A Moment Matching Approach To The Valuation Of A Volume Weighted Average Price Option

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Plan of talk

- What an option is
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- What an option is
- What an Asian Option is and more importantly what a Volume Weighted Average Price is Option
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• What an option is
• What an Asian Option is and more importantly what a Volume Weighted Average Price is Option
• Future work
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- What an Asian Option is and more importantly what a Volume Weighted Average Price is Option
- Future work
- Questions
Options, the basics

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- Managed funds/superannuation funds use puts to protect against stock declines (I hope!).
- There are many different types of options, European, American, Asian, Bermudan, Australian, Lookback, Barrier, Spread, Options on Options .....and the list continues to grow all the time as people want new products to manage their risk.
European Call, \( V_T = \max(S_T - K, 0) \)

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- Closed form solution published by Black and Scholes in 1973

\[
C = S_0 N(d_1) - Ke^{-rT} N(d_2)
\]

with

\[
d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}, \quad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}}
\]

where \( K \) is the strike price, \( S_0 \) is the price of the share at time 0, \( \sigma \) is the share’s volatility, \( T \) the time to expiry and \( N(\cdot) \) is the cumulative probability function.
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- Assumes stock evolves as Geometric Brownian motion,
  \[ dS = \mu Sdt + \sigma SdW \] (Log normal)
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- Assumes stock evolves as Geometric Brownian motion,

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dS = \mu Sdt + \sigma SdW \quad \text{(Log normal)}
\]

- The solution, remarkably, does not contain drift of the stock
Running Average - $\frac{1}{t} \int_{0}^{t} S_{\nu} d\nu$
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Asian Option, \( V_T = \left( \frac{1}{T} \int_0^T S_v dv - k \right)^+ \)

- Similar to my option is most like this one
- Cheaper than vanilla call or put.
- At the money it is about half the cost of a European. In fact volatility is about \( \frac{\sigma}{\sqrt{3}} \). Price is simply
  \( r^* = r - \frac{1}{2} \left( r - \frac{\sigma^2}{6} \right) \) and \( \sigma^* = \frac{\sigma}{\sqrt{3}} \) into the BS for the price of a Geometric Asian Option
- Very popular in currency and commodity markets
A Volume Weighted Average Price

- Assigns more weight to periods of heavy trading, than light trading

\[
\text{VWAP} (T) = \frac{\int_0^T S_u U_v du}{\int_0^T U_v du}
\]

Where \( S_t \) is the price of the stock at time \( t \) and \( U_t \) is the rate of trades of the stock at time \( t \).
A Volume Weighted Average Price

- Assigns more weight to periods of heavy trading, than light trading
- Example: Suppose a stock trades at $10 today and there are 100 trades, tomorrow it trades at $100 and there is 1 trade.
  The *volume weighted average price* is \[
  \frac{10 \times 100 + 100 \times 1}{100 + 1} = \$10.89
  \]
  while a *arithmetic weighted average price* is \[
  \frac{10 + 100}{2} = \$55.00.
  \]
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  \]

- We can write the VWAP at time T as
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  VWAP(T) = \frac{\int_0^T S_u U_v dv}{\int_0^T U_v dv}
  \]
  Where \( S_t \) is the price of the stock at time \( t \) and \( U_t \) is the rate of trades of the stock at time \( t \).
Example Of Real Stocks

![Graph showing stock price, arithmetic average, and VWAP for DELL over time.](image-url)
Example Of Real Stocks
Example Of Real Stocks

![Graph of stock price, arithmetic average, and VWAP for NAB stock over 250 days. The graph shows fluctuations in the stock price with smoother lines representing the arithmetic average and VWAP.](image-url)
Example Of Real Stocks

![Graph showing TELSTRA stock price, arithmetic average, and VWAP over time.](Image)
My Problem

To price and hedge

- \( V_T = \max \left( \frac{\int_0^T S_u U_v dv}{\int_0^T U_v dv} - K, 0 \right) \) (fixed strike) and
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with \( S \) and \( U \) being defined by the stochastic differential equations

