A NEKRASOV–OKOUNKOV FORMULA FOR MACDONALD POLYNOMIALS

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Abstract. We prove a Macdonald polynomial analogue of the celebrated Nekrasov–Okounkov hook-length formula from the theory of random partitions. As an application we obtain a proof of one of the main conjectures of Hausel and Rodriguez-Villegas from their work on mixed Hodge polynomials of the moduli space of stable Higgs bundles on Riemann surfaces.

1. Introduction

In their paper *Mixed Hodge polynomials of character varieties* [22], Hausel and Rodriguez-Villegas study the non-singular affine variety

$$\mathcal{M}_n := \big\{A_1,B_1,\dots,A_g,B_g \in \mathrm{GL}(n,\mathbb{C}) :\ (A_1,B_1)\cdots(A_g,B_g)=\zeta_n I\big\}/\!/\mathrm{GL}(n,\mathbb{C}),$$

where g is a nonnegative integer, (A,B) is shorthand for the commutator $ABA^{-1}B^{-1}$, ζ_n is a primitive nth-root of unity, and $/\!/$ is a GIT quotient by the conjugation action of $\mathrm{GL}(n,\mathbb{C})$. $\mathcal{M}_n$, which is the twisted character variety of a closed Riemann surface Σ of genus g with points the twisted homomorphisms from $\pi_1(\Sigma)$ to $\mathrm{GL}(n,\mathbb{C})$ modulo conjugation, has dimension $d_n=2n^2(g-1)+2$ $(g\geqslant 1)$. For low values of the rank, $\mathcal{M}_n$ was previously considered by Hitchin [23] ($n=2$) and Gothen [16] ($n=3$) in their work on the moduli space of stable Higgs bundles of rank n on Σ. The main focus of Hausel and Rodriguez-Villegas is to extend the computation of the two-variable mixed Hodge polynomial $H(\mathcal{M}_n;q,t)$ by Hitchin and Gothen to arbitrary n, and thus to obtain the Poincaré and E-polynomials $P(\mathcal{M}_n;t)$ and $E(\mathcal{M}_n;q)$ corresponding to the one-dimensional subfamilies

$$P(\mathcal{M}_n;t)=H(\mathcal{M}_n;1,t) \quad\text{and}\quad E(\mathcal{M}_n;q)=q^{d_n}H(\mathcal{M}_n;q^{-1},-1).$$

(For an arbitrary complex algebraic variety X the mixed Hodge polynomial is defined as the three-variable generating function $H(X;x,y,t)$ of mixed Hodge numbers $h^{p,q;t}(X)$, but since the cohomology of $\mathcal{M}_n$ is of type (p,p) [22, Corollary 4.1.11], $h^{p,q;t}(\mathcal{M}_n)$ vanishes unless unless $p=q$ and one can define $H(\mathcal{M}_n;q,t):=H(\mathcal{M}_n;q,q,t)$. In [22, Corollary 2.2.4] it is also shown that $H(\mathcal{M}_n;q,t)$ does not depend on the choice of ζ_n so that $H(\mathcal{M}_n;q,t)$ is indeed well defined.)

Determining $H(\mathcal{M}_n;q,t)$ for arbitrary rank n and genus g is a very hard problem. The breakthrough observation by Hausel and Rodriguez-Villegas is that, conjecturally, the mixed Hodge polynomial are related to Macdonald polynomials from the theory of symmetric functions, resulting in an alternative means of computing $H(\mathcal{M}_n;q,t)$ as follows. Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be an integer partition identified as usual with its Young or Ferrers diagram. For s a square (in the diagram) of λ, the

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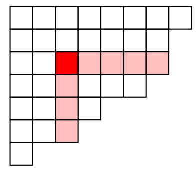
arm-length and leg-length $a(s)=a_\lambda(s)$ and $l(s)=l_\lambda(s)$ are given by the number of boxes to the right, respectively, below s. That is, if s has coordinates (i,j) then $a(s)=\lambda_i-j$ and $l(s)=\lambda'_j-i$, where λ' is the conjugate of λ. For example, the arm-length and leg-length of the square $(3,3)$ in the partition $(8,7,7,6,4,3,1)$



are 4 and 3 respectively. Hausel and Rodriguez-Villegas define the genus-g hook function $\mathcal{H}_\lambda(z,w)$ as

$$\mathcal{H}_\lambda(z,w)=\prod_{s\in\lambda}\frac{\big(z^{2a(s)+1}-w^{2l(s)+1}\big)^{2g}}{\big(z^{2a(s)+2}-w^{2l(s)}\big)\big(z^{2a(s)}-w^{2l(s)+2}\big)},$$

