Hall

Littlewood

&

$$\sum_{n_1,n_2\geqslant 0} \frac{q^{n_1^2-n_1n_2+n_2^2}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2}$$

Introduction

$\chi_\Lambda(z,q)$ character of $A^{(1)}_{n-1}$ highest weight module of highest weight Λ.

Branching rule:

$$\chi_\Lambda(z,q)\,\chi_{\Lambda'}(z,q) = \sum_{\Lambda''} b^{\Lambda''}_{\Lambda\Lambda'}(q)\,\chi_{\Lambda''}(z,q)$$

$\Lambda = \mu_0\Lambda_0 + \cdots + \mu_{n-1}\Lambda_{n-1}$, Λ_i fundamental weights

Level $\ell = \mu_0 + \cdots + \mu_{n-1} \in \mathbb{Q}$

If $\ell = \frac{n}{k-n} - n$, $\ell' = 1$, $\ell'' = \ell + \ell' = \frac{k}{k-n} - n$

then $b^{\Lambda''}_{\Lambda\Lambda'}(q)$ may be labelled by a single level $k-n$ dominant integral weight j, i.e.,

$$j = (j_0 - 1)\Lambda_0 + \cdots + (j_{n-1}-1)\Lambda_{n-1}$$

$$j_i \geq 1, \quad j_0 + \cdots + j_{n-1} = k$$

$$b_j(q) = \frac{(q^k;q^k)_\infty^{n-1}}{(q;q)_\infty^{n-1}} \prod_{a=1}^{n-1} \prod_{b=0}^{n-1} (q^{j_b+j_{b+1}+\cdots j_{a+b-1}};q^k)_\infty$$

Here $j_i = j_{i'}$ if $i \equiv i' \pmod n$ and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$
$$(a;q)_\infty = (1-a)(1-aq)\cdots\cdots \qquad |q|<1$$
$$(a_1,a_2,\ldots,a_k;q)_n = (a_1;q)_n\cdots(a_k;q)_n$$

Example 1

$A_1^{(1)}$, $k=5$, $j_0=2$, $j_1=3$ (or $j_0=3$, $j_1=2$)

$$b_j(q) = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}$$

Example 2

$A_1^{(1)}$, $k \to 2k+1$, $j_0=k$, $j_1=k+1$

$$b_j(q) = \frac{(q^k,q^{k+1},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}$$

$$b_j(q) = \frac{(q^k;q^k)_\infty^{n-1}}{(q;q)_\infty^{n-1}} \prod_{a=1}^{n-1} \prod_{b=0}^{n-1} (q^{j_b+j_{b+1}+\cdots j_{a+b-1}};q^k)_\infty$$

Here $j_i = j_{i'}$ if $i \equiv i' \pmod{n}$ and

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$$(a_1,a_2,\dots,a_k;q)_n = (a_1;q)_n \cdots (a_k;q)_n$$

Example 1

$A_1^{(1)}$, $k=5$, $j_0=2$, $j_1=3$ (or $j_0=3$, $j_1=2$)

$$b_j(q) = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty} = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}$$

first RR identity

Example 2

$A_1^{(1)}$, $k \to 2k+1$, $j_0=k$, $j_1=k+1$

$$b_j(q) = \frac{(q^k,q^{k+1},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}$$
$$= \sum \frac{q^{n_1^2+\cdots+n_{k-1}^2}}{(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-1}}}$$

AG general.

Despite > 100 years of RR identities and numerous generalizations (all related to $A_1^{(1)}$) very little is known about $b_j(q)$ for $A_{n-1}^{(1)}$.

Andrews, Schilling, SOW ('99):

$A_2^{(1)}$, $(j_0, j_1, j_2) = (2,2,3)$, $k = 2+2+3 = 7$

$$b_j(q) = \frac{(q^2, q^2, q^3, q^4, q^5, q^5, q^7, q^7; q^7)_\infty}{(q;q)_\infty^2}$$

$$= \sum_{n_1, n_2 \geq 0} \frac{q^{n_1^2 - n_1 n_2 + n_2^2}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$ is a q-binomial coefficient

Motivation

Let $x = (x_1, \ldots, x_n)$ and $P_\lambda(x;q)$ the Hall-Littlewood polynomial labelled by the partition λ.

