Extinction in metapopulations with environmental stochasticity driven by catastrophes

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look left

$\Longrightarrow$

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- $\mathbf{X} = [X \;\; Y]^T$. The state of the metapopulation:
  - X is the number of suitable patches;
  - Y is the number of occupied patches.
- r, c, e, γ, p. Rates and other parameters.

Demographic Events

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- Each occupied patch produces migrants at rate c, which may colonise empty, suitable patches

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- Each local population goes extinct at rate e

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on $S = \{(x, y) \mid x, y \in \mathbb{N}, 0 \le y \le x \le N\}$.

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$$(x, y) \longrightarrow (x - (i + j), y - j) \qquad \text{at rate}$$

$$\gamma \binom{x - y}{i} \binom{y}{j} p^{i+j} (1 - p)^{x-i-j}.$$

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Finite State-space Processes

When N is finite, we can hope to evaluate measures of interest directly.

- Extinction times (a.k.a. first passage or 'hitting' times) are almost surely finite!
- Quasi-stationary distributions exist!
  (limiting distribution conditional on non-extinction)

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  (limiting distribution conditional on non-extinction)

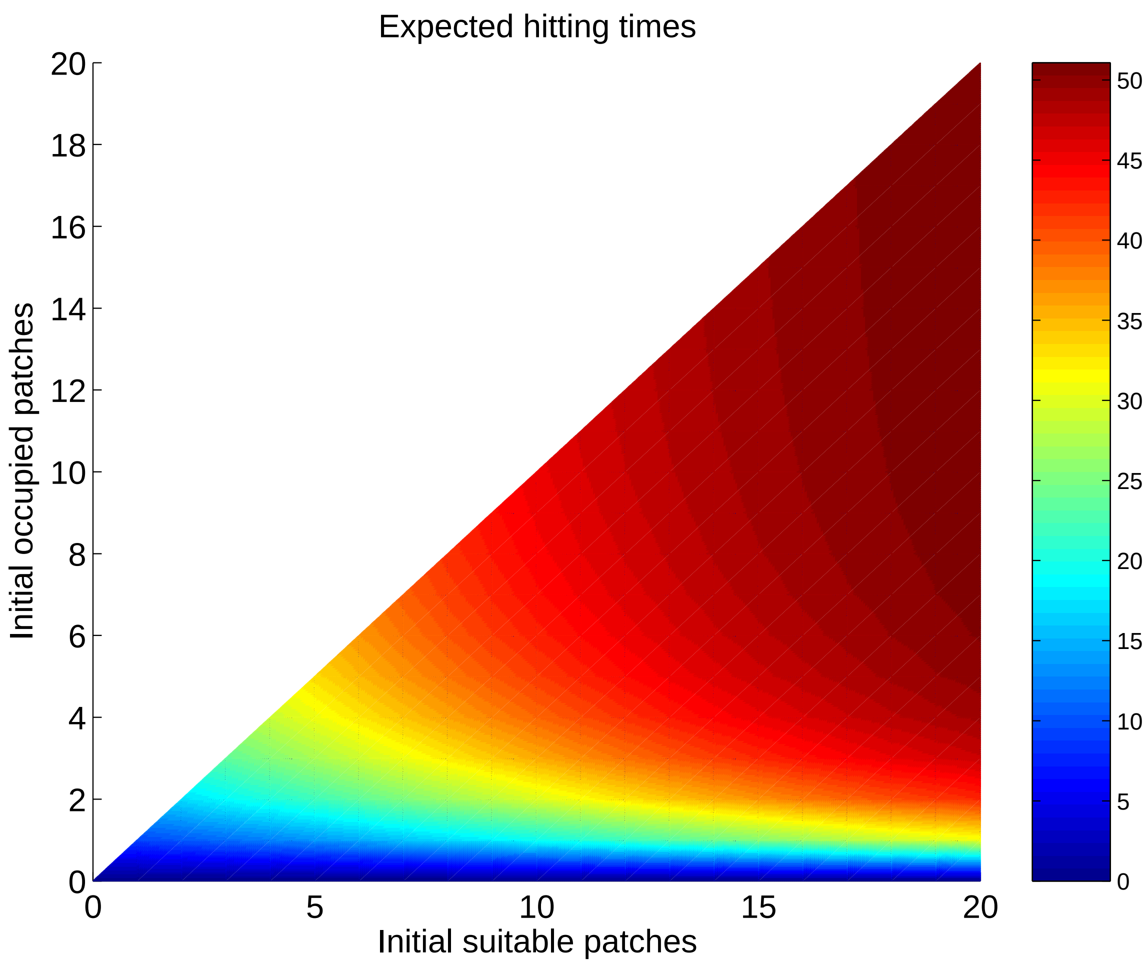
If these are easy to calculate (or approximate), we can use them to analyse the system.