- \( dS = rS \, dt + \sigma S \, dW_1 \) (stock)
- \( dU = \alpha(\mu - U) \, dt + \beta U \, dW_2 \) (trades per unit time), (Use several mean reverting models, add jumps later)

For the moment assume correlation between \( W_1 \) and \( W_2 \) is zero, relax this assumption later once we know the problem better.
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To price and hedge

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- Probabilistic approach requires us to evaluate an expectation for which we do not know the PDF
- Can solve by Monte Carlo, but slow.
An approximation

- Inspired by early work, on Asian options, assume that the volume weighted average price

\[
\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv}
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has a log normal distribution at the final time.
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- But how do we get these???
Approximations

- We write the VWAP as \[ \frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv} = \frac{Y}{Z} \]
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, Mood et al. (1974).
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- That’s all great, but how do we get all these expectations?
- From the Ito-Doeblin formula, and lots of patience
The Doeblin in the Ito-Doeblin formula
The modern theory of stochastic calculus developed from the work of Itô [92]. Not only did Itô define the integral with respect to Brownian motion, but he also developed the change-of-variable formula commonly called Itô’s rule or Itô’s formula. As demonstrated in this chapter, this formula is at the heart of a wide range of useful calculations. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French National Academy of Sciences received a document from W. Doeblin, a French soldier on the German front.
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Shreve (2004)
Now back to the problem

- We need many expectations.
Now back to the problem

- We need many expectations.
- We can find all these expectations from properties of the Itô integral.
Demonstrate the method on $\mathbb{E}(S_t)$

We have $dS = \mu S dt + \sigma S dW$
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This is really shorthand for

$$S_t - S_0 = \int_0^t \mu S \nu d\nu + \int_0^t \sigma S \nu dW \nu$$
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Taking the expectation of this we have

$$\mathbb{E}(S_t - S_0) = \mathbb{E}(\int_0^t \mu S \nu d\nu + \int_0^t \sigma S \nu dW_\nu)$$
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Now the expectation of an Ito integral is 0, so we have

$$\mathbb{E}(S_t - S_0) = \mathbb{E}(\int_0^t \mu S \nu d\nu)$$
Demonstrate the method on $\mathbb{E}(S_t)$

Then moving the expectation inside the integral

$$\mathbb{E}(S_t - S_0) = \int_0^t \mu \mathbb{E}(S_\nu) d\nu$$
Demonstrate the method on $\mathbb{E}(S_t)$

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$$\mathbb{E}(S_t - S_0) = \int_0^t \mu \mathbb{E}(S_\nu) d\nu$$

finally differentiating we have

$$\frac{d\mathbb{E}(S_t)}{dt} = \mu \mathbb{E}(S_t)$$
Demonstrate the method on $\mathbb{E}(S_t)$

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finally differentiating we have

$$\frac{d\mathbb{E}(S_t)}{dt} = \mu \mathbb{E}(S_t)$$

which is simple to solve given the initial condition.
Obtaining the expectations

- We can do this for all the expectations which we require, it is long and tedious, but doable.
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- Final system has 19 equations which are easy to solve in Matlab or Maple
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- We can do this for all the expectations which we require, it is long and tedious, but doable.
- Final system has 19 equations which are easy to solve in Matlab or Maple
- Can now use the approximations

\[
E \left( \frac{Y}{Z} \right) \approx \frac{E(Y)}{E(Z)} - \frac{Cov(Y, Z)}{(E(Z))^2} + \frac{E(Y)}{(E(Z))^3} \cdot Var(Z)
\]

\[
Var \left( \frac{Y}{Z} \right) \approx \left( \frac{E(Y)}{E(Z)} \right)^2 \left( \frac{Var(Y)}{(E(Y))^2} + \frac{Var(Z)}{(E(Z))^2} - 2 \frac{Cov(Y, Z)}{E(Y)E(Z)} \right)
\]

to find the expectation and variance at any time \( T \) for

\[
\frac{\int_0^T S_v U_v dv}{\int_0^T U_v dv}
\]
Lots of +ve eigenvalues, but the one to look out for is the combination of $\beta^2 - 2\alpha$ which appears in many places.
Now use the log normal approximation