Macdonald:

$$\sum_{\lambda} P_{2\lambda}(x;q) = \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\leqslant i<j\leqslant n} \frac{1-q\,x_i x_j}{1-x_i x_j} =: \Psi(x;q$$

Stembridge:

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant k-1}} P_{2\lambda}(x;q) = \sum_{\varepsilon \in \{-1,1\}^n} \Psi(x^{\varepsilon};q)\,(x^{1-\varepsilon})^{k-1} \qquad (1)$$

Specialization formula

$$P_\lambda(1,q,\ldots,q^{n-1}) = \frac{q^{n(\lambda)}\,(q;q)_n}{(q;q)_{n-\ell(\lambda)}\, b_\lambda(q)} \qquad (2)$$

$n(\lambda) = \sum_{i\geqslant 1} (i-1)\lambda_i$, $\ell(\lambda)$ length of λ,

$$b_\lambda(q) = \prod_{i\geqslant 1} (q;q)_{m_i(\lambda)} = \prod_{i\geqslant 1} (q;q)_{\lambda_i' - \lambda_{i+1}'}$$

(1) & (2) & $n \to \infty$ $\Rightarrow$

$$\sum_{\substack{\lambda \\ \lambda_1 < k}} \frac{q^{2n(\lambda)+|\lambda|}}{b_\lambda(q)} = \frac{(q^k, q^{k+1}, q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty}$$

$k=2$ first Rogers-Ramanujan identity

$k>2$ Andrews-Gordon generalization

<u>Question</u> Can one prove the A_2 RR identity using Hall-Littlewood polynomials?

Hall-Littlewood functions

$$P_\lambda(x;q) = \sum_{w \in S_n/S_n^\lambda} w\left(x^\lambda \prod_{\lambda_i > \lambda_j} \frac{x_i - q x_j}{x_i - x_j} \right)$$

The $P_\lambda(x;q)$ form a $\mathbb{Z}[q]$ basis of $\Lambda_n[q]$

Stability $P_\lambda(x_1,\dots,x_n,0;q) = P_\lambda(x_1,\dots,x_n;q) \Rightarrow$
Hall-Littlewood 'functions' in an infinite # of variables $x = (x_1, x_2, \dots)$, $\mathbb{Z}[q]$ basis of $\Lambda[q]$.

$P_\lambda(x;1) = m_\lambda(x)$ monomial symmetric function

$P_\lambda(x;0) = s_\lambda(x)$ Schur function

Classical $\sum_\lambda m_\lambda(x) = \prod_{i \geq 1} \frac{1}{1-x_i} \Rightarrow$

$$\sum_{\lambda,\mu} m_\lambda(x)\, m_\mu(y) = \prod_{i \geq 1} \frac{1}{(1-x_i)(1-y_i)}$$

A_2 Hall-Littlewood identity

$$\boxed{\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q) P_\mu(y;q) = \prod_{i\geq 1} \frac{1}{(1-x_i)(1-y_i)} \prod_{i,j\geq 1} \frac{1-x_i y_j}{1-q^{-1}x_i y_j}}$$

$$n(\lambda) = \sum_{i\geq 1} (i-1)\lambda_i = \sum_{i\geq 1} \binom{\lambda_i'}{2}$$

$$(\lambda|\mu) = \sum_{i\geq 1} \lambda_i \mu_i$$

$$\psi_T(q) = \prod_i \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q)$$

$$\phi_T(q) = \prod_i \phi_{\lambda^{(i)}/\lambda^{(i-1)}}(q)$$

$$\psi_{\lambda/\mu}(q) = \prod_{j \in J} \left(1 - q^{m_j(\mu)}\right)$$

$\lambda - \mu$ horizontal strip

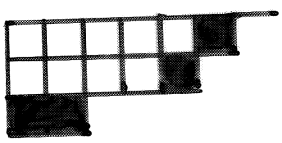
J: set of integers j such that column j of $\lambda - \mu$ is empty and column $j+1$ is nonempty

$$\phi_{\lambda/\mu}(q) = \prod_{i \in I} \left(1 - q^{m_i(\lambda)}\right)$$

$\lambda - \mu$ h.s.

