Generalised Rogers–Ramanujan identities and arithmetics

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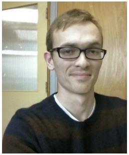
Based on joint work with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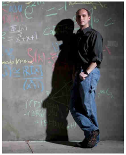
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History and motivation

The Rogers–Ramanujan identities, first discovered by Rogers in 1894, are the pair of q-series identities:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$\left(= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots\right)$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

$$\left(= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots\right)$$

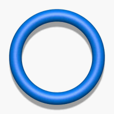
The Rogers–Ramanujan q-series have since showed up in a large number of areas of mathematics and physics, including partition theory, knot theory, conformal field theory, statistical mechanics, probability, orthogonal polynomials, representation theory, modular forms and K theory.

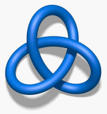
- Knot theory (Dasbach, Armond, Garoufalidis, Hikami, Khovanov, Morton, Murakami^2, ...)

A torus knot is a knot representable as a curve on the torus.

Torus knots are classified by pairs of relatively prime integers p, q, the winding numbers of the knot.

For example, the $(0, 1)$-torus knot and $(2, 3)$-torus knots are the unknot and trefoil





One popular method for recognising knots is to compute knot invariants such as the Alexander or Jones polynomial.

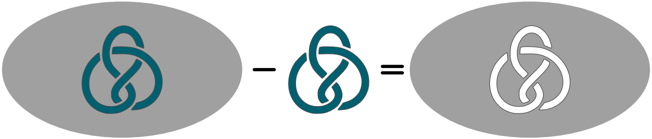
The Jones polynomials $J_K(q)$ of the unknot and of the trefoil are 1 and $q + q^3 - q^4$.

For N a nonnegative integer the coloured Jones polynomial $J_{K,N}(q)$ of a knot K is a generalisation of the Jones polynomial such that $J_{K,1}(q) = J_K(q)$.

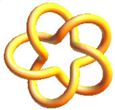
People care about the coloured Jones polynomial because of the volume conjecture (Kashaev, Murakami[2]):

$$\mathrm{Vol}(K) \stackrel{?}{=} 2\pi \lim_{N\to\infty} \frac{\log|J_{K,N}(e^{2\pi i/N})|}{N}$$

where $\mathrm{Vol}(K)$ is the hyperbolic volume of the knot complement of K in the 3-sphere.



For example, the figure-eight knot has hyperbolic volume 2.02988. . .

It is easy to compute the coloured Jones polynomial of $K =$  :

$$
\begin{aligned}
J_{K,0} &= \quad 1 \\
J_{K,1} &= -q^{-7}(1 - q + q^3 - 2q^4 + q^6 - q^7 + q^9 - q^{10} + q^{12} + q^{15}) \\
J_{K,2} &= \quad q^{-19}(1 - q - q^5 + q^7 - q^9 + \cdots - q^{30}) \\
&\vdots \\
J_{K,8} &= \quad q^{-196}(1 - q - q^4 + q^7 + q^9 - \cdots + q^{180})
\end{aligned}
$$

The tail $t_K(q) = a_0 + a_1 q + a_2 q^2 + \cdots$ of $J_{K,N}(q)$ is a q-series such that, for all N,

$$t_K(q) \equiv \pm q^a J_{K,N}(q) + O(q^{N+1})$$

Theorem. (Morton, 1995)

The tail of the $(2,5)$-torus knot is

$$H(q) \prod_{n=1}^{\infty}(1 - q^n) = 1 - q - q^4 + q^7 + \cdots$$

The arithmetics of the Rogers–Ramanujan series

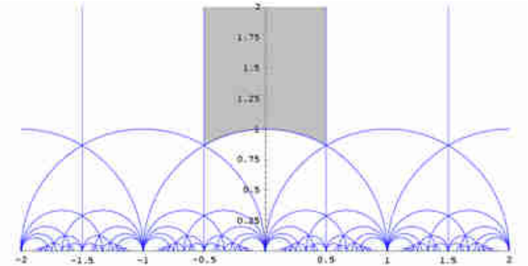
The modular group Γ is the group of fractional linear transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \qquad ad - bc = 1, \quad \tau \in \mathbb{H}$$

