# Long Live the King

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# $g$-Selberg integrals

The Selberg integral corresponds to the following $k$-dimensional generalisation of the beta integral:

$$\int_D \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le k} (t_i-t_j)^{2\gamma}\,\mathrm{d}\mathbf{t} = \prod_{i=0}^{k-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma\big(\alpha+\beta+(k+i-1)\gamma\big)\Gamma(\gamma)}$$

Here

$$D=\{\mathbf{t}\in\mathbb{R}^k,\ 0\le t_k\le\cdots\le t_1\le 1\}$$

and

$$\mathrm{d}\mathbf{t}=\mathrm{d}t_1\cdots\mathrm{d}t_k$$

The Selberg-like integral

$$\int_C \prod_{i=1}^{k}(t_i-z)^{\alpha-1}(t_i-w)^{\beta-1}\prod_{1\le i<j\le k}(t_i-t_j)^{2\gamma}\,\mathrm{d}\mathbf{t}$$

(where $C$ is a $k$-dimensional Pochhammer double loop) arises in the solution of the Knizhnik–Zamolodchikov (KZ) equation for the Lie algebra $\mathfrak{sl}_2 = \mathrm{A}_1$.

In Arrangements of hyperplanes and Lie algebra homology, (Invent. Math. **106** (1991), 139–194)) Schechtman and Varchenko solved the KZ equation for general simple Lie algebra $\mathfrak{g}$ in terms of Selberg-like integrals.

- Simple Lie algebra $\mathfrak{g}$ of rank $n$.
- Root system $\Phi$ with simple roots $\alpha_i$, $i \in [n]$.
- Fundamental weights $\Lambda_i$, $i \in [n]$.
- Bilinear symmetric form $(\cdot,\cdot)$ on the dual of the Cartan subalgebra:

$$(\alpha_i, \Lambda_j) = \delta_{ij}$$

and

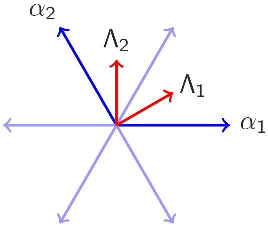
$$\Big( (\alpha_i, \alpha_j) \Big)_{i,j=1}^{n} = \text{Cartan matrix of } \mathfrak{g}$$

- Two highest weight modules $V_\lambda$ and $V_\mu$ of highest weight $\lambda$ and $\mu$ where

$$\lambda = \sum_{i=1}^{n} \lambda_i \Lambda_i \quad \text{and} \quad \mu = \sum_{i=1}^{n} \mu_i \Lambda_i$$

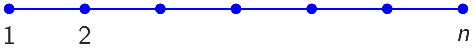
Let $\epsilon_i$ the $i$th standard unit vector in $\mathbb{R}^{n+1}$. Then the root system $A_n$ is given by

$$\{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i \leq j \leq n+1\}$$




The root system $A_2$ with fundamental weights in red.

$$\Big( (\alpha_i, \alpha_j) \Big)_{i,j=1}^{n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$




The Cartan matrix and Dynkin diagram of $A_n$.

Attach the set

$$\{t_j\}_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i}$$

of $k_i$ integration variables to the simple root $\alpha_i$ (or to the $i$th node of the Dynkin diagram of $\mathfrak{g}$) and write

$$\alpha_{t_j} = \alpha_i$$

if the variable $t_j$ is attached to the root $\alpha_i$.

Then the master function is defined as

$$\Phi_{\lambda,\mu}(\mathbf{t}) = \prod_{i=1}^{k} t_i^{-(\lambda,\alpha_{t_i})}(1-t_i)^{-(\mu,\alpha_{t_i})} \prod_{1\le i<j\le k} (t_i - t_j)^{(\alpha_{t_i},\alpha_{t_j})}$$

where $k = k_1 + \cdots + k_n$.

Let $D$ be an "appropriately chosen" real integration domain or chain in $[0,1]^k$.