I: set of integers i such that column i of $\lambda - \mu$ is nonempty and column $i+1$ is empty

Example $\lambda = (6,5,2)$, $\mu = (5,4)$

$\lambda - \mu =$  $I = \{2,6\}$, $J = \{4\}$

$$\phi_{\lambda/\mu}(q) = \left(1 - q^{m_2(\lambda)}\right)\left(1 - q^{m_6(\lambda)}\right) = (1-q)^2$$

$$\psi_{\lambda/\mu}(q) = \left(1 - q^{m_4(\mu)}\right) = (1-q)$$

Proof

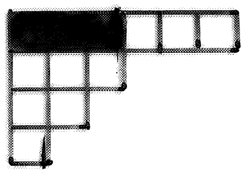
$$P_\lambda(x;q) = \sum_T \psi_T(q)\, x^T$$

$$Q_\lambda(x;q) = b_\lambda(q)\, P_\lambda(x;q) = \sum_T \phi_T(q)\, x^T$$

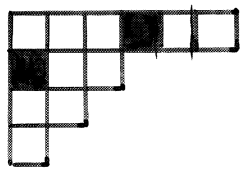
1	1	1	2	3	4
2	3	5			
3	4				
5					

$$x^T = x_1^3\, x_2^2\, x_3^3\, x_4^2\, x_5^2$$

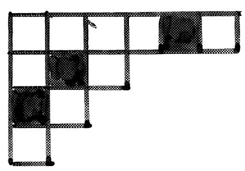
$\lambda^{(1)} = (3)$ $\qquad$ $\lambda^{(1)} - \overset{0}{\overset{\shortparallel}{\lambda^{(0)}}}$



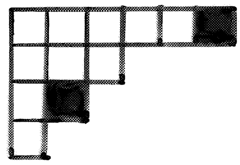
$\lambda^{(2)} = (4,1)$ $\qquad$ $\lambda^{(2)} - \lambda^{(1)}$



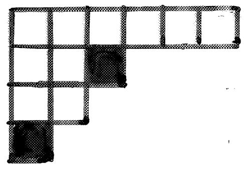
$\lambda^{(3)} = (5,2,1)$ $\qquad$ $\lambda^{(3)} - \lambda^{(2)}$



$\lambda^{(4)} = (6,2,2)$ $\qquad$ $\lambda^{(4)} - \lambda^{(3)}$



$\lambda = \lambda^{(5)} = (6,3,2,1)$ $\qquad$ $\lambda^{(5)} - \lambda^{(4)}$



For λ, μ partitions

$$\sum_{\substack{\nu \\ \nu-\mu \text{ h.s.}}} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} z^{|\nu-\mu|} \psi_{\nu/\mu}(q)$$

$$= \frac{1}{1-z} \sum_{\substack{\nu \\ \lambda-\nu \text{ h.s.}}} q^{n(\mu)+n(\nu)-(\mu'|\nu')} (z/q)^{|\lambda-\nu|} \phi_{\lambda/\nu}(q)$$

Generalized geometric series

$$\lambda = \mu = 0 \Rightarrow \begin{cases} \nu = (k) & \text{on lhs} \\ \nu = 0 & \text{on rhs} \end{cases}$$

$$\psi_{(k)/0}(q) = \phi_{0/0}(q) = 1 \Rightarrow$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