and use this to define two further families of rational functions $\{U_n(z,w)\}_{n\geqslant 1}$ and $\{\overline{H}_n(z,w)\}_{n\geqslant 1}$ by

$$\sum_\lambda \mathcal{H}_\lambda(z,w)T^{|\lambda|}=\exp\bigg(\sum_{n\geqslant 1}U_n(z,w)\frac{T^n}{n}\bigg),$$

where $|\lambda|=\lambda_1+\lambda_2+\cdots$ is the size of the partition λ, and

$$\overline{H}_n(z,w):=\frac{1}{n}(z^2-1)(1-w^2)\sum_{d|n}\mu(d)U_{n/d}(z^d,w^d), \tag{1.1}$$

with μ the Möbius function.[1]

Conjecture 1.1 ([22, Conjecture 4.2.1])**.** *The mixed Hodge polynomial of $\mathcal{M}_n$ is given by*

$$H(\mathcal{M}_n;q,t)=\big(tq^{1/2}\big)^{d_n}\,\overline{H}_n\big(q^{1/2},-t^{-1}q^{-1/2}\big). \tag{1.2}$$

This remarkable conjecture has several further implications. Since $H(\mathcal{M}_n;q,t)$ is a polynomial with positive coefficients, (1.2) implies that the rational function $\overline{H}_n(z,-w)$ also must be a polynomial with nonnegative coefficients. In the opposite direction, by $a_\lambda(s)=l_{\lambda'}(s)$ we have $\mathcal{H}_\lambda(z,w)=\mathcal{H}_{\lambda'}(w,z)$, which implies the "curious Poincaré duality" [22, Conjecture 4.2.4]

$$H(\mathcal{M}_n;q,t)=(qt)^{d_n}H\big(\mathcal{M}_n;q^{-1}t^{-2},t\big).$$

A non-rigorous, string theoretic derivation of (1.2) in the more general case of punctured Riemann surfaces [20,21] has recently been given in [8].

In the genus-0 case $\mathcal{M}_n$ has a single point for $n=1$ and no points for higher rank. Hence $H(\mathcal{M}_n;q,t)=\delta_{n,1}$ and, by (1.1), (1.2) and $\sum_{d|n}\mu(d)=\delta_{n,1}$, this gives

[1]Alternatively, $\overline{H}_n(z,w)$ may be defined by

$$\sum_\lambda \mathcal{H}_\lambda(z,w)T^{|\lambda|}=\mathrm{Exp}\bigg(\sum_{n\geqslant 1}\frac{\overline{H}_n(z,w)T^n}{(z^2-1)(1-w^2)}\bigg),$$

where Exp is a plethystic exponential [7, 15], defined for formal power series $f(z,w;T):=\sum_{n\geqslant 1}c_n(z,w)T^n$ as $\mathrm{Exp}\big(f(z,w;T)\big):=\exp\big(\sum_{r\geqslant 1}f(z^r,w^r;T^r)/r\big)$.

$\overline{H}_n(z,w) = \overline{H}_{n/d}(z^d,w^d) = w^{-2n}/(1-z^{2n})(1-w^{-2n})$. For genus 0 the conjecture is thus equivalent to the combinatorial identity

$$\sum_{\lambda} \mathcal{H}\big(q^{1/2},t^{-1/2}\big)T^{|\lambda|} = \prod_{i,j\geqslant 1} \frac{1}{1-q^{i-1}t^jT}, \tag{1.3}$$

which follows immediately as a special case of the Kaneko–Macdonald binomial theorem for Macdonald polynomials [26, 38].[2]

More interesting is the genus-1 case. Then $H(\mathcal{M}_n;q,t)=(1+qt)^2$ for all $n\geqslant 1$, which by (1.2) implies $\overline{H}_n(z,w)=(z-w)^2$. Solving (1.1) for $U_n(z,w)$ leads to

$$U_n(z,w) = \sum_{k|n} \frac{n}{k}\cdot\frac{(1-z^kw^{-k})^2}{(1-z^{2k})(1-w^{-2k})}.$$

Since $\sum_{d|n}\mu(d)\sum_{m|(n/d)} f(md)=f(1)$, Conjecture 1.1 for $g=1$ is thus equivalent to the following combinatorial identity.

Conjecture 1.2 ([22, Conjecture 4.3.2])**.** *For $g=1$,*

$$\sum_{\lambda} \mathcal{H}_\lambda\big(q^{1/2},t^{-1/2}\big)T^{|\lambda|} = \prod_{i,j,k\geqslant 1} \frac{(1-q^{i-1/2}t^{j-1/2}T^k)^2}{(1-q^{i-1}t^{j-1}T^k)(1-q^it^jT^k)}.$$

In this paper we settle this conjecture by proving the following more general combinatorial identity.

