THE UNIVERSITY OF QUEENSLAND
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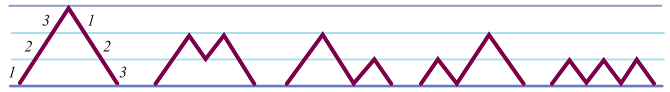
Infinity

Maths magazine Number 20 Autumn 2006

Sequences & Eugene Charles Catalan

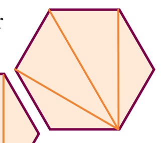
Imagine you are drawing mountains!

How many different "mountains" can you draw with 3 upstrokes and 3 downstrokes?



There are 5 as shown, but more generally how could you work out the answer if there are n upstrokes and n downstrokes?

Here is another problem. Can you work out the number of ways a convex polygon with 6 sides can be cut into triangles by connecting vertices with straight lines?



What would the answer be if the polygon had $n+2$ sides?

...How many more?

The answer is the n^{th} Catalan number named after Eugene Charles Catalan.

Eugene Charles Catalan was born in Bruges, Belgium in 1814 and died in 1894. He spent a great deal of his time studying and teaching mathematics in France. Due to his strong-left political views, he participated in France's 1848 revolution which led to the so called Second Republic. For a short period he held a seat on France's Chamber of Deputies. His mathematical work involved continued fractions, descriptive geometry, number theory and combinatorics.

There is a famous sequence of numbers which bear Catalan's name. The Catalan numbers form a sequence of natural numbers which appear in many counting problems.

The n^{th} Catalan number is given by $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$

If you're not sure what these symbols mean,

firstly $n!=n\mathrm{x}(n\text{-}1)\mathrm{x}(n\text{-}2)\mathrm{x}...\mathrm{x}2\mathrm{x}1$, and $1!=1$.

So for example, 1!=1, 2!=2x1=2, 3!=3x2x1=6 and so on.

It is also conventional to have 0!=1.

We then define, for $m \geq n$, $\binom{m}{n} = \frac{m!}{(m-n)!n!}$

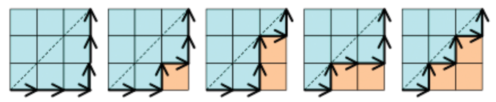
The Catalan numbers, for n=0,1,2,... are then 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, ...

Where do Catalan numbers appear?

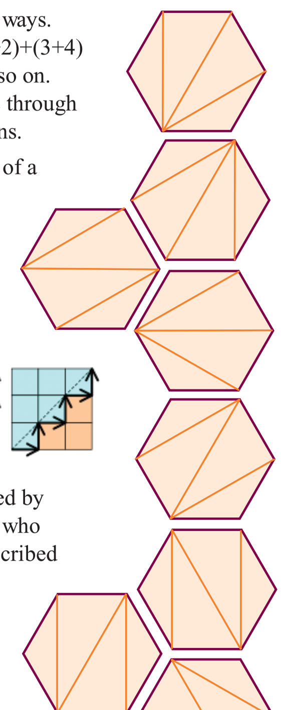
- They can be computed by a number triangle where each number is the sum of the numbers just above it and to its left. The Catalan numbers appear down the right hand side of the triangle:

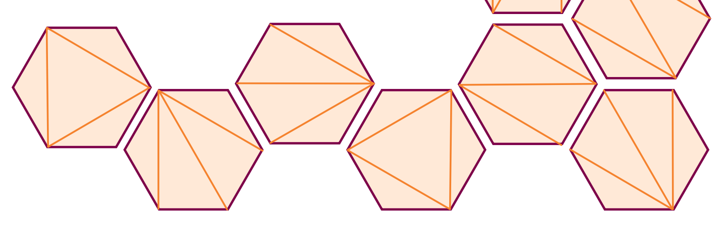
  1
  1 **1**
  1 2 **2**
  1 3 5 **5**
  1 4 9 14 **14**
  1 5 14 28 42 **42** and so on.