Now we know that the expectation and variance of our underlying $d\tilde{S} = \tilde{\mu}Sdt + \tilde{\sigma}SdW$ are

- $\mathbb{E}(\tilde{S}(t)) = S_0e^{\tilde{\mu}t}$
- $\text{Var}(\tilde{S}(t)) = S_0^2e^{2\tilde{\mu}t}(e^{\tilde{\sigma}^2t} - 1)$
Now use the log normal approximation

Now we know that the expectation and variance of our underlying \( d\tilde{S} = \tilde{\mu}Sdt + \tilde{\sigma}SdW \) are

- \[
E(\tilde{S}(t)) = S_0e^{\tilde{\mu}t}
\]
- \[
Var(\tilde{S}(t)) = S_0^2e^{2\tilde{\mu}t}(e^{\tilde{\sigma}^2t} - 1)
\]

- We can rewrite these as

\[
\tilde{\mu} = \frac{1}{t} \log \frac{E(\tilde{S}(t))}{S_0}
\]
\[
\tilde{\sigma} = \sqrt{\frac{1}{t} \log \frac{Var(\tilde{S}(t)) + (E(\tilde{S}(t)))^2}{(E(\tilde{S}(t)))^2}}
\]
How well does it work?

From Simulation and ODEs, $dS=0.2S\,dt+0.5S\,dW$

1, $dU=100(100−U)\,dt+2U\,dW$

$s_0=110$, $u_0=10$, time of VWAP is from 0 to 0.5, $1e7$ Simulations, Time split up into $1e3$ intervals

Solid - results from simulations, Dashed - results from ODEs
How well does it work?

From Simulation and ODEs, \( dS = 0.2S dt + 0.5S dW_1 \), \( dU = 110(100 - U) dt + 2U dW_2 \)

\( s_0 = 110, u_0 = 10, \) time of VWAP is from 0 to 0.5, 1e7 Simulations, Time split up into 1e3 intervals

Solid - results from simulations, Dashed - results from ODEs
PDF at final time for different $\sigma$s

$\sigma = 0.05$

PDF for the VWAP, $dS = 0.15Sdt + 0.05SdW_1$, $dU = 100(110-U)dt + 2UdW_2$

$s_0 = 110$, $u_0 = 200$, time of VWAP is from 0 to 0.5, 2e7 Simulations, Time split up into 1e3 intervals

- Log Normal Fit From Empirical
- Log Normal Fit From ODE
- Empirical
PDF at final time for different $\sigma$ s

$\sigma = 0.1$

PDF for the VWAP, $dS=0.15Sdt+0.1SdW_t$, $dU=100(110-U)dt+2UdW_t$
$s_0=110$, $u_0=200$, time of VWAP is from 0 to 0.5, $2e7$ Simulations, Time split up into $1e3$ intervals
PDF at final time for different $\sigma$s

$\sigma = 0.15$

PDF for the VWAP, $dS=0.15Sdt+0.15SdW_1$, $dU=100(110-U)dt+2 UdW_2$

$s_0=110$, $u_0=200$, time of VWAP is from 0 to 0.5, 2e7 Simulations, Time split up into 1e3 intervals
PDF at final time for different $\sigma$s

$\sigma = 0.2$

PDF for the VWAP, $dS = 0.15Sdt + 0.2SdW_1$, $dU = 100(110-U)dt + 2UdW_2$

$s_0 = 110$, $u_0 = 200$, time of VWAP is from 0 to 0.5, $2e7$ Simulations, Time split up into $1e3$ intervals

Log Normal Fit From Empirical
- Log Normal Fit From ODE
- Empirical
PDF at final time for different $\sigma$s