The modular group is generated by

$$\tau \mapsto \tau + 1 \quad \text{and} \quad \tau \mapsto -\frac{1}{\tau}$$

Geometrically,



Recall that

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)}$$

$$= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}$$

$$= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots$$

Amazingly, Ramanujan was aware of the modular properties of

$$f(\tau) = F(q) := q^{1/5}\frac{H(q)}{G(q)}, \qquad q = \mathrm{e}^{2\pi \mathrm{i}\tau},\ \tau \in \mathbb{H}$$

In his second letter to Hardy (dated 27 Feb. 1913) he stated the reciprocity theorem

$$\left(\frac{1+\sqrt{5}}{2} + f(\tau)\right)\left(\frac{1+\sqrt{5}}{2} + f\left(-\frac{1}{\tau}\right)\right) = \frac{5+\sqrt{5}}{2}$$

which implies that

$$F(\mathrm{e}^{-2\pi}) = f(\mathrm{i}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}$$

Since

$$F(q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}$$

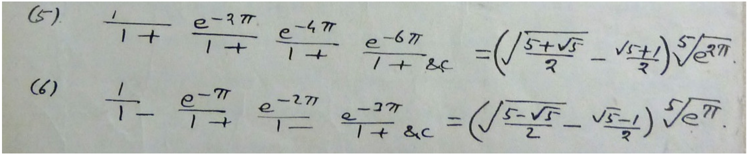
this implies the Rogers–Ramanujan continued fraction

$$\cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \cdots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2} \right) e^{2\pi/5}$$

Since

$$F(q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}$$

this implies the Rogers–Ramanujan continued fraction



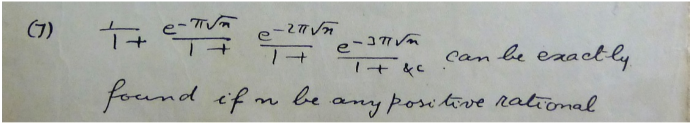
$$(5) \quad \frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+\ \&c} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

$$(6) \quad \frac{1}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+\ \&c} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$$

Of these G.H. Hardy wrote:

> *A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.*

In his first letter to Hardy (16 Jan. 1913), Ramanujan also claimed that



(7) $\cfrac{1}{1+\cfrac{e^{-\pi\sqrt{n}}}{1+\cfrac{e^{-2\pi\sqrt{n}}}{1+\cfrac{e^{-3\pi\sqrt{n}}}{1+\&c}}}}$ can be exactly found if n be any positive rational

As usual it is not exactly clear what he meant by this, but to support his claim he gave several examples of what he had in mind, such as

$$F\left(e^{-6\pi}\right) = \sqrt{c^2+1} - c$$

where

$$2c = 1 + \sqrt{5}\,\frac{\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5}}{\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5}}$$

G.N. Watson added to Ramanujan's claim by stating, without proof, that the $F\left(e^{-\pi\sqrt{n}}\right)$ are algebraic.

For example, $F\left(\mathrm{e}^{-6\pi}\right)$ is a root of

$$\begin{aligned} x^{16} &+ 38x^{15} - 240x^{14} - 300x^{13} - 235x^{12} - 726x^{11} \\ &+ 92x^{10} - 1840x^{9} - 675x^{8} + 1840x^{7} + 92x^{6} \\ &+ 726x^{5} - 235x^{4} + 300x^{3} - 240x^{2} - 38x + 1 \end{aligned}$$

Over the years many people have contributed to making the claims of Ramanujan and Watson more precise.

The first to add to the list of values for $F\left(\mathrm{e}^{-\pi\sqrt{n}}\right)$ was Ramanathan (1984 & 1985). Among other things, he proved a second reciprocity theorem:

$$\left(\left(\frac{1+\sqrt{5}}{2}\right)^5 + f^5(\tau)\right)\left(\left(\frac{1+\sqrt{5}}{2}\right)^5 + f^5\left(-\frac{1}{5\tau}\right)\right) = 5\sqrt{5}\left(\frac{5+\sqrt{5}}{2}\right)^5$$

allowing some non-integral values of n to be computed, such as $n = 4/5$.

