On the costs and decisions of controlling a population

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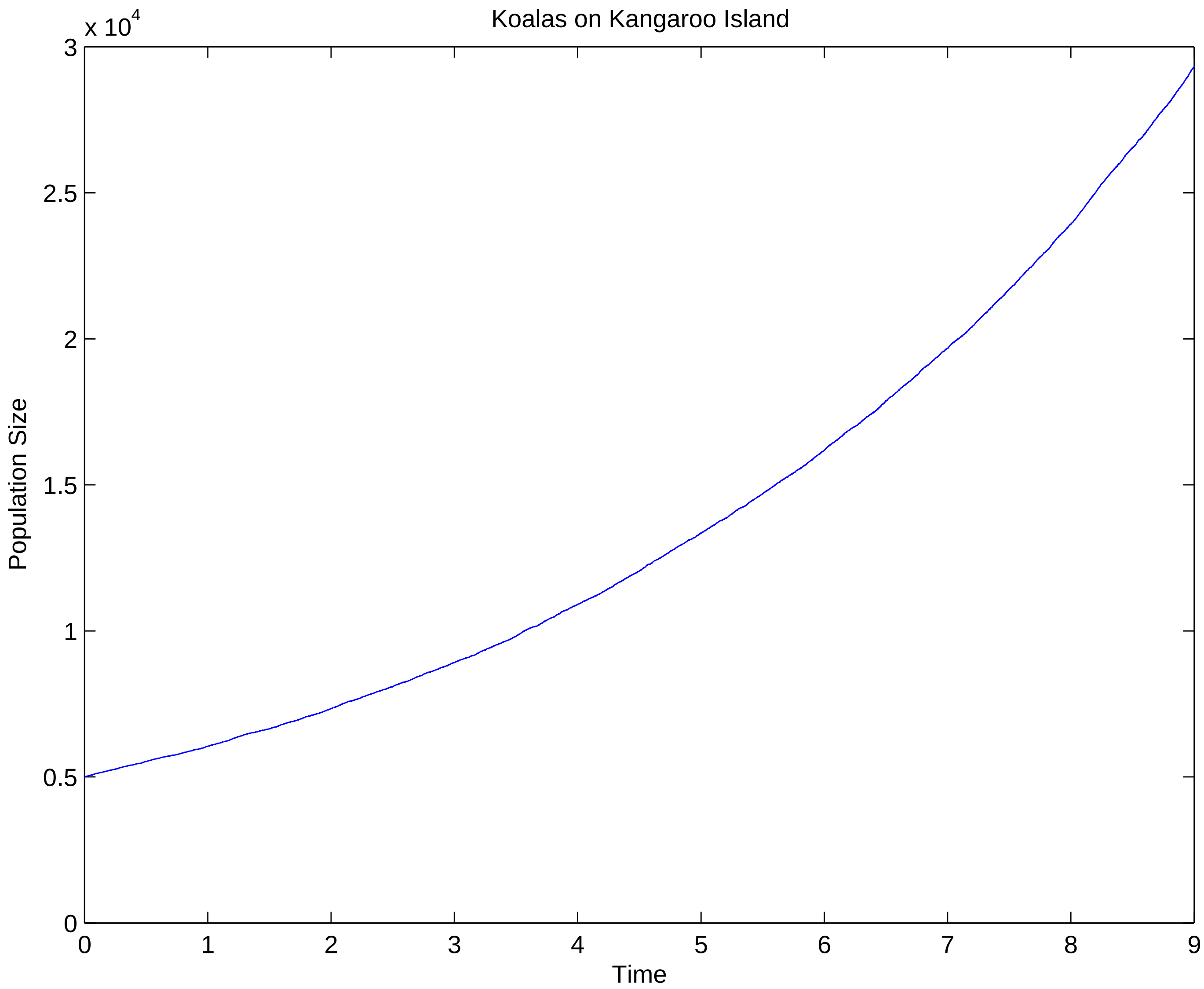
Discipline of Mathematics and MASCOS