$\sigma = 0.25$

PDF for the VWAP, $dS=0.15Sdt+0.25SdW_1$, $dU=100(110-U)dt+2UdW_2$

$s_0=110$, $u_0=200$, time of VWAP is from 0 to 0.5, 2e7 Simulations, Time split up into 1e3 intervals

Log Normal Fit From Empirical
Log Normal Fit From ODE
Empirical
PDF at final time for different $\sigma$s

$\sigma = 0.3$

PDF for the VWAP, $dS=0.15Sdt+0.3SdW$, $dU=100(110-U)dt+2UdW$

$s_0=110$, $u_0=200$, time of VWAP is from 0 to 0.5, 2e7 Simulations, Time split up into 1e3 intervals
PDF at final time for different $\sigma$s

$\sigma = 0.35$

PDF for the VWAP, $dS=0.15Sdt+0.35SdW$, $dU=100(110−U)dt+2UdW$

$s_0=110$, $u_0=200$, time of VWAP is from 0 to 0.5, 2e7 Simulations, Time split up into 1e3 intervals
PDF at final time for different $\sigma$s

$\sigma = 0.4$

PDF for the VWAP, $dS = 0.15Sdt + 0.4SdW$, $dU = 100(110 - U)dt + 2UdW$

$s_0 = 110$, $u_0 = 200$, time of VWAP is from 0 to 0.5, 2e7 Simulations, Time split up into 1e3 intervals
Comments on Result

- Approximation is better for lower $\sigma$, which is not unexpected - this method when applied to the normal Asian option which makes an approximation to $\int_0^t S_\nu d\nu$ is only good for small $\sigma$. 
Comments on Result

• Approximation is better for lower $\sigma$, which is not unexpected - this method when applied to the normal Asian option which makes an approximation to $\int_0^t S_\nu d\nu$ is only good for small $\sigma$.

• Approximation is bad for small times.
Pricing the options

This is the easy part. We can easily obtain PDEs which describe the option price from standard techniques

- Fixed strike (BC $\max(\tilde{S}_T - K, 0)$)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} (\tilde{\sigma} \tilde{S})^2 \frac{\partial^2 V}{\partial \tilde{S}^2} + (\tilde{\mu} - \lambda(t, \tilde{S})\tilde{\sigma} \tilde{S}) \frac{\partial V}{\partial \tilde{S}} - r\tilde{V} = 0
\]

We could also price American options as well as exotic products such as Barrier, Lookback....
Pricing the options

This is the easy part. We can easily obtain PDEs which describe the option price from standard techniques

- Fixed strike (BC $\max(\tilde{S}_T - K, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma \tilde{S})^2 \frac{\partial^2 V}{\partial \tilde{S}^2} + (\bar{\mu} - \lambda(t, \tilde{S})\sigma \tilde{S}) \frac{\partial V}{\partial \tilde{S}} - r\tilde{V} = 0$$

- Floating strike (BC $\max(S_T - \tilde{S}_T, 0)$)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma S \tilde{S} \frac{\partial^2 V}{\partial S \partial \tilde{S}}$$

$$+ \frac{1}{2}(\bar{\sigma} \tilde{S})^2 \frac{\partial^2 V}{\partial \tilde{S}^2} + r S \frac{\partial V}{\partial S} + (\bar{\mu} - \lambda(t, \tilde{S})\bar{\sigma} \tilde{S}) \frac{\partial V}{\partial \tilde{S}} - rV = 0$$
Pricing the options

This is the easy part. We can easily obtain PDEs which describe the option price from standard techniques

- **Fixed strike (BC \( \max(\tilde{S}_T - K, 0) \))**

\[
\frac{\partial V}{\partial t} + \frac{1}{2} (\tilde{\sigma}\tilde{S})^2 \frac{\partial^2 V}{\partial \tilde{S}^2} + (\tilde{\mu} - \lambda(t, \tilde{S})\tilde{S}) \frac{\partial V}{\partial \tilde{S}} - r\tilde{V} = 0
\]

