ON BIRTH-DEATH PROCESSES AND EXTREME ZEROS OF ORTHOGONAL POLYNOMIALS

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1. orthogonal polynomials
   - definitions, notation
   - zeros, orthogonalizing measure
   - OP’s on $[0, \infty)$

2. birth-death processes (with killing)
   - definitions, notation
   - decay rate
   - recent results

3. extreme zeros of OP’s
orthogonal polynomials

definition: \( \{P_n(x), \ n = 0, 1, \ldots\} \) (monic, \( \deg(P_n) = n \)) is orthogonal polynomial sequence (OPS) if there exists (Borel) measure \( \psi \) (of total mass 1) such that

\[
\int_{-\infty}^{\infty} P_n(x)P_m(x)\psi(dx) = k_n\delta_{nm}
\]

with \( k_n > 0 \) (\( \psi \) is not necessarily unique)

Favard’s theorem:

\( \{P_n(x), \ n = 0, 1, \ldots\} \) is OPS \iff there exist \( c_n \in \mathbb{R}, \ \lambda_n > 0 \) such that

\[
P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_nP_{n-2}(x)
P_0(x) = 1, \ P_1(x) = x - c_1
\]
orthogonal polynomials

point of departure: \( \{P_n(x), \ n = 0, 1, \ldots \} \) satisfies

\[
P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)
\]

\[
P_0(x) = 1, \quad P_1(x) = x - c_1
\]

with \( c_n \in \mathbb{R}, \ \lambda_n > 0 \)

general problem: find information on orthogonalizing measure from coefficients in recurrence relation (cf. Chihara's book)

fact: support of orthogonalizing measure is related to zeros of polynomials

approach: find information on zeros of \( P_n(x) \) from coefficients in recurrence relation
zeros of orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots\} satisfies

\[ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \]

\[ P_0(x) = 1, \quad P_1(x) = x - c_1 \]

if \( a_j > 0 \) and

\[ T_n := \begin{pmatrix}
    c_1 & \lambda_2/a_2 & 0 & \cdots & \cdots \\
    a_2 & c_2 & \lambda_3/a_3 & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \cdots & c_{n-1} & \lambda_n/a_n \\
    \vdots & \vdots & \cdots & 0 & a_n & c_n
\end{pmatrix} \]

then

\[ \det(xI_n - T_n) = P_n(x) \]

observation: zeros of \( P_n(x) \) are eigenvalues of \( T_n \)
zeros of orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots\} is OPS with zeros \(x_{ni}\)

zeros of \(P_n(x)\) real and distinct:

\[x_{n1} < x_{n2} < \ldots < x_{nn}\]

interlacing property:

\[x_{n+1,i} < x_{ni} < x_{n+1,i+1}\]

hence

\[\xi_i := \lim_{n \to \infty} x_{ni}\] and \[\sigma := \lim_{i \to \infty} \xi_i\]

exist, and

\[-\infty \leq \xi_i \leq \xi_{i+1} \leq \sigma \leq \infty\]
zeros of orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots\} \text{ is OPS with zeros } x_{ni}

let

\[ \xi_i := \lim_{n \to \infty} x_{ni} \quad \text{and} \quad \sigma := \lim_{i \to \infty} \xi_i \]

then

\[ -\infty \leq \xi_i \leq \xi_{i+1} \leq \sigma \leq \infty \]

moreover

\[ \xi_i = \xi_{i+1} \Rightarrow \xi_i = \sigma \]

similarly

\[ \eta_i := \lim_{n \to \infty} x_{n,n-i+1} \quad \text{etc} \]
zeros of orthogonal polynomials

\( \{ P_n(x), \ n = 0, 1, \ldots \} \) is OPS with zeros \( x_{ni} \) and \( \xi_i := \lim_{n \to \infty} x_{ni} \)

then

\[-\infty \leq \xi_i \leq \xi_{i+1} \leq \sigma = \lim_{i \to \infty} \xi_i \leq \infty\]

and

\[\xi_i = \xi_{i+1} \Rightarrow \xi_i = \sigma\]

if \( \xi_1 > -\infty \) there are three possibilities:

