Refined q-trinomial coefficients and character identities

S. Ole Warnaar*

Dedicated to Rodney Baxter on the occasion of his 60th birthday

Abstract

A refinement of the q-trinomial coefficients is introduced, which has a very powerful iterative property. This "$\mathcal{T}$-invariance" is applied to derive new Virasoro character identities related to the exceptional simply-laced Lie algebras $\mathrm{E}_6, \mathrm{E}_7$ and E_8.
Key words: Happy birthday; q-Trinomial coefficients; Exceptional Virasoro characters.

1 Introduction

1.1 Rodney Baxter

Rodney Baxter is justly famous for his many beautiful discoveries in mathematics and physics. The 8-vertex model, Yang–Baxter equation, corner transfer matrix and hard-hexagon model are among his most envied mathematical trophies. This paper deals with a less-well-known discovery of Rodney Baxter (made together with George Andrews), that of the q-trinomial coefficients [2]. My main aim will be to (for once) prove Baxter (and Andrews) wrong, and show that the statement

> "The literature is sparse on trinomial coefficients perhaps because they lack both depth and elegance.",

made in the introduction of [2], is not at all justified.

To have any chance of succeeding, I have omitted all proofs in this paper (which *are* lacking elegance indeed!). These will be given in a forthcoming longer paper on the same topic.

1.2 q-Trinomial coefficients

In their joined work on a generalization of the hard-hexagon model, Andrews and Baxter [2] were led to consider q-deformations of the numbers appearing in the following

*Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands; e-mail: `warnaar@wins.uva.nl`

generalized Pascal triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 3 & 2 & 1 & & \\ & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ . & . & . & . & . & . & . & . & . \end{array}$$

The generating function for the numbers appearing in the $(L+1)$th row is $(1+x+x^2)^L$, so that an explicit expression for the trinomial coefficients can be found by double application of Newton's binomial expansion. Explicitly,

$$(1+x+x^2)^L = \sum_{a=-L}^{L} \binom{L}{a}_2 x^{a+L},$$

with

$$\binom{L}{a}_2 = \sum_{k\geq 0} \binom{L}{k}\binom{L-k}{k+a}.$$

(The effective range of summation is from $\max\{0,-a\}$ to $\min\{L,\lfloor (L-a)/2\rfloor\}$ so that one indeed finds a nonzero number for $|a|\leq L$ only.)

Andrews and Baxter introduced several q-analogues of the trinomial coefficients. Here we shall restrict ourselves to the simplest two given by [2, Eq. (2.7); $B=A$]

$$\begin{bmatrix} L;q \\ a \end{bmatrix}_2 = \begin{bmatrix} L \\ a \end{bmatrix}_2 = \sum_{k\geq 0} q^{k(k+a)} \begin{bmatrix} L \\ k \end{bmatrix} \begin{bmatrix} L-k \\ k+a \end{bmatrix}$$

and

$$T(L,a;q) = T(L,a) = q^{\frac{1}{2}(L-a)(L+a)} \begin{bmatrix} L;q^{-1} \\ a \end{bmatrix}_2. \tag{1.1}$$

(This is $T_0(L,a;q^{1/2})$ of [2].) Here

$$\begin{bmatrix} n \\ a \end{bmatrix} = \begin{cases} \dfrac{(q)_n}{(q)_a (q)_{n-a}} & \text{for } 0\leq a\leq n \\ 0 & \text{otherwise,} \end{cases}$$

is a q-binomial coefficient or Gaussian polynomial, with $(a;q)_n = (a)_n = \prod_{j=0}^{n-1}(1-aq^j)$ for $n\geq 1$ and $(a;q)_0 = (a)_0 = 1$. A convenient explicit expression for $T(L,a)$ is given by [2, Eq. (2.60)]

$$T(L,a) = \sum_{\substack{n=0 \\ n+a+L \text{ even}}}^{L-|a|} \frac{q^{\frac{1}{2}n^2}(q)_L}{(q)_{\frac{L-a-n}{2}}(q)_{\frac{L+a-n}{2}}(q)_n}.$$

Some useful properties of the q-trinomial coefficients are the symmetries $\begin{bmatrix} L \\ a \end{bmatrix}_2 = \begin{bmatrix} L \\ -a \end{bmatrix}_2$ and $T(L,a) = T(L,-a)$, and the large L limits

$$\lim_{L\to\infty} \begin{bmatrix} L \\ a \end{bmatrix}_2 = \frac{1}{(q)_\infty} \tag{1.2}$$

and

$$\lim_{\substack{L\to\infty\\ L+a+\sigma \text{ even}}} T(L,a) = \sum_{\substack{n=0\\ n+\sigma \text{ even}}}^{\infty} \frac{q^{\frac{1}{2}n^2}}{(q)_n} = c_\sigma(q). \tag{1.3}$$

Here c_0 and c_1 are (normalized) level-1 string functions of $A_1^{(1)}$, which admit the alternative representations

$$\begin{aligned} c_\sigma(q) &= \frac{(-q^{1/2};q)_\infty + (-1)^\sigma (q^{1/2};q)_\infty}{2(q;q)_\infty} \\ &= \frac{q^{\frac{1}{2}\sigma}}{(q;q)_\infty (q^{3-2\sigma},q^4,q^{5+2\sigma};q^8)_\infty (q^{2+4\sigma},q^{14-4\sigma};q^{16})_\infty} \end{aligned} \tag{1.4}$$

with the convention that $(a_1,\dots,a_k;q)_n = (a_1;q)_n \dots (a_k;q)_n$.

