A Moment Matching Approach To The Valuation Of A Volume Weighted Average Price Option

Antony Stace

Department of Mathematics and MASCOS

University of Queensland

15th October 2004

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

Plan of talk

- What an option is

Plan of talk

- What an option is
- What an Asian Option is and more importantly what a Volume Weighted Average Price is Option

Plan of talk

- What an option is
- What an Asian Option is and more importantly what a Volume Weighted Average Price is Option
- Future work

Plan of talk

- What an option is
- What an Asian Option is and more importantly what a Volume Weighted Average Price is Option
- Future work
- Questions

Options, the basics

- Gives the holder the right to something, without the obligation.

Options, the basics

- Gives the holder the right to something, without the obligation.
- European Call Option: The right, but not obligation, to buy a share at a specified price(the strike) and time.

Options, the basics

- Gives the holder the right to something, without the obligation.
- European Call Option: The right, but not obligation, to buy a share at a specified price(the strike) and time.
- Used to manage risk. Used in currencies markets, commodities such as oil, electriricy...

Options, the basics

- Gives the holder the right to something, without the obligation.
- European Call Option: The right, but not obligation, to buy a share at a specified price(the strike) and time.
- Used to manage risk. Used in currencies markets, commodities such as oil, electriricy...
- Managed funds/superannuation funds use puts to protect against stock declines(I hope!).

Options, the basics

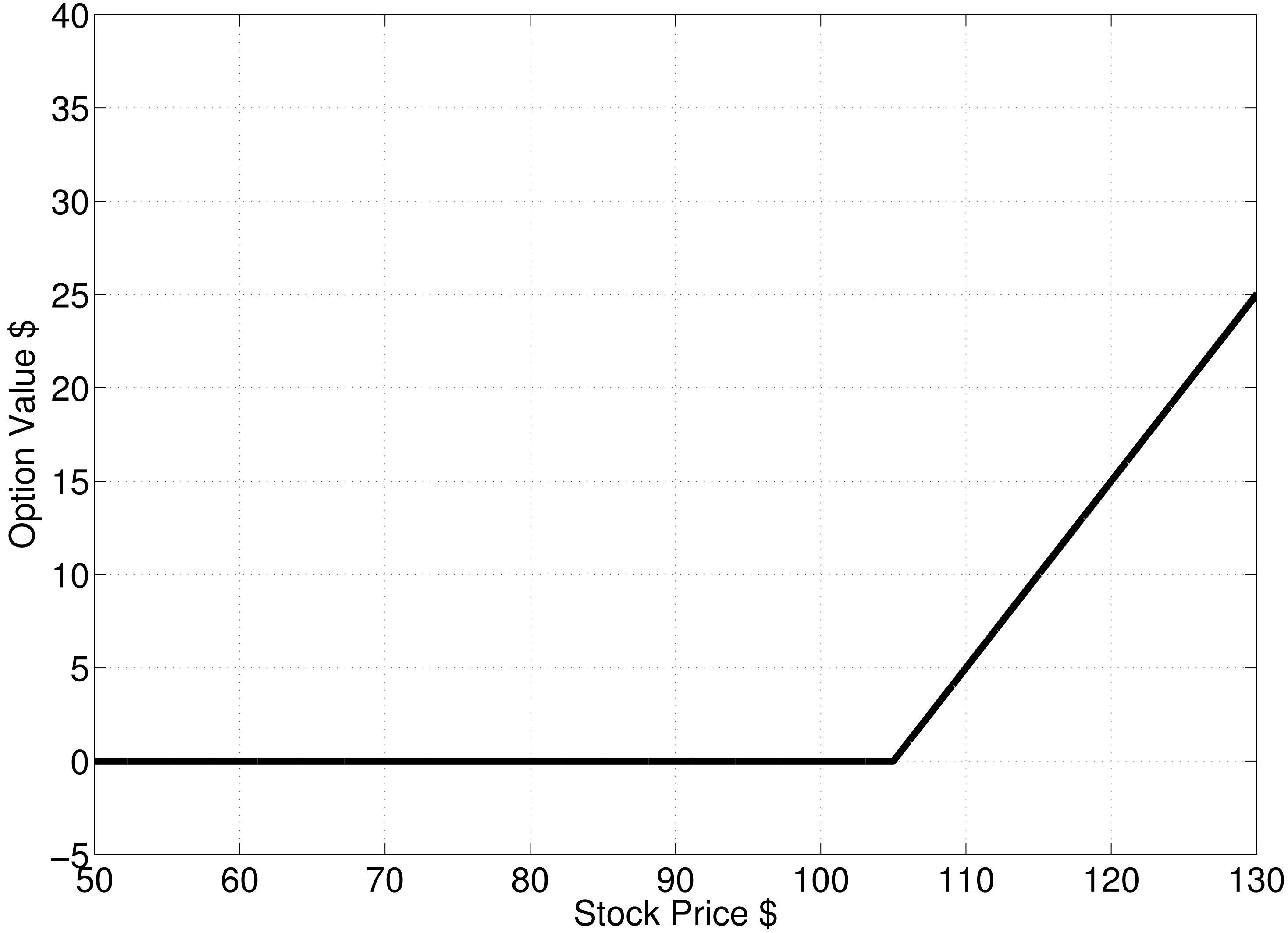
- Gives the holder the right to something, without the obligation.
- European Call Option: The right, but not obligation, to buy a share at a specified price(the strike) and time.
- Used to manage risk. Used in currencies markets, commodities such as oil, electriricy...
- Managed funds/superannuation funds use puts to protect against stock declines(I hope!).
- There are many different types of options, European, American, Asian, Bermudan, Australian, Lookback, Barrier, Spread, Options on Options .....and the list continues to grow all the time as people want new products to manage their risk.

European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is \$105

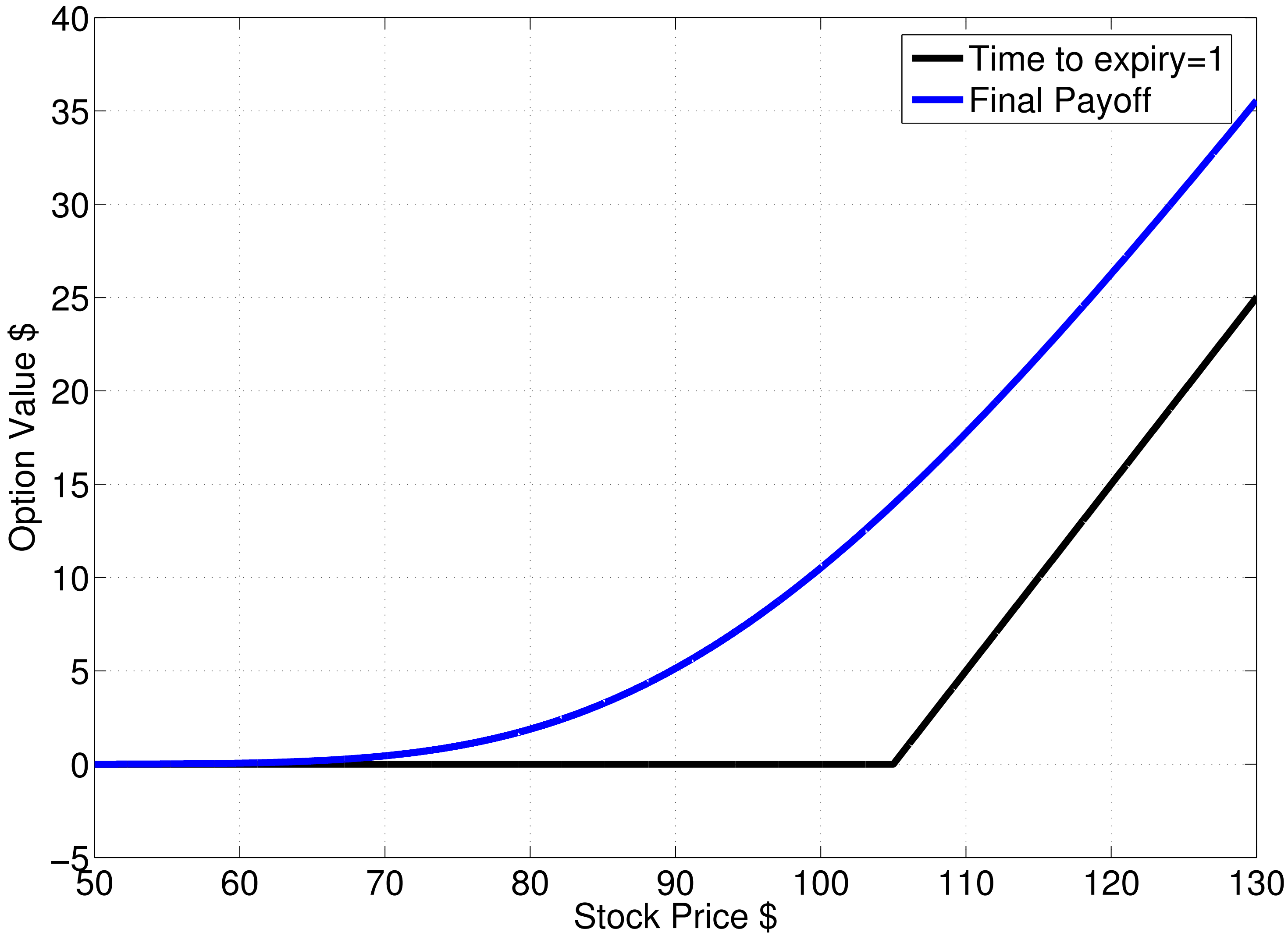
European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is \$105



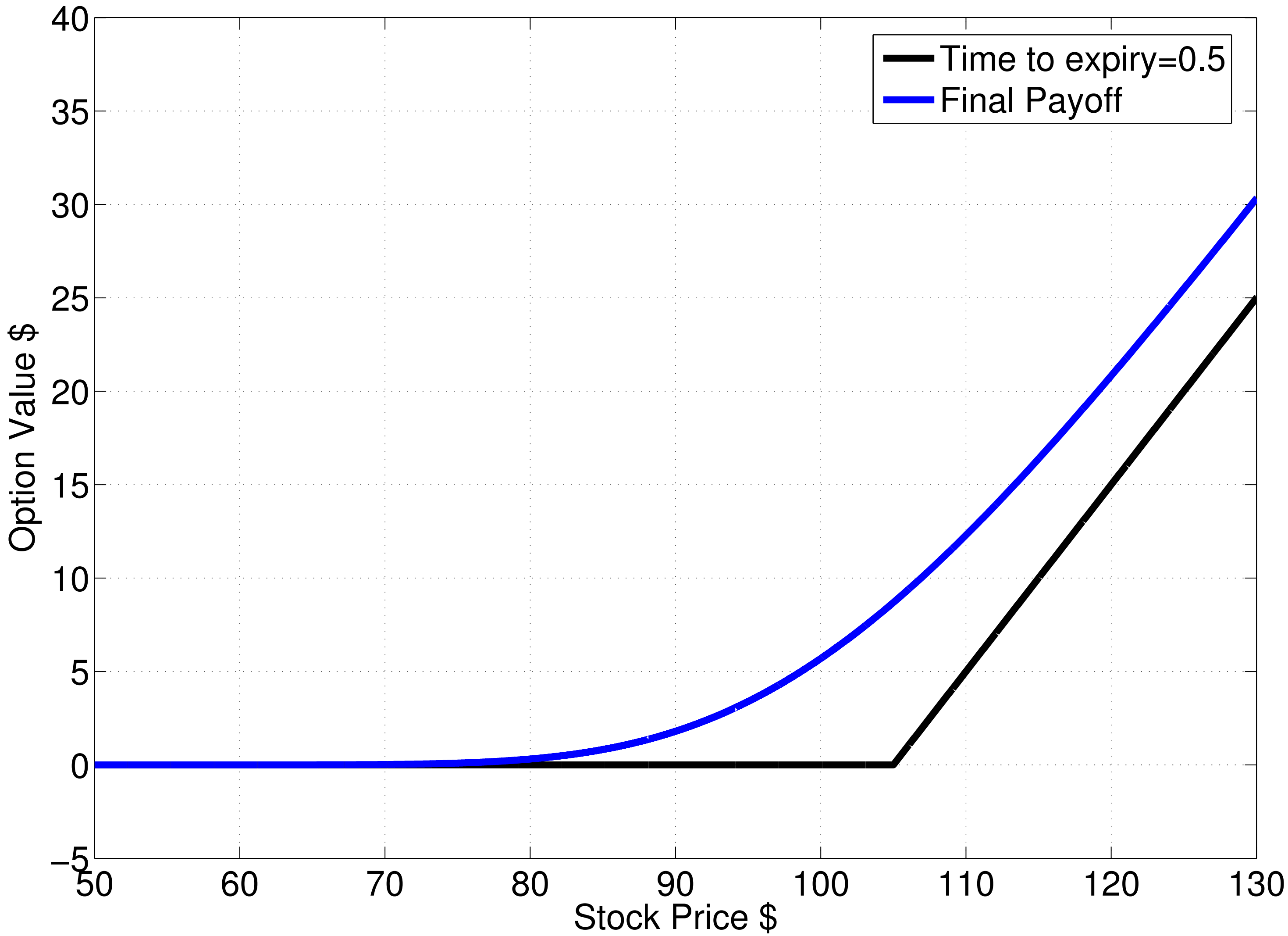
European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is $105



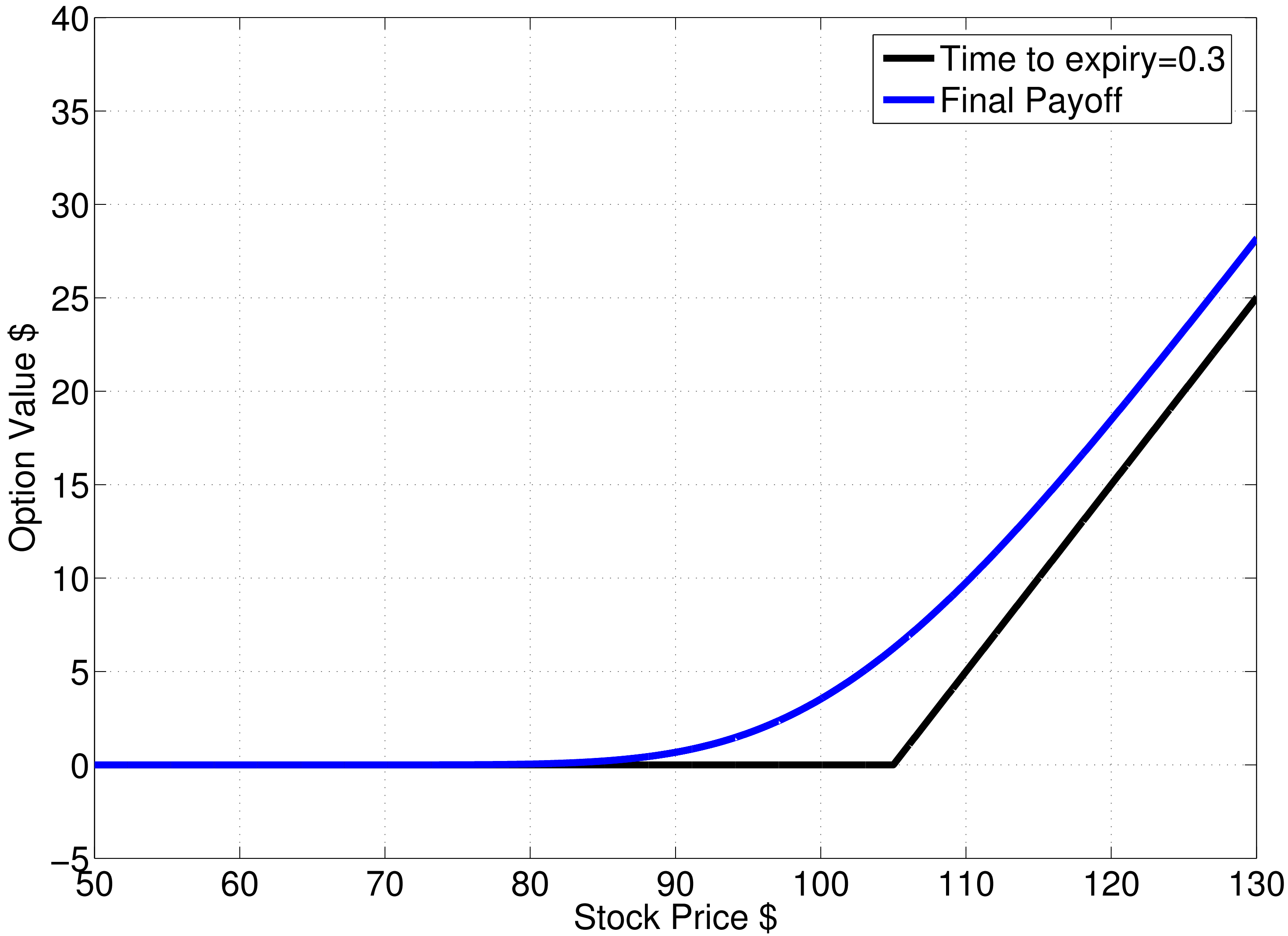
European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is \$105