Theorem 1.3 (q,t-Nekrasov–Okounkov formula)**.** *We have*

$$\begin{aligned}\sum_{\lambda} T^{|\lambda|}\prod_{s\in\lambda} &\frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})} \\ &= \prod_{i,j,k\geqslant 1}\frac{(1-uq^it^{j-1}T^k)(1-u^{-1}q^{i-1}t^jT^k)}{(1-q^{i-1}t^{j-1}T^k)(1-q^it^jT^k)}.\end{aligned} \tag{1.4}$$

For $u=(t/q)^{1/2}$ this is Conjecture 1.2 and for general u it is a q,t-analogue of the Nekrasov–Okounkov formula discovered by Nekrasov and Okounkov in their work on random partitions and Seiberg–Witten theory [41]. Indeed, if $h(s):=a(s)+l(s)+1$ is the hook-length of s and $\mathscr{H}(\lambda):=\{h(s):s\in\lambda\}$ is the multiset of hook-lengths of λ, then (1.4) for $t=q$ simplifies to

$$\sum_{\lambda} T^{|\lambda|}\prod_{h\in\mathscr{H}(\lambda)}\frac{(1-uq^h)(1-u^{-1}q^h)}{(1-q^h)^2} = \prod_{k,r\geqslant 1}\frac{(1-uq^rT^k)^r(1-u^{-1}q^rT^k)^r}{(1-q^{r-1}T^k)^r(1-q^{r+1}T^k)^r},$$

first found in [25, p. 749] and [9, Theorem 5]. Setting $u=q^z$ and letting q tend to 1 this yields the Nekrasov–Okounkov formula [41, Equation (6.12)] (see also [18, Corollary 1.9], [52])

$$\sum_{\lambda} T^{|\lambda|}\prod_{h\in\mathscr{H}(\lambda)}\Big(1-\frac{z^2}{h^2}\Big) = \prod_{k\geqslant 1}(1-T^k)^{z^2-1}. \tag{1.5}$$

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[2] Hausel and Rodriguez-Villegas prove (1.3) differently, using a duality of Garsia and Haiman [11] and the Cauchy identity for Schur functions.

In [52, Proposition 6.1] Westbury has shown that for fixed λ and p a sufficiently large integer ($p > |\lambda|$ suffices)

$$\prod_{h\in\mathscr{H}(\lambda)}\Big(\frac{p^2}{h^2}-1\Big)$$

is the dimension of the irreducible $\mathfrak{sl}(p,\mathbb{C})$-module indexed by the partition

$$\mu := (\lambda_1,\dots,\lambda_p)+(\lambda'_1-\lambda'_p,\dots,\lambda'_1-\lambda'_2,0).$$

Using the q-analogue of Weyl's dimension formula for $\mathfrak{sl}(p,\mathbb{C})$ [35, p. 124] (see also [4, Lemma 3.1]) or Stanley's hook content formula [49, Theorem 15.3] (see also [50, Lemma 7.21.2]) it is not hard to show that in the q-case

$$\prod_{h\in\mathscr{H}(\lambda)}\frac{(1-q^{h+p})(1-q^{h-p})}{(1-q^h)^2}=(-1)^{|\lambda|}q^{-l(\lambda)\binom{p}{2}}s_\mu(1,q,\dots,q^p),$$

where $l(\lambda)$ is the length of λ (the number of non-zero λ_i) and s_μ a Schur function. We did not find a similar such interpretation of the product in (1.4) in terms of Macdonald polynomials.

The remainder of this paper is organised as follows. In the next section we first review some basic material from the theory of Macdonald polynomials and interpolation Macdonald polynomials. Then we apply these polynomials to prove a number of key identities needed in our proof of Theorem 1.3. This includes the following elegant Cauchy-like identity for principally specialised skew Macdonald polynomials.

Theorem 1.4. *Let* $\rho := (0,1,2,\dots)$ *and* $q^\rho := (1,q,q^2,\dots)$. *Then*

$$\begin{aligned}(1.6)\qquad \sum_{\lambda,\mu,\nu,\tau} & a^{|\lambda|}b^{|\mu|}c^{|\nu|}d^{|\tau|}b_\nu(q,t)b_\tau(t,q)Q_{\lambda/\nu}(t^\rho;q,t)Q_{\lambda'/\tau}(q^\rho;t,q)\\ &\times Q_{\mu/\nu}(t^\rho;q;t)Q_{\mu'/\tau}(q^\rho;t,q)=\frac{1}{(abcd;abcd)_\infty}\cdot\frac{(-a,-b;q,t,abcd)_\infty}{(abc,abd;q,t,abcd)_\infty},\end{aligned}$$