Then the $\mathfrak{g}$-Selberg integral is

$$S^{\mathfrak{g}}_{\lambda,\mu;\gamma} = S^{\mathfrak{g}}_{\lambda,\mu;\gamma}(k_1,\ldots,k_n) := \int_D |\Phi_{\lambda,\mu}(\mathbf{t})|^{\gamma}\,\mathrm{d}\mathbf{t}$$

- Example 1: $g = A_1$

$$\lambda = (1-\alpha)\gamma^{-1}\Lambda_1 \quad \text{and} \quad \mu = (1-\beta)\gamma^{-1}\Lambda_1$$

$$S^{A_1}_{\lambda,\mu;\gamma} = \int_D \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le k} |t_i - t_j|^{2\gamma} \mathrm{d}\mathbf{t}$$

- Example 2: $g = A_2$

$$\lambda = (1-\alpha_1)\gamma^{-1}\Lambda_1 + (1-\alpha_2)\gamma^{-1}\Lambda_2$$

$$\mu = (1-\beta_1)\gamma^{-1}\Lambda_1 + (1-\beta_2)\gamma^{-1}\Lambda_2$$

$$S^{A_2}_{\lambda,\mu;\gamma} = \int_D \prod_{i=1}^{k_1} t_i^{\alpha_1-1}(1-t_i)^{\beta_1-1} \prod_{i=1}^{k_2} s_i^{\alpha_2-1}(1-s_i)^{\beta_2-1}$$

$$\times \prod_{1\le i<j\le k_1} |t_i - t_j|^{2\gamma} \prod_{1\le i<j\le k_2} |s_i - s_j|^{2\gamma} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} |t_i - s_j|^{-\gamma} \mathrm{d}\mathbf{t}$$

# The Mukhin–Varchenko conjecture

Mukhin and Varchenko made the following conjecture regarding the $\mathfrak{g}$-Selberg integral.

Let $\mathrm{Sing}_{\lambda,\mu}[\nu]$ denote the space of singular vectors of weight $\nu$ in $V_\lambda \otimes V_\mu$:

$$\mathrm{Sing}_{\lambda,\mu}[\nu] := \{v \in V_\lambda \otimes V_\mu : h_i v = \nu(h_i) v,\ e_i v = 0,\ 1 \le i \le n\}.$$

**Conjecture** (Mukhin–Varchenko)

If $\mathrm{Sing}_{\lambda,\mu}[\lambda + \mu - \sum_{i=1}^{n} k_i \alpha_i]$ is one-dimensional then $S^{\mathfrak{g}}_{\lambda,\mu;\gamma}$ evaluates as a product of gamma functions.

Remarks on critical points of phase functions and norms of Bethe vectors, Adv. Stud. Pure Math. **27** (2000), 239–246.

Let

$$\lambda = (1-\alpha)\gamma^{-1}\Lambda_n, \qquad \mu = (1-\beta_1)\gamma^{-1}\Lambda_1 + \cdots + (1-\beta_n)\gamma^{-1}\Lambda_n$$

**Theorem (SOW)**

If $k_1 \le k_2 \le \cdots \le k_n$ then

$$S^{\mathrm{A}_n}_{\lambda,\mu;\gamma} = \prod_{s=1}^{n}\prod_{i=1}^{k_s} \frac{\Gamma(\alpha\delta_{s,n} + (i - k_{s+1} - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}$$

$$\times \prod_{1\le s\le r\le n}\prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma(\beta_s + \cdots + \beta_r + (i+s-r-1)\gamma)}{\Gamma\big(\alpha\delta_{r,n} + \beta_s + \cdots + \beta_r + (i+s-r+k_r-k_{r+1}-2)\gamma\big)}$$

A Selberg integral for the Lie algebra $\mathrm{A}_n$, Acta Math. to appear.

# An $A_2$ Selberg integral

Let $P_\lambda(X) = P_\lambda(X; q, t)$ be a (normalised) Macdonald polynomial and let

$$(a)_\lambda = (a; q, t)_\lambda = \prod_{i \geq 1} (at^{1-i}; q)_{\lambda_i}$$

be a $q$-shifted factorial labelled by a partition.