A_2 Rogers-Ramanujan

Take $x = (a, aq, \dots, aq^{n-1}, 0, 0, \dots)$

$y = (b, bq, \dots, bq^{m-1}, 0, 0, \dots)$ in HL identity

$\Rightarrow$

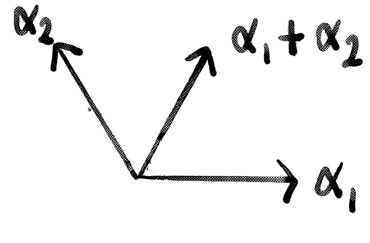
$$\sum_{\lambda,\mu} \frac{a^{|\lambda|} b^{|\mu|} q^{(\lambda'|\lambda') + (\mu'|\mu') - (\lambda'|\mu')}}{(q;q)_{n-\ell(\lambda)} (q;q)_{m-\ell(\mu)} b_\lambda(q) b_\mu(q)}$$

$$= \frac{(abq;q)_{n+m}}{(q, aq, abq;q)_n (q, bq, abq;q)_m} \qquad (3)$$

Note: $n \to \infty$ & $a \to a_1$, $b \to a_2$ $\Rightarrow$

$$\frac{1}{(a_1 q, a_2 q, a_1 a_2 q; q)_\infty} = \prod_{\alpha \in \Delta_+} \frac{1}{(a^\alpha q;q)_\infty}$$

Δ_+ set of positive roots of A_2



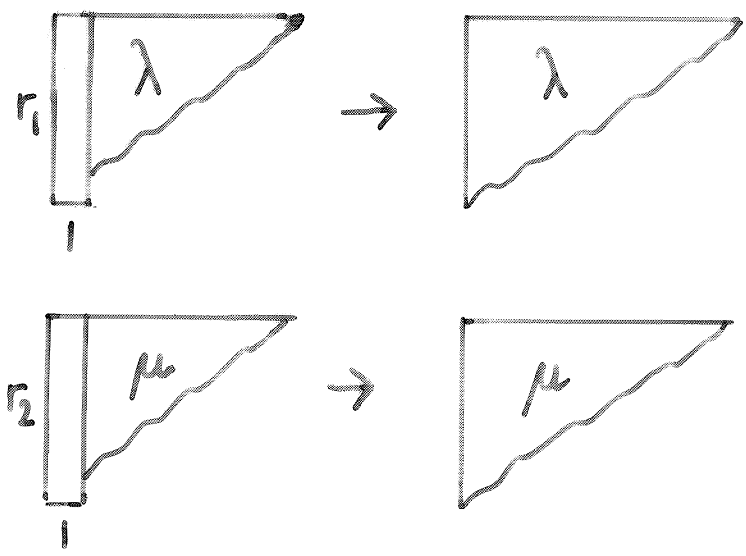
Denote the lhs of (3) by $R_{(n,m)}(a,b;q)$

Then <u>trivially</u>

$$\sum_{r_1=0}^{n}\sum_{r_2=0}^{m} \frac{q^{r_1^2-r_1r_2+r_2^2}\, a^{r_1}\, b^{r_2}}{(q;q)_{n-r_1}\,(q;q)_{m-r_2}}\, R_{(r_1,r_2)}(a,b;q)$$

$$= R_{(n,m)}(a,b;q) \qquad (4)$$

Proof:



qed

Corollary: Denote the rhs of (3) by $R_{(n,m)}(a,b;q)$
Then (4) again holds

Let $k_1+k_2+k_3=0$ and replace $r_1 \to r_1-k_1-k_2$, $r_2 \to r_2-k_1$, $a \to q^{k_2-k_3}$, $b \to q^{k_1-k_2}$, $n \to n-k_1-k_2$, $m \to m-k_1$ $\Rightarrow$

$$\sum_{r_1=0}^{n}\sum_{r_2=0}^{m} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{n-r_1}(q;q)_{m-r_2}(q;q)_{r_1+r_2}^2} \prod_{i=1}^{3} \begin{bmatrix} r_1+r_2 \\ r_1+k_i \end{bmatrix}$$

$$= \frac{q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)}}{(q;q)_{n+m}^2} \prod_{i=1}^{3} \begin{bmatrix} n+m \\ n+k_i \end{bmatrix} \qquad (5)$$

A_2 Bailey lemma (Andrews-Schilling-SOW)

