The decay parameter and the smallest Dirichlet eigenvalue of a birth-death process

Hanjun Zhang
Department of Mathematics, University of Queensland
hjz@maths.uq.edu.au

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Remark

This is joint work with Phil Pollett, University of Queensland.
Outline

- Some definitions
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- Further research
Some definitions

**Definition**  Let $E$ be a countable set, to be called the *state space*. A function $P_{ij}(t)$, $i, j \in E$, $t \geq 0$, is called a *transition function* on $E$ if

- $P_{ij}(t) \geq 0$, for all $i, j \in E$ and $t \geq 0$; and
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- $P_{ij}(0) = \delta_{ij}$ (the Kronecker delta)
- $\sum_{j \in E} P_{ij}(t) \leq 1$ for all $t \geq 0$, $i \in E$.
- $P_{ij}(s + t) = \sum_{k \in E} P_{ik}(s)P_{kj}(t)$ for all $s, t \geq 0$ and $i, j \in E$ (this is called the Chapman-Kolmogorov equation, or semigroup property).
Some definitions

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- **standard** if $\lim_{t \to 0} P_{ii}(t) = 1$ for all $i \in E$;
- **honest** if $\sum_{j \in E} P_{ij}(t) = 1$ for all $t \geq 0$, $i \in E$, and **dishonest** otherwise.
Some definitions

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All the probabilistic information about the process, insofar as it concerns only countably many time instants, is contained in the transition function and initial distribution. One could almost say that the transition function is the Markov Chain.
Some definitions

It is well known that

$$P'(0) = Q$$

exists in that

$$q_{ij} := \lim_{t \to 0} t^{-1} P_{ij}(t) \text{ exists in } [0, \infty) \text{ when } j \neq i.$$  

$$q_i := -q_{ii} := \lim_{t \to 0} t^{-1}[1 - P_{ii}(t)] \text{ exists in } [0, \infty] \text{ for every } i.$$  

Fatou’s lemma shows that

$$\sum_{j \neq i} q_{ij} \leq q_i.$$  

We then call $Q = (q_{ij}, i, j \in E)$ a $q$-matrix.
The decay parameter

Suppose we have a $q$-matrix $Q$ over $E$. Let $P$ be an arbitrary $Q$-transition function. Suppose that $E = \{0\} \cup C$, where $0$ is an absorbing state and $C = \{1, 2, \cdots\}$ is irreducible. The decay parameter $\lambda_C$ is defined by

$$\lim_{t \to \infty} -\frac{1}{t} \log P_{ij}(t) = \lambda_C.$$  

Kingman showed that this limit exists and is the same for all $i, j \in C$, and that $0 \leq \lambda_C < \infty$. 
It is called the decay parameter because there exist constants $M_{ij} > 0$ with $M_{ii} = 1$ such that

$$P_{ij}(t) \leq M_{ij}e^{-\lambda_C t}, \quad i, j \in C.$$ 

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If $\mu > \lambda_C$, there does not exist any $\mu$-invariant measure (Pollett 1986); in particularly, there does not exist any quasi-stationary distribution if $\lambda_C = 0$. 

Now we shall start with discussing the general continuous-time Markov chain \( (P_{ij}(t), \, i, j \in E, \, t \geq 0) \). Let \( C \subset E \) and \( m \) is some probability measure in \( C \) such that

\[
\sum_{i \in C} m_i P_{ij}(t) \leq m_j, \quad j \in C.
\]  

Let \( L^2(m) \) be the Hilbert space of functions on \( E \) with

\[
\|f\|^2 := \sum_{i \in E} f_i^2 m_i < \infty.
\]

Here, and in what follows, we use convention \( m_i = 0, \quad i \in E \setminus C \).
The smallest Dirichlet eigenvalue

Let \( \{P(t)\}_{t \geq 0} \) be a positive, strongly continuous, contractive and Markovian semigroup (i.e., \( P(t)1 \leq 1 \ \forall t \geq 0 \)) on \( L^2(m) \) induced by \( (P_{ij}(t), i, j \in E, t \geq 0) \).

Denote by \( L \) and \( \mathcal{D}(L) \) respectively the infinitesimal generator and its domain induced by \( \{P(t)\}_{t \geq 0} \).

