# About scale estimator for the Cauchy distribution

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# Inspirations

- Cauchy is a common distribution in physics and finance
- Also common in Agriculture, e.g. for a trait expressed as the ratio of two standardized variables (root-to-shoot ratio)
- Well-shaped: continuous, location-scale, unimodal, symmetric
- Ill-behaved: heavy-tailed, the mgf is not defined, moments do not exist

# Scale parameter of Cauchy

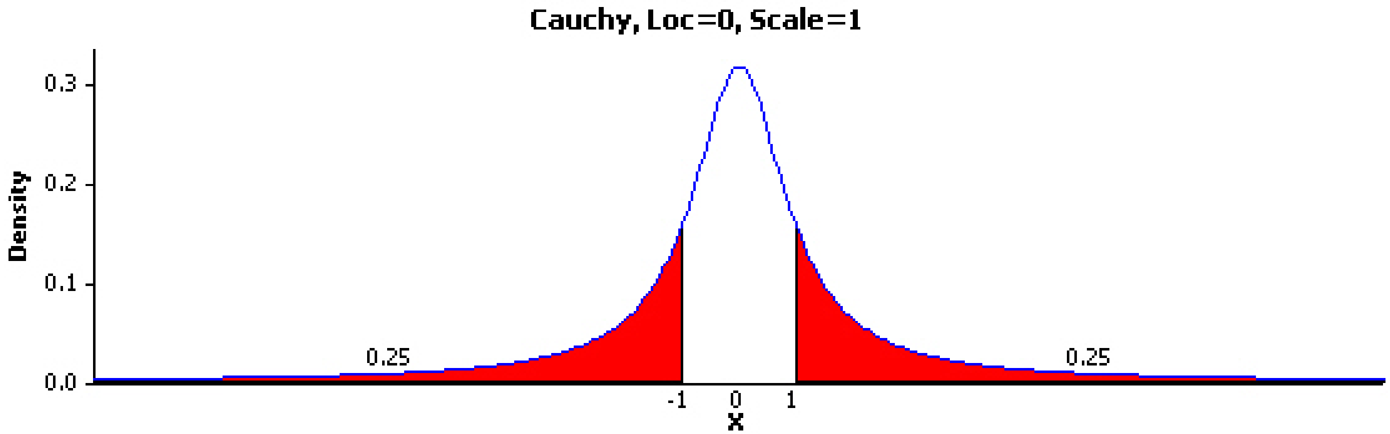
- Can be interpreted as a 'half-width-at-half-height', is also equal to half the interquartile range

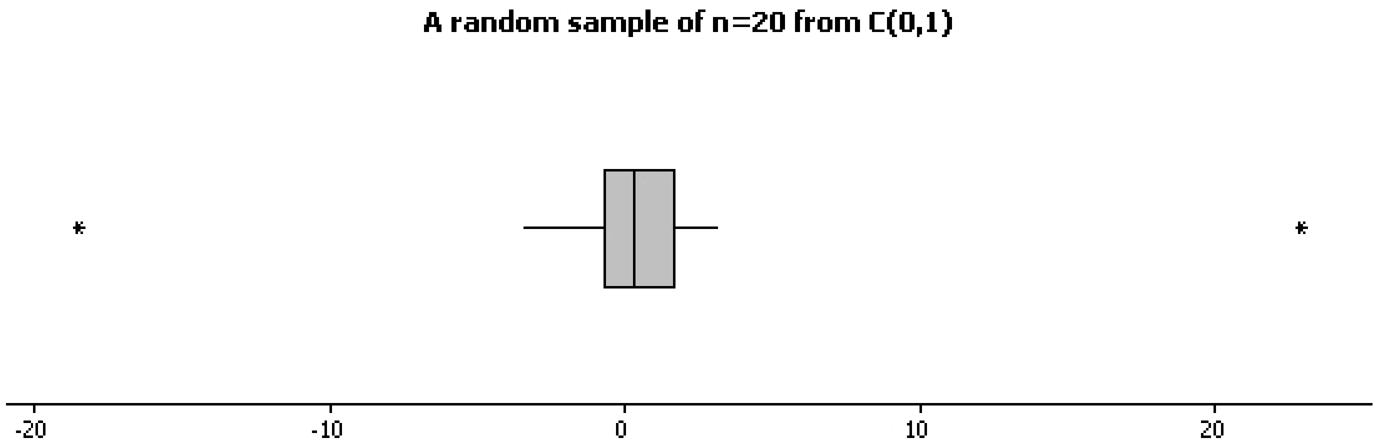
- Referred to as 'probable error'; half of the values from the distribution will lie within the interval and half outside