1. \( \xi_1 < \cdots < \xi_i < \xi_{i+1} < \cdots < \sigma = \infty \)
2. \( \xi_1 < \cdots < \xi_i < \xi_{i+1} < \cdots < \sigma < \infty \)
3. \( \xi_1 < \cdots < \xi_i = \xi_{i+k} \) for some \( i \) and all \( k > 0 \)
orthogonalizing measure

\( \{P_n(x), \ n = 0, 1, \ldots \} \) is OPS satisfying

\[
P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)
\]

\[
P_0(x) = 1, \quad P_1(x) = x - c_1
\]

**Theorem:** if \( \xi_1 > -\infty \) there exists an orthogonalizing measure \( \psi \) for \( \{P_n(x)\} \), that is,

\[
\int_{-\infty}^{\infty} P_m(x)P_n(x)\psi(dx) = k_n\delta_{mn}
\]

such that

\[
\sigma = \infty \Rightarrow \text{supp}(\psi) = \{\xi_1, \xi_2, \ldots\}
\]

\[
\sigma < \infty \Rightarrow \text{supp}(\psi) \cap (-\infty, \sigma] = \overline{\{\xi_1, \xi_2, \ldots\}}
\]

**Remark:** \( \psi \) not necessarily unique (Hamburger moment problem)
orthogonalizing measure

\[ \{P_n(x), \ n = 0, 1, \ldots \} \text{ is OPS satisfying} \]
\[
P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \\
P_0(x) = 1, \ P_1(x) = x - c_1
\]

**general problem**: find information on orthogonalizing measure from coefficients in recurrence relation

**specific problem**: find information on \( \xi_1 \) (and \( x_{n1} \), and \( \xi_2 \)) in terms of coefficients in recurrence relation

**observe**: \( \{\tilde{P}_n(x) := (-1)^n P_n(-x), \ n = 0, 1, \ldots \} \) is OPS with \( \tilde{c}_n := -c_n \) and \( \tilde{\lambda}_n := \lambda_n \), hence
\[
x_{nn} = -\tilde{x}_{n1}
\]
zeros of orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots \} is OPS satisfying

\[ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \]

\[ P_0(x) = 1, \quad P_1(x) = x - c_1 \]

with \( c_n \in \mathbb{R}, \ \lambda_n > 0 \)

\textbf{recall:} zeros of \( P_n(x) \) are eigenvalues of

\[ T_n = \begin{pmatrix}
  c_1 & \lambda_2/a_2 & 0 & \cdots & \cdots \\
  a_2 & c_2 & \lambda_3/a_3 & \cdots & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \cdots & \cdots & \cdots & c_{n-1} & \lambda_n/a_n \\
  \cdots & \cdots & \cdots & 0 & a_n & c_n
\end{pmatrix} \]

where \( a_j > 0 \)
zeros of orthogonal polynomials

**Theorem** (Gilewicz & Leopold (1985), vD (1984,1987)):

\[
x_{n1} = \max_{a > 0} \min_{1 \leq i \leq n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}
\]

\[
\xi_1 = \max_{a > 0} \inf_{i \geq 1} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}
\]

where \( \lambda_1 = 0 \), \( a = (a_1, a_2, \ldots) \)

**Proof:** *Geršgorin discs*
zeros of orthogonal polynomials

**Theorem** (vD (1987), Ismail & Li (1992):

\[ x_{n1} = \max_h \min_{1 \leq i < n} \frac{1}{2} \left\{ c_i + c_{i+1} - \sqrt{(c_{i+1} - c_i)^2 + \frac{4\lambda_{i+1}}{(1 - h_i)h_{i+1}}} \right\} \]

where \( h = (h_1, \ldots, h_n) \), \( h_1 = 0 \), \( h_n = 1 \), \( 0 < h_i < 1 \) (1 < i < n)

