– Workshop on Stochastics and Special Functions –

Combinatorics of Orthogonal Polynomials

– Perturbed Chebyshev polynomials –

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May 22, 2009

Orthogonal Polynomials: Two questions

- **Definition (Moment functional)**

  Let $\{\mu_n\}_{n \geqslant 0}$, $\mu_n \in \mathbb{C}$ be a sequence, called the *moment sequence*, and $\mathcal{L} : \mathbb{C}(x) \to \mathbb{C}$ a functional acting on polynomials $\mathbb{C}(x)$. Then $\mathcal{L}$ is called a moment functional if

  1. $\mathcal{L}$ is linear
  2. $\mathcal{L}(x^n) = \mu_n$, $n \geqslant 0$

- **Definition (Orthogonal Polynomials)**

  Let $\{P_k(x)\}_{k \geqslant 0}$ be a sequence of polynomials in x with coefficients in the field $\mathbb{C}$. The polynomials are orthogonal with respect to a moment functional $\mathcal{L}$ if, for all $n, m \geqslant 0$,

  1. $\deg(P_n) = n$
  2. $\mathcal{L}(P_n P_m) = \Lambda_n \delta_{n,m}$, $\Lambda_n \in \mathbb{C} \backslash 0$

- Question: Given a moment sequence $\{\mu_n\}_{n\geqslant 0}$ does there exist an orthogonal polynomial sequence?
- Answer: Yes, so long as the Hankel determinants do not vanish.

$$H_n = \begin{pmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & \ddots & \vdots \\ \mu_n & \cdots & \mu_{2n} \end{pmatrix}$$

then

$$P_n(x) = \frac{1}{\det H_{n-1}} \det \begin{pmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & \ddots & \vdots \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & \cdots & x^n \end{pmatrix}.$$

- Question: Give a sequence of orthogonal polynomials $\{P_k(x)\}_{k\geqslant 0}$ that satisfy the classical three term recurrence does there exists a moment sequence and linear functional which they are orthogonal with respect to?

- **Theorem (Favard)**

  Let $\{b_k\}_{k\geqslant 0}$ and $\{\lambda_k\}_{k\geqslant 1}$ be sequences with $b_k, \lambda_k \in \mathbb{C}$ and let $\{P_k(x)\}_{k\geqslant 0}$ be a sequence of polynomials satisfying

$$P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x) \qquad k \geqslant 1$$
$$P_0 = 1, \qquad P_1 = x - b_0$$

  Then there exists a unique moment functional $\mathcal{L} : x^n \to \mu_n$ such that $\{P_k(x)\}_{k\geqslant 0}$ are a monic orthogonal sequence with respect to $\mathcal{L}$ iff $\lambda_n \neq 0$ for all $n \geqslant 1$.

- Compute $\{\mu_n\}_{n\geqslant 0}$ recursively $\mathcal{L}(1) = \mu_0$, $\mathcal{L}(P_n) = 0$, $n > 0$.

Orthogonal Polynomials: Combinatorics

- We would like a combinatorial representation:
  - Combinatorial computation of polynomials and moments
  - Combinatorial proofs; Favard, orthogonality etc.
  - Orthogonal polynomial results give combinatorial answers
- Two fundamental representations

$$P_n(x) = \sum_{\pi \in \mathbb{P}_n} w(\pi)$$

$$\mu_n = \sum_{\tau \in \mathbb{M}_n(0,0)} w(\tau)$$

$$\mathbb{P}_n = \text{Set of pavings on a line of } n \text{ vertices}$$

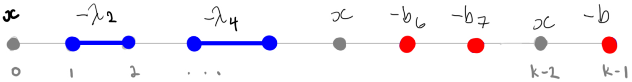
$$\mathbb{M}_n(0,0) = \text{Set of } 0 \to 0 \text{ Motzkin paths with } n \text{ steps}$$

and $w(\pi)$, $w(\tau)$ is the respective weights.