- There are C_n "mountains" you can draw with n upstrokes and n downstrokes.
- If you want to add n+1 numbers in the order they are given, you can do so by bracketing in C_n ways.
  For example, 1+2+3=(1+2)+3=1+(2+3): 2 ways.
  1+2+3+4=((1+2)+3)+4=(1+(2+3))+4=(1+2)+(3+4)
  =1+((2+3)+4)=1+(2+(3+4)): 5 ways, and so on.
  Catalan himself discovered these numbers through their connection with bracketed expressions.
- C_n is the number of paths along the edges of a grid with nxn square cells which do not cross the diagonal, are only directed to the right or upwards and which start in the bottom left hand corner and finish in the top right hand corner. The following diagrams show the case n=3:



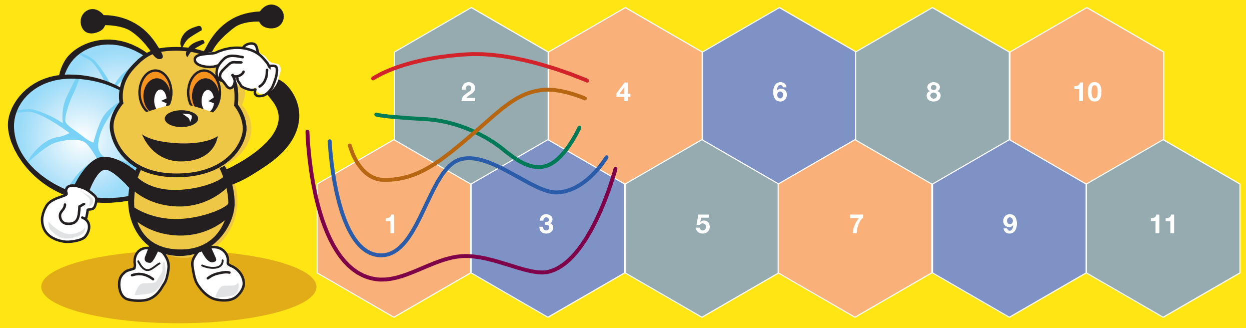
- The Catalan sequence was first encountered by Euler in the 18th Century (see Infinity 19) who was interested in the polygon problem described at the start of the article (known as Euler's Polygon Division Problem).





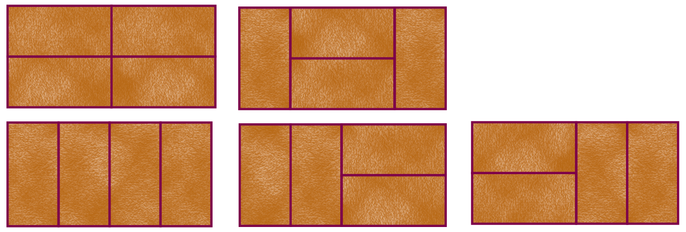
BEE-Lines

How many ways are there to reach your cell, if you can only move to the right? There is only one path to cell 1, two to cell 2, three to cell 3, but five to cell 4. How many paths are there to cell 11? (ANSWER ON PAGE 3)



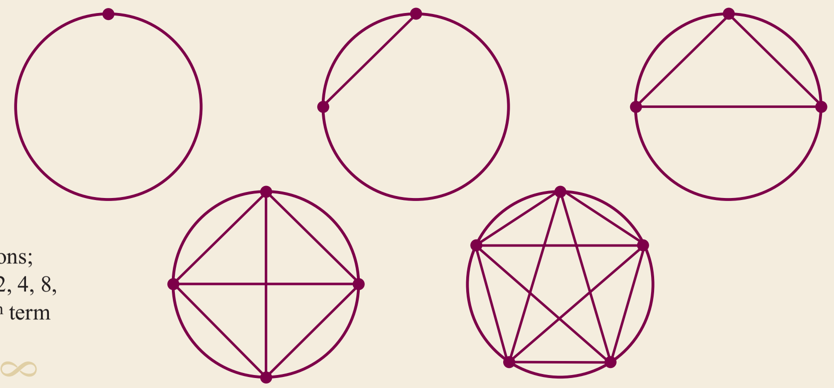
BRICKwalls

Using bricks 2 units by 1 unit, how many ways are there to build a brick wall 2 units high and 4 units long? How many ways are there to build the wall if it is 11 units long? (ANSWER ON PAGE 3)



COMPetition

Place any number of points on the edge of a circle. Connect all the points with straight lines so that no more than two lines pass through a common point inside the circle. We can make a sequence of numbers with the n^{th} term defined by the number of regions when the circle has n points on its edge. The following diagram allows you to see the first few terms in the sequence:



With 1 point, there is 1 region; 2 points = 2 regions; 3 points = 4 regions; 4 points = 8 regions; 5 points = 16 regions. The sequence so far is 1, 2, 4, 8, 16,… It seems that the sequence is just the powers of 2, so that the n^{th} term is given by 2^{n-1}, but this is not the case. What is the 7th? 8th? 15th? Can you give a general expression for the n^{th} term in the sequence?