\[ \xi_1 = \max_b \inf_{i \geq 1} \frac{1}{2} \left\{ c_i + c_{i+1} - \sqrt{(c_{i+1} - c_i)^2 + 4\lambda_{i+1}/b_i} \right\} \]

where \( b = (b_1, b_2, \ldots) \) is a chain sequence

**Proof**: ovals of Cassini
zeros of orthogonal polynomials

**Theorem** (vD (1987), Levenshtein (1995)):

\[
x_{n1} = \min_{h \geq 0} \left\{ \sum_{i=1}^{n} \left( h_i^2 c_i - 2 h_{i-1} h_i \sqrt{\lambda_{i+1}} \right) \right\}
\]

where \( h = (h_0, \ldots, h_n) \), \( h_0 = 0 \), \( \sum_{i=1}^{n} h_i^2 = 1 \)

\[
\xi_1 = \inf_{h \geq 0} \left\{ \lim_{n \to \infty} \inf \left\{ \sum_{i=1}^{n} \left( h_i^2 c_i - 2 h_{i-1} h_i \sqrt{\lambda_{i+1}} \right) \right\} \right\}
\]

where \( h = (h_0, h_1, \ldots) \), \( h_0 = 0 \), \( \sum_{i=1}^{\infty} h_i^2 = 1 \)

**Proof:** Courant-Fischer theorem/Raleigh quotients/field of values (symmetrize \( T_n \) by suitable similarity transformation)
orthogonal polynomials on \([0, \infty)\)

**Theorem:** the following are equivalent:

(i) \(\xi_1 \geq 0\)

(ii) there exist numbers \(\alpha_n > 0\) and \(\beta_{n+1} > 0\) such that \(c_1 = \alpha_1\), and, for \(n > 1\),

\[
\begin{align*}
  c_n &= \alpha_n + \beta_n \\
  \lambda_n &= \alpha_{n-1}\beta_n 
\end{align*}
\]

(iii) there exist numbers \(\alpha_n > 0\), \(\beta_{n+1} > 0\) and \(\gamma_n \geq 0\) such that \(c_1 = \alpha_1 + \gamma_1\), and, for \(n > 1\),

\[
\begin{align*}
  c_n &= \alpha_n + \beta_n + \gamma_n \\
  \lambda_n &= \alpha_{n-1}\beta_n 
\end{align*}
\]
orthogonal polynomials on \([0, \infty)\)

\[ \xi_1 \geq 0 \iff \text{there exist numbers } \alpha_n > 0 \text{ and } \beta_{n+1} > 0 \text{ such that } c_1 = \alpha_1 \text{ and, for } n > 1, \]

\[ c_n = \alpha_n + \beta_n, \quad \lambda_n = \alpha_{n-1}\beta_n \]

\[ \iff \text{there exist numbers } \alpha_n > 0, \beta_{n+1} > 0 \text{ and } \gamma_n \geq 0 \text{ such that } c_1 = \alpha_1 + \gamma_1 \text{ and, for } n > 1, \]

\[ c_n = \alpha_n + \beta_n + \gamma_n, \quad \lambda_n = \alpha_{n-1}\beta_n \]

assuming \(\psi\) is unique:

if \(\gamma_n > 0\) for some \(n\) then \(\psi(\{0\}) = 0\) and

\[ \int_{(0,\infty)} x^{-1} \psi(dx) < \infty \]

if \(\gamma_n \equiv 0\) then

\[ \psi(\{0\}) = \left\{\sum_{n} \frac{\alpha_1 \cdots \alpha_n}{\beta_2 \cdots \beta_{n+1}}\right\}^{-1} \geq 0 \]
summary OPS on $[0, \infty)$

$\{P_n(x), \ n = 0, 1, \ldots\}$ satisfies

$$P_n(x) = (x - \alpha_n - \beta_n - \gamma_n)P_{n-1}(x) - \alpha_{n-1}\beta_n P_{n-2}(x)$$

