Quasi-Birth-and-Death Processes with an Infinite Phase Space

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- State transitions are restricted to states in the same level or in the two adjacent levels (hence the name QBD).
- Transition intensities are assumed to be level-independent.

- A QBD process has a generator Q with block tri-diagonal structure

$$Q = \begin{pmatrix} \tilde{Q}_1 & Q_0 & & & \\ Q_2 & Q_1 & Q_0 & & \\ & Q_2 & Q_1 & Q_0 & \\ & & Q_2 & Q_1 & Q_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Q_0, Q_1, Q_2, and $\tilde{Q}_1$ are $(m+1) \times (m+1)$ matrices.

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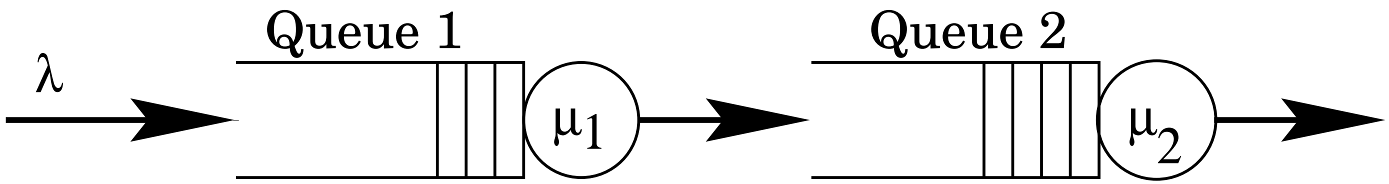

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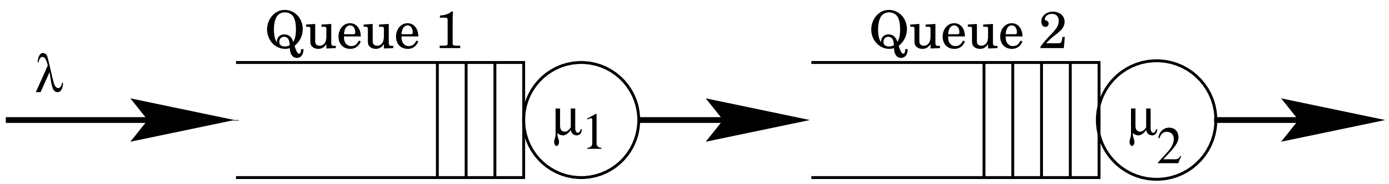

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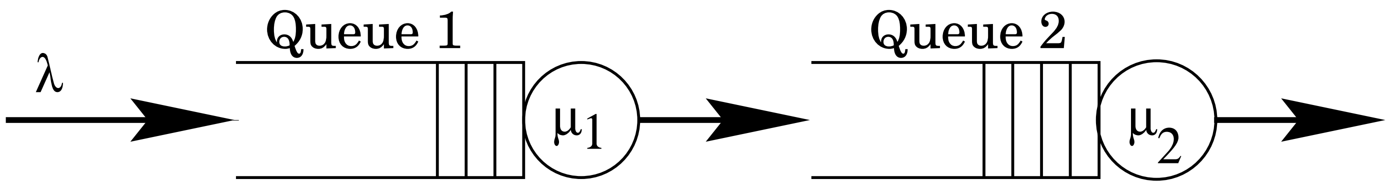

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- Let Y_t denote the number of customers in the second queue at time t (the level).