In the above, $b_\lambda(q,t)$ is Macdonald's q,t-hook function

$$b_\lambda(q,t):=\prod_{s\in\lambda}\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}$$

and

$$(a;q_1,q_2,\dots,q_m)_\infty := \prod_{i_1,\dots,i_m\geqslant 0}(1-aq_1^{i_1}\cdots q_m^{i_m}),$$

$$(a_1,\dots,a_k;q_1,q_2,\dots,q_m)_\infty := (a_1;q_1,q_2,\dots,q_m)_\infty\cdots(a_k;q_1,q_2,\dots,q_m)_\infty$$

are generalised q-shifted factorials. In Section 3 we study a function $f_{n,m}$ which may be viewed as a rational function analogue of

$$(1.7)\qquad f(u,T;q,t):=(uq;q,t)_\infty\sum_\lambda T^{|\lambda|}\prod_{s\in\lambda}\frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})},$$

see Proposition 3.3. We determine a number of hidden symmetries of $f_{n,m}$, conjecture its polynomiality, and show that up to the factor $(uq;q,t)_\infty$ the limit $\lim_{n,m\to\infty} f_{n,m}$ is given by the product side of (1.4), thus proving Theorem 1.3.

Then, in Section 4 we discuss a number of special cases of the q,t-Nekrasov–Okounkov formula, as well as a close link between our work and that of Iqbal, Kozçaz and Shabbir [24] on the topological vertex formalism. Finally, in the appendix we give an alternative proof of the q,t-Nekrasov–Okounkov formula, suggested to us by Jim Bryan, which is based on Waelder's equivariant Dijkgraaf–Moore–Verlinde–Verlinde (DMVV) formula for the Hilbert scheme of points in the plane [51].

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2. Macdonald polynomials

2.1. **Partitions.** A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly-decreasing sequence of nonnegative integers such that only finitely-many λ_i are non-zero. The positive λ_i are called the parts of λ and the number of parts, denoted $l(\lambda)$, is called the length of the partition. If $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$ we say that λ is a partition of n, and denote this by $\lambda \vdash n$. As is customary, the unique partition of 0 will be denoted by 0. We identify a partition λ with its Young diagram diagram consisting of $l(\lambda)$ left-aligned rows of squares with λ_i squares in the ith row. The conjugate of λ, denoted λ', is given by reflecting λ in the main diagonal $i = j$, i.e., its parts are the columns of λ. If μ is contained in λ, that is, $\mu_i \leqslant \lambda_i$ for all i we write $\mu \subset \lambda$. We also adopt the standard dominance order on partitions, writing $\mu \leqslant \lambda$ if and only if $\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i$ for all $i \geqslant 1$, where λ, μ are partitions such that $|\lambda| = |\mu|$. Throughout the paper we repeatedly use

$$\delta_n := (n-1, \dots, 1, 0), \qquad \rho_n := (0, 1, \dots, n-1)$$

and $\rho := (0, 1, 2, \dots)$. Of course, if $f(x)$ is a symmetric function, then $f(t^{\delta_n}) = f(t^{\rho_n})$. Apart from the arm and leg lengths of a partition defined in the introduction, we also need to arm-colength $a'(s) = a'_\lambda(s)$ and leg-colength $l'(s) = l'_\lambda(s)$ of $s \in \lambda$, given by the number of boxes in λ immediately to the left or above s, respectively. Equivalently, $a'(s) = j - 1$ and $l'(s) = i - 1$ for $s = (i, j)$. Finally we recall the following standard statistic on partitions [37]:

$$n(\lambda) := \sum_{s \in \lambda} l'(s) = \sum_{i \geqslant 1} (i-1)\lambda_i = \sum_{i \geqslant 1} \binom{\lambda'_i}{2}.$$

2.2. **Hook functions.** In the introduction we already defined the hook functions $\mathcal{H}_\lambda(z,w)$ and $b_\lambda(q,t)$. We will further need

$$\begin{aligned} c_\lambda(q,t) &:= \prod_{s\in\lambda}\big(1-q^{a(s)}t^{l(s)+1}\big) \\ c'_\lambda(q,t) &:= \prod_{s\in\lambda}\big(1-q^{a(s)+1}t^{l(s)}\big), \end{aligned}$$

so that

$$b_\lambda(q,t)=\frac{c_\lambda(q,t)}{c'_\lambda(q,t)}$$

and

$$\begin{aligned} (z;q,t)_\lambda &:= \prod_{s\in\lambda}\big(1-zq^{a'(s)}t^{-l'(s)}\big) \\ &= \prod_{i,j\geqslant 1}\frac{1-zq^{i-1}t^{j-\lambda'_i}}{1-zq^{i-1}t^j}=\prod_{i=1}^n (zt^{1-i};q)_{\lambda_i}, \end{aligned} \tag{2.1}$$

where $(z;q)_n:=(1-z)\cdots(1-zq^{n-1})$ is the usual q-shifted factorial.