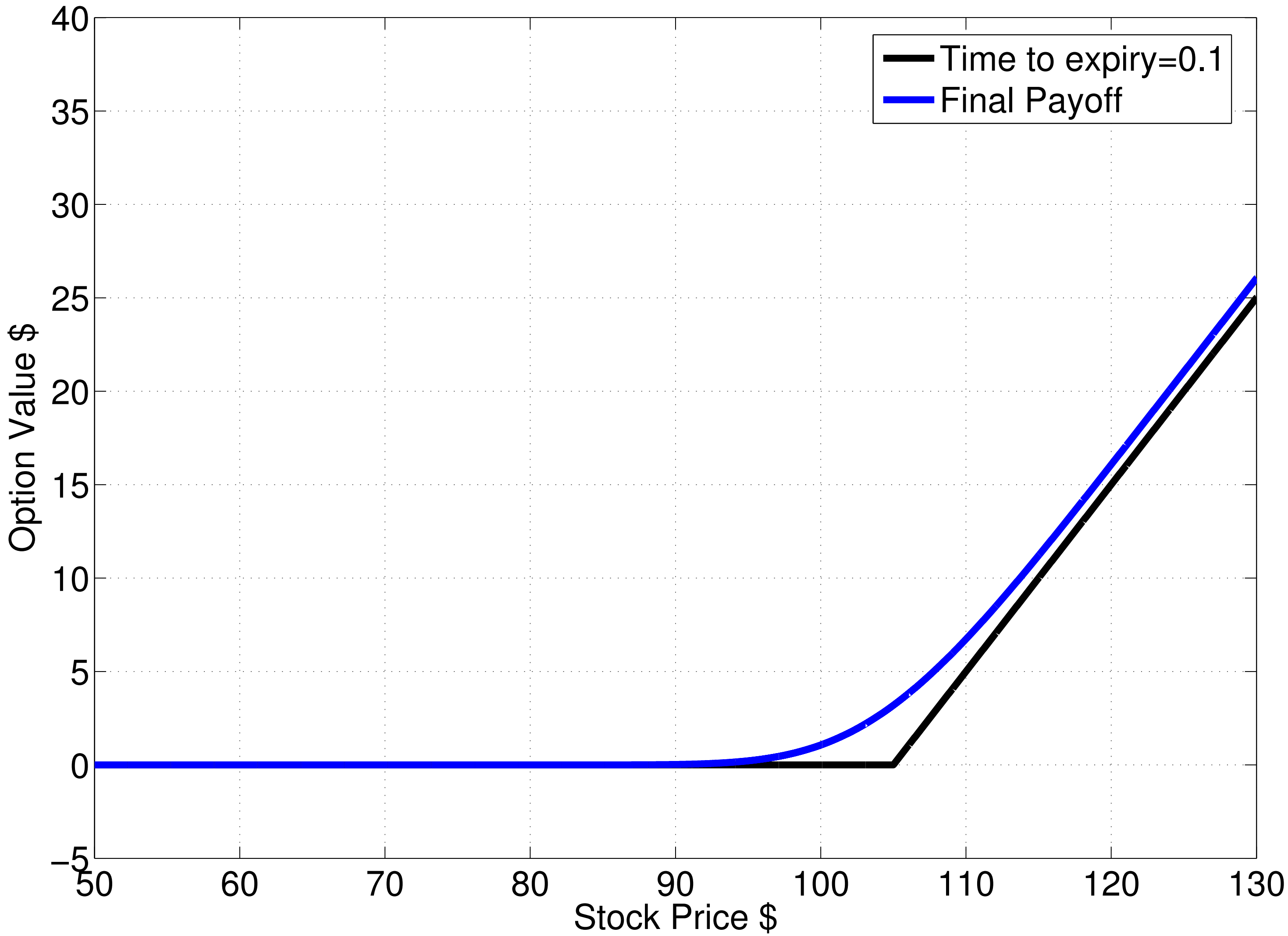
European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is \$105



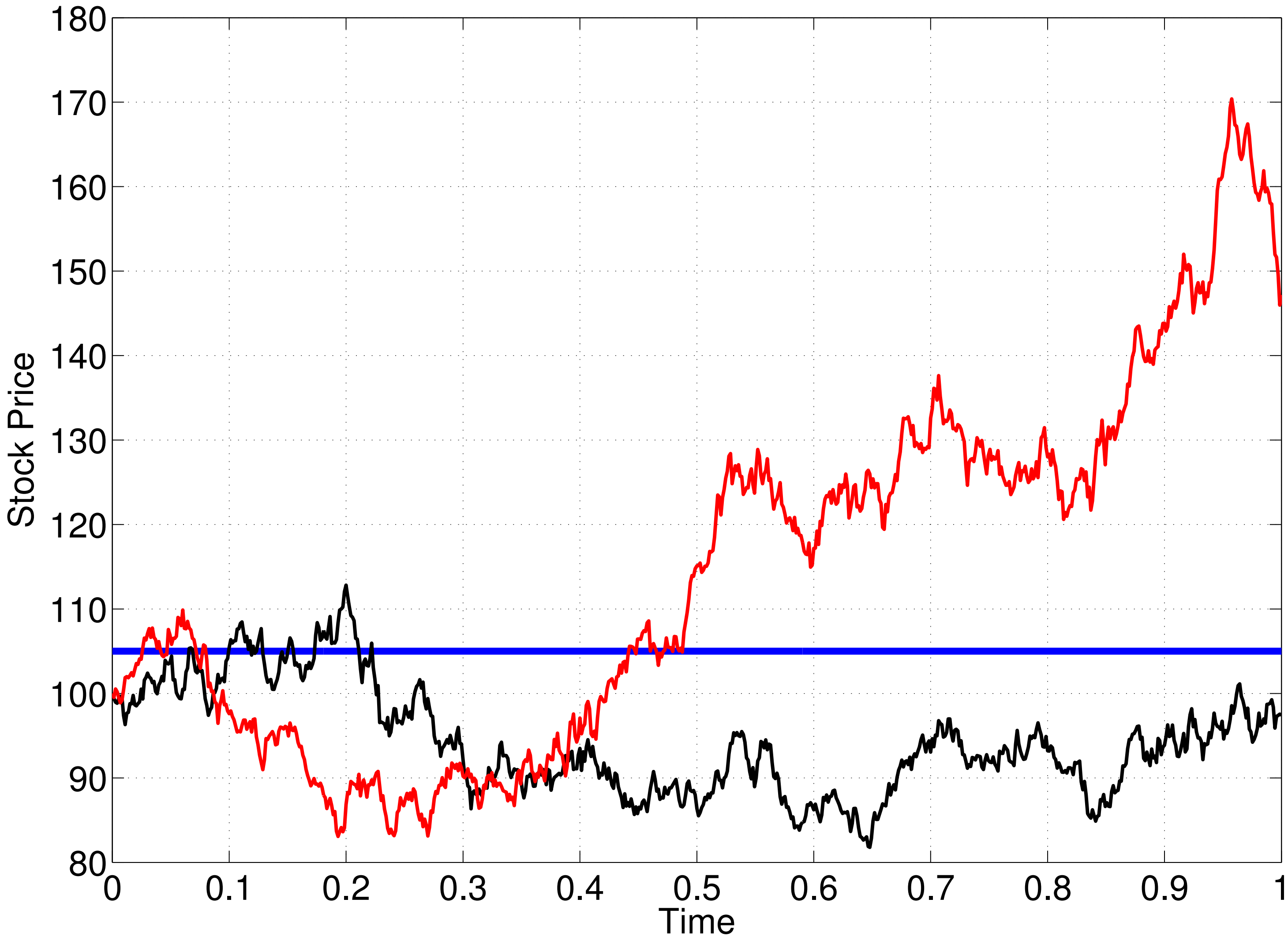
European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is $105



European Call, $V_T = \max(S_T - K, 0)$

The strike price, K, is \$105



European Call, $V_T = \max(S_T - K, 0)$

- Closed form solution published by Black and Scholes in 1973

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

with

$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

where K is the strike price, S_0 is the price of the share at time 0, σ is the share's volatility, T the time to expiry and $N(\cdot)$ is the cumulative probability function.

European Call, $V_T = \max(S_T - K, 0)$

- Closed form solution published by Black and Scholes in 1973

  $$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

  with

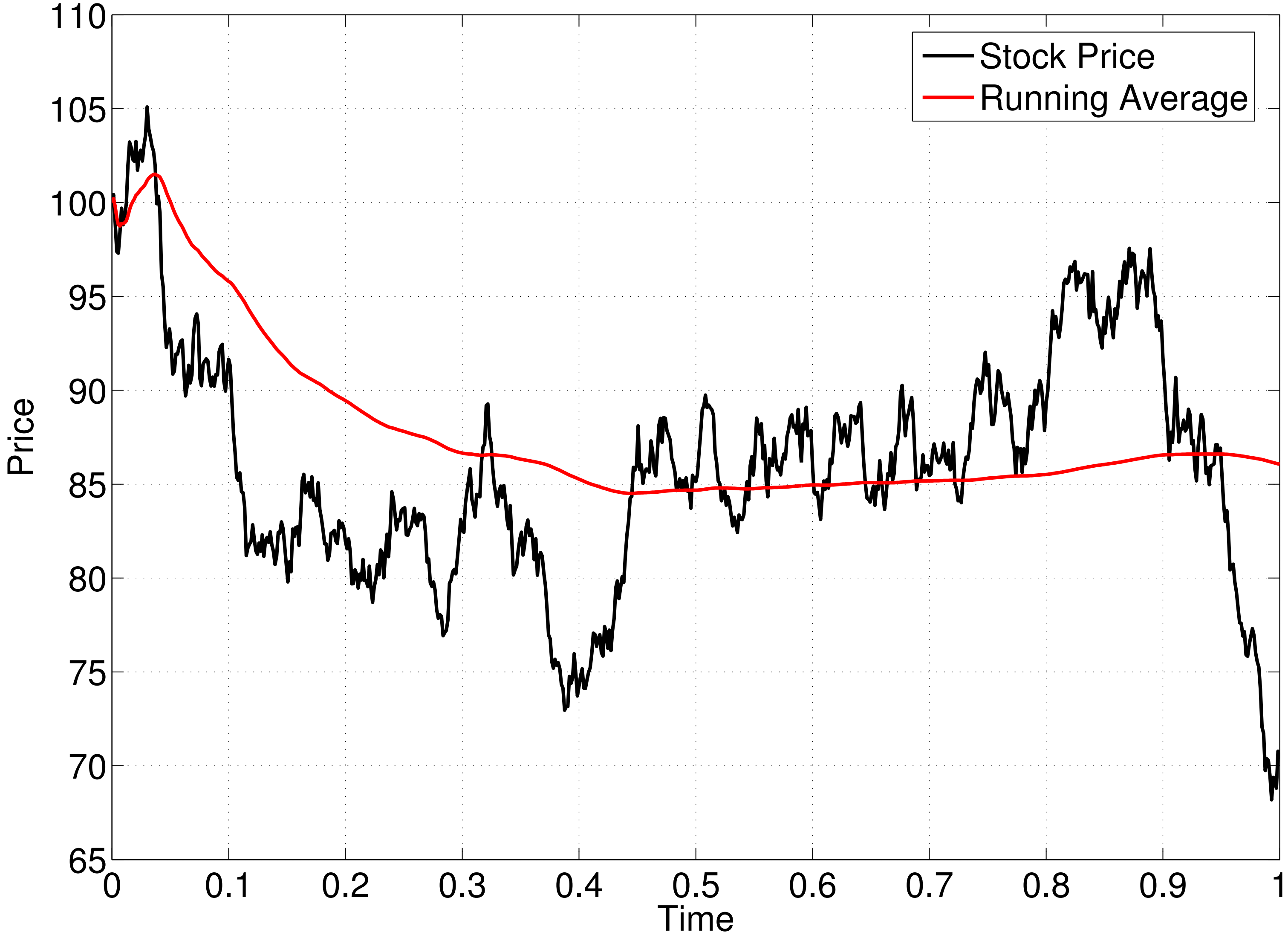
  $$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

  where K is the strike price, S_0 is the price of the share at time 0, σ is the share's volatility,T the time to expiry and $N(\cdot)$ is the cumulative probability function.
- Assumes stock evolves as Geometric Brownian motion, $dS = \mu S dt + \sigma S dW$ (Log normal)