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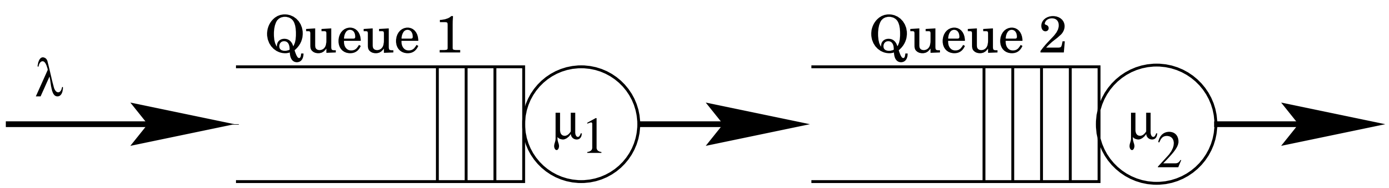
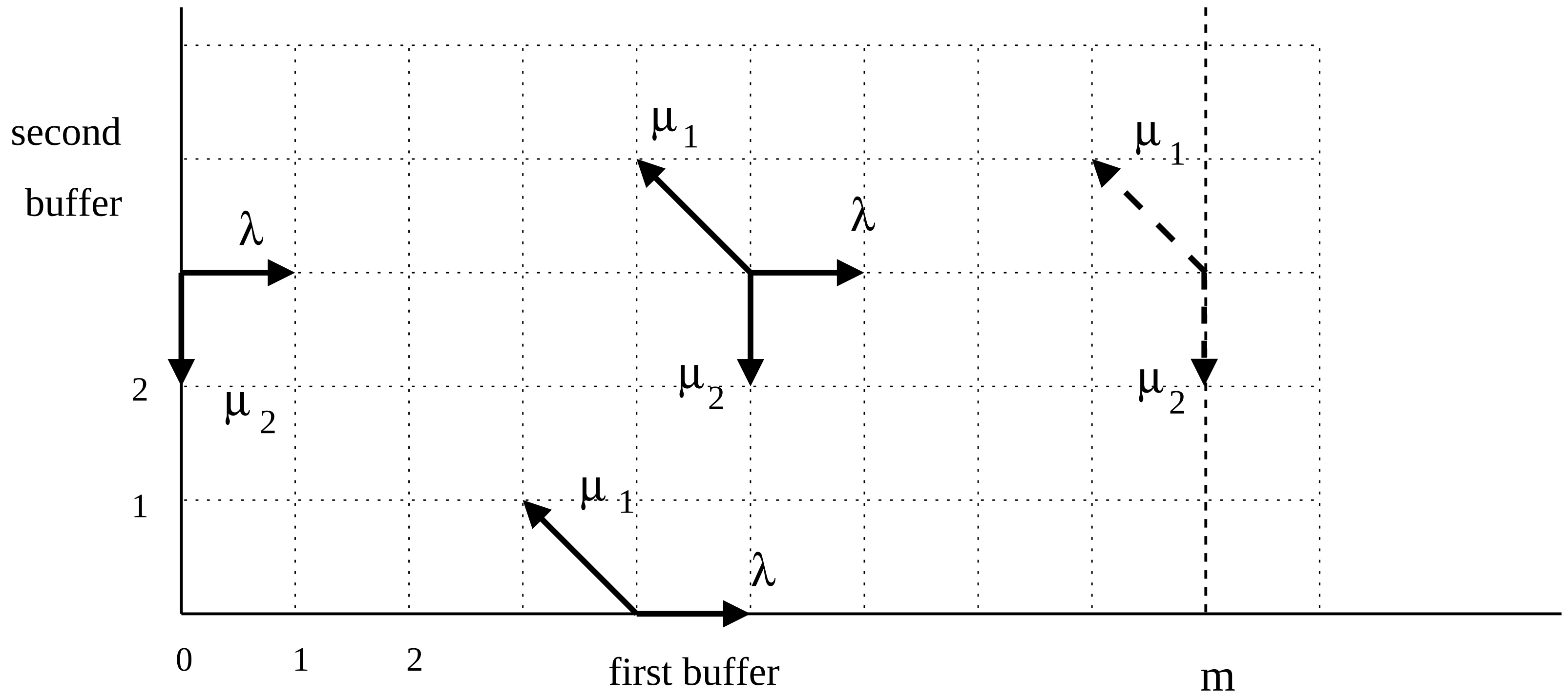


- Let J_t denote the number of customers in the first queue at time t (the phase).
- Let Y_t denote the number of customers in the second queue at time t (the level).
- First queue has capacity m (may be finite or infinite).

Example: Tandem network

Transition intensities for the tandem network



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Recall that the generator Q has a block tri-diagonal structure

$$Q = \begin{pmatrix} \tilde{Q}_1 & Q_0 & & & \\ Q_2 & Q_1 & Q_0 & & \\ & Q_2 & Q_1 & Q_0 & \\ & & Q_2 & Q_1 & Q_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Example: Tandem network

For the tandem network with infinte buffer size m, the blocks in the generator Q are given by the infinite-dimensional matrices

$$Q_0 = \begin{pmatrix} 0 & \cdots & & \\ \mu_1 & 0 & \cdots & \\ & \mu_1 & 0 & \cdots \\ & & \ddots & \ddots \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \mu_2 & & & \\ & \mu_2 & & \\ & & \mu_2 & \\ & & & \ddots \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} -(\lambda + \mu_2) & \lambda & & \\ & -(\lambda + \mu_1 + \mu_2) & \lambda & \\ & & -(\lambda + \mu_1 + \mu_2) & \lambda \\ & & & \ddots & \ddots \end{pmatrix},$$

Example: Tandem network

and

$$\tilde{Q}_1 = \begin{pmatrix} -\lambda & \lambda & & & \\ & -(\lambda+\mu_1) & \lambda & & \\ & & -(\lambda+\mu_1) & \lambda & \\ & & & \ddots & \ddots \end{pmatrix}.$$

The Matrix-Geometric Property

- Denote the limiting probabilities

$$\pi_{kj} := \lim_{t \to \infty} \mathbb{P}(Y_t = k, J_t = j).$$

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- Then

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 R^k, \quad k \geq 0$$

(assuming that the QBD is ergodic).