- A multitude of estimators, often computationally difficult

- Second component of the Cramer-von Mises goodness-of-fit statistic gives a fully efficient scale test for the Cauchy distribution (Durbin and Knott, 1972)

# Is it normal?


Cauchy, Loc=0, Scale=1
Density
0.3
0.2
0.1
0.0
0.25
0.25
-1
0
1
X


A random sample of n=20 from C(0,1)
-20
-10
0
10
20

# MLE of the scale parameter of the Cauchy distribution

Let $X$ follow an origin-centered Cauchy distribution, $C(\mu = 0, \sigma > 0)$, of the density $f$:

$$f(x) = \frac{1}{\pi\sigma}\frac{1}{1 + (x/\sigma)^2}.$$

The MLE of scale is

$$MLE(\sigma) = \left(\widehat{\sigma} : \sum \frac{\widehat{\sigma}^2}{x^2 + \widehat{\sigma}^2} = \frac{n}{2}\right).$$

The solution exists for $n > 2$ and is unique (Copas, 1975).
Clarke (1983) also showed that the MLE is consistent, robust and asymptotically efficient.

# Controversy about the distribution of $\hat{\sigma}/\sigma$

- Haas, Bain and Antle (1970) tabulated exact percentiles and also suggested that $\hat{\sigma}/\sigma$ is asymptotically normal, $N(1, 2/n)$.

- The exact 90% $\mathrm{CI}(\hat{\sigma}/\sigma) = $ (0.790, 1.267) for $n = 100$ is far from the normal approximation of (0.768, 1.232) suggested by Haas et al. (this contradicts the observation by Clarke (1983))

- Howlader and Weiss (1988) computationally demonstrated that the distribution of $\hat{\sigma}/\sigma$ is positively skewed.

- Mardia, Southworth and Taylor (1999) computationally estimated $\mathrm{E}(\hat{\sigma}/\sigma) - 1 \approx 1/n$.

# Proposition 1

The asymptotic sampling distribution of the MLE($\sigma$) for a sample of $n$ observations from a Cauchy distribution $C(\mu_0, \sigma)$ with a known location parameter $\mu_0$, is lognormal $\text{logN}(\ln \sigma, 2/n)$.

# Hyperbolic secant distribution*

- Continuous, unimodal, symmetric, slightly leptokurtic, all moments exist, location-scale distribution, HSD($\mu$,$\sigma$).

- The density of HSD($\mu$,1) is $g(x) = \frac{1}{\pi}\text{sech}(y - \mu)$

- $MLE(\mu) = \left(\widehat{\mu} : \sum \frac{\exp(2\widehat{\mu})}{\exp(2x)+\exp(2\widehat{\mu})} = \frac{n}{2}\right)$.

- $MLE(\mu)$ is asymptotically normal N($\mu$, $2/n$) (Vaughan, 2002).

*Phil remembers Gordon Smyh's excitment when Gordon re-discovered the HSD while working out a distribution of the intraclass correlation coefficient (Smyth, 1994); the HSD was firstly introduced in the 1920's.

# Cauchy as log-HSD

- The natural logarithm of the absolute value of an origin-centered Cauchy $C(0, \sigma)$ follows a hyperbolic secant distribution HSD($\ln \sigma, 1$) (Kravchuk, 2005).

- For the HSD, MLE($\ln \sigma$) is normal N($\ln \sigma, 2/n$).

- For the Cauchy distribution, MLE($\sigma$) is lognormal logN($\ln \sigma, 2/n$). This completes the proof.

# Back to the controversies

- Haas, Bain and Antle's (1970) exact 90% interval of $\widehat{\sigma}/\sigma$ for $n = 100$ (0.790, 1.267) is close to the lognormal approximation (0.792, 1.262).

- Howlader and Weiss's (1988) observed skeweness is expected for a lognormal distribution.

- Mardia, Southworth and Taylor's (1999) bias of $1/n$ is the first term of the expansion of the exact definition of the bias: $\mathrm{E}(\widehat{\sigma}/\sigma) - 1 = (\exp(1/n) - 1) = 1/n + 1/(2!n^2) + \cdots$

# Proposition 2

The natural logarithm of the absolute value of a non-centered Cauchy C($\mu \neq 0, \sigma > 0$) follows a symmetric distribution with the expected value of $\ln \sqrt{\mu^2 + \sigma^2}$.

# Outline of the working

- $X + \mu$ is $C(\mu, \sigma)$, what is the distribution of $Y = \ln|X + \mu|$?