2 A refinement of the q-trinomial coefficients

For integers L, M, a and b we define the polynomial

$$\begin{aligned} \mathcal{T}(L,M,a,b;q) &= \mathcal{T}(L,M,a,b) \\ &= \sum_{\substack{n=0\\ n+a+L \text{ even}}}^{\min\{L-|a|,M\}} q^{\frac{1}{2}n^2} \begin{bmatrix} M \\ n \end{bmatrix} \begin{bmatrix} M+b+(L-a-n)/2 \\ M+b \end{bmatrix} \begin{bmatrix} M-b+(L+a-n)/2 \\ M-b \end{bmatrix}. \end{aligned}$$

Some trivial properties of $\mathcal{T}$ are

$$\mathcal{T}(L,M,a,b) = 0 \quad \text{if } |a|>L \text{ or } |b|>M$$

(the if is not an iff), the symmetry

$$\mathcal{T}(L,M,a,b) = \mathcal{T}(L,M,-a,-b),$$

the duality

$$\mathcal{T}(L,M,a,b;1/q) = q^{ab-ML}\mathcal{T}(L,M,a,b;q) \tag{2.1}$$

and the limit

$$\lim_{M\to\infty} \mathcal{T}(L,M,a,b) = \frac{T(L,a)}{(q)_L}. \tag{2.2}$$

What is perhaps less evident is that $\mathcal{T}$ can be viewed as a refinement of both types of q-trinomial coefficients in the following sense:

$$\sum_{i=|b|}^{L-|a-b|} q^{\frac{1}{2}(i^2-b^2)} \mathcal{T}(L-i,i,a-b,b) = T(L,a) \tag{2.3}$$

and

$$\sum_{i=|b|}^{L-|a-b|} q^{\frac{1}{2}(i^2-b^2)} \mathcal{T}(i,L-i,b,a-b) = \begin{bmatrix} L \\ a \end{bmatrix}_2. \tag{2.4}$$

Here it is assumed that $a \geq b \geq 0$ or $a \leq b \leq 0$ in both formulas. We note that the second equation follows from the first by application of (1.1) and (2.1). Equation (2.3) results after taking $M \to \infty$ in Theorem 3.1 of the next section.

As an example of (2.3) let us calculate $T(4,2)$ in three different ways. When $b=0$ in (2.3) we get

$$\begin{aligned} T(4,2) &= \mathcal{T}(4,0,2,0) + q^{1/2}\mathcal{T}(3,1,2,0) + q^2\mathcal{T}(2,2,2,0) \\ &= 1 + q(1+q+q^2) + q^2(1+q+2q^2+q^3+q^4), \end{aligned}$$

when $b=1$,

$$\begin{aligned} T(4,2) &= \mathcal{T}(3,1,1,1) + q^{3/2}\mathcal{T}(2,2,1,1) + q^4\mathcal{T}(1,3,1,1) \\ &= 1 + q + q^2 + q^2(1+q)^2 + q^4(1+q+q^2) \end{aligned}$$

and, finally, when $b=2$,

$$\begin{aligned} T(4,2) &= \mathcal{T}(2,2,0,2) + q^{5/2}\mathcal{T}(1,3,0,2) + q^6\mathcal{T}(0,4,0,2) \\ &= 1 + q + 2q^2 + q^3 + q^4 + q^3(1+q+q^2) + q^6. \end{aligned}$$

Simplifying each of these three expressions correctly yields $T(4,2) = 1+q+2q^2+2q^3+2q^4+q^5+q^6$.

To conclude this section we remark that (2.4) is a bounded analogue of the following summation [3, Eq. (4.3); $-q^{-n} \to \infty$], [4, Eq. (2.10)]

$$\sum_{i=0}^{\infty} q^{\frac{1}{2}i^2} \frac{T(i,b)}{(q)_i} = \frac{q^{\frac{1}{2}b^2}}{(q)_\infty} \tag{2.5}$$

as can be seen by taking L to infinity in (2.4) using (1.2) and (2.2). Hence (2.4) should be compared with [13, Eq. (10)]

$$\sum_{i=0}^{\infty} q^{\frac{1}{2}i^2} \begin{bmatrix} L \\ i \end{bmatrix} T(i,b) = q^{\frac{1}{2}b^2} \begin{bmatrix} 2L \\ L-b \end{bmatrix}, \tag{2.6}$$

which also yields (2.5) in the large L limit.

3 $\mathcal{T}$-invariance

The important question to be addressed is whether the refined q-trinomial $\mathcal{T}$ is at all relevant. The answer to this is a clear "yes". Not only did we find that almost any result for q-trinomial coefficients has an analogue for the polynomials $\mathcal{T}$ ((2.6) appears to be an exception), but, thanks to the following theorem, $\mathcal{T}$ has perhaps even more depth than the q-trinomials.

Theorem 3.1. *For L, M, a, b integers such that $a, b \geq 0$ or $a, b \leq 0$ there holds*

$$\sum_{i=|b|}^{\min\{L-|a|,M\}} q^{\frac{1}{2}i^2} \begin{bmatrix} L+M-i \\ L \end{bmatrix} \mathcal{T}(L-i,i,a,b) = q^{\frac{1}{2}b^2}\mathcal{T}(L,M,a+b,b). \tag{3.1}$$

This is a very powerful summation formula that allows one to iterate identities involving $\mathcal{T}$, thereby generating an infinite chain of $\mathcal{T}$-identities. By the limit (2.2) this then produces an infinite chain of q-trinomial identities, and hence (by (1.1)–(1.3)) of q-series identities.