It is easy to check from the definition that

$$c'_{\lambda'}(q,t)=c_\lambda(t,q), \tag{2.2}$$

and hence

$$b_{\lambda'}(q,t)=\frac{1}{b_\lambda(t,q)}. \tag{2.3}$$

It is also an elementary exercise to verify the relation

$$c'_\lambda(1/q,1/t)=(-1)^{|\lambda|}q^{-n(\lambda')-|\lambda|}t^{-n(\lambda)}c'_\lambda(q,t). \tag{2.4}$$

2.3. **Macdonald polynomials.** Let $F=\mathbb{Q}(q,t)$ and Λ_F the ring of symmetric functions in $x=(x_1,x_2,\dots)$ with coefficients in F. Further denote by $\Lambda_{n,F}$ the analogous ring over the finite alphabet $(x_1,\dots,x_n)$. The Newton power sums p_λ and monomial symmetric functions m_λ are defined as

$$p_\lambda(x):=\prod_{i\geqslant 1}p_{\lambda_i}(x)$$

where $p_r(x):=x_1^r+x_2^r+\cdots$ and $p_0:=1$, and

$$m_\lambda=\sum_\alpha x^\alpha,$$

where the sum is over distinct permutations α of λ and $x^\alpha:=x_1^{\alpha_1}x_2^{\alpha_2}\cdots$. Both families of symmetric functions are bases for Λ_F.

Following Macdonald [37] we define the q,t-Hall scalar product on Λ_F as

$$\langle p_\lambda,p_\mu\rangle_{q,t}:=\delta_{\lambda\mu}z_\lambda\prod_{i\geqslant 1}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},$$

where $z_\lambda:=\prod_{i\geqslant 1}m_i(\lambda)!\,i^{m_i(\lambda)}$ and $m_i(\lambda):=\lambda'_i-\lambda'_{i+1}$. The Macdonald polynomials $P_\lambda(q,t)=P_\lambda(x;q,t)$ are the unique family of symmetric functions such that [37, p. 322]

$$P_\lambda(q,t)=m_\lambda+\sum_{\mu<\lambda}u_{\lambda\mu}(q,t)m_\mu,\qquad u_{\lambda\mu}\in F$$

and

$$\langle P_\lambda(q,t), P_\mu(q,t)\rangle_{q,t} = 0 \qquad \text{if} \quad \lambda \neq \mu.$$

We also require the skew Macdonald polynomials $P_{\lambda/\mu}(q,t)$ defined by

$$\langle P_{\lambda/\mu}(q,t), P_\nu(q,t)\rangle_{q,t} = \langle P_\lambda(q,t), P_\mu(q,t)P_\nu(q,t)\rangle_{q,t}.$$

The polynomial $P_{\lambda/\mu}(q,t)$ vanishes unless $\mu \subset \lambda$. Moreover, in $\Lambda_{n,F}$, $P_\lambda(q,t)$ vanishes unless $l(\lambda) \leqslant n$.

A second family of Macdonald polynomials $Q_{\lambda/\mu}(x;q,t) = Q_{\lambda/\mu}(q,t)$ may be defined by

$$Q_{\lambda/\mu}(q,t) = \frac{b_\lambda(q,t)}{b_\mu(q,t)}\, P_{\lambda/\mu}(q,t). \tag{2.5}$$

Then $\langle P_\lambda(q,t), Q_\mu(q,t)\rangle_{q,t} = \delta_{\lambda\mu}$, which is equivalent to the Cauchy identity [37, p. 324]

$$\sum_\lambda P_\lambda(x;q,t) Q_\lambda(y;q,t) = \prod_{i,j\geqslant 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \tag{2.6}$$

For Macdonald polynomials in n variables we need the principal specialisation formula [37, p. 337]

$$P_\lambda(t^{\delta_n};q,t) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a'(s)}t^{n-l'(s)}}{1-q^{a(s)}t^{l(s)+1}} = t^{n(\lambda)}\,\frac{(t^n;q,t)_\lambda}{c_\lambda(q,t)} \tag{2.7}$$

and the Macdonald–Koornwinder duality [37, p. 332]

$$P_\lambda(t^{\delta_n};q,t)P_\mu(q^\lambda t^{\delta_n};q,t) = P_\mu(t^{\delta_n};q,t)P_\lambda(q^\mu t^{\delta_n};q,t) \tag{2.8}$$

for $l(\lambda), l(\mu) \leqslant n$. Here $q^\lambda t^{\delta_n} := (q^{\lambda_1}t^{n-1}, q^{\lambda_2}t^{n-2}, \dots, q^{\lambda_n}t^0)$.