- $P(\ln|X + \mu| \leq y) = P(X \leq \exp(y) - \mu) - P(X \leq -\exp(y) - \mu)$

- $f(y) = \frac{\exp(y)}{\pi\sigma}\left(\frac{1}{1+(\exp(y)-\mu)^2/\sigma^2} + \frac{1}{1+(\exp(y)+\mu)^2/\sigma^2}\right)$

- $\int\limits_{-\infty}^{\infty} x f(x) dx = \ln\sqrt{\mu^2 + \sigma^2}$ (Cauchy's residue theorem)

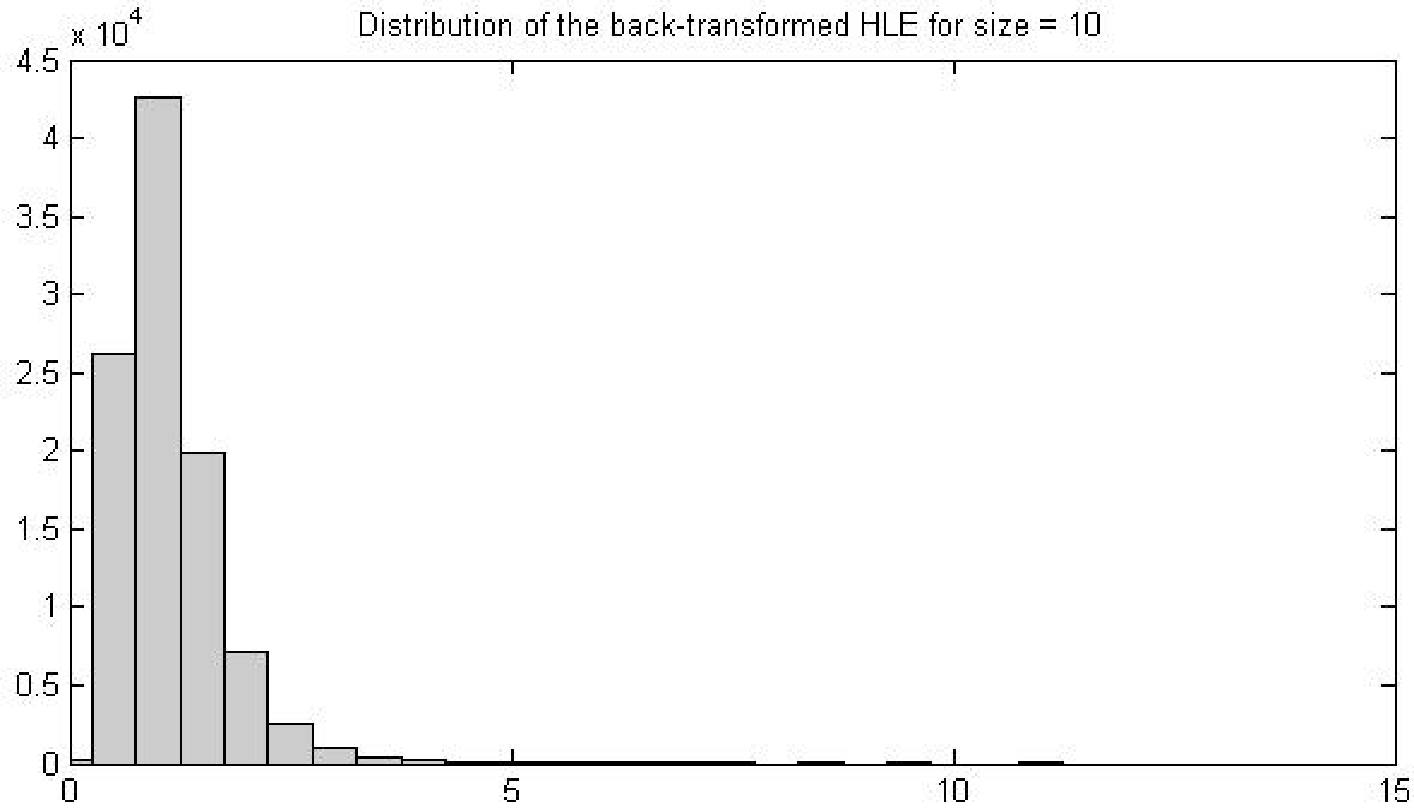
- Easy to show that $f(y) = f(2\mu - y)$.