$P_0(x) = 1, \ P_1(x) = x - \alpha_1 - \gamma_1$

with $\alpha_n > 0, \beta_{n+1} > 0$ and $\gamma_n \geq 0$

$\implies$

$\{P_n(x), \ n = 0, 1, \ldots\}$ is OPS with respect to measure $\psi$ with support in $[0, \infty)$ and

$$\xi_1 = \lim_{n \to \infty} x_{n1} = \inf \supp(\psi)$$

$$\xi_i = \lim_{n \to \infty} x_{ni} = \inf \left\{ \supp(\psi) \setminus \bigcup_{j<i} \xi_j \right\}$$

$$\sigma = \lim_{i \to \infty} \xi_i = \inf \supp(\psi)$$
**birth-death process with killing**

**definition:** *birth-death process with killing* is Markov process \( \{X(t), t \geq 0\} \) on \( \{0, 1, \ldots\} \) with coffin state 0, *birth rate* \( \alpha_n > 0 \) and *killing rate* \( \gamma_n \geq 0 \) in state \( n \geq 1 \), and *death rate* \( \beta_n > 0 \) in state \( n > 1 \).

**representation** for \( i, j > 0 \):

\[
p_{ij}(t) := \Pr\{X(t) = j \mid X(0) = i\} = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx)
\]

with

\[
\pi_1 := 1, \quad \pi_n := \frac{\alpha_1 \cdots \alpha_{n-1}}{\beta_2 \cdots \beta_n} \quad (n > 1)
\]

and

\[
\alpha_n Q_n(x) = (\alpha_n + \beta_n + \gamma_n - x) Q_{n-1}(x) - \beta_n Q_{n-2}(x)
\]

\( Q_0(x) = 1, \quad \alpha_1 Q_1(x) = \alpha_1 + \gamma_1 - x \)
birth-death processes with killing

\[ p_{i,j}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)\psi(dx) \]

\[ t = 0: \quad \delta_{i,j} = \pi_j \int_0^\infty Q_i(x)Q_j(x)\psi(dx) \]

defining

\[ P_n(x) = (-1)^n \alpha_1 \alpha_2 \ldots \alpha_n Q_n(x) \]

we have

\[ P_n(x) = (x - \alpha_n - \beta_n - \gamma_n)P_{n-1}(x) - \alpha_{n-1}\beta_nP_{n-2}(x) \]

\[ P_0(x) = 1, \quad P_1(x) = x - \alpha_1 - \gamma_1 \]

OPS with respect to measure \( \psi \) on \([0,\infty)\)!
zeros of orthogonal polynomials

**note:** let

\[
R_n = \begin{pmatrix}
0 & 0 & 0 & \ldots & \ldots \\
-\alpha_1 - \gamma_1 & \alpha_1 & 0 & \ldots & \ldots \\
\beta_2 & -\alpha_2 - \beta_2 - \gamma_2 & \alpha_2 & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\ldots & \ldots & \ldots & \ldots & \alpha_{n-1} \\
\ldots & \ldots & \ldots & \beta_n & -\alpha_n - \beta_n - \gamma_n
\end{pmatrix}
\]

truncated $q$-matrix of birth-death process with killing, then

\[
P_n(x) = \det(xI_n + R_n)
\]

so zeros of $P_n(x)$ are eigenvalues of $-R_n$
birth-death processes: decay rate

\[ p_{ij}(t) = \pi_j \int_0^{\infty} e^{-xt} Q_i(x)Q_j(x)\psi(dx) \]

hence

\[ p_j := \lim_{t \to \infty} p_{ij}(t) = \pi_j \psi(\{0\}) \]

\[ p_{ij}(t) - p_j = \pi_j \int_0^{\infty} e^{-xt} Q_i(x)Q_j(x)\psi(dx) \]

interest: decay rate

\[ \delta = \xi_1 + \xi_2\mathbb{I}_{\{\xi_1=0\}} \]