Neuts' R matrix

- The matrix R has dimensions $(m+1) \times (m+1)$, and is the minimal non-negative solution to the equation

$$Q_0 + R\,Q_1 + R^2\,Q_2 = 0.$$

- Probabilistic interpretation of R

  Let μ_i be the mean sojourn time in state (k, i) for $k \geq 1$ (this is independent of k).

  Then, R_{ij} is μ_i times the total expected time spent in state $(k+1, j)$ before first return to level k, starting from state (k, i).

The Caudal Characteristic

- For $m < \infty$ (finite phase space) the marginal stationary probability that the QBD is in level k decays geometrically with rate $\text{sp}(R) < 1$, where $\text{sp}(R)$ is the spectral radius of the matrix R.

$$\text{sp}(R) := \max_i |\lambda_i|,$$

  where λ_is are the eigenvalues of R.

- That is,

$$\lim_{k\to\infty} \frac{\sum_i \pi_{ki}}{(\text{sp}(R))^k} = \kappa,$$

  where κ is a constant.

The Perron-Frobenius Eigenvalue

- For a finite-dimensional, square, irreducible, non-negative matrix A, there exists a strictly positive eigenvalue which is simple and is greater than or equal to the modulus of all the other eigenvalues.

- This eigenvalue is called the *Perron-Frobenius eigenvalue* of A, and is equal to the spectral radius of A.

- For the case of a finite number of phases ($m < \infty$) the caudal characteristic (decay rate) is given by the Perron-Frobenius eigenvalue.

Infinite-dimensional analogue

- When m is infinite, things get more complicated.
- The infinite-dimensional analogue of the Perron-Frobenius eigenvalue is the *convergence norm*.
- The common convergence radius α , $0 \leq \alpha < \infty$, of the power series

$$\sum_{k=0}^{\infty} A^k(i,j)z^k$$

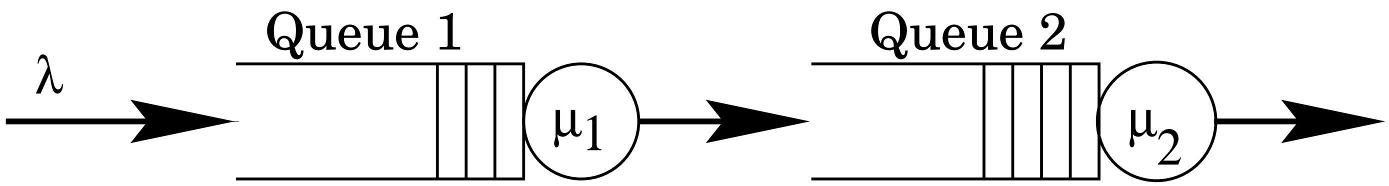
is called the *convergence parameter* of the matrix A .

- The quantity $1/\alpha$ is called the convergence norm of A, and satisfies

$$1/\alpha = \lim_{k\to\infty} \left(A^k(i,j)\right)^{1/k}$$

independently of i and j.

- Kroese, Scheinhardt, and Taylor (2003) considered the two-node tandem Jackson network with an infinite waiting room at the first queue (an infinite phase space).



- It was found that the decay rate of the stationary distribution of the “level” process is not necessarily equal to the convergence norm of the R matrix.

- This decay rate can be made to take any value from a range of admissable values, by controlling the transition structure only at level zero (i.e. by modifying the $\tilde{Q}_1$ matrix).

- The limiting behaviour of the tandem queue with a finite waiting room at the first queue, as the waiting room is increased to infinity, was considered.

- The eigenvalues of the R matrix converge to a continuum.

- The limiting value of the decay rate in the finite waiting room case is not necessarily the same as the decay rate in the infinite waiting room case.

Summary

- A QBD process is a two-dimensional Markov chain with a block-tridiagonal generator.
- Matrix-Geometric Property

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 \, R^k, \quad k \geq 0$$

- For a QBD with a finite number of phases

$$\lim_{k \to \infty} \frac{\sum_i \pi_{ki}}{(\mathsf{sp}(R))^k} = \kappa,$$

Summary

- Kroese, Scheinhardt, and Taylor (2003) found that more complicated behaviour occurs for a special case of a QBD with an infinite phase space.
- My research (to be done) will seek to generalise the results for QBDs with an infinite phase space.