The first theoretical result on the arithmetic properties of $F\left(\mathrm{e}^{-\pi\sqrt{n}}\right)$ is due to Berndt, Chan and Zhang (1996).

Theorem.

If $\tau \in \mathbb{H}$ is in an imaginary quadratic field then $f(\tau)$ is an (algebraic) unit.

By taking $\tau = \mathrm{i}\sqrt{n/4}$ this implies that $F\left(\mathrm{e}^{-\pi\sqrt{n}}\right)$ is a unit for $n \in \mathbb{Q}_+$.

This is a much stronger statement than that of Watson.

- Example

The number $f(\mathrm{i})$ has minimal polynomial

$$x^4+2x^3-6x^2-2x+1$$

so that it is an unit. But

$$q^{-1/60}G(q)\big|_{q=\mathrm{i}} = -\sqrt[4]{\frac{1+3\sqrt{5}-2\sqrt{10+2\sqrt{5}}}{10}}$$

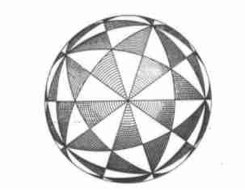
$$q^{11/60}H(q)\big|_{q=\mathrm{i}} = -\sqrt[4]{\frac{1+3\sqrt{5}+2\sqrt{10+2\sqrt{5}}}{10}}$$

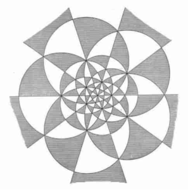
share the minimal polynomial

$$x^{16}-\frac{2}{5}x^{12}-\frac{41}{25}x^8-\frac{18}{125}x^4+\frac{1}{625}$$

and are thus algebraic numbers but not algebraic integers.

The sphere, tessellated by an icosahedron has as symmetry group the alternating group A_5 of order 60. Under stereographic projection this corresponds to the subgroup G_{60} of $\mathrm{PSL}(2, \mathbb{C})$.





In 2005 Duke used G_{60} to show that $f = f(\tau)$ satisfies the icosahedral equation

$$(f^{20} - 228f^{15} + 494f^{10} + 228f^{5} + 1)^3 + j(\tau)f^5(f^{10} + 11f^5 - 1)^5 = 0$$

where $j(\tau) = q^{-1} + 744 + 196884q + \cdots$ is Klein's j-invariant.

This implies that

Theorem.

The number $f(\tau)$ is expressible in terms of radicals over $\mathbb{Q}$ iff $j(\tau)$ is and the icosahedral equation is reducible over $\mathbb{Q}\big(e^{2\pi i\tau/5}, j(\tau)\big)$.

From the theorem of complex multiplication this immediately yields

Theorem.

If $\tau \in \mathbb{H}$ is in an imaginary quadratic field then $f(\tau)$ is a unit expressible in terms of radicals over $\mathbb{Q}$.

which settles Ramanujan's claim.

In the remainder of this talk I will discuss some generalisations of the previous results to more general Rogers–Ramanujan type identities arising from affine Lie algebras.

To keep technicalities to a minimum I will explain the main ideas in the classical instead of affine Lie algebra setting.

Plane partitions and representation theory

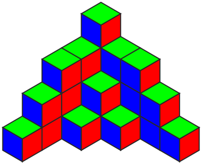
A plane partition is a two-dimensional array of nonnegative integers such that the numbers are weakly decreasing from left to right and from top to bottom, and such that finitely many numbers are positive.

Geometrically, a plane partition may also be thought of as a configuration of stacked unit cubes, such that ...

For example,

4	3	3	2	1
3	2	1		
3	1			
2				
1				

and



represent the same plane partition of 26.

A symmetric plane partition is a plane partition that is invariant under reflection in the main diagonal.