In our proof of (1.6) it will be convenient to adopt plethystic or λ-ring notation [17, 31]. In particular, for $f \in \Lambda_F$ we use $f([(a-b)/(1-t)])$, defined in terms of the power sums as

$$p_r\Big(\Big[\frac{a-b}{1-t}\Big]\Big) := \frac{a^r-b^r}{1-t^r}. \tag{2.9}$$

The map $\varepsilon_{a,b,t} : \Lambda_F \to F[a,b]$ given by $\varepsilon_{a,b,t}(f) \mapsto f([(a-b)/(1-t)])$ is a ring homomorphism, and in particular

$$P_{\lambda/\nu}\Big(\Big[\frac{a-b}{1-t}\Big];q,t\Big) = \sum_\mu P_{\lambda/\mu}\Big(\Big[\frac{a}{1-t}\Big];q,t\Big)P_{\mu/\nu}\Big(\Big[\frac{-b}{1-t}\Big];q,t\Big) \tag{2.10a}$$

$$= \sum_\mu P_{\lambda/\mu}\Big(\Big[\frac{-b}{1-t}\Big];q,t\Big)P_{\mu/\nu}\Big(\Big[\frac{a}{1-t}\Big];q,t\Big). \tag{2.10b}$$

We also note that

$$f\Big(\Big[a\,\frac{1-t^n}{1-t}\Big]\Big) = f(at^{\rho_n}) = f(at^{\delta_n}) \tag{2.11a}$$

$$f\Big(\Big[\frac{a}{1-t}\Big]\Big) = f(at^\rho). \tag{2.11b}$$

Let $\omega_{q,t}$ be the automorphism of Λ_F defined by

$$\omega_{q,t}(p_r) = (-1)^{r-1}\frac{1-q^r}{1-t^r}\, p_r.$$

Then [37, p. 327]

$$\omega_{q,t}\big(P_{\lambda/\mu}(q,t)\big)=Q_{\lambda'/\mu'}(t,q). \tag{2.12}$$

If $f\in\Lambda_F$ is homogeneous of degree r then it is readily checked using (2.9) and (2.12) that

$$\varepsilon_{a,b,t}(f)=(-1)^r\varepsilon_{b,a,q}\,\omega_{q,t}(f).$$

Applying this with $f=P_{\lambda/\mu}(q,t)$ and using (2.12) implies the duality

$$P_{\lambda/\mu}\Big(\Big[\frac{a-b}{1-t}\Big];q,t\Big)=(-1)^{|\lambda|-|\mu|}Q_{\lambda'/\mu'}\Big(\Big[\frac{b-a}{1-q}\Big];t,q\Big). \tag{2.13}$$

2.4. **Interpolation Macdonald polynomials.** In this section we work exclusively in $\Lambda_{n,F}$, and assume that $x=(x_1,\dots,x_n)$ and μ is a partition of length at most n. Then the interpolation Macdonald polynomial (or shifted Macdonald polynomial) $\bar{P}^*_\mu=\bar{P}^*_\mu(x;q,t)$ is the unique (inhomogeneous) symmetric polynomial of degree $|\mu|$ in x such that

$$\bar{P}^*_\mu(q^\lambda t^{\delta_n};q,t)=0\quad\text{for all }\lambda\text{ such that }\mu\not\subset\lambda \tag{2.14}$$

and

$$[x^\mu]\bar{P}^*_\mu(x;q,t)=1. \tag{2.15}$$

The polynomials $\bar{P}^*_\mu(x;q,t)$ were first introduced and studied by Knop, Okounkov and Sahi in [28, 29, 42, 43, 47], and the choice of defining relations differs slightly from author to author. For example, in (2.14) the "for all" condition is sometimes replaced by the weaker "for all $\lambda\neq\mu$ such that $|\lambda|\leqslant|\mu|$" and the normalisation (2.15) is sometimes replaced by

$$\bar{P}^*_\mu(q^\mu t^{\delta_n};q,t)=(-1)^{|\mu|}q^{n(\mu')}t^{(n-1)|\mu|-2n(\mu)}c'_\mu(q,t). \tag{2.16}$$

