The Role of Orthogonal Polynomials in Determining the Decay Rates of Multidimensional Queueing Processes

Peter Taylor

Department of Mathematics and Statistics,

The University of Melbourne

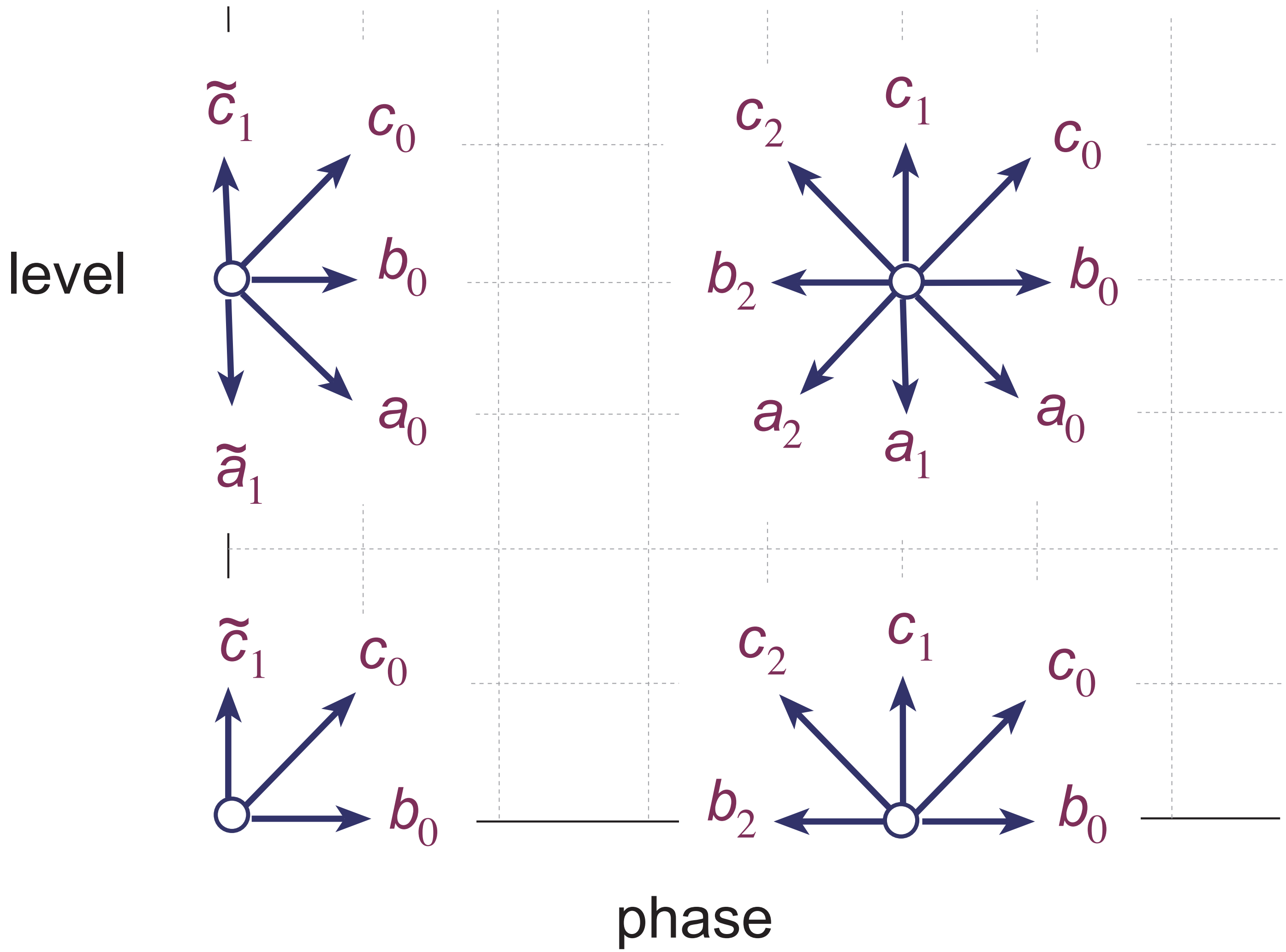
(with D Kroese, W Scheinhardt and A Motyer)

Australian Research Council
Centre of Excellence for Mathematics
and Statistics of Complex Systems

Quasi-birth-and-death processes

- A QBD process is a two-dimensional continuous-time Markov Chain $\{(Y_t, J_t), t \geq 0\}$ on the state space $\{0, 1, \ldots\} \times \{0, 1, \ldots, m\}$.
- The variable Y_t is called the *level* of the process at time t and the variable J_t is called the *phase* of the process at time t.
- The parameter m may be either finite or infinite.
- State transitions are restricted to states in the same level or in the two adjacent levels (hence the name QBD).
- Transition intensities are assumed to be level-independent away from the boundary.