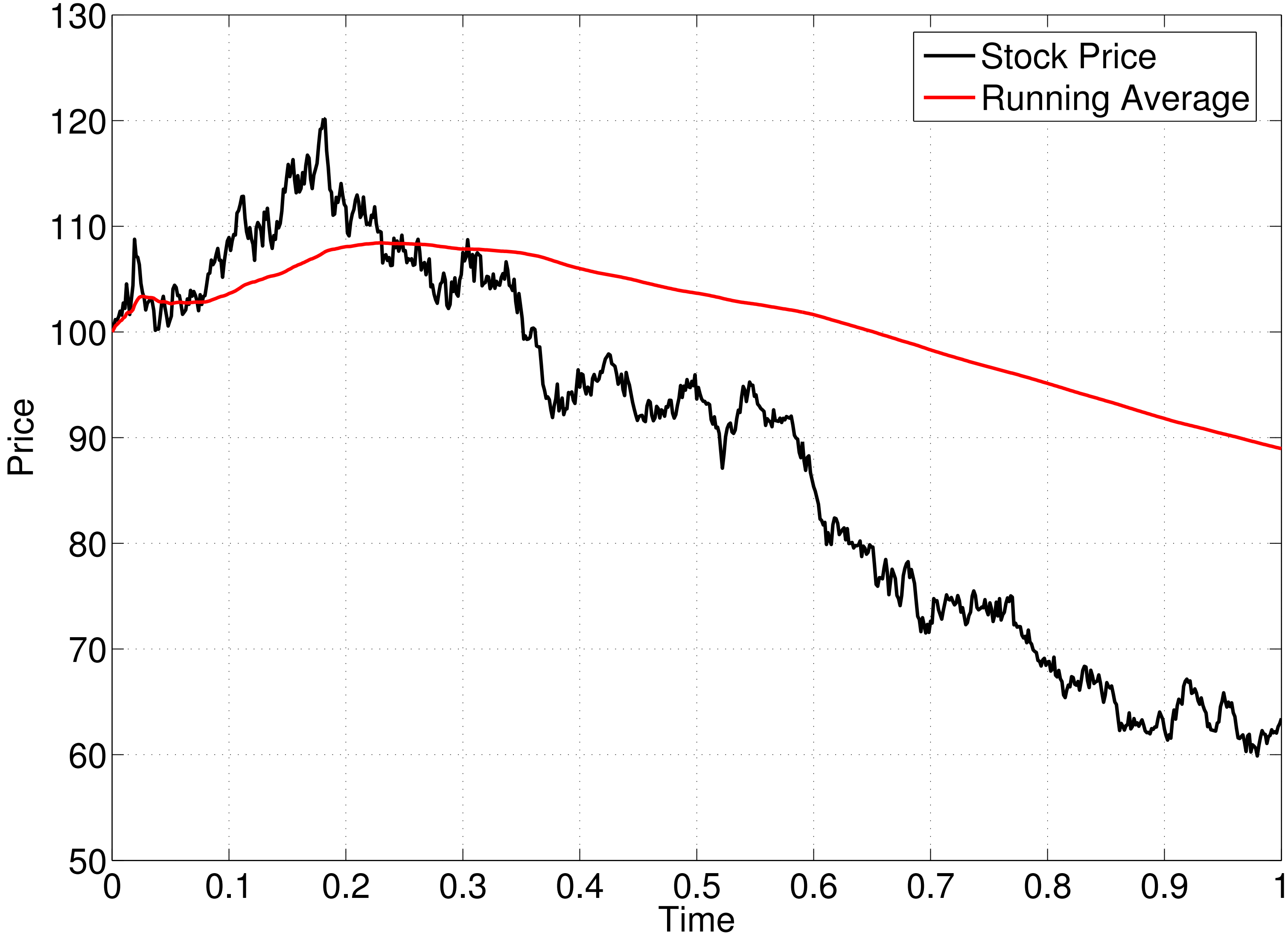
European Call, $V_T = \max(S_T - K, 0)$

- Closed form solution published by Black and Scholes in 1973
$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$
with
$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$
where K is the strike price, S_0 is the price of the share at time 0, σ is the share's volatility, T the time to expiry and $N(\cdot)$ is the cumulative probability function.
- Assumes stock evolves as Geometric Brownian motion, $dS = \mu S dt + \sigma S dW$ (Log normal)
- The solution, remarkably, does not contain drift of the stock

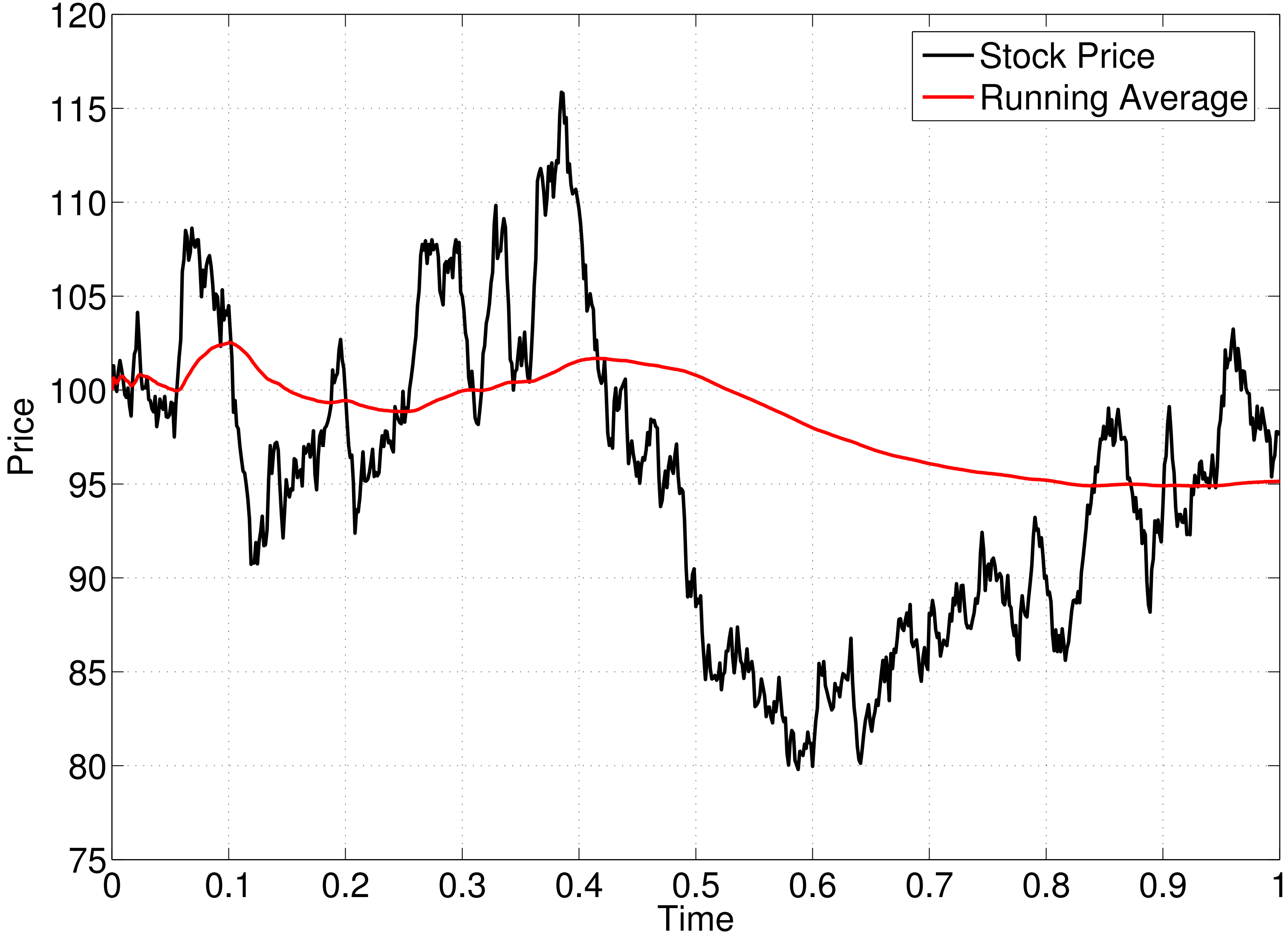
Running Average - $\frac{1}{t}\int_0^t S_\nu d\nu$



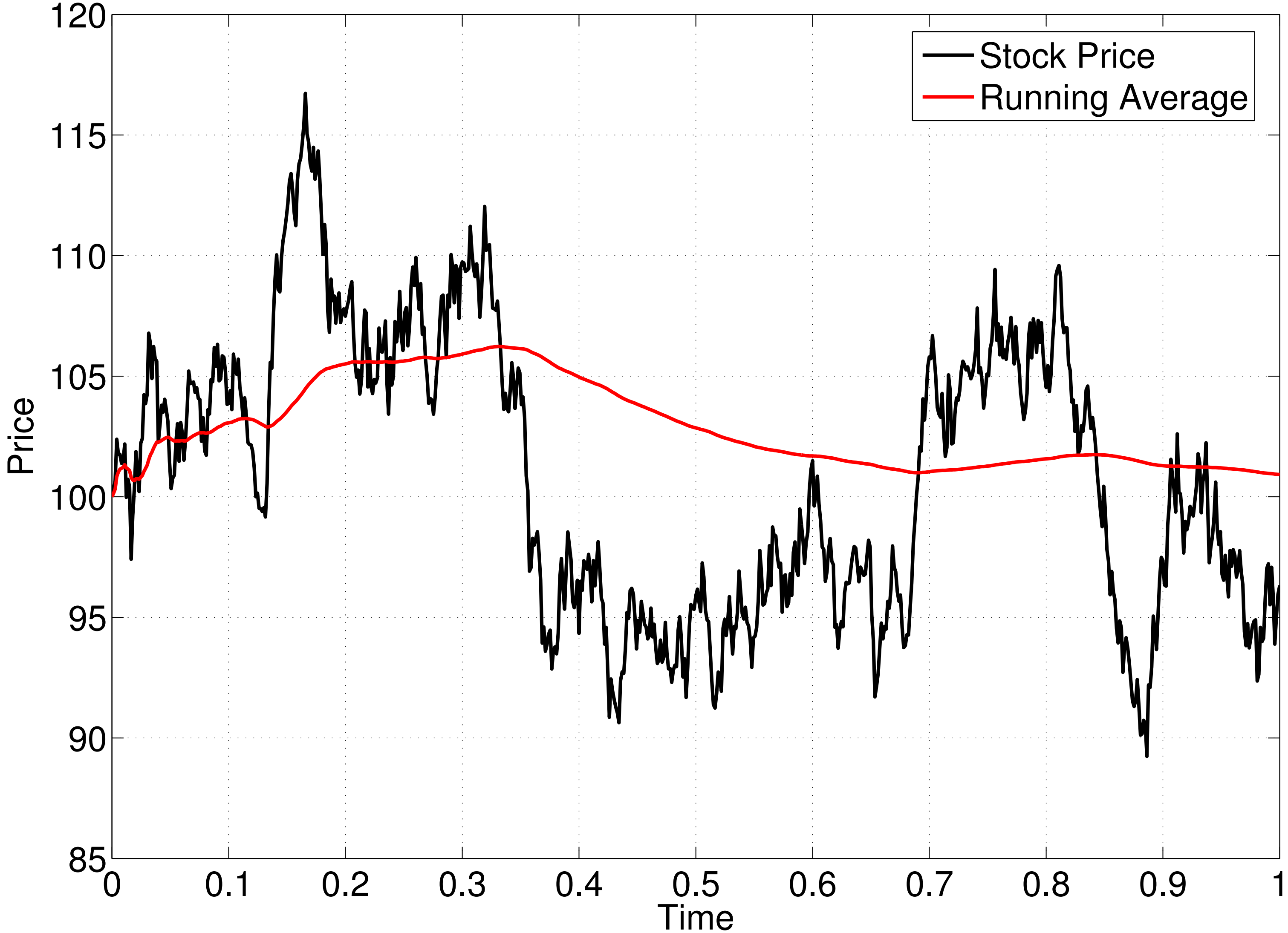
Running Average - $\frac{1}{t}\int_0^t S_\nu d\nu$



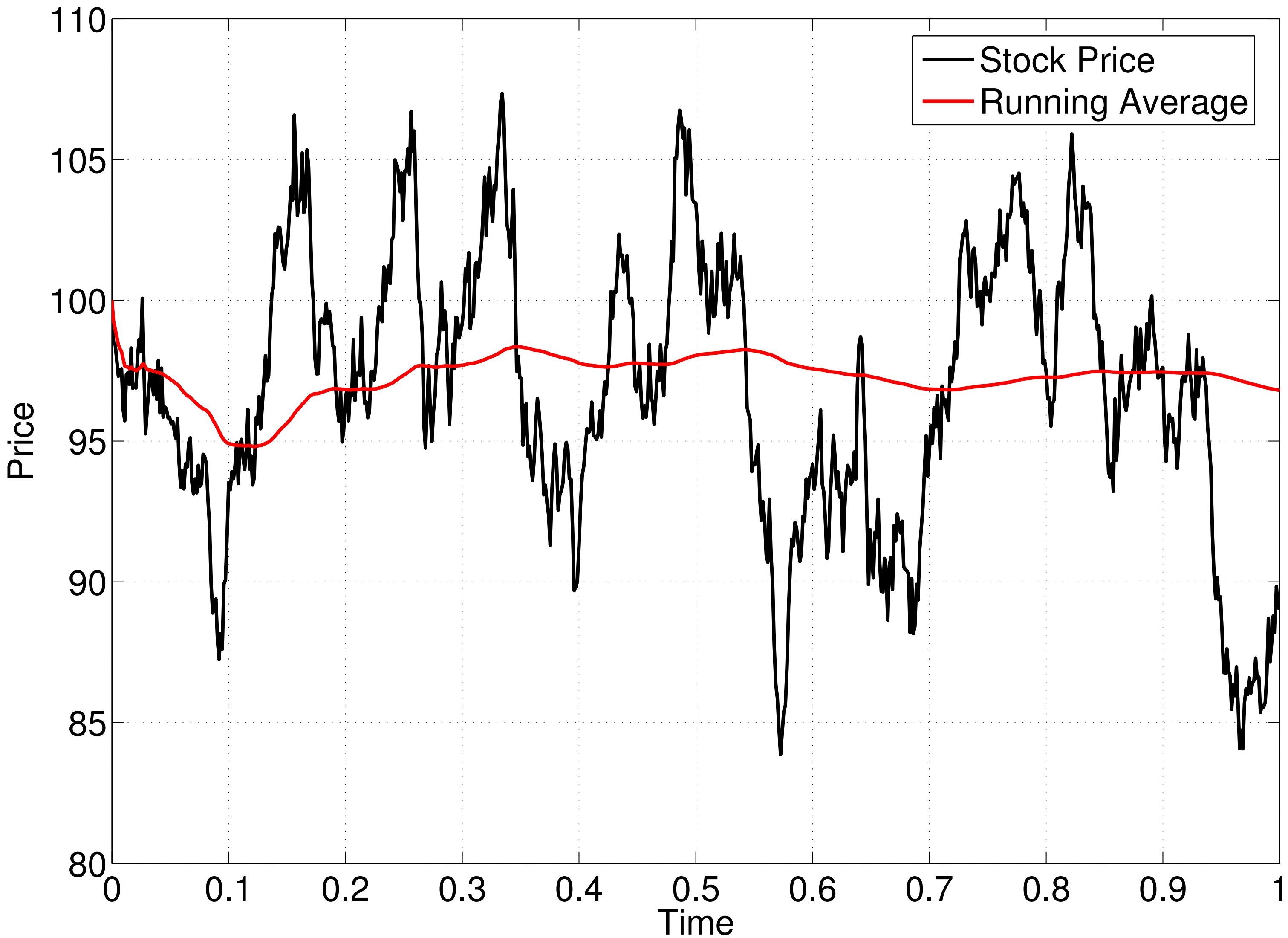
Running Average - $\frac{1}{t}\int_0^t S_\nu d\nu$