- **Floating strike (BC \( \max(S_T - \tilde{S}_T, 0) \))**

\[
\frac{\partial V}{\partial t} + \frac{1}{2} (\sigma S)^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma S \tilde{S} \frac{\partial^2 V}{\partial S \partial \tilde{S}} + \frac{1}{2} (\tilde{\sigma}\tilde{S})^2 \frac{\partial^2 V}{\partial \tilde{S}^2} + r S \frac{\partial V}{\partial S} + (\tilde{\mu} - \lambda(t, \tilde{S})\tilde{S}) \frac{\partial V}{\partial \tilde{S}} - rV = 0
\]

- We could also price American options as well as exotic products without too much more work? I.e Barrier, Lookback.....
In the case that the market price of risk is constant, the
fixed strike has the analytic solution

\[ V_{fixed}(0) = e^{(r-\tilde{\mu}+\tilde{\sigma}\lambda)T} S(0) \Phi(d_1) - Ke^{-rT} \Phi(d_2) \]

where

\[
\begin{align*}
    d_1 &= d_2 + \tilde{\sigma} \sqrt{T} \\
    d_2 &= \frac{\log(S(0)/K) + (\tilde{\mu} - \tilde{\sigma}\lambda - \frac{1}{2}\tilde{\sigma}^2)T}{\tilde{\sigma} \sqrt{T}}
\end{align*}
\]

where \( \Phi(\cdot) \) is the cumulative normal distribution function, Benth (2004).
Solutions

- In the case that the market price of risk is constant, the fixed strike has the analytic solution

\[
V_{\text{fixed}}(0) = e^{(r-\tilde{\mu}+\tilde{\sigma}\lambda)T}S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)
\]

where

\[
d_1 = d_2 + \tilde{\sigma}\sqrt{T} \quad \text{and} \quad \\
d_2 = \frac{\log(S(0)/K) + (\tilde{\mu} - \tilde{\sigma}\lambda - \frac{1}{2}\tilde{\sigma}^2)T}{\tilde{\sigma}\sqrt{T}}
\]

where \(\Phi(\cdot)\) is the cumulative normal distribution function, Benth (2004).

- Otherwise we must use a numeric technique such as finite differences, Monte Carlo, FFT, etc
An Example

Method demonstrated on the system

\[ dS = 0.1S\, dt + 0.5S\, dW_1 \]
\[ dU = 100(110 - U)\, dt + 2U\, dW_2 \]

\( U_0 = 200, \, K = 100, \, r = 10\% \) and time from 0 to 0.5
An Example

Fixed Strike VWAP Price, $K=100$, $r=0.1$, $dS=0.15Sdt+0.5SdW_t$, $dU=100(110−U)dt+2UdW_2$

$u_0=200$, time of VWAP is from 0 to 0.5
Share Purchase Plans

\[ V_T = \left( S_{T_2} - D \int_{T_0}^{T_1} S_v U_v dv \right)^+, T_1 - T_0 \text{ typically 3-10 days, } T_2 - T_1 \text{ typically 10-30 days, } D \text{ a discount factor usually 70%-90\%} \]

- We can value this using the method just described.
- Raises capital easily, no prospectus
- Aimed at small investors, max $5000
- IAG, Suncorp, AMP
- We can immediately now say how much it is worth to participate in a share purchase plan (Actually what the companies are giving away for free!!)
- I am not suggesting you do this, but since they have given you this payoff.....
Summary

- Have a way to price the option
Summary

- Have a way to price the option
- Can price exotics
Summary

- Have a way to price the option
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- FAST
Summary

- Have a way to price the option
- Can price exotics
- FAST
- Can use as a control variate in Monte Carlo
Summary

- Have a way to price the option
- Can price exotics
- FAST
- Can use as a control variate in Monte Carlo
- Can tell you how much companies are giving to you when they offer shares at a VWAP to you in a share purchase plan.
Future Work

- Find a region where this approximation is good in some sense.
- Take more moments?
- Find a practical hedge
- More Monte Carlo
Thanks

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References