Our class of interest



Quasi-birth-and-death processes

A general QBD process has a block partitioned generator Q with tri-diagonal structure

$$Q = \begin{pmatrix} \tilde{Q}_1 & Q_0 & & & \\ Q_2 & Q_1 & Q_0 & & \\ & Q_2 & Q_1 & Q_0 & \\ & & Q_2 & Q_1 & Q_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Q_0, Q_1, Q_2 and $\tilde{Q}_1$ are $(m+1) \times (m+1)$ matrices.

In our class of interest, the blocks in the generator are themselves tridiagonal and homogeneous away from the boundary. Thus, we can write

$$Q_0 = \begin{pmatrix} \tilde{c}_1 & c_0 & & & \\ c_2 & c_1 & c_0 & & \\ & c_2 & c_1 & c_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} \tilde{b}_1 & b_0 & & & \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

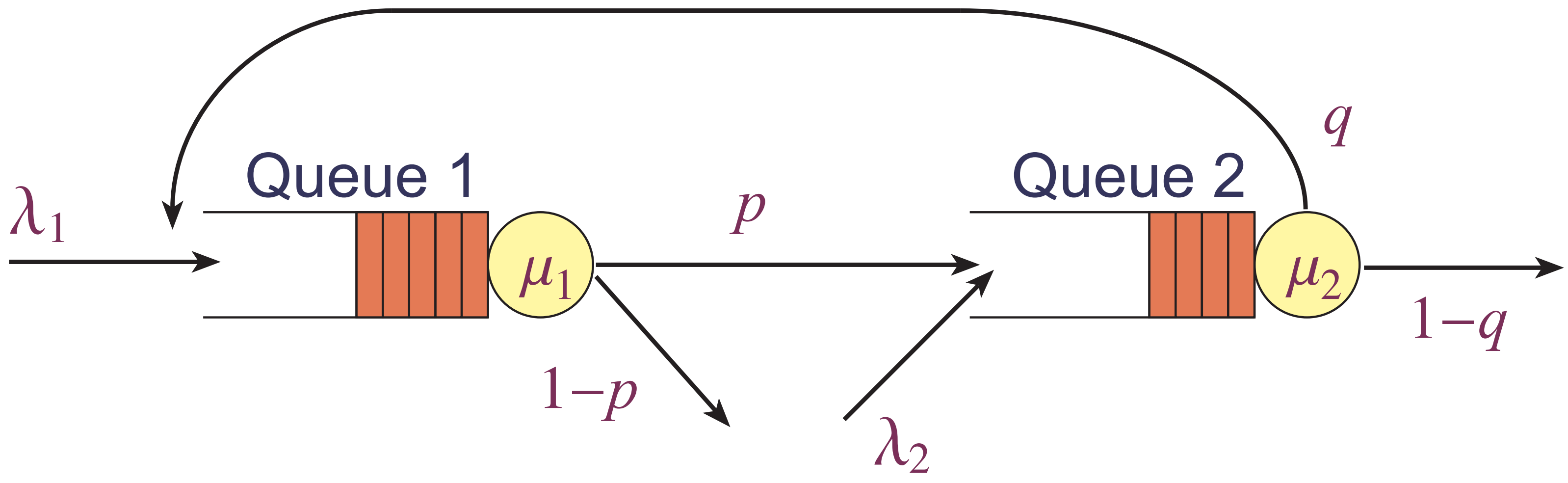
$$Q_2 = \begin{pmatrix} \tilde{a}_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ & a_2 & a_1 & a_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

and

$$\tilde{Q}_1 = \begin{pmatrix} \bar{b}_1 & b_0 & & & \\ b_2 & \hat{b}_1 & b_0 & & \\ & b_2 & \hat{b}_1 & b_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The Two-node Jackson Network

- Let J_t denote the number of customers in the first queue at time t (the phase), and
- Y_t denote the number of customers in the second queue at time t (the level).



Our Tools: 1

The Matrix-Geometric Property.

Assume that the QBD is positive recurrent and denote the limiting probabilities

$$\pi_{kj} := \lim_{t\to\infty} \mathbb{P}(Y_t = k, J_t = j).$$

With

$$\boldsymbol{\pi}_k = (\pi_{k0}, \pi_{k1}, \ldots, \pi_{km}),$$

then

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 R^k, \quad k \geq 0$$

for both finite and infinite m.