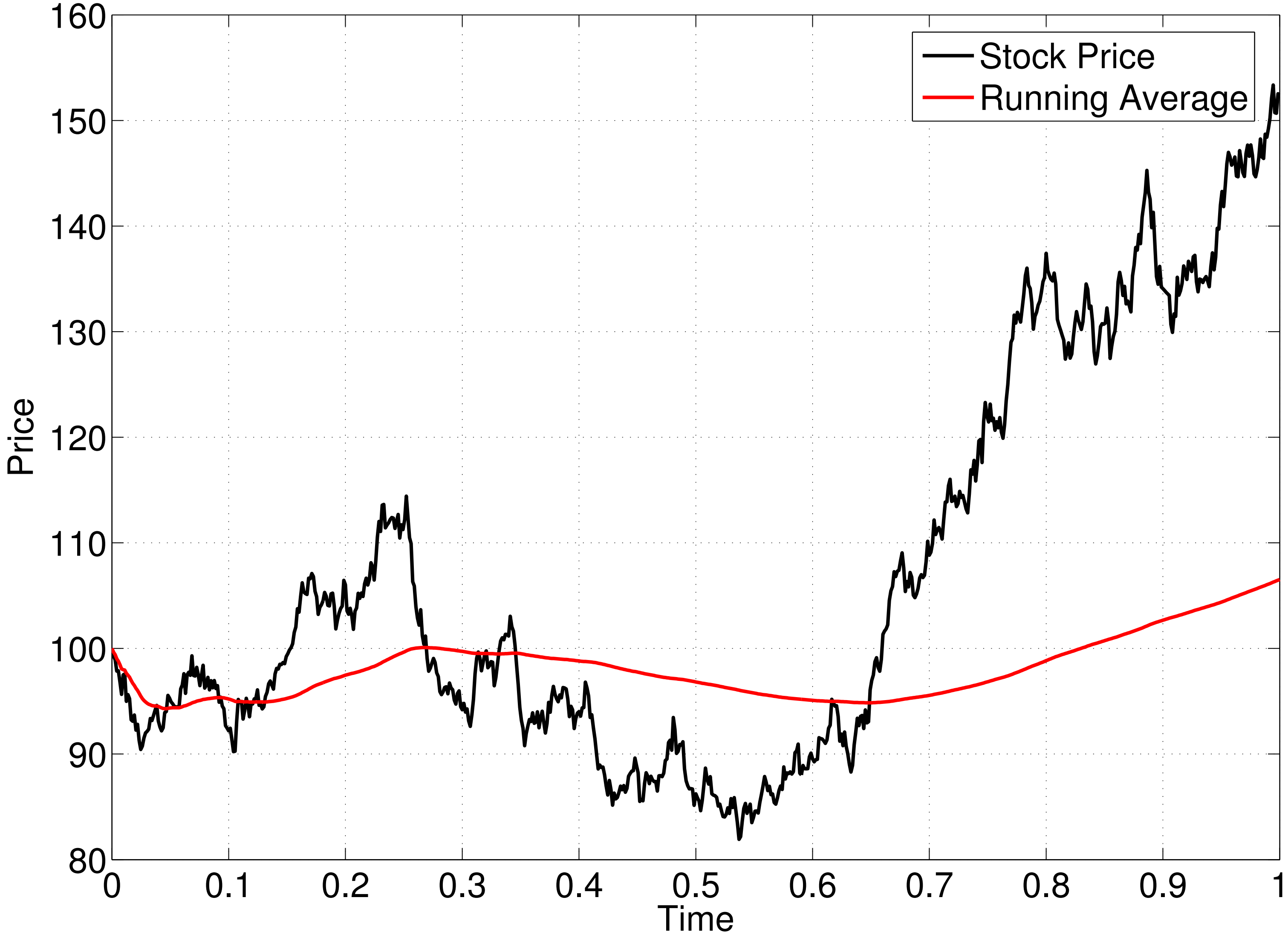
Running Average - $\frac{1}{t}\int_0^t S_\nu d\nu$



Running Average - $\frac{1}{t}\int_0^t S_\nu d\nu$



Running Average - $\frac{1}{t}\int_0^t S_\nu d\nu$



Asian Option, $V_T = \left(\frac{1}{T} \int_0^T S_v dv - k \right)^+$

- Similar to my option is most like this one
- Cheaper than vanilla call or put.
- At the money it is about half the cost of a European. In fact volatility is about $\frac{\sigma}{\sqrt{3}}$. Price is simply $r^* = r - \frac{1}{2}(r - \frac{\sigma^2}{6})$ and $\sigma^* = \frac{\sigma}{\sqrt{3}}$ into the BS for the price of a Geometric Asian Option
- Very popular in currency and commodity markets

A Volume Weighted Average Price

- Assigns more weight to periods of heavy trading, than light trading

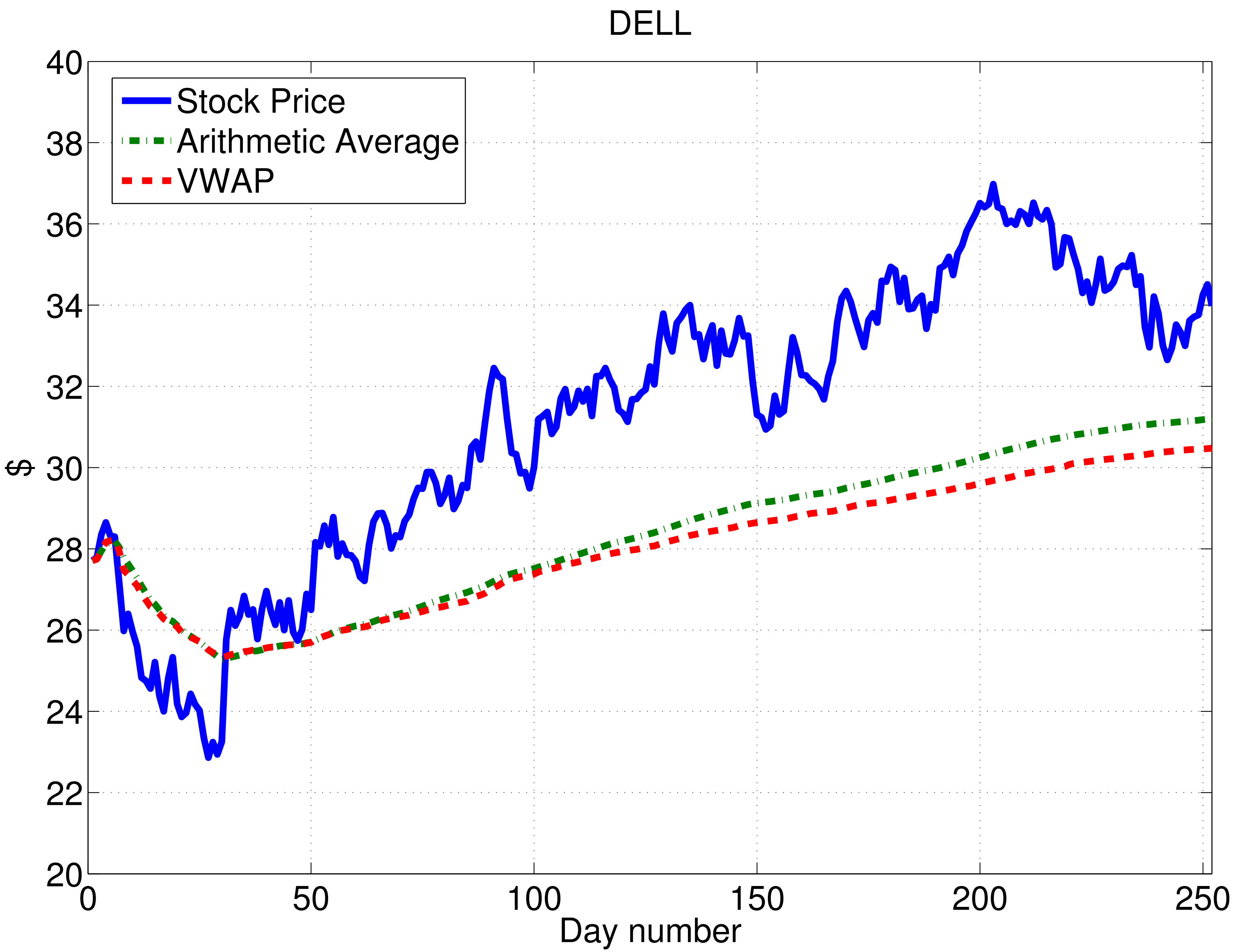
A Volume Weighted Average Price

- Assigns more weight to periods of heavy trading, than light trading
- Example: Suppose a stock trades at $10 today and there are 100 trades, tomorrow it trades at $100 and there is 1 trade.
  The *volume weighted average price* is
  $\frac{\$10\times100+\$100\times1}{100+1} = \$10.89$
  while a *arithmetic weighted average price* is
  $\frac{\$10+\$100}{2} = \$55.00$.

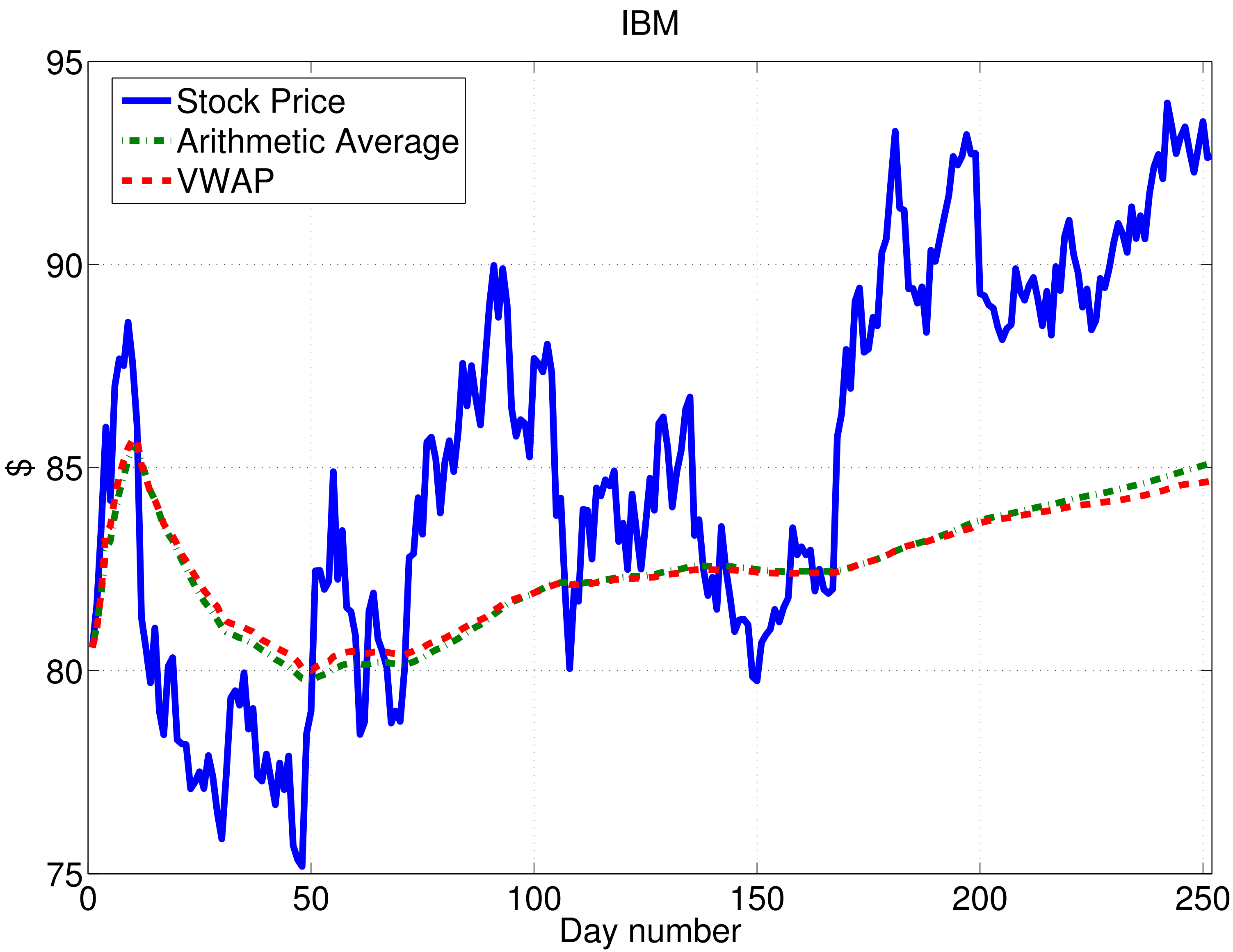
A Volume Weighted Average Price

- Assigns more weight to periods of heavy trading, than light trading
- Example: Suppose a stock trades at \$10 today and there are 100 trades, tomorrow it trades at \$100 and there is 1 trade.
  The *volume weighted average price* is
  $\frac{\$10\times100+\$100\times1}{100+1} = \$10.89$
  while a *arithmetic weighted average price* is
  $\frac{\$10+\$100}{2} = \$55.00$.
- We can write the VWAP at time T as
  $VWAP(T) = \frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv}$
  Where S_t is the price of the stock at time t and
  U_t is the rate of trades of the stock at time t.