Extinction Times

$$Q_C \boldsymbol{\tau} = -1$$

(Generally, take the minimal, non-negative solution.)

Extinction Times

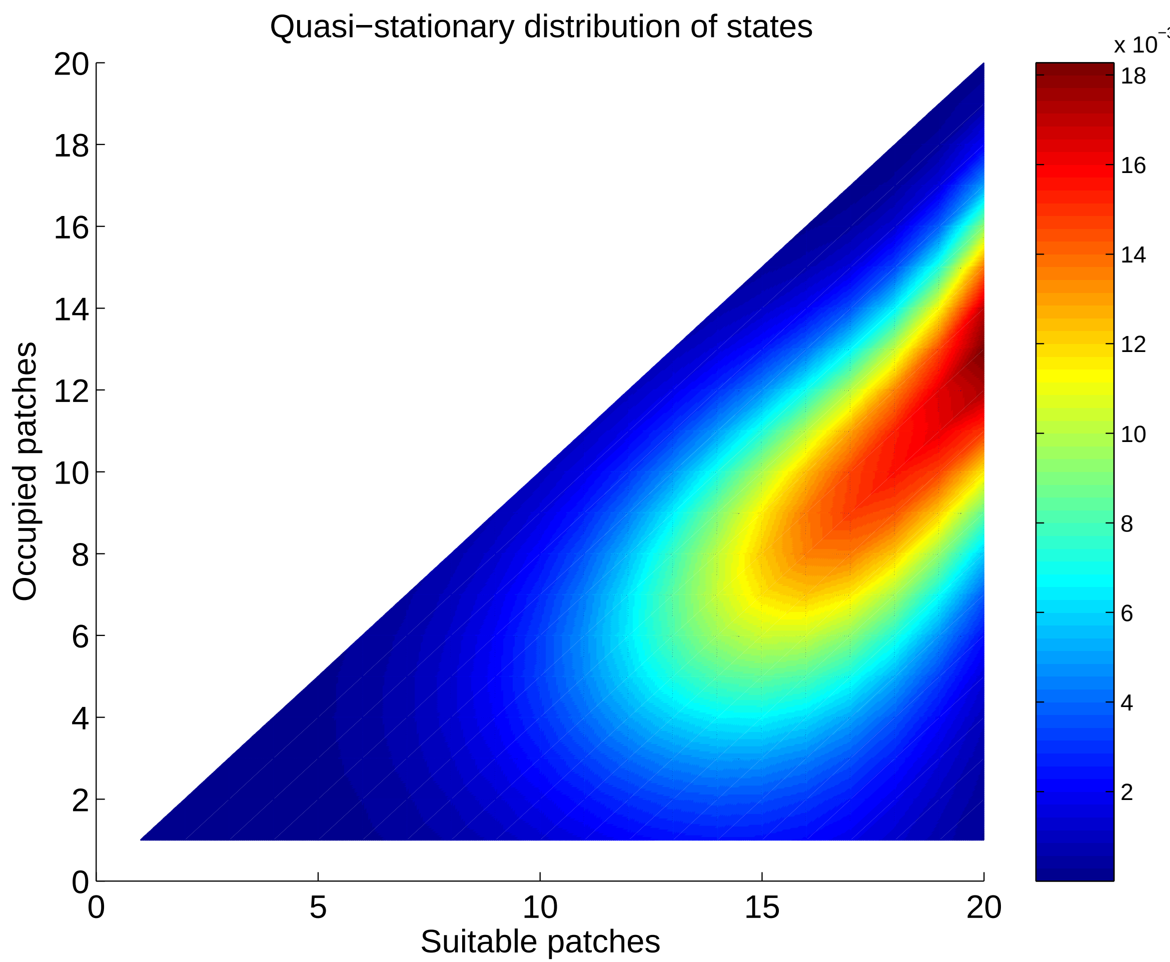


Quasi-stationary distributions

$$Q_C \mathbf{m} = -\lambda \mathbf{m}$$

(λ is the eigenvalue with max. real part.)

Quasi-stationary distributions



Difficulties as $N \uparrow$

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- To make progress, we need good approximations: *e.g.* stochastic differential equations for the limit as $N \to \infty$?

