Quasi-stationary distributions and the decay parameter

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AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics and Statistics of Complex Systems
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Introduction

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Let \(P_i(\cdot) = P_i(\cdot | X_0 = i)\) and if \(\nu\) is a finite measure on \(\mathbb{N}\), let \(P_\nu = \sum \nu_i P_i\). Here and below any unqualified sum is taken over \(\mathbb{N}\).
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Let \(P_i(\cdot) = P_i(\cdot \mid X_0 = i)\) and if \(\nu\) is a finite measure on \(\mathbb{N}\), let \(P_\nu = \sum \nu_i P_i\). Here and below any unqualified sum is taken over \(\mathbb{N}\).

Finally, assume that \(\mathbb{N}\) is irreducible and that 0 is accessible from some (and hence from every) state in \(\mathbb{N}\).
We further define

\[ T = \inf\{t \geq 0 : X(t) = 0\} \]

the absorption (hitting) time at 0. We shall only be interested in processes for which \( E_i T < \infty \) for all \( i \geq 1 \).
A quasi-stationary distribution (qs) $M = (m_i)$ is a probability measure on \{1, 2, \cdots \} with the property that, starting with $M = (m_i)$, the conditional distribution, given the event that at time $t$ the process has not been absorbed, still $M = (m_i)$. That is,

$$\frac{\sum m_i P_i(X(t) = j)}{\sum m_i P_i(X(t) \neq 0)} = m_j. \quad (1)$$
Quasi-stationary distributions

Quasi-stationary distributions for Markov processes and chains have been studied by several authors. Vere-Jones (1962), Seneta and Vere-Jones (1996) and Kingman (1963) studied the case of a general denumerable state space.
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$$
\sum m_i q_{ij} = -\mu m_j,
$$

and if for all $t > 0$,

$$
\sum m_i p_{ij}(t) = e^{-\mu t} m_j,
$$

it is called $\mu$-invariant on $\{1, 2, \cdots\}$ for $P$. 
M.G. Nair and P.K. Pollett (1993) show that if $M$ is probability distribution on $\{1, 2, \cdots \}$. Then $M$ is a quasi-stationary distribution on $\{1, 2, \cdots \}$ for $P$ if and only if, for some $\mu > 0$, $M$ is $\mu$-invariant on $\{1, 2, \cdots \}$ for $P$. 
We call $M = (m_j) \nu$-the limit conditional distribution ($\nu$-LCD) if $\nu$ is a probability measure on $\{1, 2, \cdots\}$ and each $j \geq 1$

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Trivially, any qsd $M$ is an $M$-LCD.

The $\nu$-LCD is a qsd (Vere-Jones(1996)).
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(i) determination of all qsd’s; and

(ii) solve the domain of attraction problem, namely, characterize all probability measure $\nu$ such that a given qsd $M$ is a $\nu$-LCD.

Although (i) has been addressed for several cases, details about (ii) are known only for finite Markov processes, and for the subcritical MBP.
Quasi-stationary distributions

We now discuss the existence of qsd for a general Markov Chain.
We now discuss the existence of quasi-stationary distributions (qsds) for a general Markov Chain.

P.A. Ferrari, H. Kesten, S. Martinez and P. Picco (1995) prove the following interesting result which makes no reference to this general theory. They make the following definition of asymptotic remoteness (AR) of the absorbing state: For each $t > 0$

$$\lim_{i \to \infty} P_i(T > t) = 1.$$  \hspace{1cm} (12)

In other words, $T \Rightarrow \infty$ as $i \to \infty$. 
Assume that AR condition holds, Ferrari et al. prove that a quasi-stationary distribution (qsd) exists iff

$$E_i(e^{\epsilon T}) < \infty$$

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Assume that AR condition holds, Ferrari et al. prove that a qsd exists iff

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for some \( \varepsilon > 0 \) and \( i \in \mathbb{N} \).

Indeed this condition is necessary with, or without, AR condition.
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Assume that AR condition holds, Ferrari et al. prove that a qsd exists iff

\[ E_i(e^{\epsilon T}) < \infty \]  \hspace{1cm} (15)

for some \( \epsilon > 0 \) and \( i \in \mathbb{N} \).

Indeed this condition is necessary with, or without, AR condition.

T. G. Pakes (1994) investigates what happens in a number of examples when AR condition fails. In fact, he examines quite closely two examples which violate AR condition but which nevertheless can have a qsd, showing AR condition is far from being a necessary condition, though it seems essential for the proofs of Ferrari et al.’s theorem.
Quasi-stationary distributions

First we have the following

**Proposition 1** The following statements are equivalent:

1. Equation (22) holds, that is,

   \[ E_i(e^{\epsilon T}) < \infty \]

   for some \( \epsilon > 0 \) and \( i \in \mathbb{N} \).