In 1898 MacMahon conjectured that the number of plane partitions that fit in a box of size $n \times n \times m$ is given by

$$\mathrm{SPP}_{n,m}(q) = \prod_{i=1}^{n} \frac{1-q^{m+2i-1}}{1-q^{2i-1}} \prod_{1 \le i < j \le n} \frac{1-q^{2(m+i+j-1)}}{1-q^{2(i+j-1)}}$$

It took 80 years before the conjecture was (independently) proved by Andrews and Macdonald.

Below I will sketch a streamlined version of Macdonald's proof.

Let $V(\Lambda)$ be an integrable highest-weight module of a complex semi-simple Lie algebra $\mathfrak{g}$.

By the Weyl character formula, the character of $V(\Lambda)$ is expressible in terms of the associated root system as

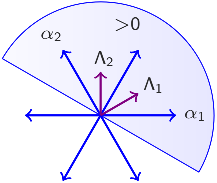
$$\mathrm{ch}\, V(\Lambda) = \frac{\sum_{w\in W} \mathrm{sgn}(w) \mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})}$$

Here W is the Weyl group and ρ the Weyl vector (half the sum of the positive roots).

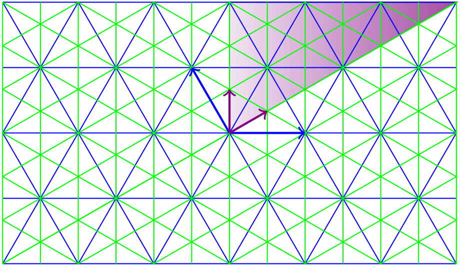
- Example

$\mathrm{A}_2 = \mathrm{sl}(3,\mathbb{C})$ has a 15-dimensional module $V(2\Lambda_1+\Lambda_2)$ with character

$$\begin{aligned}\mathrm{ch}\, V = {} & \mathrm{e}^{2\Lambda_1+\Lambda_2} + \mathrm{e}^{3\Lambda_1-\Lambda_2} + \mathrm{e}^{2\Lambda_1-2\Lambda_2} + \mathrm{e}^{\Lambda_1-3\Lambda_2} + \mathrm{e}^{-\Lambda_1-2\Lambda_2} + \mathrm{e}^{-2\Lambda_1} \\ & + \mathrm{e}^{-3\Lambda_1+2\Lambda_2} + \mathrm{e}^{-2\Lambda_1+3\Lambda_2} + \mathrm{e}^{2\Lambda_2} + 2\mathrm{e}^{\Lambda_1} + 2\mathrm{e}^{-\Lambda_2} + 2\mathrm{e}^{-\Lambda_1+\Lambda_2}\end{aligned}$$



The A_2 root system and fundamental weights.



The root lattice, weight lattice and set of dominant weights.

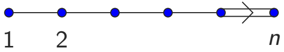
Let F_q be the homomorphism

$$F_q : \begin{cases} \mathbb{C}[\mathrm{e}^{-\alpha_1}, \ldots, \mathrm{e}^{-\alpha_n}] \to \mathbb{C}[q] \\ \mathrm{e}^{-\alpha_i} \mapsto q^{\langle \rho, \alpha_i \rangle} \end{cases}$$

Applied to the Weyl character formula this gives the q-dimension formula

$$\dim_q V(\Lambda) := F_q\big(\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda)\big) = \prod_{\alpha > 0} \frac{1 - q^{\langle \Lambda + \rho, \alpha \rangle}}{1 - q^{\langle \rho, \alpha \rangle}}$$

Macdonald noted that if $\mathfrak{g} = \mathrm{B}_n$:



then

$$\dim_{q^2} V(m\Lambda_n) = \prod_{i=1}^{n} \frac{1 - q^{m+2i-1}}{1 - q^{2i-1}} \prod_{1 \le i < j \le n} \frac{1 - q^{2(m+i+j-1)}}{1 - q^{2(i+j-1)}}$$

$$= \mathrm{SPP}_{n,m}(q)$$

Following an idea of Reshetikhin and Okounkov, given a symmetric plane partition

$$\begin{array}{|c|c|c|c|c|}\hline 4 & 3 & 3 & 2 & 1 \\ \hline 3 & 2 & 1 \\ \cline{1-3} 3 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

we can read off the diagonal slices to obtain a sequence of interlacing partitions

$$(4,2),\ (3,1)^2,\ (3)^2,\ (2)^2,\ (1)^2$$

These partitions can be reassembled into a column-strict plane partition all of whose parts are odd