Below we have collected a number of results from the theory of interpolation Macdonald polynomials needed in our proof of the q,t-Nekrasov–Okounkov formula. In [47, Theorem 1.1] Sahi showed that the top-homogeneous degree term of $\bar{P}^*_\mu(x;q,t)$ is the Macdonald polynomial $P_\mu(x;q,t)$. In other words,

$$\lim_{a\to\infty}a^{-|\mu|}\bar{P}^*_\mu(ax;q,t)=P_\mu(x;q,t). \tag{2.17}$$

For μ a partition of length at most n, the interpolation Macdonald polynomials satisfy the stability property

$$\bar{P}^*_\mu(tx_1,\dots,tx_n,1;q,t)=t^{|\mu|}\bar{P}^*_\mu(x_1,\dots,x_n;q,t). \tag{2.18}$$

Okounkov [42] used this to define the q,t-binomial coefficients

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t}:=\frac{\bar{P}^*_\mu(q^\lambda t^{\delta_n};q,t)}{\bar{P}^*_\mu(q^\mu t^{\delta_n};q,t)}. \tag{2.19}$$

Thanks to (2.18) the left-hand side is independent of n as long as we take $n\geqslant l(\lambda),l(\mu)$. It follows from the vanishing property (2.14) that $\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]_{q,t}=0$ unless $\mu\subset\lambda$. From a duality of $\bar{P}^*_\mu(x;q,t)$ given in [43, Theorem IV] Okounkov inferred the duality [42, Equation (2.12)]

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t}=\begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_{1/t,1/q}. \tag{2.20}$$

Finally we need the binomial theorem [42] for interpolation Macdonald polynomials, given by

$$\sum_{\nu} a^{|\nu|} \begin{bmatrix}\lambda\\ \nu\end{bmatrix}_{1/q,1/t} \frac{\bar{P}^*_{\lambda}(at^{-\delta_n};q,t)}{\bar{P}^*_{\nu}(at^{-\delta_n};q,t)}\, \bar{P}^*_{\nu}(x;1/q,1/t) = \bar{P}^*_{\lambda}(ax;q,t). \tag{2.21}$$

To conclude this section we apply the binomial theorem to prove the following sum over the product of two skew Macdonald polynomials.

Proposition 2.1. *For λ and μ partitions,*

$$\begin{aligned}\sum_{\nu} q^{-n(\lambda')-n(\mu')-|\nu|} t^{n(\lambda)+n(\mu)} b_{\nu}(t,q) Q_{\lambda'/\nu}(q^{\rho};t,q) Q_{\mu'/\nu}(q^{\rho};t,q)&\\ = P_{\mu}(t^{\rho};q,t) P_{\lambda}(q^{-\mu}t^{\rho};q,t).&\end{aligned} \tag{2.22}$$

Proof. Let λ and μ be partitions of length at most n. If we specialise $x = q^{-\mu}t^{-\delta_n}$ in the binomial theorem (2.21) and use definition (2.19) of the q,t-binomial coefficient we obtain

$$\begin{aligned}\sum_{\nu} a^{|\nu|} \begin{bmatrix}\lambda\\ \nu\end{bmatrix}_{1/q,1/t} \begin{bmatrix}\mu\\ \nu\end{bmatrix}_{1/q,1/t} \frac{\bar{P}^*_{\lambda}(at^{-\delta_n};q,t)}{\bar{P}^*_{\nu}(at^{-\delta_n};q,t)}\, \bar{P}^*_{\nu}(q^{-\nu}t^{-\delta_n};1/q,1/t)&\\ = \bar{P}^*_{\lambda}(aq^{-\mu}t^{-\delta_n};q,t).&\end{aligned}$$

By the duality (2.20) we may replace $\left[\begin{smallmatrix}\lambda\\ \nu\end{smallmatrix}\right]_{1/q,1/t}\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]_{1/q,1/t}$ by $\left[\begin{smallmatrix}\lambda'\\ \nu'\end{smallmatrix}\right]_{t,q}\left[\begin{smallmatrix}\mu'\\ \nu'\end{smallmatrix}\right]_{t,q}$. Also replacing a by at^{n-1}, then multiplying both sides by $a^{-|\lambda|}$, and finally letting a tend to infinity using (2.17) results in

$$\begin{aligned}\sum_{\nu} t^{(n-1)|\nu|} \begin{bmatrix}\lambda'\\ \nu'\end{bmatrix}_{t,q} \begin{bmatrix}\mu'\\ \nu'\end{bmatrix}_{t,q} \frac{P_{\lambda}(t^{\delta_n};q,t)}{P_{\nu}(t^{\delta_n};q,t)}\, \bar{P}^*_{\nu}(q^{-\nu}t^{-\delta_n};1/q,1/t)&\\ = P_{\lambda}(q^{-\mu}t^{\rho_n};q,t).&\end{aligned} \tag{2.23}$$