4 Three exceptional examples

Taking simple $\mathcal{T}$-identities such as

$$\sum_{j\in\mathbb{Z}} q^{j(j+1)}\{\mathcal{T}(L,M,2j,j)-\mathcal{T}(L,M,2j+2,j)\}=\delta_{L,0}\delta_{M,0}$$

or

$$\sum_{j\in\mathbb{Z}} (-1)^j q^{j(j+1)/2}\{\mathcal{T}(L,M,j,j)-\mathcal{T}(L,M,j,j+1)\}=\delta_{M,0}$$

as input, and using the $\mathcal{T}$-invariance of Theorem 3.1 to iterate these, we have proved large classes of identities for doubly bounded analogues of Virasoro characters. Many of the limiting character identities are known and many are new. In this paper we shall, however, not prove a single identity using (3.1). Instead, we shall only try to demonstrate the power of the theorem to generate new identities. As input we take three identities for which, at present, we have not a clue to a proof. However, accepting these initial conjectures, six beautiful series of identities follow, which would have been almost impossible to conceive without Theorem 3.1.

4.1 Some preliminaries

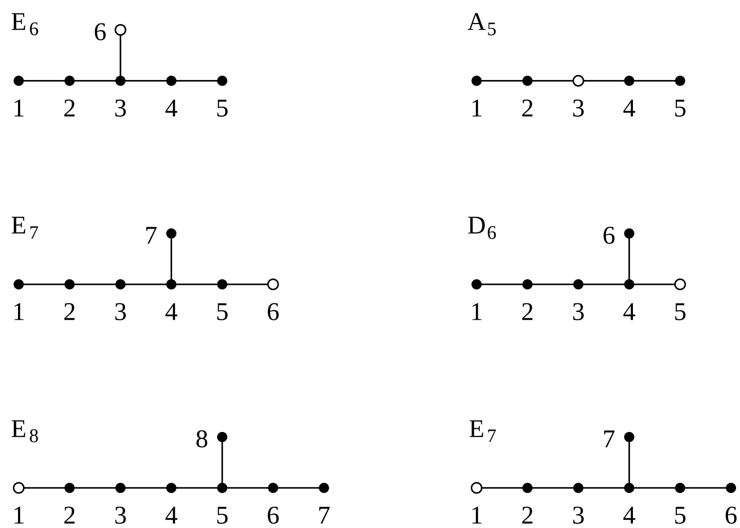

Figure 1: Dynkin diagrams of the Lie algebras E_n, A_5 and D_6 with labelling of vertices as used in the text. Removing the marked vertex (and its corresponding edge) in a diagram of the first column yields the diagram to its right. Conversely, adding a vertex plus edge to the marked vertices in the second column yields the graphs of the first column.

Let $\mathfrak{g}$ be any of the simply laced Lie algebras whose Dynkin diagram is shown in Figure 1. Given the Dynkin diagram of $\mathfrak{g}$ together with its labelling of vertices, we define a corresponding incidence matrix $\mathcal{I}_{\mathfrak{g}}$ with entries

$$(\mathcal{I}_{\mathfrak{g}})_{i,j}=\begin{cases}1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge}\\ 0 & \text{otherwise,}\end{cases}$$

where $i,j=1,\dots,r_{\mathfrak{g}}$ with $r_{\mathfrak{g}}$ the rank of $\mathfrak{g}$ (i.e., the number of vertices of the diagram). For any $\mathfrak{g}$ we define an (m,n)-system as the set of $r_{\mathfrak{g}}$ linear coupled equations

$$m+n=\frac{1}{2}(\mathcal{I}_{\mathfrak{g}}m+Ne_i). \tag{4.1}$$

Here N is a nonnegative integer, m,n,e_i are vectors in $\mathbb{Z}_+^{r_{\mathfrak{g}}}$, with e_i the unit vector ($(e_i)_j=\delta_{i,j}$) associated with the ith vertex of the Dynkin diagram of $\mathfrak{g}$. Only those labels i will occur that correspond to the marked vertices (drawn as open circles) in Figure 1. This fixes i for all $\mathfrak{g}$ other than E_7.

For given N and i, m determines n and vice versa. If $C_{\mathfrak{g}}$ is the Cartan matrix of $\mathfrak{g}$, i.e., $C_{\mathfrak{g}}=2I-\mathcal{I}_{\mathfrak{g}}$, we find explicitly that

$$n=\frac{1}{2}(Ne_i-C_{\mathfrak{g}}m) \quad \text{and} \quad m=C_{\mathfrak{g}}^{-1}(Ne_i-2n).$$

Note though that not all m (n) with integer entries will also yield an n (m) with integer entries.

As an example let $\mathfrak{g}=\mathrm{E}_7$, $N=6$ and $i=1$. Then the only admissible solutions to (4.1) are

$$\begin{aligned}
&m=5e_1+4e_2+3e_3+2e_4+e_7, && n=e_5\\
&m=3e_1+4e_2+5e_3+6e_4+4e_5+2e_6+3e_7, && n=2e_1\\
&m=5e_1+4e_2+5e_3+6e_4+4e_5+2e_6+3e_7, && n=e_2\\
&m=7e_1+8e_2+9e_3+10e_4+6e_5+2e_6+5e_7, && n=e_6\\
&m=9e_1+12e_2+15e_3+18e_4+12e_5+6e_6+9e_7, && n=0
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&m=0, && n=3e_1\\
&m=2e_1, && n=e_1+e_2\\
&m=4e_1+2e_2, && n=e_3\\
&m=4e_1+4e_2+4e_3+4e_4+2e_5+2e_7, && n=e_1+e_6\\
&m=6e_1+6e_2+6e_3+6e_4+4e_5+2e_6+2e_7, && n=e_7\\
&m=6e_1+8e_2+10e_3+12e_4+8e_5+4e_6+6e_7, && n=e_1,
\end{aligned} \tag{4.3}$$

where the first (second) set of solutions meets the criterion that $n_1+n_3+n_7$ is even (odd).