Crossnumber

a number crossword

Each answer below (across and down) is a positive integer. No black squares this time! If you can do 1, 2, 3, 4 and 5 down, you won't need 6, 8 and 9 across, but these can serve as a check for your work! And there is no clue for 7 across; can you make one up?

1	2	3	4	5
6				
7				
8				
9				

ACROSS

1 The number of hundreds of seconds in January.
6 7 x 11 x 229
7 This is one for you! Find a suitable clue for this answer (when you've completed the rest of the puzzle).
8 This one is a product of the prime number 17419 with another small prime.
9 This is a product of the prime 2503 with a small prime to the 4th power.

DOWN

1 If it takes six boys six days to peel 8748 kg of potatoes working a nine hour day, how many kilograms of potatoes can 11 boys peel in 8 nine-hour days? (Assume a constant rate of potato peeling!)
2 The product of the first, third, 19th and 26th prime numbers.
3 The number of seconds in (8/9)ths of a twenty-four hour day.
4 123 x 678.
5 You have a bag of 18 dominoes, all of them different. You pull out 8 of them all at once and put them in a heap. How many different such sets of 8 dominoes can you pull out of the collection of 18?

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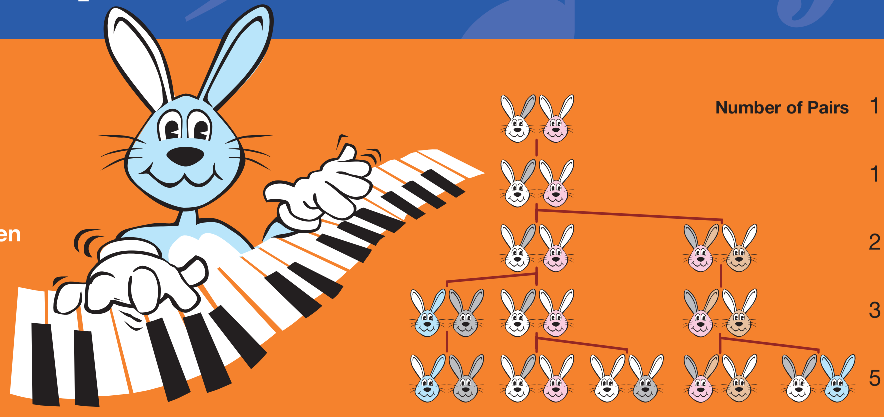
Music, the Fibonacci Sequence and the Golden Section

Andrew Garton (Honours student in Mathematics with Professor Phil Pollett)

Mathematicians have been interested in music and vice versa since the time of Pythagoras, but the first conscious use of Fibonacci sequences and the golden section as a compositional device is disputed. They are certainly used extensively by Béla Bartók (1881-1945) and Claude Debussy in the early twentieth century, but there are some indications that even Beethoven employed them in some of his string quartets.



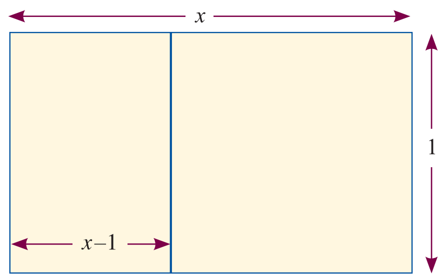
The Fibonacci sequence is the endless sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233,... It is the most famous example of an additive sequence, where, apart from the first two numbers, each number in the sequence is always the sum of the previous two.

The sequence arose in the 13th Century when the Italian mathematician Leonardo Pisano, better known as Fibonacci (short for Filius Bonacci, which means "son of Bonacci" in Latin), investigated a problem on the breeding of rabbits. (See Infinity issue 13.) ∞

THE GOLDEN RATIO IS INTIMATELY CONNECTED TO THE FIBONACCI SEQUENCE.