Apply (5) to A_2 Euler identity ($k-1$ times)

$$\sum_{k_1+k_2+k_3=0} \sum_{w\in S_3} \varepsilon(w) \prod_{i=1}^{3} q^{\frac{3}{2}k_i^2+\frac{1}{2}(3k_i-w_i+i)^2-w_ik_i} \begin{bmatrix} n+m \\ n+3k_i-w_i+i \end{bmatrix}$$

$$= \begin{bmatrix} n+m \\ n \end{bmatrix}$$

and then let $n, m \to \infty$

$$\Rightarrow \sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu)<k}} \frac{q^{(\lambda|\lambda)+(\mu|\mu)-(\lambda|\mu)}}{b_{\lambda'}(q)\, b_{\mu'}(q)\, (q;q)_{\lambda_{k-1}+\mu_{k-1}}}$$

$$= \frac{(q^k, q^k, q^{k+1}, q^{2k}, q^{2k+1}, q^{2k+1}, q^{3k+1}, q^{3k+1}; q^{3k+1})_\infty}{(q;q)_\infty^3}$$

$k=2$ (first) A_2 Rogers-Ramanujan identity

$k>2$ A_2 Andrews-Gordon identity

(ASW)

Encore (or the rewards of reviewing)

Γ quiver (directed graph) of n vertices

$\Delta \subset \mathbb{Z}^n$ root system associated to Γ

$\alpha \in \mathbb{N}^n - \{0\}$

$A_\Gamma(\alpha, p)$ # of isomorphism classes of absolutely indecomposable reps. of dimension vector α over the algebraic closure of a field of p elements

$A_\Gamma(\alpha, p)$
1) inde^pe^ndent of orientation of the arrows (Kac)
2) $\in \mathbb{Z}[p]$ (Kac)
3) $\in \mathbb{N}[p]$ (Kac conjecture 1)
4) $A_\Gamma(\alpha, 0) = \text{mult}(\alpha)$ (Kac conj. 2)

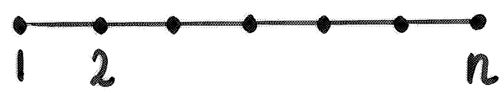
Let $A_\Gamma(\alpha,p) = t_0(\alpha)+t_1(\alpha)p+\dots+t_d(\alpha)p^d$

Hua:

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \frac{q^{\sum_{i=1}^{n}(\lambda^{(i)'}|\lambda^{(i)'}) - \sum_{1\le i\le j\le n} A_{ij}(\lambda^{(i)'}|\lambda^{(j)'})}\, a_1^{|\lambda^{(1)}|}\cdots a_n^{|\lambda^{(n)}|}}{b_{\lambda^{(1)}}(q)\cdots b_{\lambda^{(n)}}(q)}$$

$$= \prod_{\alpha\in\Delta^+}\prod_{j=0}^{d} \frac{1}{(a^\alpha q^{1-j};q)_\infty}$$

A_{ij} # edges between i and j

<u>Example</u> Γ : 

A_n

$A_\Gamma(\alpha,p)=1 \Rightarrow$

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} q^{n(\lambda^{(1)})+\dots+n(\lambda^{(n)})-(\lambda^{(1)'}|\lambda^{(2)'})-\dots-(\lambda^{(n-1)'}|\lambda^{(n)'})}$$

$$\times P_{\lambda^{(1)}}(1,q,q^2,\dots;q)\cdots P_{\lambda^{(n)}}(1,q,q^2,\dots;q)$$

$$= \prod_{\alpha\in\Delta^+}\frac{1}{(a^\alpha q;q)_\infty}$$

Challenge

Compute

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} q^{n(\lambda^{(1)})+\dots+n(\lambda^{(n)})-(\lambda^{(1)\prime}|\lambda^{(2)\prime})-\dots-(\lambda^{(n-1)\prime}|\lambda^{(n)\prime})}$$

$$\times P_{\lambda^{(1)}}(x;q)\,P_{\lambda^{(2)}}(y;q)\dots P_{\lambda^{(n)}}(z;q)$$