University of Queensland

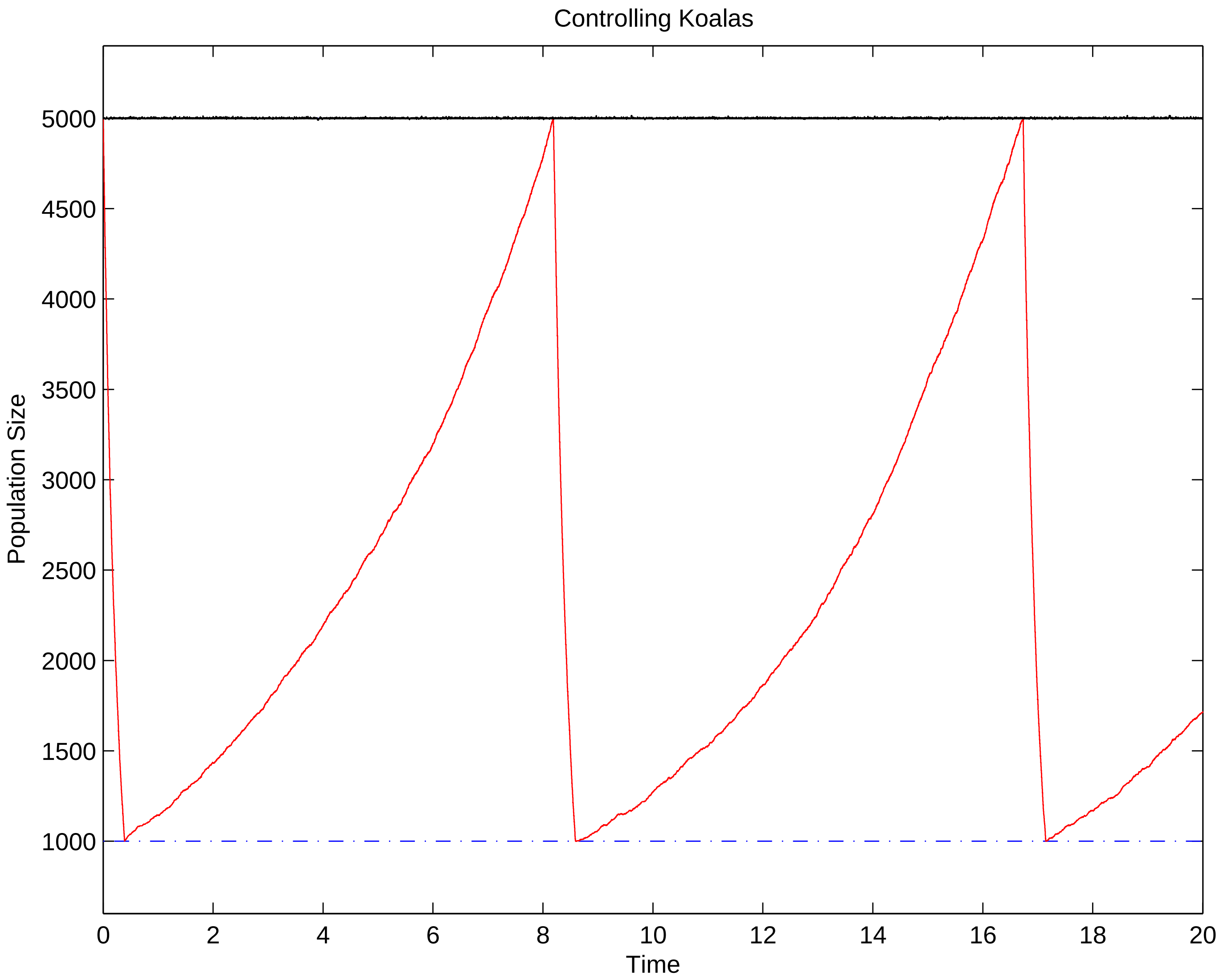
Australian Research Council
Centre of Excellence for Mathematics
and Statistics of Complex Systems

