Volume Weighted Asian Option
Antony William Stace
Department of Mathematics, University of Queensland
Poster day, 26th September 2003
aws@maths.uq.edu.au

Abstract
The Volume Weighted Asian Option (VWAO) uses a volume weighted average of the underlying to define the contract. Unlike the vanilla Asian Option, the VWAO assigns more weight to days of heavy trading than to days of light trading. This poster presents a model which describes the VWAO. Following this, several results concerning the nature of the VWAO are shown. Finally, an application of the VWAO model is demonstrated via the valuation of a share purchase plan which uses a volume weighted average.

What is an Option?
- An agreement between two parties whereby one party (issuer/writer) has the rights to perform a specified transaction having specified terms with the other party (holder).
- Most common are those involving stocks, in particular: Call option: Gives the holder the right to buy a stock at a later time for a set price. Mathematically, we say that the holder has a payoff at expiry time of max(S_T−Strike, 0). An example is a call option on BP shares to buy 1 BP share in 6 months time for $100. If the stock price is above this at the time then a profit will be made to the holder of the option.
- Put option: Gives the holder the right to sell a stock at a later time for a set price.
- Close form solutions for the price of the call and put option have been known since 1973.

What is an Asian Option?
An agreement between two parties whereby one party (issuer/writer) has the rights to perform a specified transaction having specified terms with the other party (holder). A party holder has the rights to perform a specified transaction being the rate of transactions and volume risk and can be determined from market data.

Volume Weighted Asian Option
This uses a volume weighted average,
\[ \text{Average} = \frac{\int S(t) U(t) dt}{\int U(t) dt} \]
with \( U \) being the rate of transactions and \( S \) being the stock price.

Example: Suppose a stock trades at $10 today and there are 100 trades, tomorrow it trades at $10 and there is 1 trade. The volume weighted average is \( \text{average} = \$10.89 \) while a time weighted average is \( \text{time average} = \$55.00 \).

We assume the stock follows geometric Brownian motion
\[ dS = \mu S dt + \sigma S dW, \]
and the volume process follows
\[ dV = \alpha(U) dt + \beta(U) dW. \]

Mean reverting models are used for the volume model, \( dV = \alpha(U) - \beta(U) dW \), \( dV = \alpha(U) + \beta(U) dW \), and \( dV = \alpha(U) + \beta(U) dW \), with correlation between \( W_1 \) and \( W_2 \) being \( p \).

PDE Formulation
Introduce the variables
\[ W(t) = \int_0^t S(r)U(r) dr \]
and
\[ Z(t) = \int_0^t U(r) dr, \]
so that the average is simply given by \( \frac{W(t)}{Z(t)} \). Now assume the value of the option takes the form \( V(t, S, U, W, Z) \). It is the independent variables \( t, S, U, W, Z \).

Form a portfolio
\[ \Pi = V - \Delta S - \Delta \Delta \]
and then do similar analysis to stochastic volatility models to get the PDE
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma S \frac{\partial V}{\partial S} + r S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial S} - r \frac{\partial V}{\partial S} - r \frac{\partial V}{\partial S} = 0, \]
with the hedging ratios
\[ \Delta_1 = \frac{\partial V}{\partial S}, \quad \Delta_2 = \frac{\partial V}{\partial S} - \Delta \frac{\partial V}{\partial S}, \]
so that the stochastic terms \( dS \) and \( dU \) vanish and the portfolio is riskless. The \( \lambda \) term is the market price of volume risk and can be determined from market data. For simplicity we set \( \lambda = KU \). This is a complex PDE to solve.

An Asymptotic Approach
Using the model
\[ dU = \alpha(U) dt + \beta(U) dW _2, \]
for volume it can be shown that for the invariant distribution \( V(U) = \int_0^U \frac{1}{\sqrt{v}} dv \) substituting into the PDE and writing \( r = \frac{1}{\sqrt{v}} : \]
\[ \frac{1}{2} \sigma_1 \sigma_2 \left( \frac{1}{\mu - \beta(U)} \right) \frac{\partial V}{\partial U} + \lambda \left( \frac{1}{\mu - \beta(U)} \right) \frac{\partial V}{\partial U} = 0. \]

Conclusions and Further Work
A model has been formulated which captures the essential features of an option based on a Volume Weighted Average. Asymptotic methods show that when the rate of volume is fast mean reverting, the pricing equation reverts to that of the regular Asian Option. The model has also been used to value a share purchase plan. Future work includes:
- Developing more sophisticated Monte Carlo schemes.
- Simplify the PDE, the includes looking at the use of Lie symmetry and similarity reductions.
- Addition of jumps to the volume model, since clearly real volume data exhibits jumps.