note: if \( \psi(\{0\}) = 0 \) then \( \xi_1 > 0 \) or \( \xi_1 = \xi_2 = 0 \), so that \( \delta = \xi_1 \);
if \( \psi(\{0\}) > 0 \) then \( \xi_1 = 0 \), so that \( \delta = \xi_2 \)
birth-death processes: decay rate

given: birth rates $\alpha_n$, death rates $\beta_{n+1}$ and killing rates $\gamma_n$, $n \geq 1$

problem: determine decay rate $\delta = \xi_1 + \xi_2 \mathbb{I}_{\{\xi_1 = 0\}}$

recall: if $\gamma_n \equiv 0$ then $\psi(\{0\}) \geq 0$ is known; if $\gamma_n > 0$ for some $n$ then $\psi(\{0\}) = 0$

if $\psi(\{0\}) = 0$ (and hence $\delta = \xi_1$):

Q1: $\xi_1 = \ ?$

Q2: $\xi_1 > 0 \ ?$

if $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (and hence $\xi_1 = 0$ and $\delta = \xi_2$):

Q3: $\xi_2 = \ ?$ (spectral gap)

Q4: $\xi_2 > 0 \ ?$
birth-death processes: decay rate

**problem:** determine \( \delta = \xi_1 + \xi_2 \mathbb{I}_{\{\xi_1=0\}} \)

**approach** if \( \psi(\{0\}) = 0 \) (hence \( \delta = \xi_1 \)): representations for \( \xi_1 \)

**approach** if \( \gamma_n \equiv 0 \) and \( \psi(\{0\}) > 0 \) (hence \( \delta = \xi_2 \)): dual process

**definition:** given \( \alpha_n, \beta_{n+1} \) and \( \gamma_n \equiv 0 \) the dual process has rates

\[
\tilde{\alpha}_n := \beta_{n+1}, \quad \tilde{\beta}_{n+1} := \alpha_{n+1}, \quad \tilde{\gamma}_1 := \alpha_1, \quad \tilde{\gamma}_n := 0 \quad (n > 1)
\]

then \( \{\tilde{P}_n(x)\} \) OPS w.r.t \( \tilde{\psi} : \)

\[
\alpha_1 \tilde{\psi}([0, x]) = \int_0^x u \psi(du), \quad x \geq 0
\]

hence

\[
\xi_2 = \tilde{\xi}_1
\]
birth-death processes: decay rate

recall: (Geršgorin discs)

\[ \xi_1 = \max_{a>0} \inf_i \left\{ c_i - a_i + 1 - \frac{\lambda_i}{a_i} \right\} \]

where \( c_1 = \alpha_1 + \gamma_1 \) and

\[ c_i = \alpha_i + \beta_i + \gamma_i, \quad \lambda_i = \alpha_{i-1} \beta_i \quad (i > 1) \]

so, if \( \gamma_i \equiv 0 \) and \( \psi(\{0\}) > 0 \) (hence \( \delta = \xi_2 \)) then

\[ \delta = \xi_2 = \tilde{\xi}_1 = \max_{a>0} \inf_i \left\{ \tilde{\alpha}_i + \tilde{\beta}_i + \tilde{\gamma}_i - a_i + 1 - \frac{\tilde{\alpha}_{i-1} \tilde{\beta}_i}{a_i} \right\} \]

\[ = \max_{a>0} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_i + 1 - \frac{\alpha_i \beta_i}{a_i} \right\} \]
decay rate: more recent results

setting: $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (so that $\xi_1 = 0$ and $\delta = \xi_2$)

Geršgorin + duality:

$$\delta = \max_{a > 0} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$


$$\delta = \max_{a > 0} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$

$$= \min_{a > 0} \sup_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$

where $\beta_1 = 0$, $a = (a_1, a_2, \ldots)$
**Setting:** $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (so that $\xi_1 = 0$ and $\delta = \xi_2$)