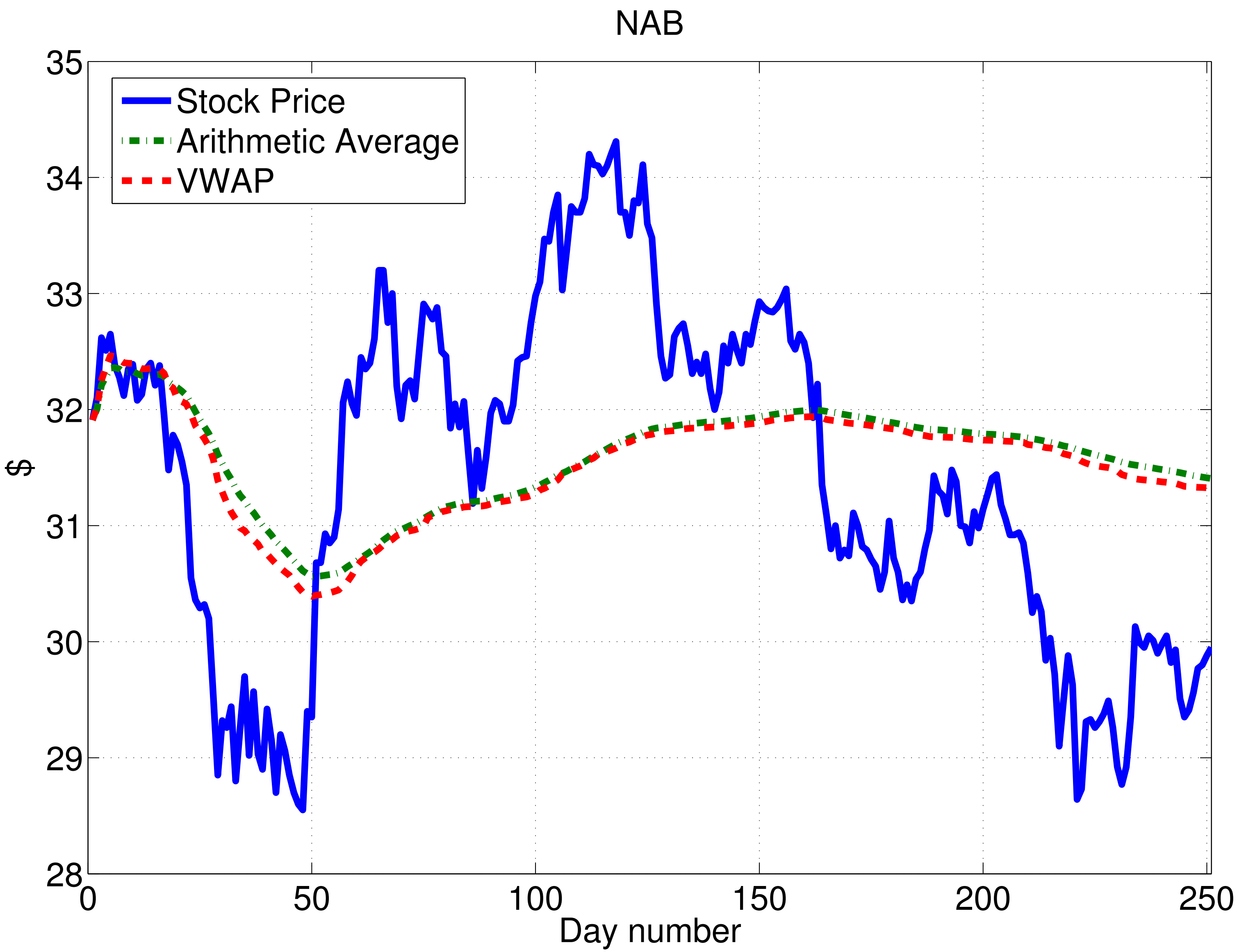
Example Of Real Stocks



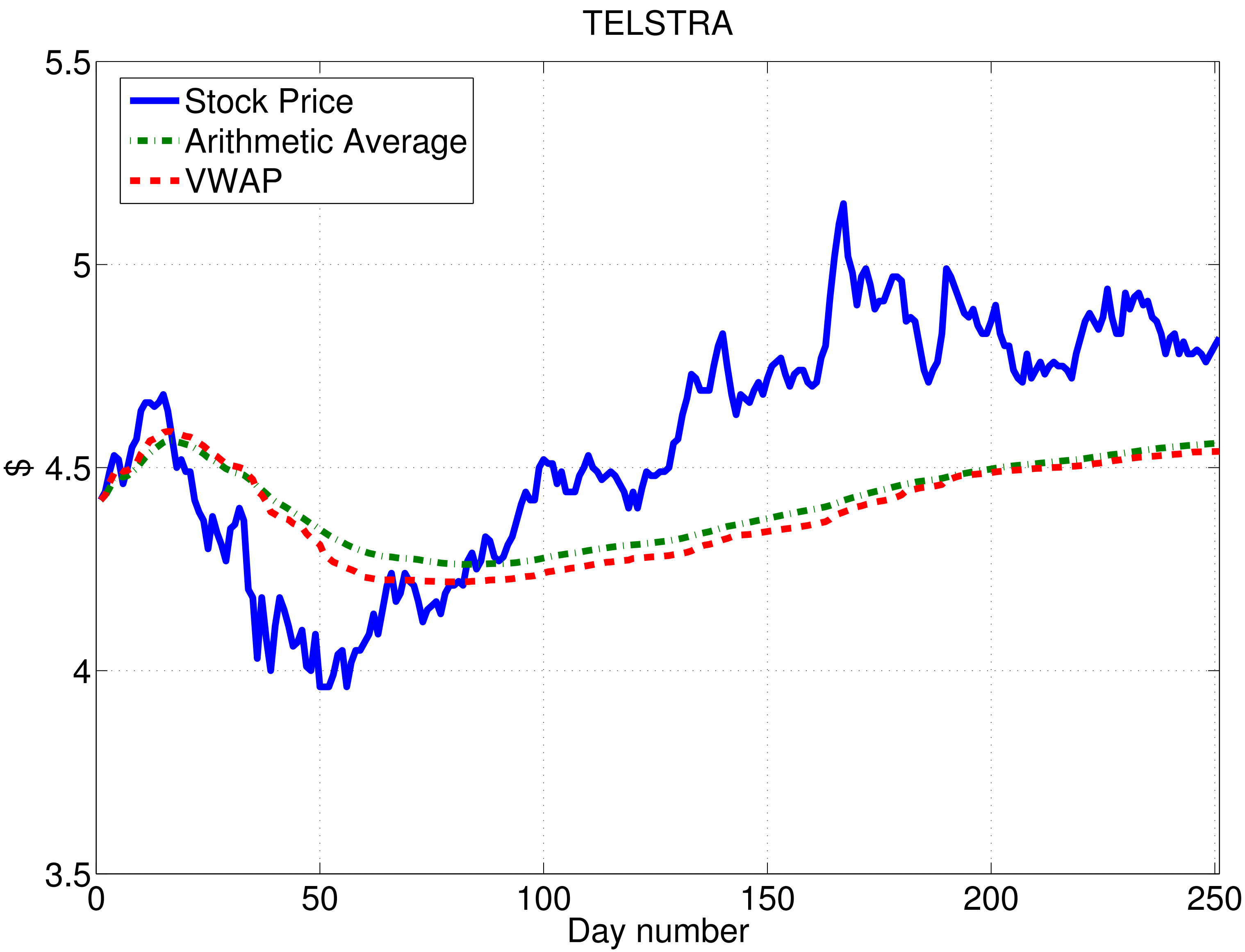
Example Of Real Stocks



Example Of Real Stocks



Example Of Real Stocks



My Problem

To price and hedge

- $V_T = \max\left(\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} - K, 0\right)$ (fixed strike) and

My Problem

To price and hedge

- $V_T = \max\left(\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} - K, 0\right)$ (fixed strike) and
- $V_T = \max\left(\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} - S_T, 0\right)$ (floating strike)

My Problem

To price and hedge

- $V_T = \max\left(\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} - K, 0\right)$ (fixed strike) and
- $V_T = \max\left(\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} - S_T, 0\right)$ (floating strike)

with S and U being defined by the stochastic differential equations

- $dS = rSdt + \sigma S dW_1$ (stock)
- $dU = \alpha(\mu - U)dt + \beta U dW_2$ (trades per unit time), (Use several mean reverting models, add jumps later)

For the moment assume correlation between W_1 and W_2 is zero, relax this assumption later once we know the problem better.

My Problem

To price and hedge

- PDE method has 4 state variables, not realistic to solve

My Problem

To price and hedge

- PDE method has 4 state variables, not realistic to solve
- Probabilistic approach requires us to evaluate an expectation for which we do not know the PDF

My Problem

To price and hedge

- PDE method has 4 state variables, not realistic to solve
- Probabilistic approach requires us to evaluate an expectation for which we do not know the PDF
- Can solve by Monte Carlo, but slow.

An approximation

- Inspired by early work, on Asian options, assume that the volume weighted average price

$$\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} \tag{1}$$

has a log normal distribution at the final time.

An approximation

- Inspired by early work, on Asian options, assume that the volume weighted average price

$$\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} \tag{1}$$

has a log normal distribution at the final time.

- We know

$$d\widetilde{S} = \widetilde{\mu}\widetilde{S}dt + \widetilde{\sigma}\widetilde{S}dW$$

has a log normal distribution,

An approximation

- Inspired by early work, on Asian options, assume that the volume weighted average price

$$\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} \tag{1}$$

has a log normal distribution at the final time.

- We know

$$d\widetilde{S} = \widetilde{\mu}\widetilde{S}dt + \widetilde{\sigma}\widetilde{S}dW$$

has a log normal distribution,

- So all we need to do is find $\widetilde{\mu}$ and $\widetilde{\sigma}$ which will match the expectation and variance of (1) and then we will have a pretty standard equation to solve.

An approximation

- Inspired by early work, on Asian options, assume that the volume weighted average price

$$\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} \tag{1}$$

has a log normal distribution at the final time.

- We know

$$d\widetilde{S} = \widetilde{\mu}\widetilde{S}dt + \widetilde{\sigma}\widetilde{S}dW$$

has a log normal distribution,

- So all we need to do is find $\widetilde{\mu}$ and $\widetilde{\sigma}$ which will match the expectation and variance of (1) and then we will have a pretty standard equation to solve.
- But how do we get these???

Approximations

- We write the VWAP as $\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z}$

Approximations

- We write the VWAP as $\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z}$
- Simple approximations to the expectation and variance of $\frac{Y}{Z}$ where Y and Z are random variables.

Approximations

- We write the VWAP as $\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z}$
- Simple approximations to the expectation and variance of $\frac{Y}{Z}$ where Y and Z are random variables.
- $$E\left(\frac{Y}{Z}\right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y,Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} Var(Z)$$

Approximations

- We write the VWAP as $\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z}$
- Simple approximations to the expectation and variance of $\frac{Y}{Z}$ where Y and Z are random variables.
- $$E\left(\frac{Y}{Z}\right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y,Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} Var(Z)$$
- $$Var\left(\frac{Y}{Z}\right) \approx \left(\frac{E(Y)}{E(Z)}\right)^2 \left(\frac{Var(Y)}{(E(Y))^2} + \frac{Var(Z)}{(E(Z))^2} - 2\frac{Cov(Y,Z)}{E(Y)E(Z)}\right)$$

  , Mood et al. (1974).

Approximations

- We write the VWAP as $\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z}$
- Simple approximations to the expectation and variance of $\frac{Y}{Z}$ where Y and Z are random variables.
- $$E\left(\frac{Y}{Z}\right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y,Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} Var(Z)$$
- $$Var\left(\frac{Y}{Z}\right) \approx \left(\frac{E(Y)}{E(Z)}\right)^2 \left(\frac{Var(Y)}{(E(Y))^2} + \frac{Var(Z)}{(E(Z))^2} - 2\frac{Cov(Y,Z)}{E(Y)E(Z)}\right)$$

  , Mood et al. (1974).
- Thats all great, but how do we get all these expectations?

Approximations

- We write the VWAP as $\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z}$
- Simple approximations to the expectation and variance of $\frac{Y}{Z}$ where Y and Z are random variables.
- $$E\left(\frac{Y}{Z}\right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y,Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} Var(Z)$$
- $$Var\left(\frac{Y}{Z}\right) \approx \left(\frac{E(Y)}{E(Z)}\right)^2 \left(\frac{Var(Y)}{(E(Y))^2} + \frac{Var(Z)}{(E(Z))^2} - 2\frac{Cov(Y,Z)}{E(Y)E(Z)}\right)$$

  , Mood et al. (1974).
- Thats all great, but how do we get all these expectations?
- From the Ito-Doeblin formula, and lots of patience

The Doeblin in the Ito-Doeblin formula

The Doeblin in the Ito-Doeblin formula

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front.

The Doeblin in the Ito-Doeblin formula

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remained sealed until

The Doeblin in the Ito-Doeblin formula

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remained sealed until May

The Doeblin in the Ito-Doeblin formula

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remained sealed until May 2000.

The Doeblin in the Ito-Doeblin formula

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remained sealed until May 2000. When it was opened, the document was found to contain a construction of the stochastic integral slightly different from Itô's and a clear statement of the change-of-variable formula.

The Doeblin in the Ito-Doeblin formula

The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called *Itô's rule* or *Itô's formula*. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remained sealed until May 2000. When it was opened, the document was found to contain a construction of the stochastic integral slightly different from Itô's and a clear statement of the change-of-variable formula. Doeblin's work [52], Yor's [166] analysis of the work, and a detailed history by Bru [24] of the context of the work appeared in the December 2000 issue of *Comptes Rendus de L'Académie des Sciences*. An English translation of this material is [25].

Shreve (2004)

Now back to the problem

- We need many expectations.

Now back to the problem

- We need many expectations.
- We can find all these expectations from properties of the Ito integral.

Demonstrate the method on $\mathbb{E}(S_t)$

We have $dS = \mu S dt + \sigma S dW$

Demonstrate the method on $\mathbb{E}(S_t)$

We have $dS = \mu S dt + \sigma S dW$
This is really shorthand for

$$S_t - S_0 = \int_0^t \mu S_\nu d\nu + \int_0^t \sigma S_\nu dW_\nu$$

Demonstrate the method on $\mathbb{E}(S_t)$

We have $dS = \mu S dt + \sigma S dW$
This is really shorthand for

$$S_t - S_0 = \int_0^t \mu S_\nu d\nu + \int_0^t \sigma S_\nu dW_\nu$$

Taking the expectation of this we have

$$\mathbb{E}(S_t - S_0) = \mathbb{E}(\int_0^t \mu S_\nu d\nu + \int_0^t \sigma S_\nu dW_\nu)$$

Demonstrate the method on $\mathbb{E}(S_t)$

We have $dS = \mu S dt + \sigma S dW$
This is really shorthand for

$$S_t - S_0 = \int_0^t \mu S_\nu d\nu + \int_0^t \sigma S_\nu dW_\nu$$

Taking the expectation of this we have

$$\mathbb{E}(S_t - S_0) = \mathbb{E}(\int_0^t \mu S_\nu d\nu + \int_0^t \sigma S_\nu dW_\nu)$$

Now the expectation of an Ito integral is 0, so we have

$$\mathbb{E}(S_t - S_0) = \mathbb{E}(\int_0^t \mu S_\nu d\nu)$$

Demonstrate the method on $\mathbb{E}(S_t)$

Then moving the expectation inside the integral

$$\mathbb{E}(S_t - S_0) = \int_0^t \mu \mathbb{E}(S_\nu) d\nu)$$

Demonstrate the method on $\mathbb{E}(S_t)$

Then moving the expectation inside the integral

$$\mathbb{E}(S_t - S_0) = \int_0^t \mu \mathbb{E}(S_\nu) d\nu)$$

finally differentiating we have

$$\frac{d\mathbb{E}(S_t)}{dt} = \mu \mathbb{E}(S_t)$$

Demonstrate the method on $\mathbb{E}(S_t)$

Then moving the expectation inside the integral

$$\mathbb{E}(S_t - S_0) = \int_0^t \mu \mathbb{E}(S_\nu) d\nu)$$

finally differentiating we have

$$\frac{d\mathbb{E}(S_t)}{dt} = \mu \mathbb{E}(S_t)$$

which is simple to solve given the initial condition.

Obtaining the expectations

- We can do this for all the expectations which we require, it is long and tedious, but doable.

Obtaining the expectations

- We can do this for all the expectations which we require, it is long and tedious, but doable.
- Final system has 19 equations which are easy to solve in Matlab or Maple

Obtaining the expectations

- We can do this for all the expectations which we require, it is long and tedious, but doable.
- Final system has 19 equations which are easy to solve in Matlab or Maple
- Can now use the approximations

$$E\left(\frac{Y}{Z}\right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y,Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} Var(Z)$$

$$Var\left(\frac{Y}{Z}\right) \approx \left(\frac{E(Y)}{E(Z)}\right)^2 \left(\frac{Var(Y)}{(E(Y))^2} + \frac{Var(Z)}{(E(Z))^2} - 2\frac{Cov(Y,Z)}{E(Y)E(Z)}\right)$$

to find the expectation and variance at any time T for

$$\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv}$$

Eigenvalues

Eigenvalue	Number of times occurring
$2\mu + \sigma^2$	1
$2\mu - 2\alpha + \sigma^2 + \beta^2$	1
$2\mu - \alpha + \sigma^2$	1
$\mu - 2\alpha + \beta^2$	1
$\beta^2 - 2\alpha$	1
μ	3
$\mu - \alpha$	3
$-\alpha$	3
0	5

Lots of +ve eigenvalues, but the one to look out for is the combination of $\beta^2 - 2\alpha$ which appears in many places.