Finally we need polynomials associated with the algebras $\mathfrak{g}$ depicted in the second column of Figure 1 as follows. Let $p=3,5,1$ for $\mathfrak{g}=\mathrm{A}_5,\mathrm{D}_6,\mathrm{E}_7$, respectively, so that p corresponds to the marked vertex of $\mathfrak{g}$. For M a nonnegative integer, $\sigma=0,1$ and (m,n)-system

$$m+n=\frac{1}{2}(\mathcal{I}_{\mathfrak{g}}m+2Me_p) \tag{4.4}$$

we define

$$F_{M;\sigma}^{\mathrm{A}_5}(q)=\sum_{\substack{n\in\mathbb{Z}_+^5\\ n_1+n_4\equiv n_2+n_5 \pmod 3\\ n_1+n_3+n_5+\sigma \text{ even}}} q^{nC_{\mathrm{A}_5}^{-1}n}\begin{bmatrix} m+n\\ n\end{bmatrix}$$

and

$$F^{\mathrm{D}_6}_{M;\sigma}(q) = \sum_{\substack{n\in\mathbb{Z}^6_+\\ n_1+n_3+n_6 \text{ even}\\ n_1+n_3+n_5+\sigma \text{ even}}} q^{nC^{-1}_{\mathrm{D}6}n} \begin{bmatrix} m+n\\ n\end{bmatrix}$$

and

$$F^{\mathrm{E}_7}_{M;\sigma}(q) = \sum_{\substack{n\in\mathbb{Z}^7_+\\ n_1+n_3+n_7+\sigma \text{ even}}} q^{nC^{-1}_{\mathrm{E}7}n} \begin{bmatrix} m+n\\ n\end{bmatrix}.$$

Here we have used the abbreviations $nC^{-1}_{\mathfrak{g}}n = \sum_{i,j=1}^{r_{\mathfrak{g}}}(C^{-1}_{\mathfrak{g}})_{i,j}n_in_j$ and $\left[\begin{smallmatrix} m+n\\ n\end{smallmatrix}\right] = \prod_{j=1}^{r_{\mathfrak{g}}}\left[\begin{smallmatrix} m_j+n_j\\ n_j\end{smallmatrix}\right]$ for $m,n\in\mathbb{Z}^{r_{\mathfrak{g}}}$. Similarly we will write $(q)_n=\prod_{j=1}^{r_{\mathfrak{g}}}(q)_{n_j}$.

As example consider again $\mathfrak{g}=\mathrm{E}_7$ and choose $M=3$. Then the only contributing terms to $F^{\mathrm{E}_7}_{3;0}(q)$ and $F^{\mathrm{E}_7}_{3;1}(q)$ correspond to the solutions of (4.1) listed in (4.2) and (4.3), respectively. Hence

$$F^{\mathrm{E}_7}_{3;0}(q) = q^6 + q^6\left[\begin{smallmatrix} 5\\ 2\end{smallmatrix}\right] + q^4\left[\begin{smallmatrix} 5\\ 1\end{smallmatrix}\right] + q^2\left[\begin{smallmatrix} 3\\ 1\end{smallmatrix}\right] + 1$$

and

$$F^{\mathrm{E}_7}_{3;1}(q) = q^{27/2} + q^{19/2}\left[\begin{smallmatrix} 3\\ 1\end{smallmatrix}\right] + q^{15/2} + q^{11/2}\left[\begin{smallmatrix} 5\\ 1\end{smallmatrix}\right] + q^{7/2}\left[\begin{smallmatrix} 3\\ 1\end{smallmatrix}\right] + q^{3/2}\left[\begin{smallmatrix} 7\\ 1\end{smallmatrix}\right].$$

4.2 An (E_7,E_8) series

Our first conjecture is the following polynomial identity involving E_7:

$$\sum_{j\in\mathbb{Z}}\Big\{q^{\frac{1}{2}j(15j+2)}\mathcal{T}(L,M,3j,5j) - q^{\frac{1}{2}(3j+1)(5j+1)}\mathcal{T}(L,M,3j+1,5j+1)\Big\}$$
$$= \sum_{\substack{n\in\mathbb{Z}^7_+\\ n_1+n_3+n_7+L \text{ even}}} q^{nC^{-1}_{\mathrm{E}_7}n}\begin{bmatrix}\frac{1}{2}(L+M+m_1)\\ 2M\end{bmatrix}\begin{bmatrix} m+n\\ n\end{bmatrix}, \qquad (4.5)$$

with (m,n)-system (4.4) and $\mathfrak{g}=\mathrm{E}_7$ (so that $p=1$). The restriction on the sum in the right side guaranties that, given $n\in\mathbb{Z}^7_+$, $L+M+m_1$ is even. Using Theorem 3.1 to iterate this conjecture we obtain an infinite series of polynomial identities. These identities are best expressed by turning E_7 into E_8 by the mechanism described in the caption of Figure 1. Specifically, for $k\geq 1$,

$$\sum_{j\in\mathbb{Z}}\Big\{q^{\frac{1}{2}j(5(5k+3)j+2)}\mathcal{T}(L,M,(5k+3)j,5j)$$
$$- q^{\frac{1}{2}(5j+1)((5k+3)j+k+1)}\mathcal{T}(L,M,(5k+3)j+k+1,5j+1)\Big\}$$
$$= \sum_{r\in\mathbb{Z}^{k-1}_+}\Big(\prod_{a=0}^{k-2} q^{\frac{1}{2}(r_a-r_{a+1})^2}\left[\begin{smallmatrix} r_{a-1}-r_a+r_{a+1}\\ r_a\end{smallmatrix}\right]\Big)\sum_{n\in\mathbb{Z}^8_+} q^{\frac{1}{4}mC_{\mathrm{E}_8}m}\left[\begin{smallmatrix} r_{k-2}-\frac{1}{2}m_1\\ r_{k-1}\end{smallmatrix}\right]\left[\begin{smallmatrix} m+n\\ n\end{smallmatrix}\right], \qquad (4.6)$$

