\mu - 2\alpha + \beta^2)E(U^2 S), E(U^2_0 S_0) = U^2_0 S_0 \\
\frac{d}{dt} E(Z SU) &= \alpha E(Z S) + (\mu - \alpha)E(Z SU) + E(SU^2), E(Z_0 S_0 U_0) = 0 \\
\frac{d}{dt} E(Z S) &= E(SU) + E(Z SU), E(Z_0 S_0) = 0 \\
\frac{d}{dt} E(Z U) &= \alpha U_{\text{mean}} E(Z) - \alpha E(Z U) + E(U^2), E(Z_0 U_0) = 0 \\
\frac{d}{dt} E(Z^2) &= 2E(ZU), E(Z^2_0) = 0 \\
\frac{d}{dt} E(U^2) &= 2\alpha U_{\text{mean}} E(U) + (\beta^2 - 2\alpha)E(U^2), E(U^2_0) = U^2_0
\end{align*}
\]
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References