Take a rectangle with side length 1 and x. Divide it into a square of side length 1 and a smaller rectangle of side length $(x-1)$ and 1.
If the smaller rectangle has the same proportions as the large one, that is $1/x = (x-1)/1$ then we say x, or more precisely $x/1$, is the golden ratio.
The equation $1/x = (x-1)/1$ can be rewritten as

$$x^2 - x - 1 = 0, \text{ which has solution } x = \frac{1+\sqrt{5}}{2} \approx 1.618034$$



Now if you take the ratio of two consecutive numbers in the Fibonacci sequence or indeed any additive sequence:

$$A_{n+1} = A_n + A_{n-1}, \quad \text{then} \quad \frac{A_{n+1}}{A_n} = 1 + \frac{1}{A_n / A_{n-1}}$$

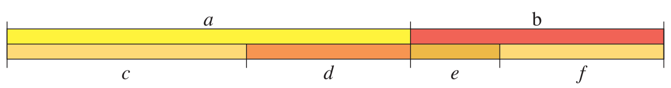
This means that the limit of this ratio,

$\phi = \lim_{n\to\infty} \frac{A_{n+1}}{A_n}$, satisfies $\phi = 1 + 1/\phi$ and is a solution to the quadratic i.e.,

ϕ is the golden ratio. So the ratio of two consecutive numbers in an additive sequence approximates the golden ratio; now the further along the sequence, the better the approximation. For instance 55/34 is within 0.03% of ϕ.

5/3	~1.666666
8/5	1.6
13/8	1.625
21/13	~1.615384
34/21	~1.619048
55/34	~1.617647
89/55	~1.618182
144/89	~1.617978
233/144	~1.618056

One property used in music, which both the Golden Ratio and the Fibonacci sequence exhibit, is additivity. For any golden section, the resulting proportions can be subdivided again by the golden section and so on.



In the diagram above,
$\phi = a/b = c/d = f/e$, which in turn implies other structures.
Here $d/e = \phi$. (See if you can prove this.)

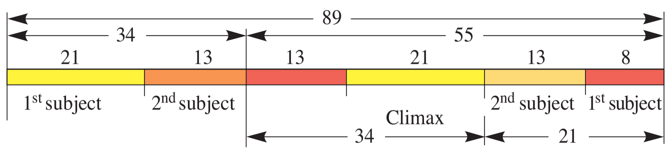
Bartók's interest in natural occurrences of the golden ratio and the Fibonacci approximations to the golden ratio (see Infinity 13) led him to employ these ratios as fundamental structural building blocks for some of his works.

In many of his works the climax of the composition or movement is placed at the approximate golden section of that piece.

The third movement of *Music for Strings, Percussion and Celeste* is particularly interesting because in its very first three bars it offers a clue to the structure of the entire movement. It begins with the rhythmic sequence 1, 1, 2, 3, 5, 8, 5, 3, 2, 1, 1; counting up and then down the Fibonacci sequence.

If one unit is taken as 4 crotchets, the structure of the rest of the piece is based on Fibonacci numbers.

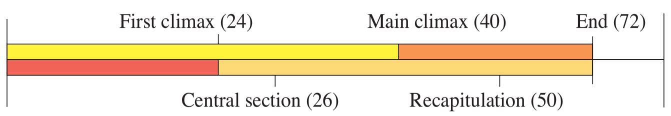
Structure of third movement of *Music for Strings, Percussion and Celeste.*



The structure of the compositions of Claude Debussy is often very difficult to define in terms of common musical analysis. The forms of his pieces clearly contain considerable aesthetic logic, but this cannot be easily explained by pre-twentieth century analytical techniques. However, once the golden section is considered, the structures of some of the works become very clear and rational.

Take *Clair de Lune*, discounting the coda; the main climax occurs at a ratio of 40:65 = 8:13, the first climax at a ratio of 24:40 = 3:5 and approximately at the negative golden mean (65-24):65.

Structure of *Clair de Lune* (bar numbers are in brackets)



Aside from Debussy and Bartók many other composers have used additive sequences, the other most common being the Lucas sequence that starts with 1 and 3. The mathematical properties of the Golden Section and these additive sequences are exceptionally suited to applications in composition, creating structure of a less obvious kind than normally sought. ∞

PROfiles

LUCY FITZGERALD

Lucy had a BCom from Bond University before embarking on a BSc at the University of Queensland (UQ). "In commencing my maths degree, I hoped to find a more challenging dimension to my existing accounting experience."

Lucy studied part-time at UQ while working as an accountant for BDO Kendalls and received her BSc (Maths) in July 2004 and a BSc (Hons) in December 2005. She is now working with BDO Corporate Finance in Sydney.