with $r_0=L$, $r_{-1}=L+M$ and (m,n)-system given by

$$m+n=\frac{1}{2}(\mathcal{I}_{\mathrm{E}_8}m + r_{k-1}e_1). \qquad (4.7)$$

We now use this result to obtain q-series identities. First we let M tend to infinity, which, by (2.2), turns (4.6) into an identity for q-trinomial coefficients. Then we either send L to infinity using (1.3), or we first replace $q \to 1/q$ and then send L to infinity using (1.2). Omitting the actual calculations (which sometimes require variable changes to remove L-dependent terms in the exponent of q) we find two families of q-series identities, one of E_7 type and one of E_8 type. To present these identities in a neat form we recall the bosonic representation of the Virasoro characters [7, 11]

$$\chi_{r,s}^{(p,p')}(q) = \frac{q^{\frac{(p'r-ps)^2-1}{4pp'}}}{(q)_\infty} \sum_{j=-\infty}^{\infty} \Big\{ q^{j(pp'j+p'r-ps)} - q^{(pj+r)(p'j+s)} \Big\}, \tag{4.8}$$

for coprime integers $2 \le p < p'$ and $r = 1, \dots, p-1$, $s = 1, \dots, p'-1$. The somewhat unusual normalization of the Virasoro characters is chosen to simplify subsequent equations. Besides the Virasoro characters we also need the following (subset) of the branching functions corresponding to the coset $(\mathrm{A}_1^{(1)} \oplus \mathrm{A}_1^{(1)}, \mathrm{A}_1^{(1)})$ at levels $2p/(p'-p)-2, 2$ and $2p/(p'-p)$ [9]:

$$B_{r,s;\sigma}^{(p,p')}(q) = \frac{q^{\frac{(p'r-ps)^2-4}{8pp'}}}{(q)_\infty} \sum_{j=-\infty}^{\infty} \Big\{ q^{j(pp'j+p'r-ps)} c_{pj+\frac{1}{2}(r-s)+\sigma}(q) - q^{(pj+r)(p'j+s)} c_{pj+\frac{1}{2}(r+s)+\sigma}(q) \Big\}, \tag{4.9}$$

for integers p, p', r, s in the same ranges as above, such that $p'-p$ and $r-s$ are even, and $\gcd((p'-p)/2, p') = 1$. The integer σ takes the values 0 or 1 and the c_j are the string functions of equations (1.3) and (1.4) with the obvious identification of $c_{2j+\sigma}$ with c_σ.

The various large L and M limits now lead to the following list of identities.

- Taking $L+\sigma$ even in (4.6), sending M and L to infinity using that $c_\sigma(q) = \chi_{\sigma+1,1}^{(3,4)}(q)/(q)_\infty$ yields for $k \ge 1$

$$\sum_{n_1,\dots,n_k \ge 0} \frac{q^{\frac{1}{2}(N_1^2+\dots+N_k^2)} F_{n_k;m_\sigma}^{\mathrm{E}_7}(q)}{(q)_{n_1} \cdots (q)_{n_{k-1}} (q)_{2n_k}} = \begin{cases} \chi_{\sigma+1,1}^{(3,4)}(q) \chi_{1,(k+1)/2}^{(5,(5k+3)/2)}(q) & k \text{ odd} \\ B_{1,k+1;\sigma}^{(5,5k+3)}(q) & k \text{ even,} \end{cases}$$

where the following definitions have been employed:

$$N_a = n_a + \dots + n_k \quad \text{and} \quad m_\sigma \equiv \sigma + \sum_{\substack{a=1 \\ a \text{ odd}}}^{k} n_a \pmod 2, \tag{4.10}$$

for $m_\sigma \in \{0,1\}$. There is a corresponding "$k=0$" identity obtained by taking the same limit as above, but now in the initial conjecture (4.5). Using the (from a q-series point of view nontrivial) relation $B_{1,1;\sigma}^{(3,5)} = \chi_{2\sigma+1,1}^{(4,5)}$, which follows from a symmetry of the $\mathrm{A}_1^{(1)}$ branching functions, we find the well-known E_7

conjecture [10]

$$\chi^{(4,5)}_{2\sigma+1,1}(q) = \sum_{\substack{n\in\mathbb{Z}^7_+ \\ n_1+n_3+n_7+\sigma \text{ even}}} \frac{q^{nC^{-1}_{\mathrm{E}_7}n}}{(q)_n}. \tag{4.11}$$

- If in (4.6) we send M to infinity, replace $q \to 1/q$ and then take the limit of large L we obtain for $k \geq 2$,

$$\sum_{r\in\mathbb{Z}^{k-1}_+} {\sum_{m\in\mathbb{Z}^8_+}}' \frac{q^{\frac{1}{2}\sum_{a=1}^{k-1}(r_a-r_{a-1})^2}}{(q)_{r_1}} \Bigl(\prod_{a=2}^{k-1} \begin{bmatrix} r_{a-1}-r_a+r_{a+1} \\ r_a \end{bmatrix}\Bigr) q^{\frac{1}{4}mC_{\mathrm{E}_8}m} \begin{bmatrix} m+n \\ m \end{bmatrix}$$
$$= \begin{cases} \chi^{((5k+3)/2,5k-2)}_{(k+1)/2,k}(q) & k \text{ odd} \\ \chi^{(5k/2-1,5k+3)}_{k/2,k+1}(q) & k \text{ even,} \end{cases} \tag{4.12}$$

with $r_0 = 0$, $r_k = r_{k-1} - m_1/2$ and (m, n)-system (4.7). The prime in the sum over m denotes the restriction $m_2 \equiv m_4 \equiv m_8 \equiv r_{k-1} \pmod 2$ with all other m_i being even. When $k = 1$ the resulting character formula takes a somewhat different form, and one obtains the E_8 identity [10, 14]

$$\chi^{(3,4)}_{1,1}(q) = \sum_{n\in\mathbb{Z}^8_+} \frac{q^{nC^{-1}_{\mathrm{E}_8}n}}{(q)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty (q^2,q^{14};q^{16})_\infty}. \tag{4.13}$$