AMSI and ICE-EM Winter School in Mathematics and Computational Biology - 2004

Koalas (*Phascolarctos cinereus*)



Controlling Koalas



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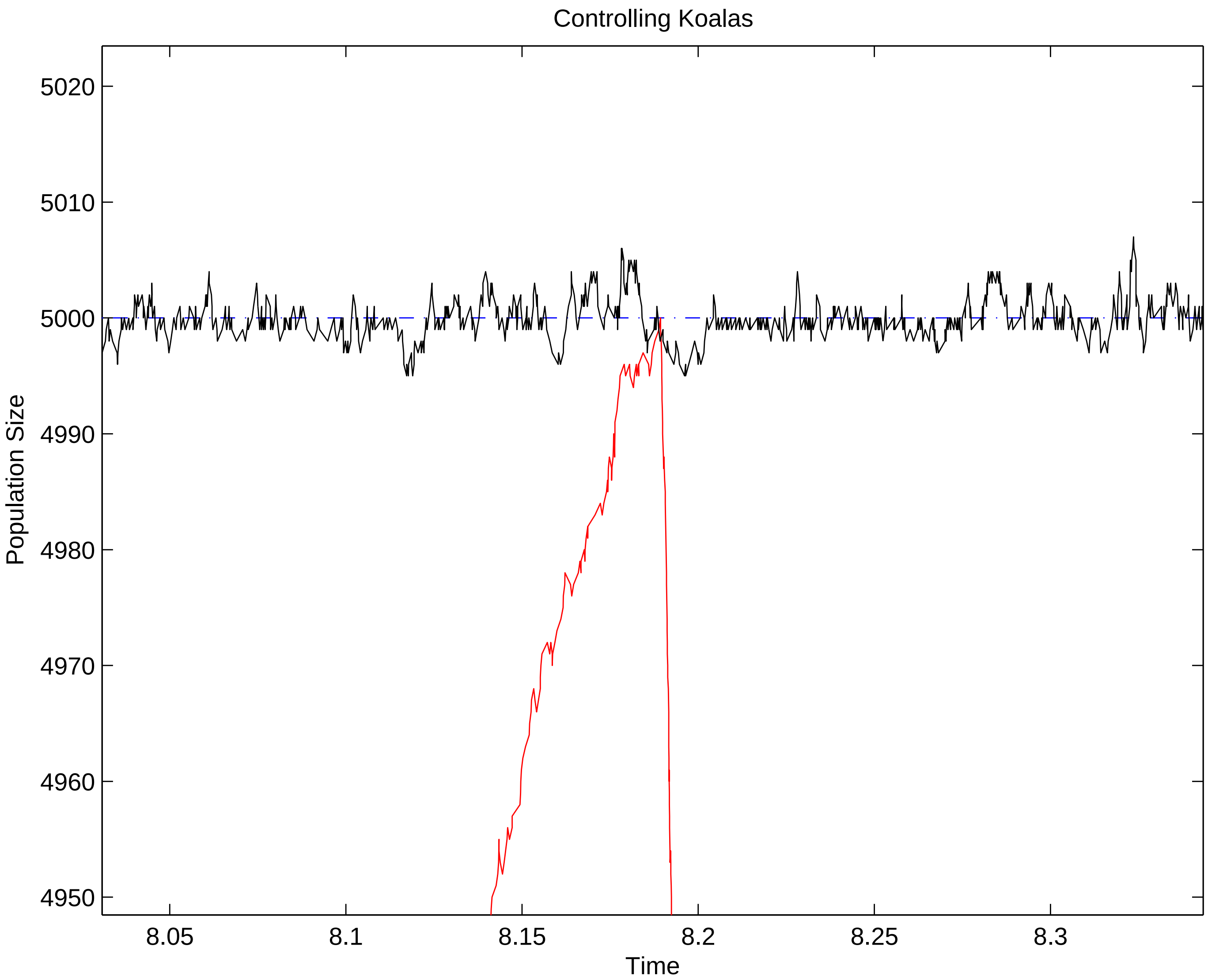