2. There is \( \lambda \) with \( 0 < \lambda < \inf_{i \geq 1} q_i \) (here \( q_i \equiv -q_{ii} \)) for which the system

   \[
   \sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda)x_i - 1, \quad i \geq 1, \quad x_0 = 0
   \]

   has a finite non-negative solution.
Secondly we can obtain that the following condition

\[ \lim_{i \to \infty} E_i T = \infty \]  \hspace{1cm} (17)

can substitute for the AR condition (that is, for each \( t > 0 \)
\[ \lim_{i \to \infty} P_i(T > t) = 1 \) which preserves the main result of
Ferrari et all (1995).
Remarks: 1. It is easy to prove that AR condition $\Rightarrow (17)$. So condition (17) is weaker than AR condition.
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2. As we know, the mean extinction time $E_i T$ is the minimal non-negative solution of the system

$$\sum_{j \geq 0} q_{ij} z_j = -1, \quad i \geq 1, \quad z_0 = 0.$$  \hspace{1cm} (19)

So condition (17) is easier to check than AR condition.
Our main result is as follows

**Theorem 1** Assume that $Q$ is stable, conservative and regular, and that $Q$ restricted to $\{1, 2, \cdots\}$ is irreducible. Assume further that (17) holds, that is

$$\lim_{i \to \infty} E_i T = \infty$$

and that $P_i(T < \infty) = 1$ for some (and hence all) $i$. Then a necessary and sufficient condition for the existence of a qsd is that there is $\lambda$ with $0 < \lambda < \inf_{i \geq 1} q_i$ for which the system (16) (that is, $\sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda) x_i - 1$, $i \geq 1$, $x_0 = 0$) has a finite non-negative solution.
The decay parameter

Suppose we have a $q$-matrix $Q$ over $E$. Let $P$ be an arbitrary $Q$-transition function. Suppose that $E = \{0\} \cup C$, where 0 is an absorbing state and $C = \{1, 2, \cdots\}$ is irreducible. The decay parameter $\lambda$ is defined by

$$\lambda = \lim_{t \to \infty} -\frac{1}{t} \log P_{ij}(t).$$

Kingman showed that this limit exists and is the same for all $i, j \in C$, and that $0 \leq \lambda < \infty$. 
The decay parameter

It is called the **decay parameter** because there exist constants \( M_{ij} > 0 \) with \( M_{ii} = 1 \) such that

\[
P_{ij}(t) \leq M_{ij} e^{-\lambda t}, \quad i, j \in C.
\]

Note, in particular, that \( P_{ii}(t) \leq e^{-\lambda t} \).
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$$\lambda = \sup \{ \alpha : P_{ij}(t) = O(\exp[-\alpha t]) \text{ as } t \to \infty \forall i, j \in C \}.$$
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$$\lambda = \sup \{ \alpha : P_{ij}(t) = O(\exp[-\alpha t]) \text{ as } t \to \infty \forall i, j \in C \}.$$ 

If $\mu > \lambda$, there does not exist any $\mu$-invariant measure (Pollett 1986); in particularly, there does not exist any qsd if $\lambda = 0$. 
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- to give criteria for decay parameter $\lambda$ to be positive in terms of the rates $(q_{ij})$;

- to determine the value of $\lambda$, or at least bounds for $\lambda$, in terms of the rates $(q_{ij})$. 
The decay parameter

Example 1 Markov branching process (MBP).
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We shall adopt the usual notation (Anderson (1991)) in prescribing MBP, that is, let $p_k, k \geq 0$, denote a sequence of non-negative numbers such that $\sum_{k=0}^{\infty} p_k = 1$, and let

$$p(s) = \sum_{k=0}^{\infty} p_k s^k, \quad 0 \leq s \leq 1,$$

denote the probability generating function of this sequence.
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$$m = p'(1) = \sum_{k=0}^{\infty} kp_k.$$
The decay parameter

MBP with generator $Q$ is given by

$$q_{ij} = \begin{cases} 
0 & \text{if } j < i - 1 \\
-ia(1 - p_i) & \text{if } j = i \\
ipa_{j-i+1} & \text{if } j \geq i - 1, j \neq i.
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It is well known that $P_i(T < \infty) = 1$ iff $m \leq 1$. 
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And if $m \leq 1$, then the decay parameter is $\lambda = (1 - m)a$. 
The decay parameter

**Example 2**: The birth and death process
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We shall adopt the usual notation in prescribing birth rates \( \lambda_i > 0 \) \( (i \geq 1) \), with \( \lambda_0 = 0 \), and death rates \( \mu_i > 0 \) \( (i \geq 1) \). Now define by \( \pi_1 = 1 \) and

\[
\pi_n = \prod_{k=2}^{n} \frac{\lambda_{k-1}}{\mu_k}, \quad n \geq 2.
\]

We will assume the process is absorbed with probability 1, that is,

\[
\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty.
\]
The decay parameter

In order to state our main results, we need the following notation:

\[ Q_n = \left( \frac{1}{\pi_1 \mu_1} + \sum_{j=1}^{n-1} \frac{1}{\pi_j \lambda_j} \right) \sum_{j=n}^{\infty} \pi_j, \quad n \geq 1, \]

and

\[ S_0 = \sup_{n \geq 1} Q_n. \]
The decay parameter

Phil Pollett and Hanjun Zhang have obtained the following

**Theorem 2** \((4S_0)^{-1} \leq \lambda \leq S_0^{-1}\).

And, hence,

\[ \lambda > 0 \quad if \ and \ only \ if \quad S_0 < \infty. \]
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**Proposition 2** If

\[ E_i(e^{\epsilon T}) < \infty \]  