4.3 A (D_6,E_7) series

In our second conjecture the role of E_7 is taken over by D_6,

$$\sum_{j\in\mathbb{Z}} \Bigl\{ q^{\frac{1}{2}j(24j+2)} \mathcal{T}(L,M,4j,6j) - q^{\frac{1}{2}(4j+1)(6j+1)} \mathcal{T}(L,M,4j+1,6j+1) \Bigr\}$$
$$= \sum_{\substack{n\in\mathbb{Z}^6_+ \\ n_1+n_3+n_6 \text{ even} \\ n_1+n_3+n_5+L \text{ even}}} q^{nC^{-1}_{\mathrm{D}_6}n} \begin{bmatrix} \frac{1}{2}(L+M+m_5) \\ 2M \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix}, \tag{4.14}$$

with (m, n)-system (4.4) where $\mathfrak{g} = \mathrm{D}_6$ (and $p = 5$). The restriction that $n_1+n_3+n_5+L$ is even is necessary for $L+M+m_5$ to be even. The additional constraint on $n_1+n_3+n_6$ is to avoid an extra

$$q \sum_{j\in\mathbb{Z}} \Bigl\{ q^{\frac{1}{2}j(24j+14)} \mathcal{T}(L,M,4j+1,6j+2) - q^{\frac{1}{2}(4j+3)(6j+1)} \mathcal{T}(L,M,4j+2,6j+3) \Bigr\}$$

on the left-hand side. Of course, this means we have actually two conjectures, but the case when $n_1 + n_3 + n_6$ is odd will not be pursued here.

Using Theorem 3.1 to iterate the D_6 conjecture we obtain an infinite series of polynomial identities involving E_7. Specifically, for $k \geq 1$ there holds

$$\sum_{j\in\mathbb{Z}}\Bigl\{q^{\frac{1}{2}j(6(6k+4)j+2)}\mathcal{T}(L,M,(6k+4)j,6j) \\ - q^{\frac{1}{2}(6j+1)((6k+4)j+k+1)}\mathcal{T}(L,M,(6k+4)j+k+1,6j+1)\Bigr\} \\ = \sum_{r\in\mathbb{Z}_+^{k-1}}\Bigl(\prod_{a=0}^{k-2}q^{\frac{1}{2}(r_a-r_{a+1})^2}\begin{bmatrix}r_{a-1}-r_a+r_{a+1}\\ r_a\end{bmatrix}\Bigr) \\ \times \sum_{\substack{n\in\mathbb{Z}_+^7\\ n_1+n_3+n_7 \text{ even}}} q^{\frac{1}{4}mC_{E_7}m}\begin{bmatrix}r_{k-2}-\frac{1}{2}m_6\\ r_{k-1}\end{bmatrix}\begin{bmatrix}m+n\\ n\end{bmatrix}, \quad (4.15)$$

where $r_0 = L$, $r_{-1} = L + M$ and

$$m + n = \frac{1}{2}(\mathcal{I}_{E_7}m + r_{k-1}e_6). \quad (4.16)$$

As before we consider the various large L and M limits.

- Taking $L+\sigma$ even in (4.15) and sending M and L to infinity yields for $k \geq 1$ that

$$\sum_{n_1,\dots,n_k\geq 0}\frac{q^{\frac{1}{2}(N_1^2+\cdots+N_k^2)}F^{D_6}_{n_k;m_\sigma}(q)}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q)_{2n_k}} = \begin{cases}\chi^{(3,4)}_{\sigma+1,1}(q)\chi^{(6,3k+2)}_{1,(k+1)/2}(q) & k \text{ odd}\\ B^{(6,6k+4)}_{1,k+1;\sigma+k/2}(q) & k \text{ even,}\end{cases}$$

  with the notation of equation (4.10) and with the identification of $B^{(p,p')}_{r,s;\sigma+2j}$ with $B^{(p,p')}_{r,s;\sigma}$.

  The $k = 0$ case, corresponding to the above limit taken in (4.14) yields

$$B^{(4,6)}_{1,1;\sigma} = \sum_{\substack{n\in\mathbb{Z}_+^6\\ n_1+n_3+n_6 \text{ even}\\ n_1+n_3+n_5\equiv\sigma \text{ even}}}\frac{q^{nC^{-1}_{D_6}n}}{(q)_n}.$$

  Although we were unable to prove this, it appears that the above branching function admits the following simplification

$$B^{(4,6)}_{1,1;\sigma}(q) = \begin{cases}\dfrac{1}{(q)_\infty}\Bigl(\displaystyle\sum_{j=0}^{\infty}(-q)^{j^2}+\sum_{j=1}^{\infty}q^{6j^2}\Bigr) & \sigma = 0\\ \dfrac{q^{3/2}}{(q)_\infty}\displaystyle\sum_{j=0}^{\infty}q^{6j(j+1)} = \frac{q^{3/2}(q^{24};q^{24})_\infty}{(q^{12};q^{24})_\infty(q;q)_\infty} & \sigma = 1.\end{cases}$$

  For $\sigma = 1$ this implies a new identity of the Rogers–Ramanujan type for the algebra D_6.