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with initial conditions $\mathbf{U}(s,0) = \lim_{N\to\infty} X(s,0)/N$.
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$$\mathbf{a}(\mathbf{U}) = \begin{bmatrix} \partial u/\partial t \\ \partial v/\partial t \end{bmatrix} = \begin{bmatrix} r(1-u) \\ cv(u-v) - ev \end{bmatrix},$$

with initial conditions $\mathbf{U}(s,0) = \lim_{N\to\infty} X(s,0)/N$. (Kurtz, 1970)

Catastrophes in the Limit

Treating catastrophes as a separate component...

- The arrival rate of catastrophes is unaffected by scaling.
- As $N \to \infty$, if T_1 is a catastrophe time,

$$\frac{\mathbf{X}(s, T_1)}{N} \xrightarrow{P} (1-p)\mathbf{U}(s, T_1-).$$

A Stochastic Integral Equation

The limiting, scaled process:

$$d\mathbf{U}(s,t) = \mathbf{a}(\mathbf{U}(s,t))dt + \int_{\mathbf{M}} \mathbf{c}(\mathbf{U}(s,t),\mathbf{m})\,\mathcal{P}[d\mathbf{m}, dt; \gamma]$$

- $\mathbf{c}(\mathbf{U}, d\mathbf{m})$ describes effect of catastrophes
- Poisson random measure $\mathcal{P}$ describes arrival of catastrophes and their magnitudes, $\mathbf{m}$.
- Generalised Itô fomula gives first passage times. (Gihman & Skorohod, 1972)

First Passage Times

First passage times, $\tau_G(\mathbf{U}_0)$, into a closed set $S \backslash G$ (i.e. out of G), starting from $\mathbf{U}_0$, are a twice continuously differentiable solution, $g(\mathbf{U})$, to

$$(Lg)(\mathbf{U}) = -1, \mathbf{U} \in G$$
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- In the present case, $(Lh)(\mathbf{U})$ is given by

$$(Lh)(\mathbf{U}) = \nabla h(\mathbf{U}) \cdot \mathbf{a}(\mathbf{U}) - \gamma h(\mathbf{U}) + \gamma h\big((1-p)\mathbf{U}\big).$$

Solving First Passage Times

Slightly different conditions:

- $g(\mathbf{U})$ should be *continuous* along all trajectories $\mathbf{U}(s,t)$, and piecewise smooth along other smooth paths.
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Solve in 'steps': G_n is the region from which *at least* n catastrophes are needed to leave G.

Solving First Passage Times

The solution has the form

$$e^{-\gamma t} g\big(\mathbf{U}(s,t)\big) = -\int_0^t \gamma e^{-\gamma r} g\big((1-p)\mathbf{U}(s,r)\big)\, dr \\ - \gamma^{-1}\left[1 - e^{-\gamma t}\right] + C_1(s),$$

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but we want a bounded solution, so set

$$C_1(s) = \int_0^\infty \gamma e^{-\gamma r} g\big((1-p)\mathbf{U}(s,r)\big)\, dr + \gamma^{-1}.$$

Solving First Passage Times

Hence (along trajectories that remain within G)

$$g\big(\mathbf{U}(s,t)\big) = \frac{1}{\gamma} + e^{\gamma t}\int_t^\infty \gamma e^{-\gamma r} g\big((1-p)\mathbf{U}(s,r)\big)dr.$$

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Clearly, $C_1(s) = g(s,0)$. We can also confirm:

- if $G = G_1$, $g(\mathbf{U}) = \gamma^{-1}$ for all $\mathbf{U}$ on trajectories remaining in G;
- if $\mathbf{U}_\infty = \lim_{t\to\infty} \mathbf{U}(s,t)$ is in G, then $g(s,\infty-) = \gamma^{-1} + g\big((1-p)\mathbf{U}_\infty\big)$.