Outline

- Stochastic Models

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- Selection of Rates and Reduction Level

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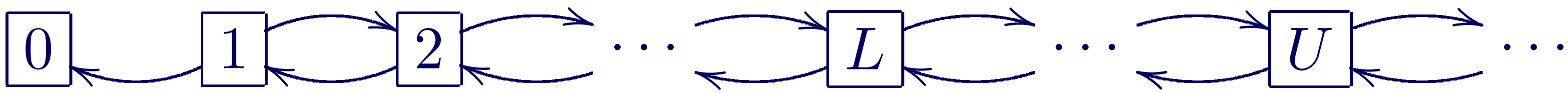
- Selection of Rates and Reduction Level
- Selection of Control Policy
- - Extinction times and total costs

Models - Controlled Populations

The birth-and-death process - Transition Diagram



Models - Controlled Populations

The birth-and-death process is a continuous-time Markov chain taking values in $S = \{0, 1, \dots\}$ with non-zero transition rates

$$q(x, x+1) = \lambda_x$$

and

$$q(x, x-1) = \mu_x$$

where λ_x and μ_x are the population birth and death rates respectively.

Models - Controlled Populations

The **linear** birth-and-death process is a continuous-time Markov chain taking values in $S = \{0, 1, \dots\}$ with non-zero transition rates

$$q(x, x+1) = \lambda x$$

and

$$q(x, x-1) = \mu x$$

where λ and μ are the per individual birth and death rates respectively.

Models - Controlled Populations

Linear birth-and-death process with **suppression** and **constant culling**

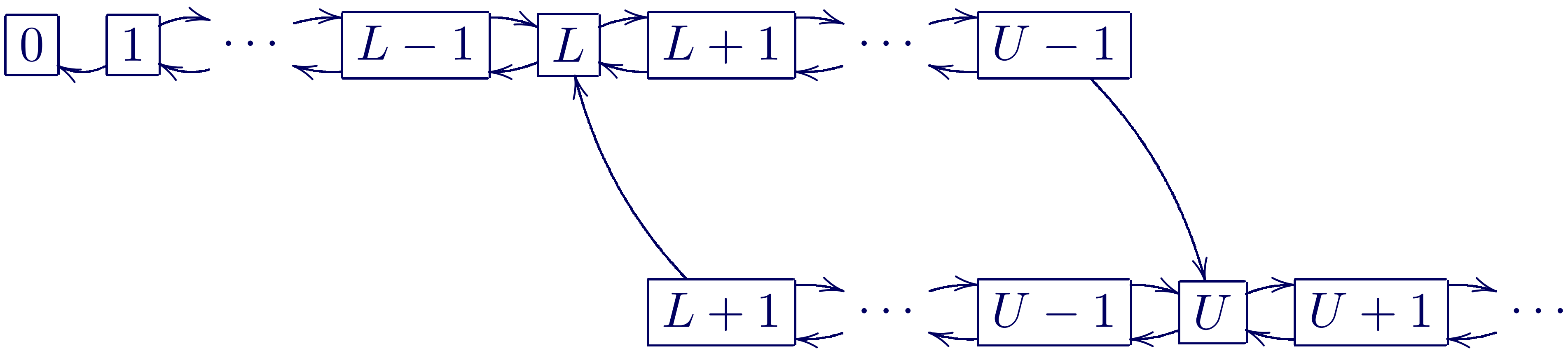
$$q(x, x+1) = \lambda x \qquad \text{for all } x$$

$$q(x, x-1) = \begin{cases} \mu x & x \leq U \\ \mu x + \kappa & x > U \end{cases}$$

where κ is the rate of culling (control).