Now use the log normal approximation

Now we know that the expectation and variance of our underlying $d\widetilde{S} = \widetilde{\mu} S dt + \widetilde{\sigma} S dW$ are

- $$\begin{aligned} \mathbb{E}(\widetilde{S}(t)) &= S_0 e^{\widetilde{\mu} t} \\ Var(\widetilde{S}(t)) &= S_0^2 e^{2\widetilde{\mu} t}(e^{\widetilde{\sigma}^2 t} - 1) \end{aligned}$$ and

Now use the log normal approximation

Now we know that the expectation and variance of our underlying $d\widetilde{S} = \widetilde{\mu} S dt + \widetilde{\sigma} S dW$ are

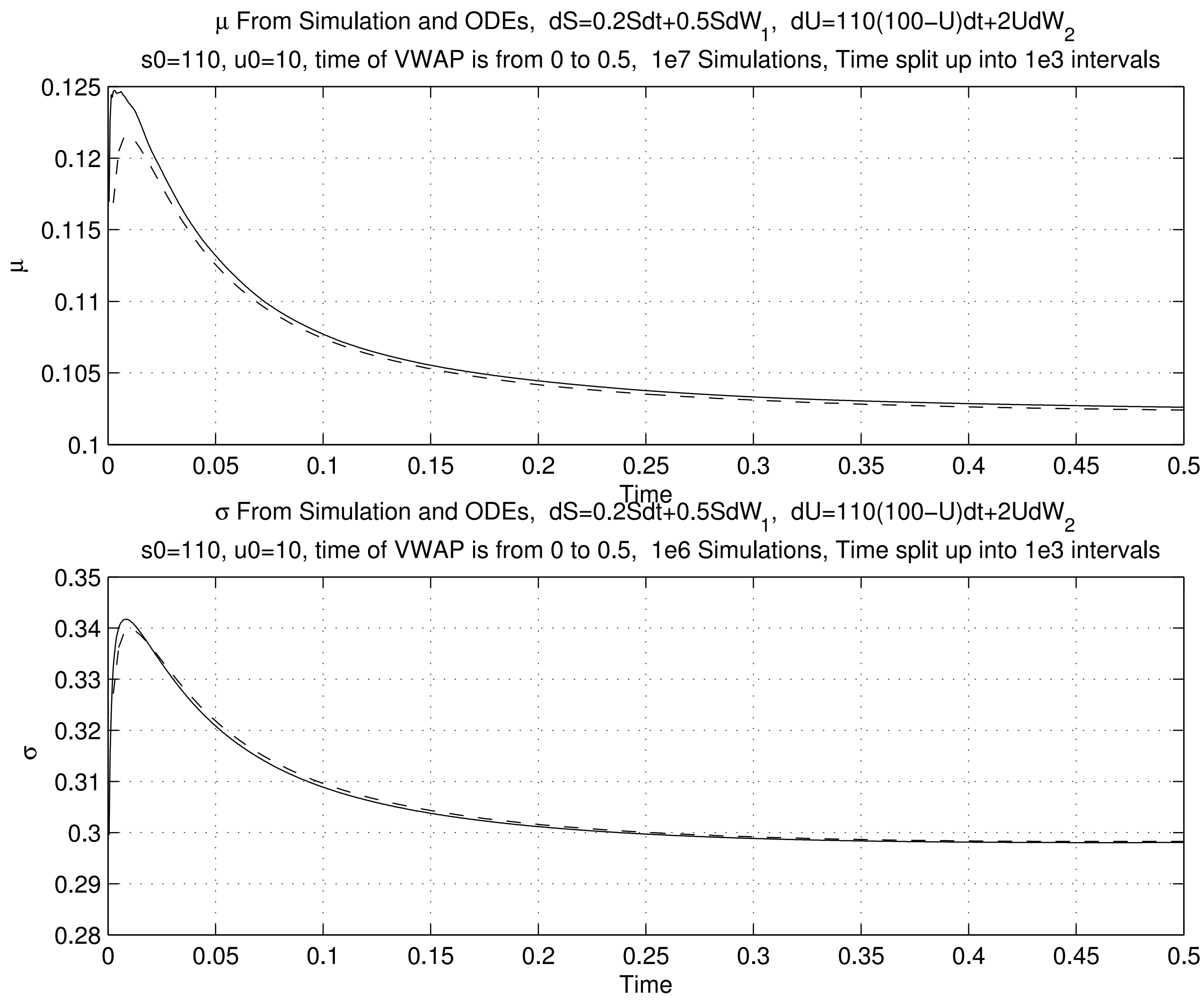
-
$$\begin{aligned} \mathbb{E}(\widetilde{S}(t)) &= S_0 e^{\widetilde{\mu} t} \\ Var(\widetilde{S}(t)) &= S_0^2 e^{2\widetilde{\mu} t}(e^{\widetilde{\sigma}^2 t} - 1) \end{aligned}$$
and

- We can rewrite these as

$$\begin{aligned} \widetilde{\mu} &= \frac{1}{t} \log \frac{\mathbb{E}(\widetilde{S}(t))}{S_0} \\ \widetilde{\sigma} &= \sqrt{\frac{1}{t} \log \frac{Var(\widetilde{S}(t)) + (\mathbb{E}(\widetilde{S}(t)))^2}{(\mathbb{E}(\widetilde{S}(t)))^2}} \end{aligned}$$

How well does it work?

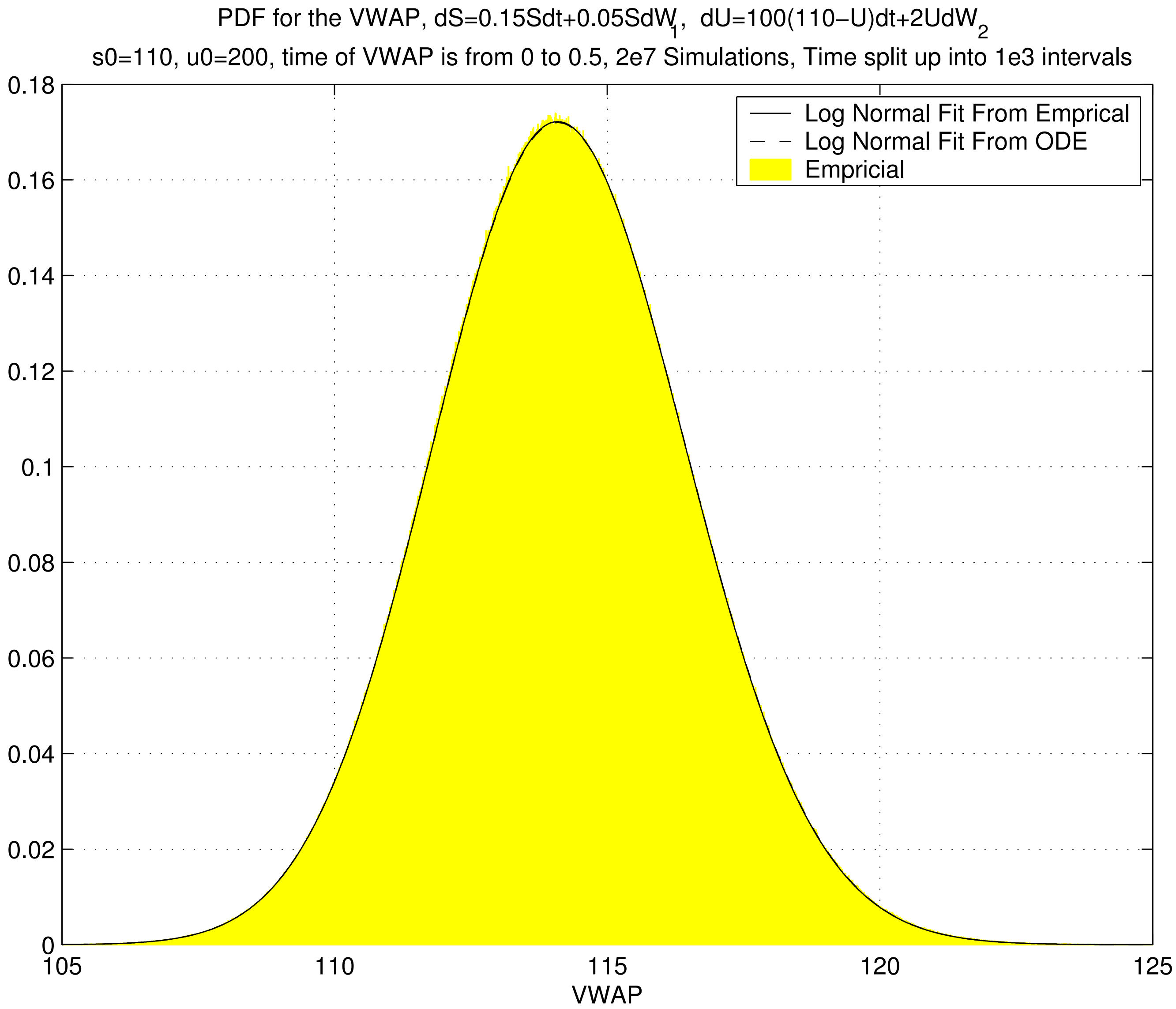
How well does it work?