The $(m+1) \times (m+1)$ matrix R is the minimal non-negative solution to the equation

$$Q_0 + R\,Q_1 + R^2\,Q_2 = 0.$$

The stationary distribution $\boldsymbol{\pi}_0$ at level zero must satisfy

$$\boldsymbol{\pi}_0\,(\tilde{Q}_1 + R\,Q_2) = 0.$$

Definition

If there exists a positive scalar z and a positive row vector $\boldsymbol{w} \in \ell^1$ such that

$$\lim_{k\to\infty} \frac{\boldsymbol{\pi}_k}{z^k} = \boldsymbol{w}$$

elementwise, then we say that the stationary distribution *decays at rate* z.

For $m < \infty$, it follows immediately from the matrix-geometric property that

$$\lim_{k \to \infty} \frac{\sum_i \pi_{ki}}{(\mathrm{sp}(R))^k} = \kappa,$$

where κ is a constant. That is, the level process has tail decay rate $\mathrm{sp}(R) < 1$.

For the class of processes considered here, $m = \infty$ so this result is not applicable.

Our Tools: 2

Theorem (Kroese, Scheinhardt, Taylor, 2004)

Consider an irreducible QBD process. If there exists a nonnegative vector $\boldsymbol{w} \in \ell^1$ and a nonnegative number $z < 1$ such that

$$\boldsymbol{w}\,R = z\boldsymbol{w},$$

and

$$\boldsymbol{w}(\tilde{Q}_1 + R\,Q_2) = \mathbf{0},$$

then the QBD process is positive recurrent with $\boldsymbol{\pi}_0 = \boldsymbol{w}$, and for all k,

$$\frac{\boldsymbol{\pi}_k}{z^k} = \boldsymbol{w}.$$

Our Tools: 3

Theorem (Ramaswami and Taylor, 1996)

Let $q_n = -Q_1(n,n)$. If the complex variable z and the vector $\boldsymbol{w} = \{w_n\}$ are such that $|z| < 1$ and $\sum_n |w_n| q_n < \infty$, then

$$\boldsymbol{w}\,(Q_0 + zQ_1 + z^2 Q_2) = \mathbf{0}$$

implies that

$$\boldsymbol{w}\,R = z\boldsymbol{w}.$$

Due to our assumptions about the structure of the generator Q, the condition $\sum_n |w_n| q_n < \infty$ is equivalent to $\boldsymbol{w} \in \ell^1$ and the equation

$$\boldsymbol{w}\left(Q_0 + zQ_1 + z^2 Q_2\right) = \mathbf{0}$$

can be written as a homogeneous second order recurrence relation in the w_n with coefficients that depend on z.

Thus

$$w_0\tilde{\gamma}_1(z) + w_1\gamma_2(z) = 0$$

and, for $n \geq 0$,

$$w_n\gamma_0(z) + w_{n+1}\gamma_1(z) + w_{n+2}\gamma_2(z) = 0,$$

where

$$\tilde{\gamma}_1(z) = \tilde{c}_1 + \tilde{b}_1 z + \tilde{a}_1 z^2$$

and, for $i = 0, 1, 2,$

$$\gamma_i(z) = c_i + b_i z + a_i z^2.$$

Our Tools: 4

It is elementary to find conditions on z such that $\boldsymbol{w} \in \ell^1$.
It is harder to find conditions on z such that $\boldsymbol{w}$ is nonnegative.
We use the theory of orthogonal polynomials for this purpose.
Introducing a new variable x, we can generalise our equations for w_n to

$$\begin{aligned} P_0(x;z) &= 1, \\ \gamma_2(z)P_1(x;z) &= x - \tilde{\gamma}_1(z), \\ \gamma_2(z)P_n(x;z) &= (x - \gamma_1(z))P_{n-1}(x;z) - \gamma_0(z)P_{n-2}(x;z), \quad n \geq 2. \end{aligned}$$

When $x = 0$, the equations are the same as those for w_n, scaled so that $w_0 = 1$, and so $w_n = P_n(0;z)$.

The $P_n(0;z)$ are positive for all n if and only if the zeros of all the $P_n(x;z)$, considered as polynomials in x, are less than zero.
Let

$$T_n(x) = \left(\sqrt{\frac{\gamma_2(z)}{\gamma_0(z)}}\right)^n P_n\left(2x\sqrt{\gamma_0(z)\gamma_2(z)} + \gamma_1(z); z\right).$$

The $T_n(x)$s satisfy the recursion which defines *perturbed Chebyshev polynomials*. The behaviour of their zeros has been well-studied, and we can translate this into information about the behaviour of the zeros of the $P_n(x)$s.