# A 'new' estimator of location

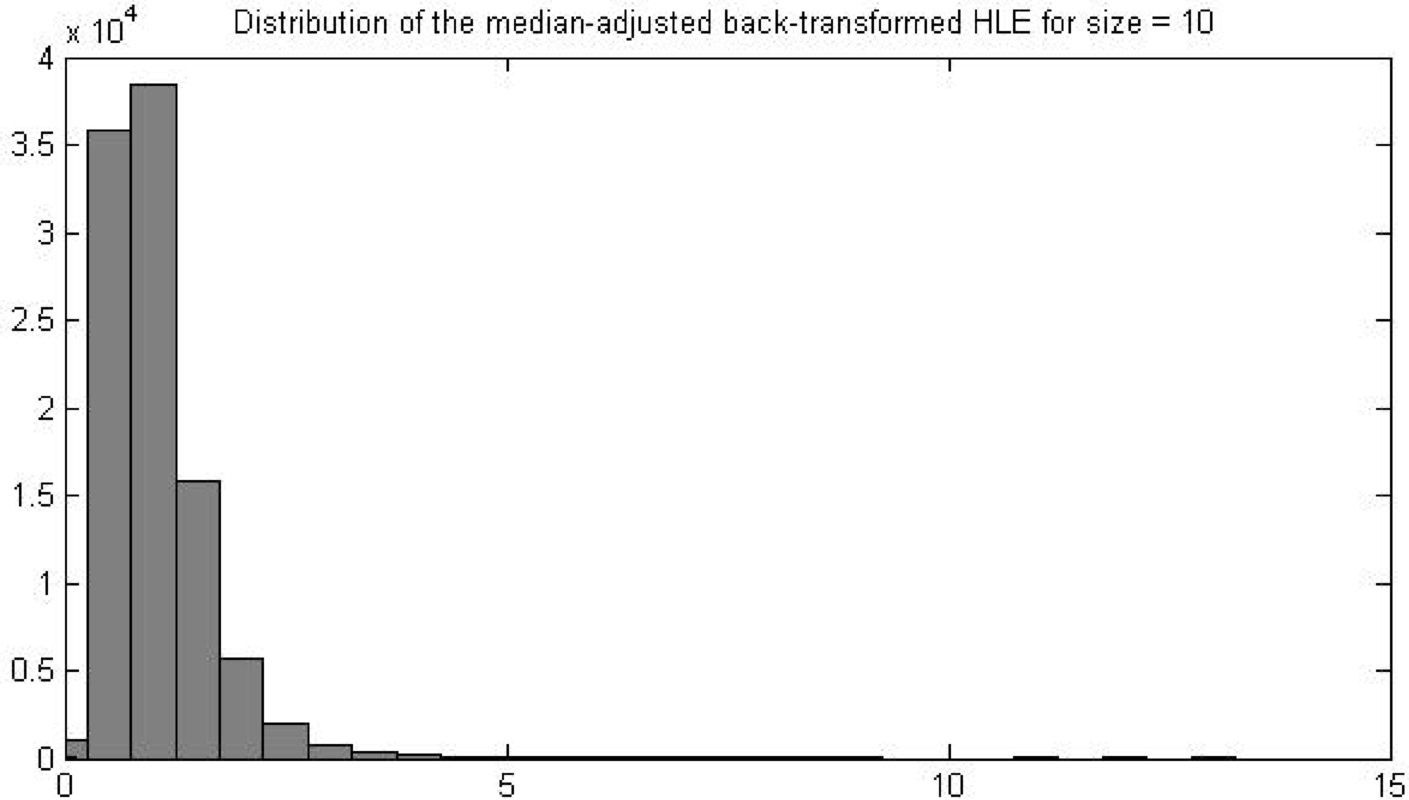
A common Hodges-Lehmann estimator can be used for estimating the log of the scale parameter.

- The estimator is asymptotically 98% efficient and behaves well on small samples.
- The estimator is normal, N($\ln \sigma, 2/(0.986n)$), its exp-transformation is, obviously, lognormal
- If the location parameter is not known, the median estimator should be used first to adjust the sample. For small samples, the variance of this median-adjusted Hodges-Lehmann estimator becomes slightly larger.

# Distribution of Hodges-Lehmann-type scale estimator for Cauchy, n=10



## Distribution of median-adjusted Hodges-Lehmann-type scale estimator, n=10



# Conclusion

It is easy to estimate the scale parameter of the Cauchy distribution in the case of known as well as unknown location parameter.
The paper is currently under review with *Comm. Statistic. Theory Methods*.
In the paper we are also additionally examining the behaviour of both MLE($\sigma$) and Hodges-Lehmann estimator for very small samples, $n = 3$ and $n = 4$.

# References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