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 \\ \cline{1-2}\end{array}$$

Following an idea of Reshetikhin and Okounkov, given a symmetric plane partition

4	3	3	2	1
3	2	1		
3	1			
2				
1				

we can read off the diagonal slices to obtain a sequence of interlacing partitions

$$(4, 2),\ (3, 1)^2,\ (3)^2,\ (2)^2,\ (1)^2$$

These partitions can be reassembled into a column-strict plane partition all of whose parts are odd

3	3	3	1
3	1		

Following an idea of Reshetikhin and Okounkov, given a symmetric plane partition

4	3	3	2	1
3	2	1		
3	1			
2				
1				

we can read off the diagonal slices to obtain a sequence of interlacing partitions

$$(4, 2),\ (3, 1)^2,\ (3)^2,\ (2)^2,\ (1)^2$$

These partitions can be reassembled into a column-strict plane partition all of whose parts are odd

5	5	5	1
3	1		

Following an idea of Reshetikhin and Okounkov, given a symmetric plane partition

$$\begin{array}{|c|c|c|c|c|}\hline 4 & 3 & 3 & 2 & 1 \\ \hline 3 & 2 & 1 \\ \cline{1-3} 3 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

we can read off the diagonal slices to obtain a sequence of interlacing partitions

$$(4, 2),\ (3, 1)^2,\ (3)^2,\ (2)^2,\ (1)^2$$

These partitions can be reassembled into a column-strict plane partition all of whose parts are odd

$$\begin{array}{|c|c|c|c|}\hline 7 & 7 & 5 & 1 \\ \hline 3 & 1 \\ \cline{1-2} \end{array}$$

Following an idea of Reshetikhin and Okounkov, given a symmetric plane partition

4	3	3	2	1
3	2	1		
3	1			
2				
1				

we can read off the diagonal slices to obtain a sequence of interlacing partitions

$$(4, 2),\ (3, 1)^2,\ (3)^2,\ (2)^2,\ (1)^2$$

These partitions can be reassembled into a column-strict plane partition all of whose parts are odd

9	7	5	1
3	1		

The above map implies a bijection between symmetric plane partition in $B(n, n, m)$ and column-strict plane partitions into odd parts in $B(n, m, 2n-1)$.

$$\mathrm{SPP}_{n,m}(q) = \sum_{\lambda \subseteq (m^n)} s_\lambda\left(q^{2n-1}, \ldots, q^3, q\right)$$

Here

$$s_\lambda(x) = \sum_T x^T$$

is the Schur function.

Theorem. (Macdonald)
For V a B_n-module of highest-weight $m\Lambda_n$ and

$$x_i = \mathrm{e}^{-\alpha_i - \cdots - \alpha_n}$$

we have

$$\mathrm{e}^{m\Lambda_n} \operatorname{ch} V(m\Lambda_n) = \sum_{\lambda \subseteq (m^n)} s_\lambda(x_1, \ldots, x_n)$$

Since $F_{q^2}(x_i) = q^{2n-2i+1}$ it follows that

$$\mathrm{SPP}_{n,m}(q) = \sum_{\lambda \subseteq (m^n)} s_\lambda(q^{2n-1}, \ldots, q^3, q)$$

$$= \dim_{q^2} V(m\Lambda_n)$$

proving MacMahon's conjecture.

Generalised Rogers–Ramanujan identities

From the point of view of representation theory, Macdonald's theorem may be viewed as a decomposition or branching formula for the character of $\mathrm{SO}(2n+1,\mathbb{C})$ indexed by $m\Lambda_n$ into characters of $\mathrm{GL}(n,\mathbb{C})$.