"As an analyst, I assist in the preparation and review of financial models, preparation of financial valuations, provision of Independent Expert Reports, and due diligence investigations. While it is not necessary to have a mathematical background in this job, the logic and reasoning I learned through every course in the maths degree prove invaluable on every project. I can develop an efficient approach to solving problems and produce technically accurate work. This additional level of understanding is a distinct advantage in my role." ∞

DANIEL HORSLEY

After completing his Honours and receiving a University Medal for this in 2001, Daniel embarked on a PhD in combinatorics. Daniel made the news in November 2005 for solving, with his supervisor Darryn Bryant, a 30 year old mathematics problem, which was first posed in the 1970s by Professor Lindner in the USA.

Daniel says: "*At the moment I am really enjoying my PhD! What I like best is working on a problem and playing around with different ideas on how to solve it. I particularly like combinatorics because you can often understand a problem within a few minutes of first hearing it and start fiddling with it straight away. You either have a solution to a problem or you don't.*"

Daniel's advice to students: "*I think people should not get turned off maths by their high school experience. There is a lot of great maths around which has nothing to do with calculus or trigonometry.*" ∞

EDITorial

Thank you for all your entries to our competition. Congratulations go to Ben Leavy and Matt Lepahe of Iona College for the best solutions to the infinity 19 competition.

Check out the On-Line Encyclopedia of Integer Sequences at
http://www.research.att.com/~njas/sequences/index.html

Just type in your own sequence and you can find out if it has a name and if there is a general expression that defines it. Also there are lots of Fibonacci puzzles at this site;
http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibpuzzles.html

Our sincere apologies to readers for a couple of errors in the last issue of Infinity on page 4. The top row of the Latin square A should read 4 3 8 and not 4 3 3.

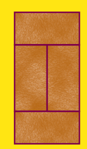
Also the Latin square D is not colored correctly.

Here is the correct coloring.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Many thanks to all those who sent in corrections.

ANSWERS to BEE-Lines and BRICKwalls (page 4)...



The answer to both BEE-Lines and Brickwalls is 12th Fibonacci number 89.

The elusive nature of the Golden Ratio!

Of all the irrationals the Golden Ratio is the hardest to approximate by a rational. The best way to see this is through continued fractions. Any rational or irrational can be represented as a continued fraction, which looks like this!

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

,where a_0 is an integer, and for i=1, 2, 3... the a_i are positive integers. If the series is truncated, the result is a rational number.

Every infinite continued fraction represents an irrational number. Now there is a systematic way to generate the a_i's. The number a_0 is the integer part of x; that is, the largest integer less than or equal to x.
If you let $x_1 = \frac{1}{x - a_0}$, then a_1 is the integer part of x_1.

Proceeding recursively, define $x_{i+1} = \frac{1}{x_i - a_i}$ and define a_{i+1} to be the integer part of x_{i+1}.

So for instance, for $\pi = 3.1415...$, $a_0 = 3$ and $x_1 = \frac{1}{0.1415...} = 7.0625...$, so that $a_1 = 7$.

And $x_2 = \frac{1}{0.0625...} = 15.996...$, so that $a_2 = 15$, etc.

$$\text{So } \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{a_3 + \dots}}}.$$

You can use this to get rational approximations to π. For instance, truncating after a_1 gives $\pi = 22/7$,
or truncating after a_2 gives $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106}$.

Actually, as you might expect, because 15 is such a large number the first approximation is pretty good; it's only 0.04% out!

Now for the Golden ratio all the a_i's are 1!

$$\Phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

See if you can prove this! (Hint; show that $\Phi = 1 + \frac{1}{\Phi}$).

Here the rational approximations look rather familiar.

Truncating the continued fraction after a_1 gives $\Phi \approx \frac{3}{2}$;

truncating after a_2 gives $\Phi \approx \frac{5}{3}$; and after a_3 gives $\Phi \approx \frac{8}{5}$.

Look familiar? These are ratios of consecutive Fibonacci numbers. But the approximations they give are really quite poor.

Here $\Phi \approx \frac{3}{2}$ is over 7% out and even $\Phi \approx \frac{5}{3}$ is over 3% out.

In fact you can prove that of all the irrational numbers the Golden Ratio is the hardest to approximate by a rational. As it turns out this can be quite a useful property in other areas of mathematics and physics such as the dynamics of the solar system. So perhaps being elusive has its advantages! ∞

Infinity Team
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