(23)

for some \( \epsilon > 0 \) and \( i \in \mathbb{N} \), then the decay parameter \( \lambda > 0 \).
The decay parameter

For a general Markov chain, we have the following

**Proposition 2** If

\[ E_i(e^{\epsilon T}) < \infty \]  \hspace{1cm} (24)

for some \( \epsilon > 0 \) and \( i \in \mathbb{N} \), then the decay parameter \( \lambda > 0 \).

By using **Proposition 1**, we get
The decay parameter

**Theorem 3** If there is $\lambda$ with $0 < \lambda < \inf_{i \geq 1} q_i$ for which the system

$$\sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda)x_i - 1, \quad i \geq 1, \quad x_0 = 0$$

(25)

has a finite non-negative solution, then the decay parameter $\lambda > 0$. 
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I guess the above condition is necessary for the decay parameter $\lambda > 0$. 
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We have the following interesting result:

**Corollary** If $\sup_{i \geq 1} E_i T < \infty$, then $\lambda > 0$. 
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\end{cases}$$

where, as above, $p_k, k \geq 0$, denote a sequence of non-negative numbers such that $\sum_{k=0}^{\infty} p_k = 1$. 
And let
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It can be seen that the ordinary MBP corresponds to the special case of \( \nu = 1 \).
R.R. Chen obtained the following conclusions

(i) If $\nu > 1$, then $Q$ is regular if and only if $m \leq 1$. 
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(iii) Assume the given GMBP $Q$ is regular. Then the extinction probability of the corresponding GMBP is 1 if and only if $m \leq 1$.

Recall that a conservative $Q$ is called regular if the Feller minimal $Q$-process is honest and thus there exists unique $Q$-process.
Anyue Chen (2002) obtained the following conclusion

If assume that the probability of eventual extinction is 1, i.e., assume $m \leq 1$. Then for all $i \geq 1$, $E_i T$ are finite if and only if

$$\int_0^1 \frac{1 - y}{p(s) - s} (-\ln y)^{\nu - 1} dy < \infty.$$  \hspace{1cm} (28)
Anyue Chen (2002) obtained the following conclusion:

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$$\int_0^1 \frac{1 - y}{p(s) - s} (-\ln y)^{\nu - 1} dy < \infty. \quad (30)$$

Moreover, if (30) is true, then for all $i \geq 1$

$$E_i T = \frac{1}{\Gamma(\nu)} \int_0^1 \frac{1 - y}{a(p(s) - s)} (-\ln y)^{\nu - 1} dy < \infty. \quad (31)$$

Where $\Gamma(\nu)$ is the gamma function.
The GMBP

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**Theorem 3** (i) If $m < 1$, and $\nu \geq 1$, then the decay parameter $\lambda > 0$.

(ii) If $m = 1$ and $\nu \geq 2$, then the decay parameter $\lambda > 0$. 
The GMBP

We here talk about the positivity of the decay parameter and the existence of qsd. The following conclusions are obtained

**Theorem 3**

(i) If $m < 1$, and $\nu \geq 1$, then the decay parameter $\lambda > 0$.

(ii) If $m = 1$ and $\nu \geq 2$, then the decay parameter $\lambda > 0$.

(iii) If $m = 1$, $1 < \nu \leq 2$ and $\sum_{k=1}^{\infty} k^2 p_k < \infty$, then there exists a qsd.
Theorem 1 Assume that $Q$ is stable, conservative and regular, and that $Q$ restricted to $\{1, 2, \cdots \}$ is irreducible. Assume further that (17) holds, that is

$$\lim_{i \to \infty} E_i T = \infty$$

and that $P_i(T < \infty) = 1$ for some (and hence all) $i$. Then a necessary and sufficient condition for the existence of a qsd is that there is $\lambda$ with $0 < \lambda < \inf_{i \geq 1} q_i$ for which the system (16), that is,

$$\sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda) x_i - 1, \quad i \geq 1, \quad x_0 = 0$$

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Conclusion

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has a finite non-negative solution, then the decay parameter $\lambda > 0$.

**Corollary** If $\sup_{i \geq 1} E_i T < \infty$, then $\lambda > 0$. 
For every birth-death process satisfying (20), that is,

\[ \sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty. \]

we have

\[ (4S_0)^{-1} \leq \lambda \leq S_0^{-1}, \]

and hence

\[ \lambda > 0 \quad if \ and \ only \ if \quad S_0 < \infty. \]
Further research

- We wish to prove $\sup_{i \geq 1} E_i T < \infty$, then there exists a qsd.
Further research

- We wish to prove $\sup_{i \geq 1} E_i T < \infty$, then there exists a qsd.

- We will obtain some formulae for the values of the decay parameter in general Markov processes.
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