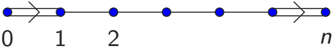
If we could prove infinite-dimensional analogues of Macdonald's decomposition formula, we should expect Rogers–Ramanujan identities instead of plane partition identities to result after specialisation.

Indeed, the Weyl–Kac formula for the characters of integrable highest-weight modules $V(\Lambda)$ of affine Lie algebras

$$\mathrm{ch}\, V(\Lambda) = \frac{\sum_{w\in W} \mathrm{sgn}(w) \mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})^{\mathrm{mult}(\alpha)}}$$

again implies "q-dimension formulas" which are completely factorised.

For example, let $\mathfrak{g}$ be the twisted affine Lie algebra $\mathrm{A}_{2n}^{(2)\dagger}$ with Dynkin diagram



Parametrise the weight Λ of the highest-weight module $V(\Lambda)$ by

$$\Lambda = (\lambda_0 - \lambda_1)\Lambda_0 + \cdots + (\lambda_{n-1} - \lambda_n)\Lambda_{n-1} + 2\lambda_n\Lambda_n$$

where $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ is a (half-)partition, and define the homomorphism F_q by

$$F_q : \begin{cases} \mathbb{C}[[\mathrm{e}^{-\alpha_0}, \ldots, \mathrm{e}^{-\alpha_n}]] \to \mathbb{C}[[q]] \\ \mathrm{e}^{-\alpha_i} \mapsto q \qquad 0 \le i \le n-1 \\ \mathrm{e}^{-\alpha_n} \mapsto -1 \end{cases}$$

Then

$$F_q\big(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda)\big) = \frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^n \theta\big(q^{\lambda_0-\lambda_i+i};q^\kappa\big) \times \prod_{1\le i<j\le n}\theta\big(q^{\lambda_i-\lambda_j-i+j}\big)\theta\big(q^{\lambda_i+\lambda_j-i-j+2n+1};q^\kappa\big)$$

where $\kappa = 2n+2\lambda_0+1$ and

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots$$

$$\theta(a;q) = (1-a)(1-q/a)(1-aq)(1-q^2/a)\cdots$$

In particular, for $n=1$ and $\Lambda = 2\Lambda_1$ or $\Lambda=\Lambda_0$ we obtain the Rogers–Ramanujan q-series

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty}\,\theta(q;q^5) \quad \text{and} \quad \frac{(q^5;q^5)_\infty}{(q;q)_\infty}\,\theta(q^2;q^5)$$

The big question now is:

Can we decompose the characters of $\mathrm{A}_{2n}^{(2)}$ a là Macdonald to obtain a "Rogers–Ramanujan sum side" (after we apply F_q)?

The modified Hall–Littlewood polynomials are a t-analogue of the Schur functions.

For example,

$$P'_{(3,1)}(x;t) = \frac{s_{(3,1)}(x) + t\, s_{(4)}(x)}{(1-t)^2}$$

Up to a trivial overall factor $b_\mu(t) = \prod_{i\geq 1}(t;t)_{m_i(\mu)}$ the coefficients of the P'_μ in its Schur expansion are given by the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$:

$$P'_\mu(x;t) = \frac{1}{b_\mu(t)} \sum_\lambda K_{\lambda\mu}(t) s_\lambda(x)$$

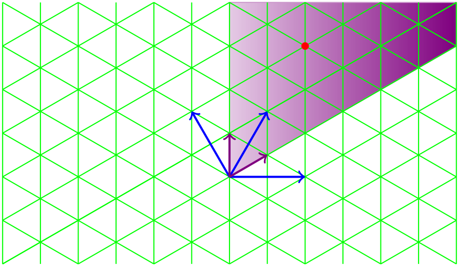
A t-analogue of Kostant's formula for weight multiplicities may be used to compute the latter

$$K_{\lambda\mu}(t) = \sum_{w\in W} \operatorname{sgn}(w) P\big(w(\lambda+\rho) - (\mu+\rho); t\big)$$

(where now we use the language of weights of $\mathfrak{sl}_{n+1}$ instead of partitions).