Let \( A = E \setminus C \). We say that \( P(t) \) is \textit{exponentially ergodic} on \( A \) in the \( L^2(m) \) norm if there is a positive \( \alpha \) such that

\[
\|P(t)f\| \leq e^{-\alpha t} \|f\|, \quad f \vert_A = 0, \ \forall t \geq 0, \tag{2}
\]

and

\[
P_{ij}(t) = 0, \quad i \in A, \ j \in E, \ \forall t \geq 0. \tag{3}
\]
The smallest Dirichlet eigenvalue

One may define

$$\text{gap}(L) = \inf \{- (Lf, f) : f \in \mathcal{D}(L), f|_A = 0, \|f\| = 1\}. \quad (4)$$

Our first step is to show that (2) and (4) are closely linked. To do so, let $D(f)$ denote the limit

$$\lim_{t \to 0} \frac{1}{t}(f - P(t)f, f) \geq \lim_{t \to 0} \frac{1}{2t} \sum_{i \in E} m_i(P(t)[f - f_i]^2)(i) \geq 0,$$

provided the limit exists. Next, define

$$\text{gap}(D) = \inf \{D(f) : f \in \mathcal{D}(D), f|_A = 0, \|f\| = 1\}. \quad (5)$$
Finally, define

\[
\delta(t) = -\sup \{ \log \| P(t) f \| : f|_A = 0, \| f \| = 1 \}. \tag{6}
\]

Since

\[
\| P(t + s) f \| \leq e^{-\delta(t)} \| P(s) f \| \leq e^{-\delta(t)-\delta(s)} \| f \|
\]

by the contraction and semigroup properties, it follows that \( \delta(t) \) is superadditive and \( \delta(0) = 0 \). Hence, the limit

\[
\delta := \lim_{t \to 0} \frac{\delta(t)}{t} = \inf_{t > 0} \frac{\delta(t)}{t} \tag{7}
\]

is well defined.
The smallest Dirichlet eigenvalue

**Theorem 1** \( \delta = \text{gap}(D) = \text{gap}(L) \).
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We now turn to study the relationship between the \( L^2 \)–exponential decay rates and the decay parameter. We recall that

\[ \lambda_C = \sup \{ \alpha : P_{ij}(t) = O(\exp[-\alpha t]) \text{ as } t \to \infty \forall i, j \in C \} . \]
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**Theorem 2** \[ \text{gap}(D) \leq \lambda_C. \]
The smallest Dirichlet eigenvalue

Three books on the relationship between Dirichlet forms and Markov processes.


The birth-death process

We shall adopt the usual notation in prescribing birth rates \( \lambda_i > 0 \ (i \geq 1) \), with \( \lambda_0 = 0 \), and death rates \( \mu_i > 0 \ (i \geq 1) \) on \( E \). Now define by \( \pi_1 = 1 \) and

\[
\pi_n = \prod_{k=2}^{n} \frac{\lambda_{k-1}}{\mu_k}, \quad n \geq 2. 
\]

We will assume the process is absorbed with probability 1, that is,

\[
\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty. 
\]
The birth-death process

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These are open problems. Many authors worked on them, for example, Callaert (1971), (1974), Callaert and Keilson (1973a), (1973b) and Van Doorn (1985), (1991).
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In order to state our main results, we need the following notation:

\[
Q_n = \left( \frac{1}{\pi_1 \mu_1} + \sum_{j=1}^{n-1} \frac{1}{\pi_j \lambda_j} \right) \sum_{j=n}^{\infty} \pi_j, \quad n \geq 1,
\]

and

\[
S_0 = \sup_{n \geq 1} Q_n.
\]
The birth-death process

**Theorem 4** \((4S_0)^{-1} \leq \lambda_C \leq S_0^{-1}\).

And, hence,

\[ \lambda_C > 0 \quad \text{if and only if} \quad S_0 < \infty. \]
Let \( \{X(t), t \geq 0\} \) be the birth-death process with \( P_{ij}(t) \) as its transition function.

The first-passage (hitting) time into state 0 is defined by

\[
\tau_0 = \inf \{ t \geq 0 : X(t) = 0 \}
\]

\((\tau_0 = 0 \text{ if } X(0) = 0)\).

We denote the moment generating function of \( \tau_0 \), given \( X(0) = i \), by

\[
G_i(\alpha) = E_i[e^{\alpha \tau_0}].
\]

It can be shown that the abscissa of convergence of \( G_i(\alpha) \) is the same for all \( i > 0 \); we denote this by \( \alpha^* \).
The birth-death process

**Theorem 5** \( \lambda_C = \alpha^* \).

In order to prove this theorem, we use the stochastic monotonicity of birth-death process and a very useful method from modern probability—Coupling. I will not give details here.
Conclusion

The decay parameter plays an important role in studying properties of Markov chains. It has a close relationship with the important concept of the smallest Dirichlet eigenvalue. They are also linked to another important concept: the abscissa of convergence of the moment generating function of the hitting time.
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For a general Markov Chain, we have

\[ \delta = \text{gap}(D) = \text{gap}(L) \leq \lambda_C. \]
For every birth-death process satisfying (8), we have

\[ \text{gap}(D) = \lambda_C = \alpha^* \]

and

\[ (4S_0)^{-1} \leq \lambda_C \leq S_0^{-1}, \]

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Further research

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- We will discuss the relationship between the decay parameter and the smallest Dirichlet eigenvalue in general Markov processes.
Acknowledgement

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