Models - Controlled Populations

Transition Diagram for **Reduction** Regime Models



Models - Controlled Populations

Linear birth-and-death process with **reduction** and **per-capita culling**

$$q((x,0),(x+1,0)) = \lambda x \qquad x < U-1$$

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$$q((x,1),(x+1,1)) = \lambda x \qquad x \in \{L+1, L+2, \ldots\}$$

$$q((x,1),(x-1,1)) = (\mu+\psi)x \qquad x \in \{L+2, L+3, \ldots\}$$

$$q((L+1,1),(L,0)) = (\mu+\psi)(L+1)$$

where ψ is the rate of culling (control).

Some Decisions of Controlling

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- What rate of culling to use?

  i.e. How large should κ and/or ψ be?

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- Reduction per-koala culling rate: $\psi =$?
- Suppression constant culling rate: $\kappa =$?

Choice of Reduction Level L

- Probability of the population “persisting”.

Choice of Reduction Level L

- Probability of the population reaching the culling level U before 0 starting from reduction level L

$$\alpha_L = \Pr(\text{hit } U \text{ before } 0 \mid X(0) = L) \geq \rho.$$

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- Expected time between culling phases.

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$$\alpha_L = \Pr(\text{hit } U \text{ before } 0 \mid X(0) = L) \geq \rho.$$

- Expected time to hit U starting from L conditional on hitting U before 0

$$\mathrm{E}(T_U \mid X(0) = L, \text{ hit } U \text{ before } 0).$$

Choice of Reduction Level L

For a birth-and-death process

$$\alpha_i = \Pr(\text{hit } U \text{ before } 0 | X(0) = i) = \frac{s_i}{s_U}$$

where $s_0 = 0$, $s_1 = 1$ and for $2 \le i \le U$

$$s_i = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{\mu_k}{\lambda_k}.$$

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$$s_i = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{\mu_k}{\lambda_k}.$$

Therefore we have

$$\alpha_i = \frac{1 - \left(\frac{\mu}{\lambda}\right)^i}{1 - \left(\frac{\mu}{\lambda}\right)^U}.$$

Choice of Reduction Level L

After choosing a suitable minimum probability ρ, the minimum reduction level L is given by

$$L := \left\lceil \frac{\ln\{1 - \rho[1 - (\mu/\lambda)^U]\}}{\ln(\mu/\lambda)} \right\rceil.$$

Koalas - Minimum L

L	ρ
4	0.9876543209877
6	0.9986282578875
8	0.9998475842097
10	0.9999830649122
12	0.9999981183236
14	0.9999997909248
16	0.9999999767694
18	0.9999999974188
20	0.9999999997132

Expected Phase Times

- Phase 1 - Time between culling phases.

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$$\tau_L = \mathrm{E}(T_U \,|\, \text{hit } U \text{ before } 0, X(0) = L) = \sum_{i=L}^{U-1} \frac{1}{\lambda_i s_i s_{i+1} \pi_i} \sum_{j=1}^{i} s_j^2 \pi_j$$

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where $s_0 = 0$, $s_1 = 1$ and for $2 \leq i \leq U$,

$$s_i = 1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{\mu_k}{\lambda_k}$$

and $\pi_1 = 1$, $\pi_j = \prod_{i=2}^{j} \frac{\lambda_{i-1}}{\mu_i}$ for $j \geq 2$.

Koalas - Expected Phase 1 Time

L	Expected Time (yrs)
20	27.868
500	11.522
1000	8.051
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$$L = 1000.$$

Expected Phase Times

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For our model

$$\tau_U^L = \frac{1}{\mu + \psi} \sum_{k=L+1}^{U} \sum_{j=0}^{\infty} \frac{1}{j+k} \left(\frac{\lambda}{\mu + \psi} \right)^j .$$

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For a birth-death process

$$c_U = \sum_{k=L+1}^{U} \frac{1}{\mu_k \pi_k} \sum_{j=k}^{\infty} f_j \pi_j$$

where $\pi_j = \prod_{i=L+1}^{j} \frac{\lambda_{i-1}}{\mu_i}$ and f_j is the cost per unit time of culling a population of size j.