Miclo (1999), Chen (2000):

$$\delta > 0 \iff \sup_i \left\{ \left( \sum_{j \leq i} \frac{1}{\alpha_j \pi_j} \right) \left( \sum_{j > i} \pi_j \right) \right\} < \infty$$

where

$$\pi_1 = 1, \quad \pi_j = \frac{\alpha_1 \cdots \alpha_{j-1}}{\beta_2 \cdots \beta_j} \quad (j > 1)$$

**Recall:** $\psi(\{0\}) > 0 \iff \sum_j \pi_j < \infty$
implications for OP’s

translation Miclo-Chen result: explicit criterion for positivity of spectral gap if $\xi_1$ is known

**Theorem:** let

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$

$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

and suppose $\xi_1 > -\infty$ then, defining $\pi_1 = 1, \alpha_1 = c_1 - \xi_1$ and, for $n > 1$,

$$\beta_n = \frac{\lambda_n}{\alpha_{n-1}}, \quad \alpha_n = c_n - \xi_1 - \beta_n$$

$$\pi_n = (\alpha_1 \ldots \alpha_{n-1})/(\beta_2 \ldots \beta_n)$$

we have

$$\xi_2 > \xi_1 \iff \sup_i \left\{ \left( \sum_{j \leq i} \frac{1}{\alpha_j \pi_j} \right) \left( \sum_{j > i} \pi_j \right) \right\} < \infty$$
decay rate: more recent results

setting: $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (so that $\xi_1 = 0$ and $\delta = \xi_2$)

Geršgorin + duality:

$$\delta = \max_{a > 0} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$


$$\delta = \max_{a > 0} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\} = \min_{a > 0} \sup_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$

where $\beta_1 = 0$, $a = (a_1, a_2, \ldots)$
extreme zeros of orthogonal polynomials

Granovsky-Zeifman-Chen result suggestive of

**Theorem:** let

\[ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \]
\[ P_0(x) = 1, \quad P_1(x) = x - c_1 \]

then, not only

\[ \xi_1 = \max_{a > 0} \inf_i \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\} \]

but also

\[ \xi_1 = \min_{a > 0} \sup_i \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\} \]
more generally:
Granovsky-Zeifman-Chen result suggestive of

**Theorem:** let

\[ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \]
\[ P_0(x) = 1, \quad P_1(x) = x - c_1 \]

then, not only

\[ x_{n1} = \max_{a > 0} \min_{i \leq n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\} \]

but also

\[ x_{n1} = \min_{a > 0} \max_{i \leq n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\} \]
extreme zeros of orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots\} is OPS satisfying

\[ P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \]
\[ P_0(x) = 1, \quad P_1(x) = x - c_1 \]

with \( c_n \in \mathbb{R}, \ \lambda_n > 0 \)

recall: zeros of \( P_n(x) \) are eigenvalues of

\[
T_n = \begin{pmatrix}
    c_1 & \lambda_2/a_2 & 0 & \cdots & \cdots \\
    a_2 & c_2 & \lambda_3/a_3 & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \cdots & \cdots & \cdots & c_{n-1} & \lambda_{n}/a_n \\
    \cdots & \cdots & \cdots & 0 & a_n & c_n
\end{pmatrix}
\]

where \( a_j > 0 \)
extreme zeros of orthogonal polynomials

**Theorem:**

\[ x_{nn} = \max_{a > 0} \min_{i \leq n} \left\{ c_i + a_{i+1} + \frac{\lambda_i}{a_i} \right\} \]

**Proof:** Perron-Frobenius theory for positive matrices

\[ \tilde{T}_n := T_n + dI \] positive for \( d \) sufficiently large, corresponds to \( \tilde{c}_n := c_n + d, \tilde{\lambda}_n := \lambda_n \), and has eigenvalues \( \tilde{x}_{ni} = x_{ni} + d \)

**Collatz-Wielandt:**

\[ x_{nn} = \max_{x > 0} \min_{i} \frac{(T_n x)_i}{x_i} = \min_{x > 0} \max_{i} \frac{(T_n x)_i}{x_i} \]

**Corollary:**

\[ x_{n1} = \min_{a > 0} \max_{i \leq n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\} \]