Solid - results from simulations, Dashed - results from ODEs

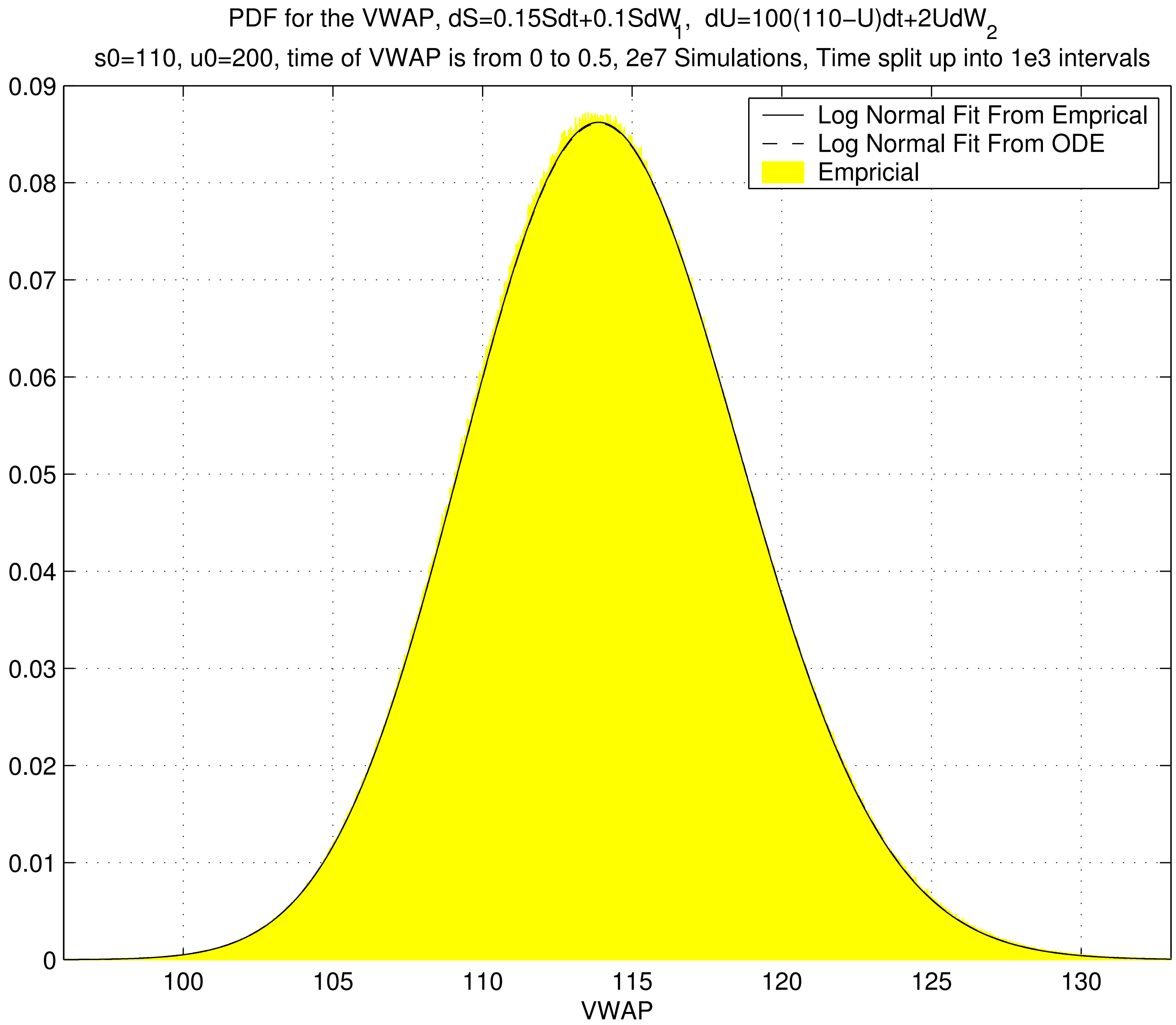
PDF at final time for different σs

$\sigma = 0.05$



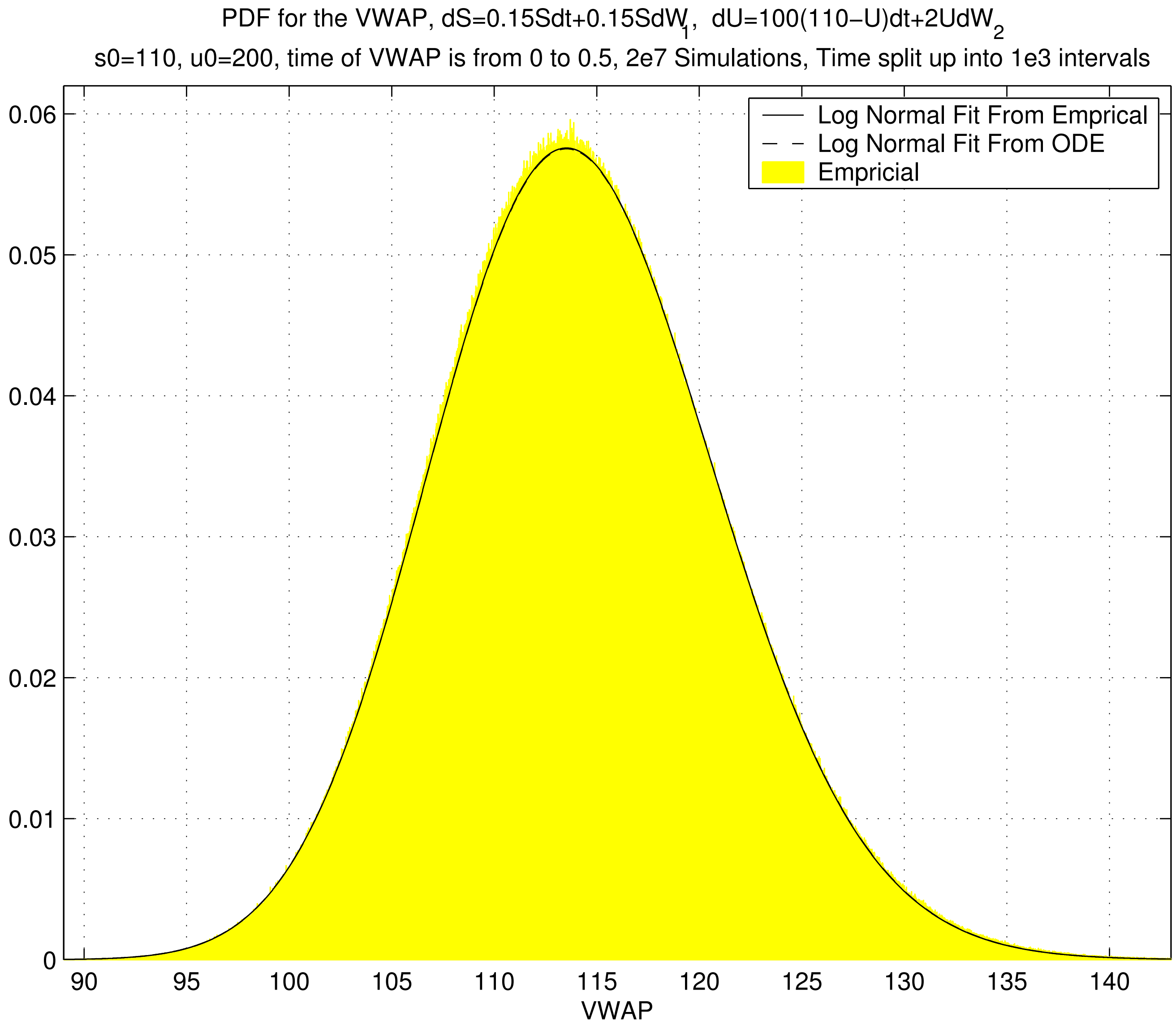
PDF at final time for different σs

$\sigma = 0.1$



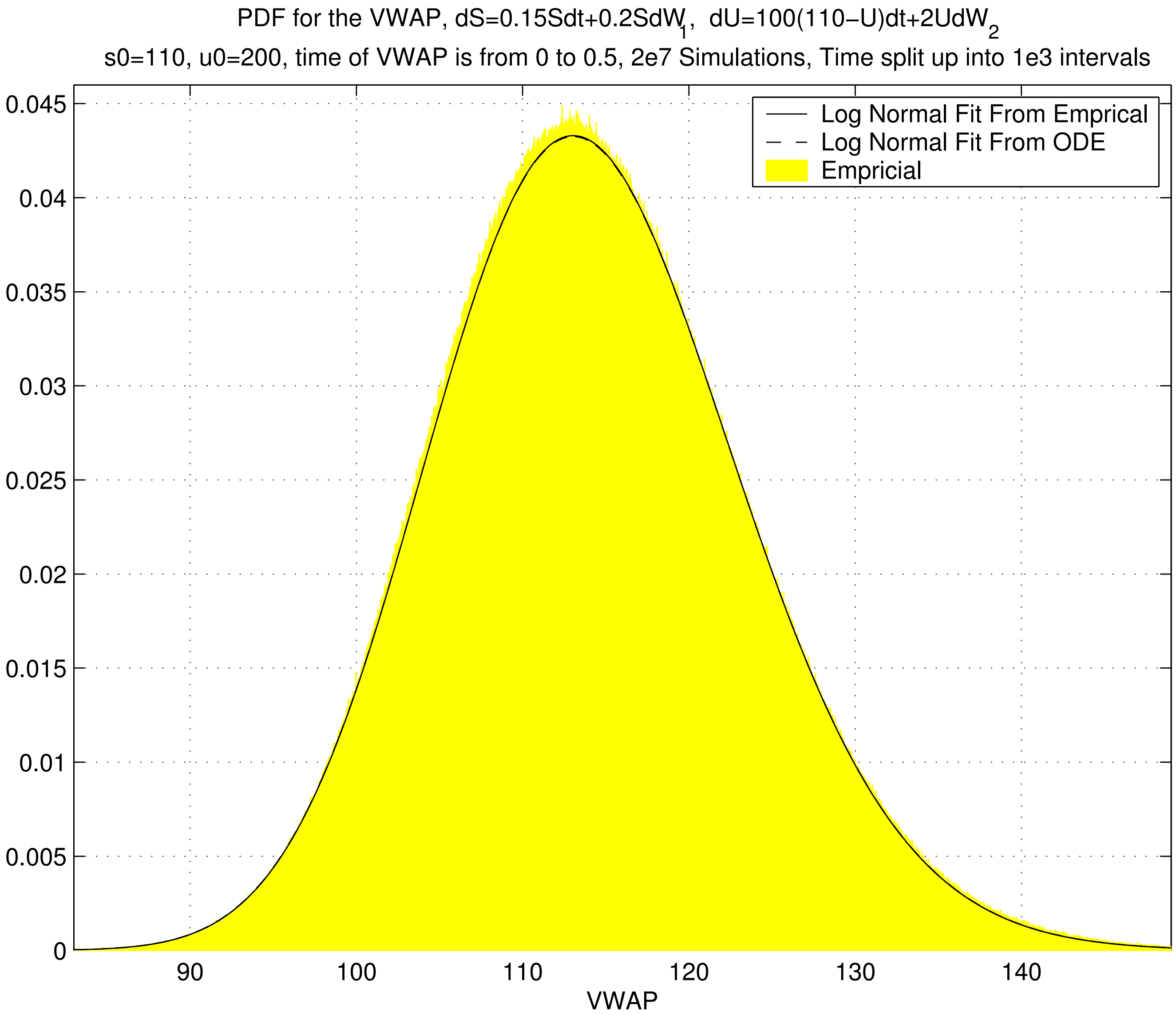
PDF at final time for different σs

$\sigma = 0.15$



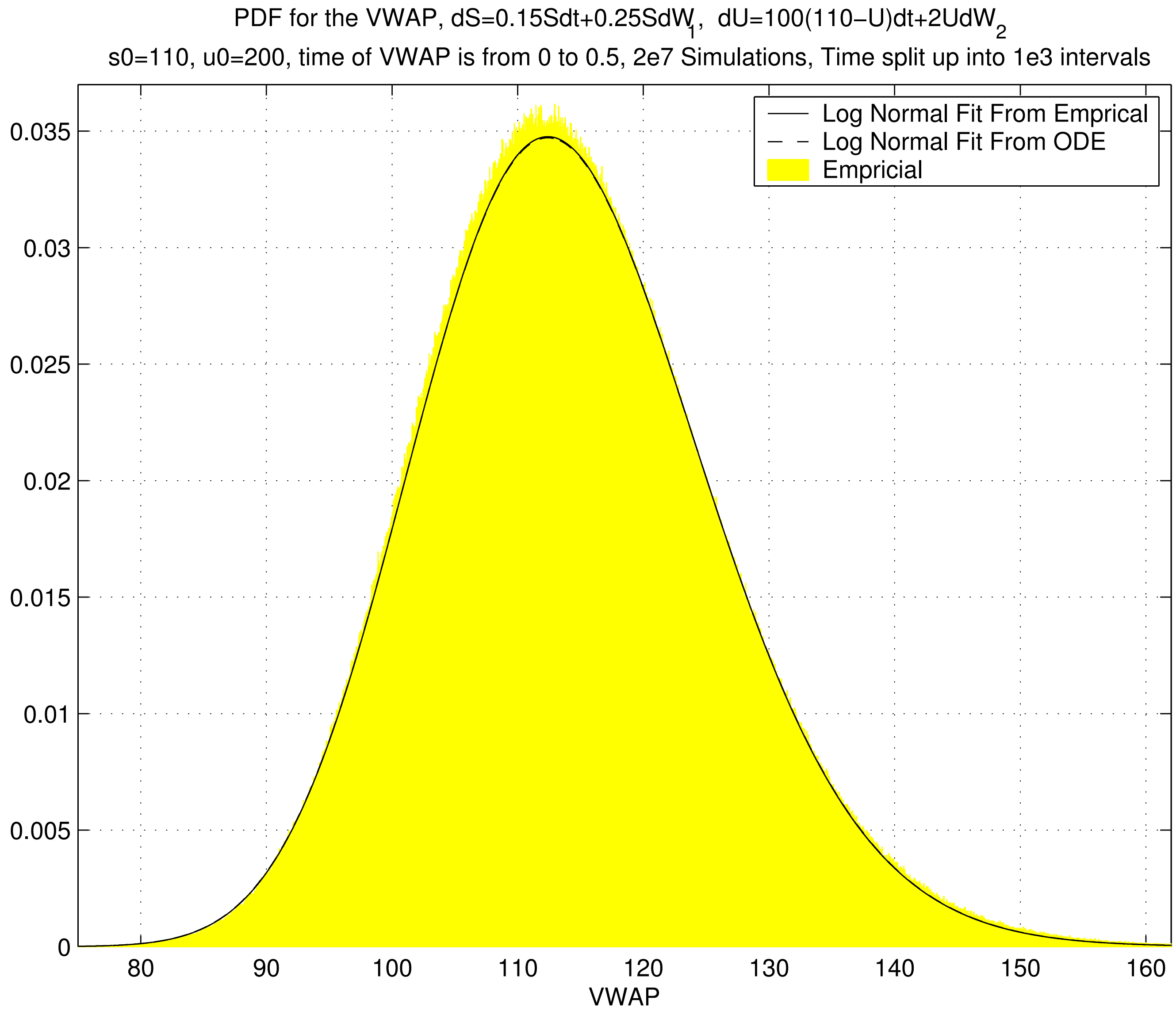
PDF at final time for different σs

$\sigma = 0.2$



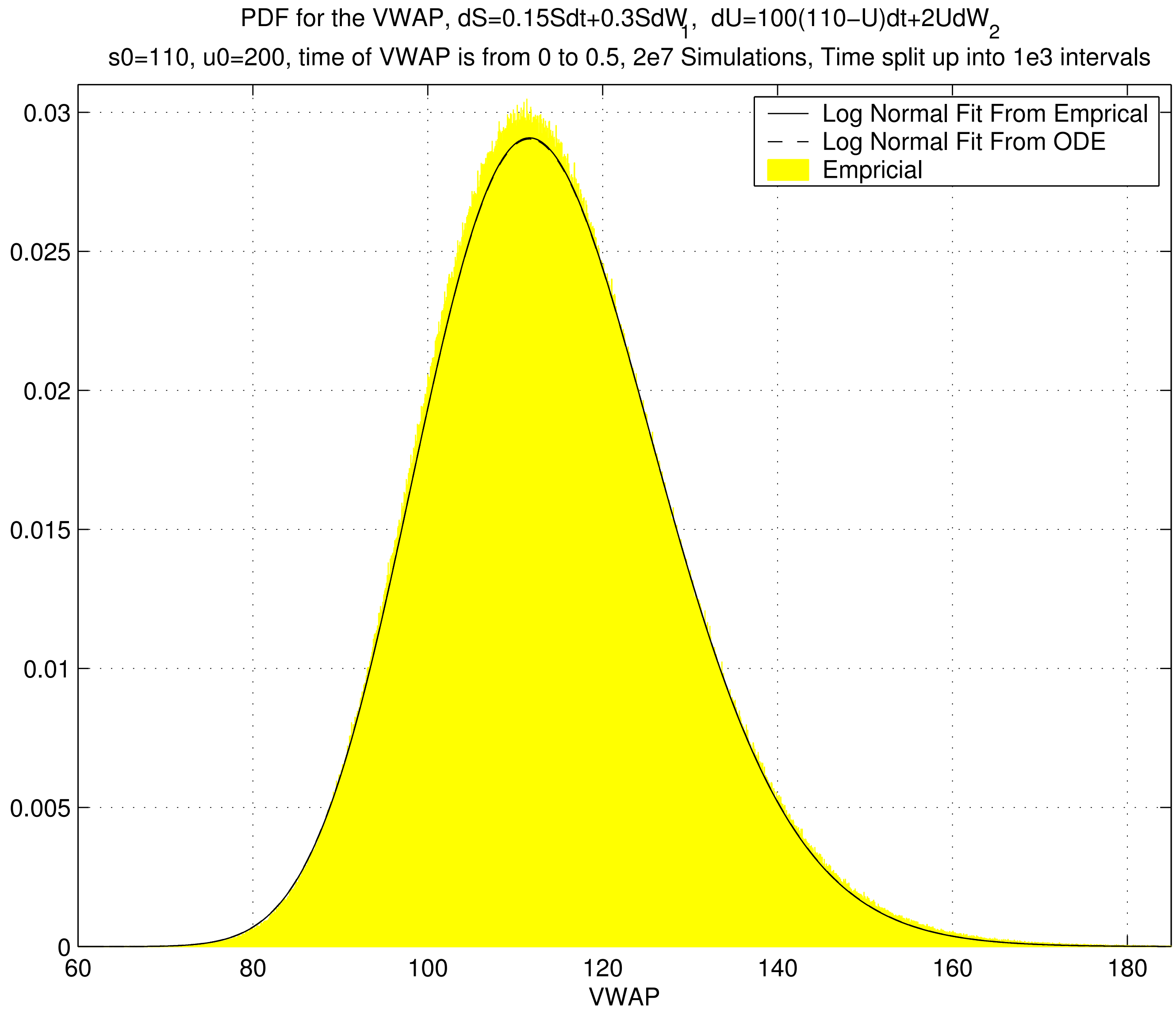
PDF at final time for different σs

$\sigma = 0.25$



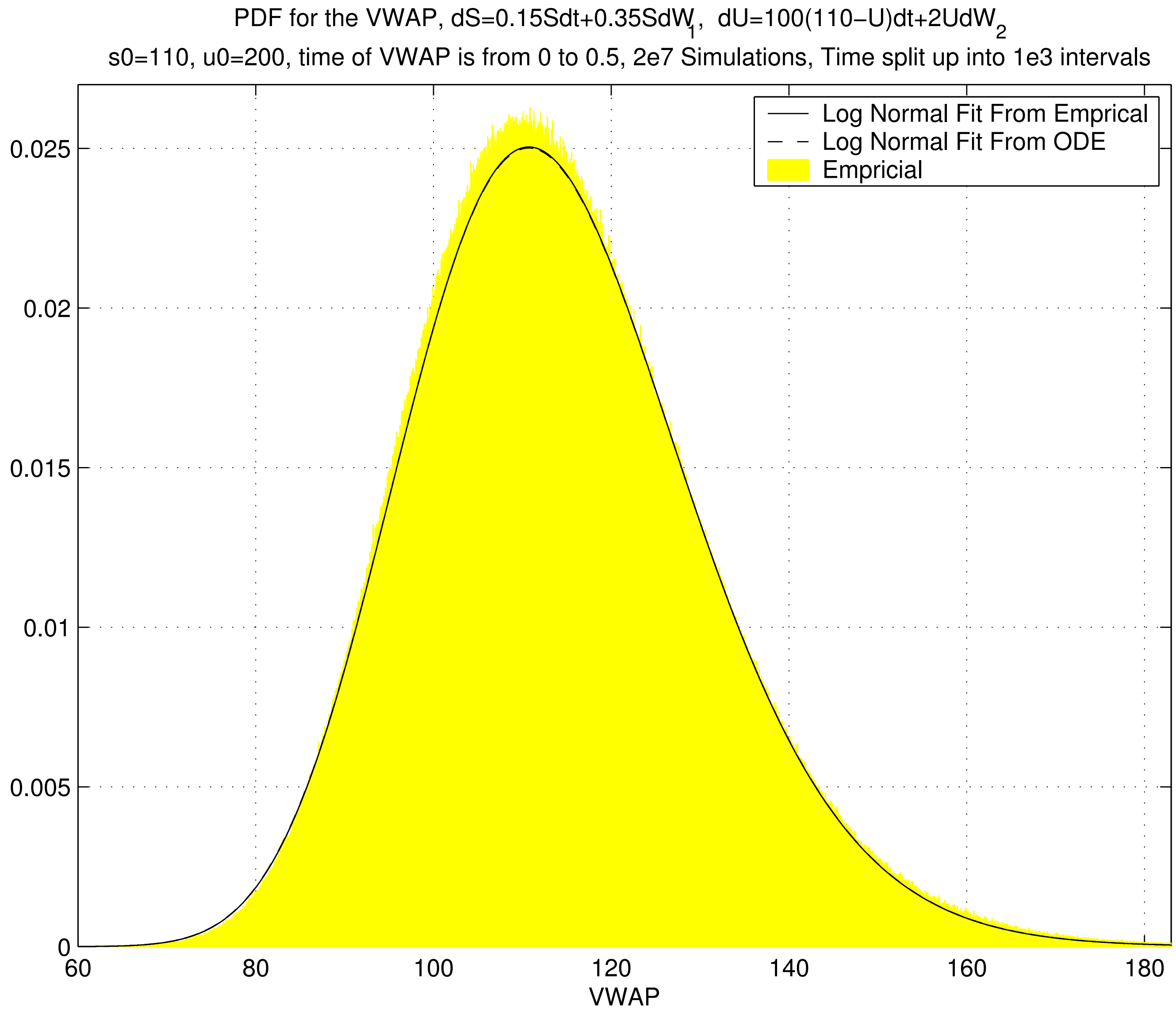
PDF at final time for different σs

$\sigma = 0.3$



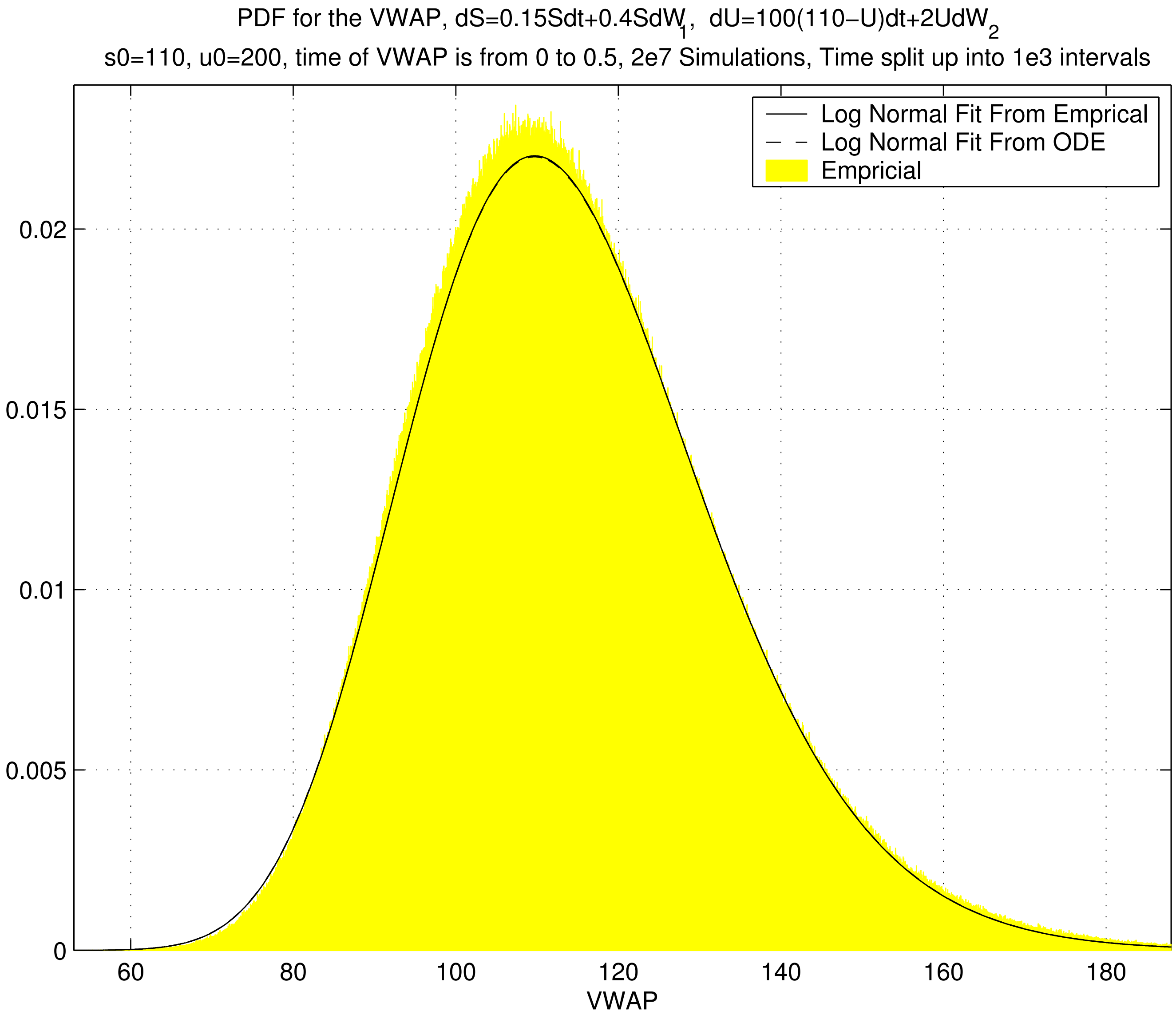
PDF at final time for different σs

$\sigma = 0.35$



PDF at final time for different σs

$\sigma = 0.4$



Comments on Result

- Approximation is better for lower σ, which is not unexpected - this method when applied to the normal Asian option which makes an approximation to $\int_0^t S_\nu d\nu$ is only good for small σ.

Comments on Result

- Approximation is better for lower σ, which is not unexpected - this method when applied to the normal Asian option which makes an approximation to $\int_0^t S_\nu d\nu$ is only good for small σ.
- Approximation is bad for small times.

Pricing the options

This is the easy part. We can easily obtain PDEs which describe the option price from standard techniques

- Fixed strike(BC $\max(\widetilde{S}_T - K, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\widetilde{\sigma}\widetilde{S})^2 \frac{\partial^2 V}{\partial \widetilde{S}^2} + (\widetilde{\mu} - \lambda(t, \widetilde{S})\widetilde{\sigma}\widetilde{S})\frac{\partial V}{\partial \widetilde{S}} - r\widetilde{V} = 0$$

Pricing the options

This is the easy part. We can easily obtain PDEs which describe the option price from standard techniques

- Fixed strike(BC $\max(\widetilde{S}_T - K, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\widetilde{\sigma}\widetilde{S})^2 \frac{\partial^2 V}{\partial \widetilde{S}^2} + (\widetilde{\mu} - \lambda(t, \widetilde{S})\widetilde{\sigma}\widetilde{S})\frac{\partial V}{\partial \widetilde{S}} - r\widetilde{V} = 0$$

- Floating strike(BC $\max(S_T - \widetilde{S}_T, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S\widetilde{\sigma}\widetilde{S}\frac{\partial^2 V}{\partial S \partial \widetilde{S}}$$
$$+\frac{1}{2}(\widetilde{\sigma}\widetilde{S})^2 \frac{\partial^2 V}{\partial \widetilde{S}^2} + rS\frac{\partial V}{\partial S} + (\widetilde{\mu} - \lambda(t, \widetilde{S})\widetilde{\sigma}\widetilde{S})\frac{\partial V}{\partial \widetilde{S}} - rV = 0$$

Pricing the options

This is the easy part. We can easily obtain PDEs which describe the option price from standard techniques

- Fixed strike(BC $\max(\widetilde{S}_T - K, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\widetilde{\sigma}\widetilde{S})^2 \frac{\partial^2 V}{\partial \widetilde{S}^2} + (\widetilde{\mu} - \lambda(t, \widetilde{S})\widetilde{\sigma}\widetilde{S})\frac{\partial V}{\partial \widetilde{S}} - r\widetilde{V} = 0$$

- Floating strike(BC $\max(S_T - \widetilde{S}_T, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S\widetilde{\sigma}\widetilde{S}\frac{\partial^2 V}{\partial S \partial \widetilde{S}}$$

$$+\frac{1}{2}(\widetilde{\sigma}\widetilde{S})^2 \frac{\partial^2 V}{\partial \widetilde{S}^2} + rS\frac{\partial V}{\partial S} + (\widetilde{\mu} - \lambda(t, \widetilde{S})\widetilde{\sigma}\widetilde{S})\frac{\partial V}{\partial \widetilde{S}} - rV = 0$$