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Minimising with respect to ψ

$$\psi = \left(\frac{1+\delta}{\delta}\right)(\lambda-\mu).$$

$\delta = 0.05 \implies \psi = 4.2$ and $\tau_U^L = 246 \text{ hrs} \implies 41 \text{ days}.$

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- - Probability of remaining in “control".

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$$\alpha_{U+1} = \Pr\left(\text{hit } \left\lceil \frac{\kappa}{\lambda - \mu} \right\rceil \text{ before } U | X(0) = U + 1\right) \leq \rho.$$

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For a birth-death process

$$\alpha_{U+1} = \frac{s_{U+1}}{s_{\left\lceil \frac{\kappa}{\lambda-\mu} \right\rceil}}$$

and $s_{U+1} = 1$, $s_i = 1 + \sum_{j=U+1}^{i-1} \prod_{k=U+1}^{j} \frac{\mu_k}{\lambda_k}$ for $i > U + 1$.

Koalas - Choice of Culling Rates

κ	α_{U+1}
1010	1.7825×10^{-2}
1020	6.280×10^{-3}
1050	1.5501×10^{-4}
1070	3.3711×10^{-6}
1100	1.1707×10^{-9}
1120	1.3957×10^{-12}
1200	6.3347×10^{-29}

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$$\kappa = 1120.$$

Summary of Koala Models

General

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- $\mu = 0.1$
- $U = 5,000$

Suppression model with constant culling

- $\kappa = 1,120$

Reduction model with per-capita culling

- $L = 1,000$
- $\psi = 4.2$

Choice of Control Policy

How do we choose the “best" control regime?

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- Extinction Probabilities

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How do we choose the “best" control regime?