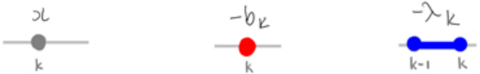
Combinatorial Objects

Definitions of pavings and paths – by example

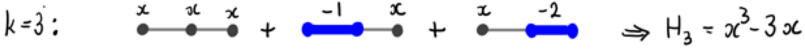
- Paving: Line segment with n vertices by:
  - ‘monomers’ – weights $\{b_k\}_{k\geqslant 0}$
  - ‘dimers’ – weights $\{\lambda_k\}_{k\geqslant 0}$



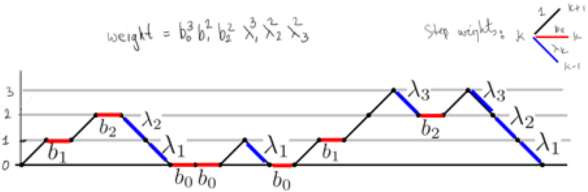
weight $= x^2 \cdot b_6 b_7 \lambda_2 \lambda_4$



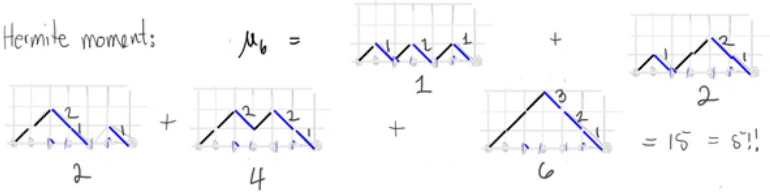
- Example: Hermite polynomials: $b_k = 0$, $\lambda_k = k$



- Motzkin paths: n steps: 'up', 'down' and 'horizontal'
  - down steps – weights $\{b_k\}_{k \geqslant 0}$
  - horizontal steps – weights $\{\lambda_k\}_{k \geqslant 0}$



- Example: Hermite polynomials: $b_k = 0$, $\lambda_k = k$



In general $\mu_n = (n-1)!!$ n even ie. number of fixed point free involutions – bijection.

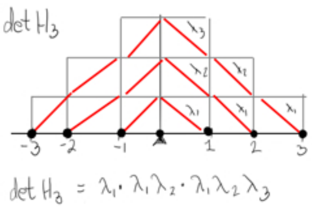
Dual representation

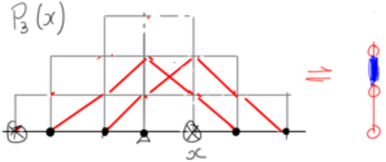
Hankel determinants

$$H_n = \begin{pmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & \ddots & \vdots \\ \mu_n & \cdots & \mu_{2n} \end{pmatrix} \qquad P_n(x) \sim \det \begin{pmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & \ddots & \vdots \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & \cdots & x^n \end{pmatrix}.$$

Combinatorially the KMLGV theorem gives:

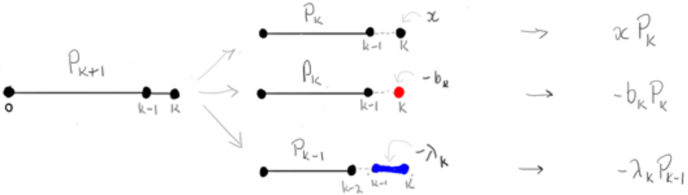
$\det H_n$ = number of n 'non-intersecting' Motzkin paths from $-k$ to k.





Combinatorial proofs

- $P_{k+1}(x) = (x - b_k)P_k(x) - \lambda_k P_{k-1}(x)$
- $P_k(x) = \sum_{\pi \in \mathbb{P}_n} w(\pi)$



Continued fraction connection

- Generating function for moments ie. Motzkin paths:

$$M(t) = \sum_{n \geqslant 0} \mu_n t^n.$$

- Jacobi continued fractions:

$$M(t) = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cdots}}}$$

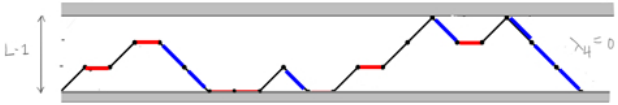
- Convergents – truncate at level L: set $\lambda_L \to 0$ and simplify

$$M(t) = \frac{N_{L-1}(t)}{D_L(t)}, \qquad N_k(t) = \hat{P}_k^*(t), \qquad D_n(t) = \hat{P}_n(t)$$

$\hat{P}_n$ is the reciprocal orthogonal polynomial: $\hat{P}_n(t) = t^{-n} P_n(1/t)$

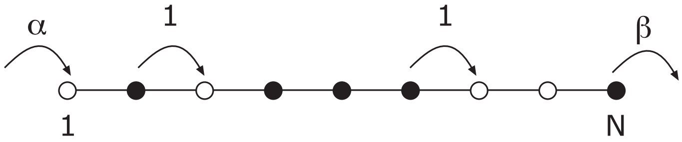
Setting $\lambda_L \to 0$ means Motzkin paths cannot have any steps at or above height L.