Let

$$
\begin{aligned}
\tau(z) &= \gamma_1(z) + 2\sqrt{\gamma_0(z)\gamma_2(z)}, \\
\chi(z) &= \tilde{\gamma}_1(z) + \frac{\gamma_0(z)\gamma_2(z)}{\tilde{\gamma}_1(z) - \gamma_1(z)}, \\
\chi_1(z) &= \begin{cases} \tau(z) & \text{if } (\tilde{\gamma}_1(z) - \gamma_1(z))^2 \le \gamma_0(z)\gamma_2(z), \\ \chi(z) & \text{otherwise.} \end{cases}
\end{aligned}
$$

For $z > 0$, $P_n(x; z)$ is positive for all n if and only if $\chi_1(z) \le x$. Thus, the vector $\boldsymbol{w} = (P_n(0; z))$ is positive if and only if $\chi_1(z) \le 0$.

The condition $\chi_1(z) \le 0$ turns out to be expressible in terms of inequalities involving polynomials in z of degree, at most, four.

Summary

- We use elementary techniques and the theory of orthogonal polynomials to derive the set of values of $z \in [0, 1]$ for which

$$\boldsymbol{w}\left(Q_0 + zQ_1 + z^2 Q_2\right) = \mathbf{0}$$

  has a solution $\boldsymbol{w} \in \ell^1$.

- This gives us the set of $\boldsymbol{w} \in \ell^1$ and $z \in [0, 1]$ for which $\boldsymbol{w}\,R = z\boldsymbol{w}$.

- Provided that $\boldsymbol{w}(\tilde{Q}_1 + zQ_2) = \mathbf{0}$, then $\boldsymbol{\pi}_0 = \boldsymbol{w}$, and for all k,

$$\frac{\boldsymbol{\pi}_k}{z^k} = \boldsymbol{w}.$$

  Thus the decay rate is z.

Summary

We have a lot of flexibility in modifying $\tilde{Q}_1$ and so, for a large range of $\boldsymbol{w}$ and z, it is reasonable to think that we can do so in order that

$$\boldsymbol{w}(\tilde{Q}_1 + zQ_2) = \mathbf{0}$$

is satisfied.
Indeed, for a number of examples we have shown how to modify $\tilde{Q}_1$ so that this equation is satisfied for all $\boldsymbol{w}$ and z compatible with

$$\boldsymbol{w}\, R = z\boldsymbol{w}.$$

Thus we have shown that, by altering the transition rates at level zero, we can obtain any decay rate in the calculated range.

The two-node Jackson network

The stationary distribution has the product form

$$\pi(n_1, n_2) = (1-\rho_1)(1-\rho_2)\rho_1^{n_1}\rho_2^{n_2}, \quad n_1, n_2 \geq 0,$$

where ρ_1 and ρ_2 are solutions to the well-known traffic equations. The stability condition is $\rho_1 < 1$, $\rho_2 < 1$.

The decay rate at queue 2 is always ρ_2. However, the ranges of possible decay rates, obtainable by varying the transition rates when queue 2 is empty can be non-trivial for different parameter values.

The two-node Jackson network

λ_1	λ_2	μ_1	μ_2	p	q	ρ_1	ρ_2	Possible decay rates
1	0	1.5	2	1	0	0.667	0.500	[0.477, 0.750)
1	0	2	1.5	1	0	0.500	0.667	[0.667, 1)
0	1	1.5	2	0	1	0.667	0.500	{0.500}
0	1	2	1.5	0	1	0.500	0.667	{0.667}
1	1	2	2	0.1	0.8	0.978	0.598	[0.597, 0.600)
1	1	2	2	0.8	0.1	0.598	0.978	[0.978, 1)
1	1	2	2	0.4	0.4	0.833	0.833	[0.833, 0.900)
1	1	10	10	0.5	0.5	0.200	0.200	[0.200, 0.600)
1	5	10	15	0.4	0.9	0.859	0.563	[0.558, 0.600)
5	1	15	10	0.9	0.4	0.563	0.859	[0.859, 1)

Finite Truncations

We can extend our orthogonal polynomial analysis to show that the limiting value of the decay rate at queue 2 when queue 1 is truncated to size m and then m is allowed to approach infinity is always the infimum of the interval of possible decay rates.

We see from our table that this is not always ρ_2. In these cases, there is a ‘discontinuity at infinity’ with respect to the parameter m.