- Extinction Probabilities
- Extinction Times
- Total Costs

Koalas - Extinction Probabilities

Linear Birth-death Suppression Model with Constant Culling

$$\rho_L \approx 1.$$

Linear Birth-death Reduction Model with Per-capita Culling

$$\rho_L = 1.$$

Koalas - Extinction Times

Linear Birth-death Suppression Model with Constant Culling

$$\gamma_L \approx 2.52 \times 10^{2798} \text{ years.}$$

Linear Birth-death Reduction Model with Per-capita Culling

$$\gamma_L \approx 1.07 \times 10^{478} \text{ years.}$$

Total Costs

Cost Functions

Linear Birth-death Suppression Model with Constant Culling

$$f_j = K1_{\{j>U\}} + M.$$

Linear Birth-death Reduction Model with Per-capita Culling

$$f_{(j,0)} = N \text{ and } f_{(j,1)} = Cj + N.$$

Total Costs

Cost Functions

Linear Birth-death Suppression Model with Constant Culling

$$f_j = K1_{\{j>U\}} + M.$$

$$K = \$50,000 \text{ and } M = \$10,000.$$

Linear Birth-death Reduction Model with Per-capita Culling

$$f_{(j,0)} = N \text{ and } f_{(j,1)} = Cj + N.$$

$$C = \$100 \text{ and } N = \$7,000.$$

Total Costs

Linear Birth-death Suppression Model with Constant Culling

$$c_L \approx \$9.18 \times 10^{2802}.$$

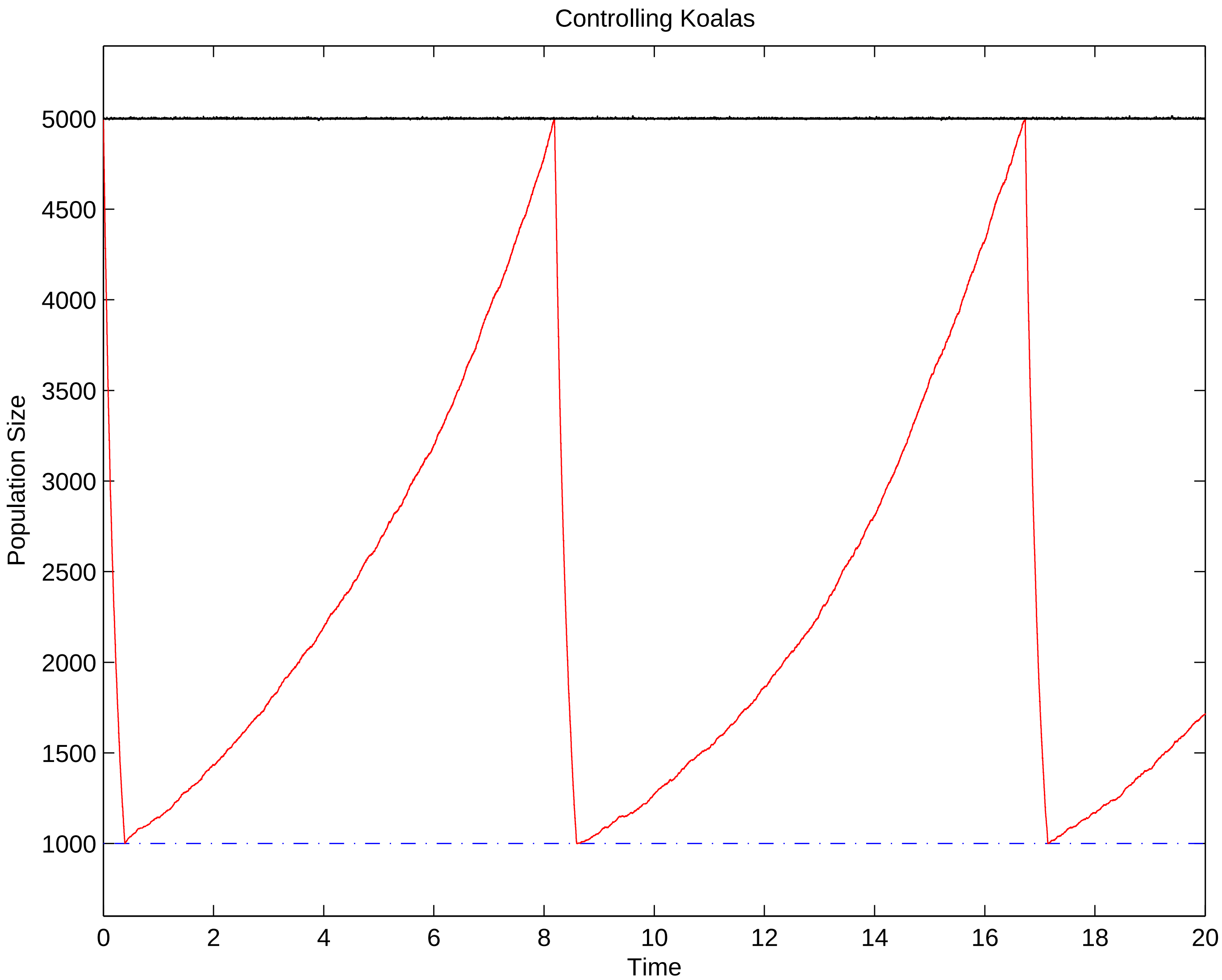
Linear Birth-death Reduction Model with Per-capita Culling

$$c_L \approx \$8.39 \times 10^{481}.$$

Choice of Control Policy

Decision Tool	*Supp. & Const.*	*Red. & Per-capita*
Cost/Time	$\$36,470$ per year	$\$7,841$ per year

Conclusion



Controlling Koalas
5000
4500
4000
3500
3000
2500
2000
1500
1000
Population Size
0
2
4
6
8
10
12
14
16
18
20
Time

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