- We could also price American options as well as exotic products without too much more work? Ie Barrier, Lookback.....

Solutions

- In the case that the market price of risk is constant, the fixed strike has the analytic solution

$$V_{fixed}(0) = e^{(r-\widetilde{\mu}+\widetilde{\sigma}\lambda)T} S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where

$$\begin{aligned} d_1 &= d_2 + \widetilde{\sigma}\sqrt{T} \qquad \text{and} \\ d_2 &= \frac{\log(S(0)/K) + (\widetilde{\mu} - \widetilde{\sigma}\lambda - \frac{1}{2}\widetilde{\sigma}^2)T}{\widetilde{\sigma}\sqrt{T}} \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative normal distribution function, Benth (2004).

Solutions

- In the case that the market price of risk is constant, the fixed strike has the analytic solution

$$V_{fixed}(0) = e^{(r-\widetilde{\mu}+\widetilde{\sigma}\lambda)T} S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

  where

$$d_1 = d_2 + \widetilde{\sigma}\sqrt{T} \quad \text{and}$$

$$d_2 = \frac{\log(S(0)/K) + (\widetilde{\mu} - \widetilde{\sigma}\lambda - \frac{1}{2}\widetilde{\sigma}^2)T}{\widetilde{\sigma}\sqrt{T}}$$

  where $\Phi(\cdot)$ is the cumulative normal distribution function, Benth (2004).
- Otherwise we must use a numeric technique such as finite differences, Monte Carlo, FFT, etc

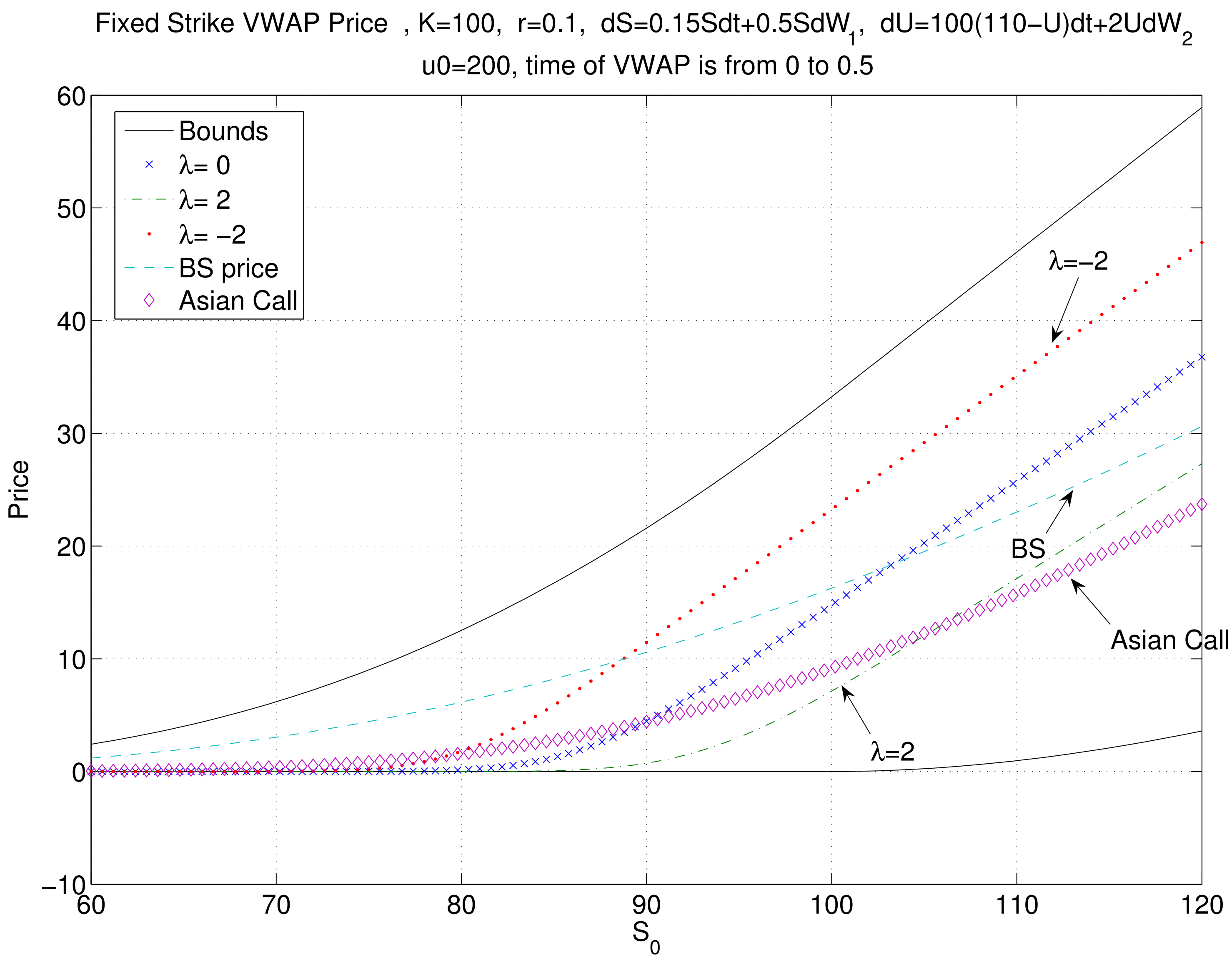
An Example

Method demonstrated on the system

$$\begin{aligned} dS &= 0.1 S dt + 0.5 S dW_1 \\ dU &= 100(110 - U)dt + 2U dW_2 \end{aligned}$$

$U_0 = 200, K = 100, r = 10\%$ and time from 0 to 0.5

An Example


Fixed Strike VWAP Price , K=100, r=0.1, dS=0.15Sdt+0.5SdW1, dU=100(110−U)dt+2UdW2
u0=200, time of VWAP is from 0 to 0.5
Bounds
λ= 0
λ= 2
λ= −2
BS price
Asian Call
λ=−2
BS
Asian Call
λ=2
Price
S0
60
50
40
30
20
10
0
−10
60
70
80
90
100
110
120

Share Purchase Plans

$V_T = \left(S_{T_2} - D \frac{\int_{T_0}^{T_1} S_v U_v dv}{\int_{T_0}^{T_1} U_v dv} \right)^+$,$T_1 - T_0$ typically 3-10 days, $T_2 - T_1$ typically 10-30 days, D a discount factor usually 70%-90%

- We can value this using the method just described.
- Raises capital easily, no prospectus
- Aimed at small investors, max $5000
- IAG, Suncorp, AMP
- We can immediately now say how much it is worth to participate in a share purchase plan(Actually what the companies are giving away for free!!)
- I am not suggesting you do this, but since they have given you this payoff.....

Summary

- Have a way to price the option

Summary

- Have a way to price the option
- Can price exotics

Summary

- Have a way to price the option
- Can price exotics
- FAST

Summary

- Have a way to price the option
- Can price exotics
- FAST
- Can use as a control variate in Monte Carlo

Summary

- Have a way to price the option
- Can price exotics
- FAST
- Can use as a control variate in Monte Carlo
- Can tell you how much companies are giving to you when they offer shares at a VWAP to you in a share purchase plan.

Future Work

- Find a region where this approximation is good is some sense.
- Take more moments?
- Find a practical hedge
- More Monte Carlo

Thanks

- MASCOS for financial assistance.
- Dr Chandler for comments and suggestions.
- Thanks for Josh for helping with the tex.

References

Benth, F. E. (2004), *Option Theory with Stochastic Analysis An Introduction to Mathematical Finance*, Springer.

Mood, A. M., Graybill, F. A. & Boes, D. C. (1974), *Introduction To The Theory Of Statistics, Third Edition*, McGraw-Hill.

Shreve, S. (2004), *Stochastic Calculus for Finance II Continuous-Time Models*, Springer.