- If we send M to infinity, replace $q\to 1/q$ and then take the limit of large L we obtain for all $k\geq 2$

$$\sum_{r\in\mathbb{Z}_+^{k-1}}\sideset{}{'}\sum_{m\in\mathbb{Z}_+^7}\frac{q^{\frac{1}{2}\sum_{a=1}^{k-1}(r_a-r_{a-1})^2}}{(q)_{r_1}}\Bigl(\prod_{a=2}^{k-1}\bigl[\begin{smallmatrix}r_{a-1}-r_a+r_{a+1}\\ r_a\end{smallmatrix}\bigr]\Bigr)q^{\frac{1}{4}mC_{\mathrm{E}_7}m}\bigl[\begin{smallmatrix}m+n\\ m\end{smallmatrix}\bigr]
=\begin{cases}\chi^{(3k+2,6k-2)}_{(k+1)/2,k}(q) & k\text{ odd}\\ \chi^{(3k-1,6k+4)}_{k/2,k+1}(q) & k\text{ even},\end{cases}\qquad(4.17)$$

with $r_0=0$, $r_k=r_{k-1}-m_6/2$ and (m,n)-system (4.16). The prime in the sum over m denotes the restriction that $m_1\equiv m_3\equiv m_5\equiv r_{k-1}\pmod 2$ and that all other m_i are even. The identity corresponding to $k=1$ is given by (4.11) with $\sigma=0$.

4.4 An (A_5,E_6) series

Our final conjecture is somewhat more involved than the previous two, as it is not possible to disentangle the two terms on the left-hand side below by an appropriate summation restriction on n,

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\Bigl\{q^{\frac{1}{2}j(48j+2)}\mathcal{T}(L,M,6j,8j)-q^{\frac{1}{2}(6j+1)(8j+1)}\mathcal{T}(L,M,6j+1,8j+1)\Bigr\}\\
&+q^3\sum_{j\in\mathbb{Z}}\Bigl\{q^{\frac{1}{2}j(48j+34)}\mathcal{T}(L,M,6j+2,8j+3)-q^{\frac{1}{2}(6j+1)(8j+7)}\mathcal{T}(L,M,6j+3,8j+4)\Bigr\}\\
&\qquad=\sum_{\substack{n\in\mathbb{Z}_+^5\\ n_1+n_4\equiv n_2+n_5\pmod 3\\ n_1+n_3+n_5\equiv L\pmod 2}}q^{nC^{-1}_{\mathrm{A}_5}n}\begin{bmatrix}\frac{1}{2}(L+M+m_3)\\ 2M\end{bmatrix}\begin{bmatrix}m+n\\ n\end{bmatrix},\qquad(4.18)\end{aligned}$$

with (m,n)-system (4.4) where $\mathfrak{g}=\mathrm{A}_5$ (and $p=3$). Iterating this last conjecture using Theorem 3.1 one finds a series of E_6-type polynomial identities as follows ($k\geq 1$)

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\Bigl\{q^{\frac{1}{2}j(8(8k+6)j+2)}\mathcal{T}(L,M,(8k+6)j,8j)\\
&\qquad-q^{\frac{1}{2}(8j+1)((8k+6)j+k+1)}\mathcal{T}(L,M,(8k+6)j+k+1,8j+1)\Bigr\}\\
&+q^{\frac{3}{2}(3k+2)}\sum_{j\in\mathbb{Z}}\Bigl\{q^{\frac{1}{2}j(8(8k+6)j+48k+34)}\mathcal{T}(L,M,(8k+6)j+3k+2,8j+3)\\
&\qquad-q^{\frac{1}{2}(8j+7)((8k+6)j+k+1)}\mathcal{T}(L,M,(8k+6)j+4k+3,8j+4)\Bigr\}\\
&\qquad=\sum_{r\in\mathbb{Z}_+^{k-1}}\Bigl(\prod_{a=0}^{k-2}q^{\frac{1}{2}(r_a-r_{a+1})^2}\bigl[\begin{smallmatrix}r_{a-1}-r_a+r_{a+1}\\ r_a\end{smallmatrix}\bigr]\Bigr)\\
&\qquad\quad\times\sum_{\substack{n\in\mathbb{Z}_+^6\\ n_1+n_4\equiv n_2+n_5\pmod 3}}q^{\frac{1}{4}mC_{\mathrm{E}_6}m}\bigl[\begin{smallmatrix}r_{k-2}-\frac{1}{2}m_6\\ r_{k-1}\end{smallmatrix}\bigr]\bigl[\begin{smallmatrix}m+n\\ n\end{smallmatrix}\bigr],\end{aligned}$$

where $r_0=L$, $r_{-1}=L+M$ and $m+n=\frac{1}{2}(\mathcal{I}_{\mathrm{E}_6}m+r_{k-1}e_6)$. Fortunately, the limiting character identities that follow from this monster are more manageable.