**Theorem:** $A_2$ Cauchy-type identity (SOW)

Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$. For $abt = q$,

$$\sum_{\lambda,\mu} P_\lambda(X) P_\mu(Y) \, (at^m)_\lambda (bt^n)_\mu \prod_{i=1}^{n} \prod_{j=1}^{m} t^{-\mu_j} \frac{(at^{j-i})_{\lambda_i - \mu_j}}{(at^{j-i+1})_{\lambda_i - \mu_j}}$$

$$= \prod_{i=1}^{n} \frac{(ax_i)_\infty}{(x_i)_\infty} \prod_{i=1}^{m} \frac{(by_i)_\infty}{(y_i)_\infty} \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(x_i y_j)_\infty}{(t^{-1} x_i y_j)_\infty}.$$

Special cases:

- $m = 0$: $q$-binomial theorem for Macdonald polynomials

$$\sum_{\lambda} (a)_\lambda \mathrm{P}_\lambda(X) = \prod_{i=1}^{n} \frac{(ax_i)_\infty}{(x_i)_\infty}$$

- $n = 0$: $q$-binomial theorem for Macdonald polynomials

$$\sum_{\lambda} (a)_\lambda \mathrm{P}_\lambda(Y) = \prod_{i=1}^{m} \frac{(ay_i)_\infty}{(y_i)_\infty}$$

- $a = b = 1,\ t = q,\ Y \mapsto qY$: Cauchy identity for Schur functions

$$\sum_{\lambda} s_\lambda(X) s_\lambda(Y) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j}$$

## Corollary 1: $A_2$ Selberg integral

Let

$$\lambda = (1-\alpha_1)\gamma^{-1}\Lambda_1 + (1-\alpha_2)\gamma^{-1}\Lambda_2$$

$$\mu = (1-\beta_1)\gamma^{-1}\Lambda_1 + (1-\beta_2)\gamma^{-1}\Lambda_2$$

such that

$$\alpha_1 + \alpha_2 = \gamma + 1$$

Then

$$S^{A_2}_{\lambda,\mu;\gamma} = \prod_{i=1}^{k_1} \frac{\Gamma(\alpha_1 + (i-k_2-1)\gamma)\Gamma(\beta_1 + (i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha_1+\beta_1+(i+k_1-k_2-2)\gamma)\Gamma(\gamma)}$$
$$\times \prod_{i=1}^{k_2} \frac{\Gamma(\alpha_2 + (i-1)\gamma)\Gamma(\beta_2 + (i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\alpha_2+\beta_2+(i+k_2-k_1-2)\gamma)\Gamma(\gamma)}$$
$$\times \prod_{i=1}^{k_1} \frac{\Gamma(\beta_1+\beta_2+(i-2)\gamma)}{\Gamma(\beta_1+\beta_2+(i+k_2-2)\gamma)}$$

Define the $n$-dimensional $q$-integral

$$S^{(n,m)}(\alpha_1,\alpha_2,\beta;k) := \int_{[0,1]^n} \prod_{i=1}^{n} x_i^{\alpha_1-1}(x_i q)_{\beta-(n-1)k-1} \times \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(x_i q)_{\alpha_2+\beta+(m-n-j)k-1}}{(x_i q)_{\alpha_2+\beta+(m-n-j+1)k-1}} \prod_{1\le i<j\le n} x_i^{2k}(q^{1-k}x_j/x_i)_{2k}\,\mathrm{d}_q x.$$

For $m=0$:

$$S^{(n,0)}(\alpha_1,\alpha_2,\beta;k) = \int_{[0,1]^n} \prod_{i=1}^{n} x_i^{\alpha_1-1}(x_i q)_{\beta-(n-1)k-1} \times \prod_{1\le i<j\le n} x_i^{2k}(q^{1-k}x_j/x_i)_{2k}\,\mathrm{d}_q x$$

this is the $q$-Selberg integral of Askey, Habsieger and Kadell.

## Corollary 2: $q$-Selberg integral transform

$$S^{(n,m)}(\alpha_1,\alpha_2,\beta;k) = q^{\zeta}\, S^{(m,n)}(\alpha_2,\alpha_1,\beta;k) \times \prod_{i=1}^{n} \frac{\Gamma_q(\beta-(i-1)k)\Gamma_q(\alpha_1+(n-i)k)\Gamma_q(ik+1)}{\Gamma_q(\alpha_1+\beta+(n-m-i)k)\Gamma_q(k+1)} \times \prod_{i=1}^{m} \frac{\Gamma_q(\alpha_2+\beta+(m-n-i)k)\Gamma_q(k+1)}{\Gamma_q(\beta-(i-1)k)\Gamma_q(\alpha_2+(m-i)k)\Gamma_q(ik+1)}$$

where

$$\zeta = 2k^2\binom{n}{3} - 2k^2\binom{m}{3} + \alpha_1 k\binom{n}{2} - \alpha_2 k\binom{m}{2}$$

# Summary

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