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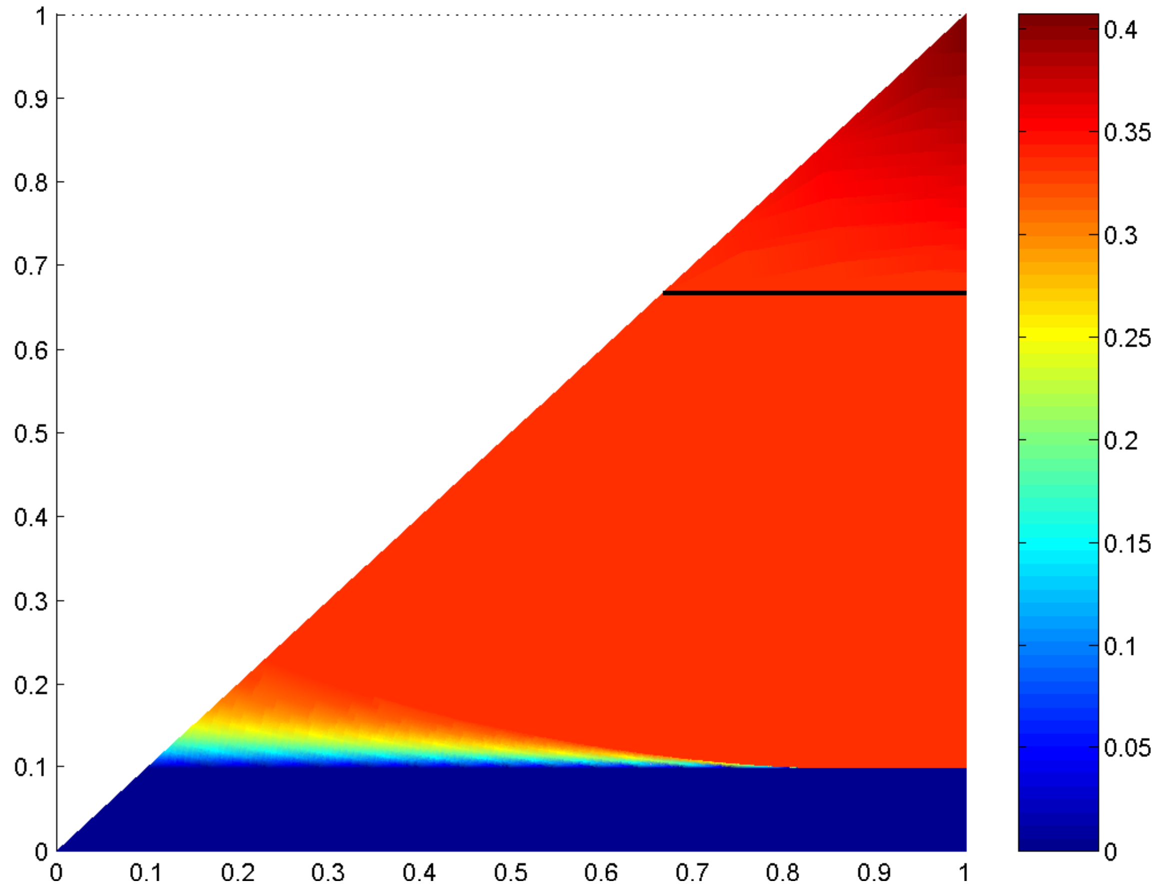
- $g\big(\mathbf{U}(s,t)\big) = \gamma^{-1}$, for all trajectories not leaving G_1 in finite time,
- solutions for trajectories heading out of G using the deterministic hitting time and a truncated exponential law, and
- a system of DEs $[\partial u/\partial t, \partial v/\partial t, \partial g/\partial t]$ gives first passage times for trajectories starting on higher steps.

Solutions: A Special Case



Solutions: General Case

The general case is a little more difficult.

- Recall that

$$g\big(\mathbf{U}(s,t)\big) = \frac{1}{\gamma} + e^{\gamma t}\int_t^{\infty} \gamma e^{-\gamma r} g\big((1-p)\mathbf{U}(s,r)\big)\,dr.$$

Solutions: General Case

The general case is a little more difficult.

- Define a mapping $K : H \longrightarrow H$,

$$K\big(f(s,t)\big) := \frac{1}{\gamma} + e^{\gamma t} \int_t^\infty \gamma e^{-\gamma r} f\left(\left(1-p\right)\mathbf{U}(s,r)\right) dr,$$

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with H being the set of bounded functions $f : G \longrightarrow \mathbb{R}_+$ under the condition

$$f\big(\mathbf{U}(s,t)\big) \geq \frac{1}{\gamma} + e^{\gamma t} \int_t^\infty \gamma e^{-\gamma r} f\left((1-p)\mathbf{U}(s,r)\right) dr.$$

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- $f \geq K(f)$, $f \in H$, so we might hope that the iterative application of K would lead to a fixed point, but...
- $H \neq \emptyset$ is equivalent to the existence of a solution, h, to

$$(Lh)(\mathbf{U}) \leq -1, \mathbf{U} \in G$$
$$h(\mathbf{U}) \geq 0, \mathbf{U} \notin G,$$

$\equiv$ to a condition from Gihman & Skorohod for the existence of a solution $\tau_G \leq h$.

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(iii) then $h(u)$ satisfies the inequality condition for all v such that $(u,v) \in S$.

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