- $M(t) = \dfrac{N_{L-1}(t)}{D_L(t)}$ generates Motzkin paths in strip of width n

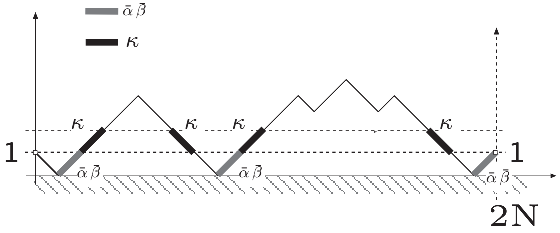


Applications

Asymmetric Simple Exclusion Process (ASEP).



- Stationary State normalisation: $Z_N(\alpha, \beta)$.

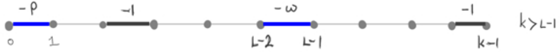


$\bar{\alpha} = 1/\alpha$, $\bar{\beta} = 1/\beta$, $\kappa^2 = 1 - (1-\bar{\alpha})(1-\bar{\beta})$:
Motzkin with $\lambda_1 = \bar{\alpha}\bar{\beta}$ and $\lambda_2 = \kappa^2$, $\lambda_k = 1$, $k > 2$.

Perturbed Chebyshev

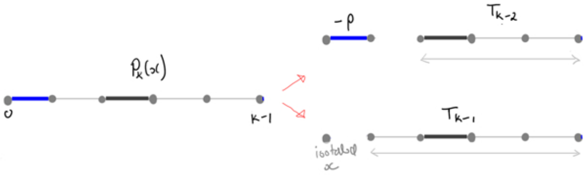
More generally, other problems (polymers interacting with a strip) require $b_k = 0$ and $\lambda_1 = -\rho$ and $\lambda_L = -\omega$, $\lambda_k = -1$, $k \neq 1, L$. (joint work with J-A. Osborn)

- If $\lambda_k = 1$, $k \geqslant 1$ then Chebyshev of the first kind.
- Thus we have the following paving problem to solve.



- But first $k < L - 1$

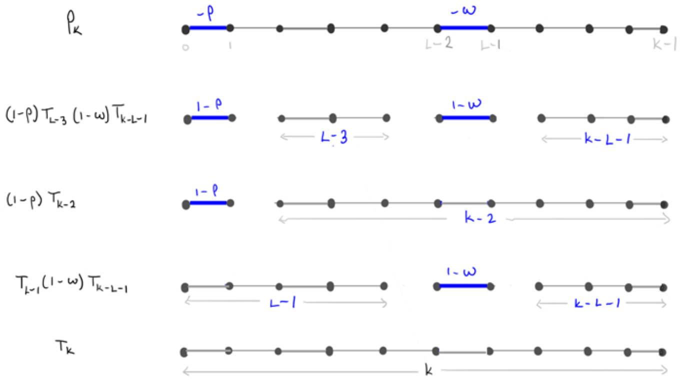
- Set of pavings partitions into two:



$$P_k = (-\rho)\,T_{k-2} + x\,T_{k-1}$$
$$= (1-\rho)\,T_{k-2} + T_k \qquad \text{using} \quad x\,T_{k-1} = T_k + T_{k-2}$$

- or combinatorially

- Thus for the two 'defect' case $k > L - 1$. There are four cases:



Thus we have,

Proposition

The orthogonal polynomial sequence P_n with $\lambda_1 = \rho$, $\lambda_{L-1} = \omega$ and $\lambda_k = 1$, $k \neq 1, L-1$ is given by

$$P_k = T_k + (1-\rho)\,T_{k-2} + (1-\omega)\,T_{L-1}\,T_{k-L-1} \\ + (1-\rho)(1-\omega)\,T_{L-3}\,T_{k-L-1}$$

where $T_k(x)$ are the Chebyshev (like) polynomials

$$T_{k+1} = xT_k(x) - T_{k-1}(x), \qquad P_1 = x, \quad P_0 = 1.$$

Thank You.