- Taking $L+\sigma$ even and letting M and L tend to infinity yields for positive k,

$$\sum_{n_1,\dots,n_k\geq 0}\frac{q^{\frac{1}{2}(N_1^2+\cdots+N_k^2)}F^{\mathrm{D}_5}_{n_k;m_\sigma}(q)}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q)_{2n_k}}$$
$$=\begin{cases}\chi^{(3,4)}_{\sigma+1,1}(q)\chi^{(8,4k+3)}_{1,(k+1)/2}(q)+\chi^{(3,4)}_{2-\sigma,1}(q)\chi^{(8,4k+3)}_{7,(k+1)/2}(q) & k\text{ odd}\\ B^{(8,8k+6)}_{1,k+1;\sigma+k/2}(q)+B^{(8,8k+6)}_{7,k+1;\sigma+k/2+1}(q) & k\text{ even,}\end{cases}$$

  with the notation of equation (4.10). The analogous limit taken in (4.18) leads to

$$B^{(6,8)}_{1,1;\sigma}(q)+B^{(6,8)}_{1,7;1-\sigma}(q)=\sum_{\substack{n\in\mathbb{Z}_+^5\\ n_1+n_4\equiv n_2+n_5 \pmod 3\\ n_1+n_3+n_5\equiv\sigma \pmod 2}}\frac{q^{nC^{-1}_{\mathrm{A}_5}n}}{(q)_n}.$$

- If we send M to infinity, replace $q\to 1/q$ and then take the limit of large L we obtain for $k\geq 2$,

$$\sum_{r\in\mathbb{Z}_+^{k-1}}\sideset{}{'}\sum_{m\in\mathbb{Z}_+^6}\frac{q^{\frac{1}{2}\sum_{a=1}^{k-1}(r_a-r_{a-1})^2}}{(q)_{r_1}}\Bigl(\prod_{a=2}^{k-1}\begin{bmatrix}r_{a-1}-r_a+r_{a+1}\\ r_a\end{bmatrix}\Bigr)q^{\frac{1}{4}mC_{\mathrm{E}_6}m}\begin{bmatrix}m+n\\ m\end{bmatrix}$$
$$=\begin{cases}\chi^{(4k+3,8k-2)}_{(k+1)/2,k}(q)+\chi^{(4k+3,8k-2)}_{(k+1)/2,7k-2}(q) & k\text{ odd}\\ \chi^{(4k-1,8k+6)}_{k/2,k+1}(q)+\chi^{(4k-1,8k+6)}_{7k/2-1,k+1}(q) & k\text{ even,}\end{cases}\qquad (4.19)$$

  where $m+n=\frac{1}{2}(\mathcal{I}_{\mathrm{E}_6}m+r_{k-1}e_6)$ and $r_0=0$, $r_k=r_{k-1}-m_6/2$. The prime in the sum over m denotes the restriction that $m_1\equiv m_3\equiv m_5\equiv r_{k-1} \pmod 2$ and that all other m_i are even. Again $k=1$ is special, corresponding to the E_6 conjecture of [10],

$$\chi^{(6,7)}_{1,1}(q)+\chi^{(6,7)}_{5,1}(q)=\sum_{\substack{n\in\mathbb{Z}_+^6\\ n_1+n_4\equiv n_2+n_5 \pmod 3}}\frac{q^{nC^{-1}_{\mathrm{E}_6}n}}{(q)_n}.\qquad (4.20)$$

5 Discussion

We hope that the examples presented in the previous section support our claim that q-trinomial coefficients, and their refinement introduced in this paper are mathematical objects of both depth and elegance.

It is quite intriguing to observe that the $\mathcal{T}$-invariance of Theorem 3.1 also appears to have physical significance. In a very recent paper by Dorey, Dunning and Tateo [6], new families of renormalization group flows between $c<1$ conformal field theories were proposed. Labelling such a theory by $M(p,p')$, in accordance with definition (4.8) of the Virasoro characters, Dorey *et al.* conjecture the following flows

$$M(p,p') + \phi_{21} \longrightarrow M(p'-p,p') \qquad p < p' < 2p \qquad (5.1a)$$
$$M(p,p') + \phi_{15} \longrightarrow M(p,4p-p') \qquad 2p < p' < 3p \qquad (5.1b)$$
$$M(p,p') + \phi_{15} \longrightarrow M(4p-p',p) \qquad 3p < p' < 4p, \qquad (5.1c)$$

where ϕ_{rs} is the perturbing operator of the $M(p,p')$ theory. Using these three flows we find the following chain ending in $M(3,4)$:

$$M(3,4) \stackrel{\text{(c)}}{\longleftarrow} M(4,13) \stackrel{\text{(a)}}{\longleftarrow} M(9,13) \stackrel{\text{(b)}}{\longleftarrow} M(9,23) \stackrel{\text{(a)}}{\longleftarrow} M(14,23) \stackrel{\text{(b)}}{\longleftarrow} \dots,$$

where (a) denotes equation (5.1a) etc. But this flow diagram coincides with the chain of character identities given in (4.12) and (4.13)! In particular, $M(3,4)$ corresponds to the E_8 identity of (4.13), $M(5n-1,10n+3)$ $(n \geq 1)$ corresponds to (4.12) for $k=2n$, and $M(5n+4,10n+3)$ $(n \geq 1)$ to (4.12) for $k=2n+1$.

In much the same way the flow diagram

$$M(4,5) \stackrel{\text{(c)}}{\longleftarrow} M(5,16) \stackrel{\text{(a)}}{\longleftarrow} M(11,16) \stackrel{\text{(b)}}{\longleftarrow} M(11,28) \stackrel{\text{(a)}}{\longleftarrow} M(17,28) \stackrel{\text{(b)}}{\longleftarrow} \dots$$

is in accordance with the chain of character identities given by (4.17) and (4.11), and

$$M(6,7) \stackrel{\text{(c)}}{\longleftarrow} M(7,22) \stackrel{\text{(a)}}{\longleftarrow} M(15,22) \stackrel{\text{(b)}}{\longleftarrow} M(15,38) \stackrel{\text{(a)}}{\longleftarrow} M(23,38) \stackrel{\text{(b)}}{\longleftarrow} \dots$$

is in one-to-one correspondence with (4.19) and (4.20).

To conclude we mention that more general applications of our refined q-trinomial coefficients will be presented in a future paper. Therein we will also discuss the $\mathcal{T}$-invariance in the broader context of the Bailey lemma [1], trinomial Bailey lemma [3] and (generalized) Burge transform [5, 8, 12].

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