The t-analogue of the Kostant partition functions $P(\mu;t)$ t-counts the number of ways of writing a weight $\mu \in P$ as a sum of positive roots:

$$\prod_{\alpha>0} \frac{1}{1-te^{\alpha}} = \sum_{\mu} P(\mu;t)e^{\mu}$$



$$P(2\rho;t) = P(2\Lambda_1+2\Lambda_2;t) = \underbrace{t^2}_{2(\alpha_1+\alpha_2)} + \underbrace{t^3}_{(\alpha_1)+(\alpha_2)+(\alpha_1+\alpha_2)} + \underbrace{t^4}_{2(\alpha_1)+2(\alpha_2)}$$

Theorem. ($\mathrm{A}_{2n}^{(2)}$ decomposition formula)

Let $V(m\Lambda_0)$ be an $\mathrm{A}_{2n}^{(2)}$ integrable highest-weight module, and set

$$x_i = \mathrm{e}^{-\alpha_i - \cdots - \alpha_{n-1}} \quad (1 \le i \le n-1)$$
$$t = \mathrm{e}^{-\alpha_0 - 2\alpha_1 - \cdots - 2\alpha_n}$$

Provided we specialise $\mathrm{e}^{-\alpha_n} \mapsto -1$,

$$\mathrm{e}^{-m\Lambda_0} \operatorname{ch} V(m\Lambda_0) = \sum_{\substack{\lambda \\ \lambda_1 \le m}} t^{|\lambda|} P'_{2\lambda}(x_1^{\pm}, \dots, x_{n-1}^{\pm}, 1; t)$$

- "Proof". Use a q, t-analogue of

$$\int_{S \in \mathrm{Sp}(2n)} s_\mu(S) \mathrm{d}S = 0$$

unless μ' is an even partition, due to Rains and Vazirani. $\square$

Recall

$$F_q : \begin{cases} \mathbb{C}[[\mathrm{e}^{-\alpha_0}, \dots, \mathrm{e}^{-\alpha_n}]] \to \mathbb{C}[[q]] \\ \mathrm{e}^{-\alpha_i} \mapsto q \qquad 0 \le i \le n-1 \\ \mathrm{e}^{-\alpha_n} \mapsto -1 \end{cases}$$

This corresponds to $x_i \mapsto q^{n-i}$ and $t \mapsto q^{2n-1}$.
Therefore, the decomposition formula thus specialises to

$$\begin{aligned} F_q\big(\mathrm{e}^{-m\Lambda_0} \operatorname{ch} V(m\Lambda_0)\big) &= \sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{(2n-1)|\lambda|} P'_{2\lambda}\big(q^{1-n}, \dots, q^{n-1}; q^{2n-1}\big) \\ &= \sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}\big(1, \dots, q^{2n-2}; q^{2n-1}\big) \\ &= \sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P_{2\lambda}\big(1, q, q^2, \dots; q^{2n-1}\big) \end{aligned}$$

But

$$F_q\big(\mathrm{e}^{-m\Lambda_0}\,\mathrm{ch}\,V(m\Lambda_0)\big)=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{i+m};q^\kappa\big)\prod_{1\le i<j\le n}\theta\big(q^{j-i},q^{i+j-1};q^\kappa\big)$$

for $\kappa=2m+2n+1$, so that

Theorem. (First $\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identity)

$$\begin{aligned}G_{m,n}(q)&:=\sum_{\substack{\lambda\\ \lambda_1\le m}}q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big)\\&=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{i+m};q^\kappa\big)\prod_{1\le i<j\le n}\theta\big(q^{j-i},q^{i+j-1};q^\kappa\big)\\&=\frac{(q^\kappa;q^\kappa)_\infty^m}{(q;q)_\infty^m}\prod_{i=1}^m\theta\big(q^{i+1};q^\kappa\big)\prod_{1\le i<j\le m}\theta\big(q^{j-i},q^{i+j+1};q^\kappa\big)\end{aligned}$$

Also the second Rogers–Ramanujan identity generalises:

(Second $\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identity)

$$\begin{aligned} H_{m,n}(q) &:= \sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{2|\lambda|} P_{2\lambda}\bigl(1, q, q^2, \dots; q^{2n-1}\bigr) \\ &= \frac{(q^\kappa; q^\kappa)_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^{n} \theta\bigl(q^i; q^\kappa\bigr) \prod_{1 \le i < j \le n} \theta\bigl(q^{j-i}, q^{i+j}; q^\kappa\bigr) \\ &= \frac{(q^\kappa; q^\kappa)_\infty^m}{(q;q)_\infty^m} \prod_{i=1}^{m} \theta\bigl(q^i; q^\kappa\bigr) \prod_{1 \le i < j \le m} \theta\bigl(q^{j-i}, q^{i+j}; q^\kappa\bigr) \end{aligned}$$

For m, n positive integers, define

$$\phi_{m,n} := \frac{mn(4mn - 4m + 2n - 3)}{12(2m + 2n + 1)}$$

and

$$\psi_{m,n} := \frac{mn(4mn + 2m + 2n + 3)}{12(2m + 2n + 1)}$$

- Example

$$\phi_{1,1} = -\frac{1}{60}, \qquad \psi_{1,1} = \frac{11}{60}, \qquad \psi_{1,1} - \phi_{1,1} = \frac{1}{5}$$

Let $q = e^{2\pi i\tau}$, where the τ is a quadratic irrational in $\mathbb{H}$.

Call τ as a CM point with discriminant $-D < 0$, where $-D$ is the discriminant of the minimal polynomial of τ.

Define $\mathcal{G}_{m,n}(\tau) := q^{\phi_{m,n}} G_{m,n}(q)$, $\mathcal{H}_{m,n}(\tau) := q^{\psi_{m,n}} H_{m,n}(q)$ and $f_{m,n}(\tau) = \mathcal{H}_{m,n}(\tau)/\mathcal{G}_{m,n}(\tau)$.

Theorem.

For $\kappa = 2m + 2n + 1$, let $\kappa\tau$ be a CM point with discriminant $-D < 0$. Then

- $1/\mathcal{G}_{m,n}(\tau)$ and $1/\mathcal{H}_{m,n}(\tau)$ are algebraic integers;
- $\mathcal{G}_{m,n}(\tau)$ and $\mathcal{H}_{m,n}(\tau)$ are units over $\mathbb{Z}[1/\kappa]$;
- $f_{m,n}(\tau)$ is an algebraic unit.

- Example

$$\mathcal{G}_{2,2}(\tau) = q^{1/3} \sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{|\lambda|} P_{2\lambda}\big(1, q, \dots; q^3\big) = q^{1/3} \prod_{j=1}^{\infty} \frac{(1-q^{9j})}{(1-q^j)}$$

$$\mathcal{H}_{2,2}(\tau) = q \sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{2|\lambda|} P_{2\lambda}\big(1, q, \dots; q^3\big) = q \prod_{j=1}^{\infty} \frac{(1-q^{9j})(1-q^{9j-1})(1-q^{9j-8})}{(1-q^j)(1-q^{9j-4})(1-q^{9j-5})}$$

$\mathcal{G}_{2,2}(\mathrm{i}/3)$ and $\mathcal{H}_{2,2}(\mathrm{i}/3)$ have minimal polynomials

$$x^2 - \frac{1}{3}$$

$$\frac{1}{19683}(19683x^{18} - 80919x^{12} - 39366x^9 + 11016x^6 - 486x^3 - 1)$$

and are thus algebraic numbers.

- Example (continued)

$\sqrt{3}\,\mathcal{G}_{2,2}(\mathrm{i}/3)$ and $\sqrt{3}\,\mathcal{H}_{2,2}(\mathrm{i}/3)$ have minimal polynomials

$$x-1$$
$$x^{18}+6x^{15}-93x^{12}-304x^{9}+420x^{6}-102x^{3}+1$$

and are thus units.

$f_{2,2}(\mathrm{i}/3)$ has minimal polynomial

$$x^{18}-102x^{15}+420x^{12}-304x^{9}-93x^{6}+6x^{3}+1$$

and is thus a unit.

The End