Next we use [33, p. 323] (see also [12, Equation (3.13)])

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t} = t^{n(\mu)-n(\lambda)} \frac{c'_{\lambda}(q,t)}{c'_{\mu}(q,t)}\, Q_{\lambda/\mu}(t^{\rho};q,t) \tag{2.24}$$

as well as the principal specialisation formula (2.7) and normalisation formula (2.16). This allows (2.23) to be rewritten as

$$\begin{aligned}\sum_{\nu} (-1)^{|\nu|} q^{n(\nu')-n(\lambda')-n(\mu')} t^{n(\nu)+n(\lambda)} \frac{(t^n;q,t)_{\lambda}}{(t^n;q,t)_{\nu}}&\\ \times \frac{c_{\nu}(q,t)c'_{\nu}(1/q,1/t)c'_{\lambda'}(t,q)c'_{\mu'}(t,q)}{c_{\lambda}(q,t)(c'_{\nu'}(t,q))^2}\, Q_{\lambda'/\nu'}(q^{\rho};t,q) Q_{\mu'/\nu'}(q^{\rho};t,q)&\\ = P_{\lambda}(q^{-\mu}t^{\rho_n};q,t).&\end{aligned}$$

Simplifying this using (2.2)–(2.4) yields

$$\begin{aligned}\sum_{\nu} q^{-n(\lambda')-n(\mu')-|\nu|} t^{n(\lambda)} \frac{(t^n;q,t)_{\lambda}}{(t^n;q,t)_{\nu}}&\\ \times \frac{c_{\mu}(q,t)}{b_{\nu}(q,t)}\, Q_{\lambda'/\nu'}(q^{\rho};t,q) Q_{\mu'/\nu'}(q^{\rho};t,q) = P_{\lambda}(q^{-\mu}t^{\rho_n};q,t).&\end{aligned}$$

Multiplying both sides by $P_\mu(t^{\rho_n};q,t) = P_\mu(t^{\delta_n};q,t)$ and once again using the principal specialisation formula (2.7), we finally arrive at

$$\sum_\nu q^{-n(\lambda')-n(\mu')-|\nu|} t^{n(\lambda)+n(\mu)} \frac{(t^n;q,t)_\lambda (t^n;q,t)_\mu}{(t^n;q,t)_\nu\, b_\nu(q,t)} \\ \times Q_{\lambda'/\nu'}(q^\rho;t,q) Q_{\mu'/\nu'}(q^\rho;t,q) = P_\mu(t^{\rho_n};q,t) P_\lambda(q^{-\mu}t^{\rho_n};q,t). \tag{2.25}$$

The identity (2.22) follows in the large-n limit, up to the variable change $\nu \mapsto \nu'$ and the use of $b_{\nu'}(q,t)b_\nu(t,q)=1$, see (2.3). □

2.5. **Proof of Theorem 1.4.** In this section we establish the Cauchy-like identity (1.6) which will be key in our subsequent proof of Theorem 1.3. In fact, we will prove a slightly less-symmetric but equivalent form obtained by the simultaneous substitution

$$(a,b,c,d) \mapsto (Tab, cd, 1/ac, 1/bd).$$

Theorem 2.2. *We have*

$$\sum_{\lambda,\mu,\nu,\tau} T^{|\lambda|} b_\nu(q,t) b_\tau(t,q) Q_{\lambda/\nu}(at^\rho;q,t) Q_{\lambda'/\tau}(bq^\rho;t,q) \\ \times Q_{\mu/\nu}(ct^\rho;q,t) Q_{\mu'/\tau}(dq^\rho;t,q) = \frac{1}{(T;T)_\infty} \cdot \frac{(-abT,-cd;q,t,T)_\infty}{(acT,bdT;q,t,T)_\infty}. \tag{2.26}$$

Before we prove this we need the following q,t-analogue of a Schur function identity from page 94 of [37].

Proposition 2.3. *We have*

$$\sum_{\lambda,\nu} T^{|\lambda|} P_{\lambda/\nu}(x;q,t) Q_{\lambda/\nu}(y;q,t) = \frac{1}{(T;T)_\infty} \prod_{i,j\geqslant 1} \frac{(tTx_iy_j;q,T)_\infty}{(Tx_iy_j;q,T)_\infty}. \tag{2.27}$$

Proof. Denote the left-hand side of (2.27) by $f(x,y)$ and recall the generalisation of the Cauchy identity (2.6) to skew functions [37, p. 352]

$$\sum_\lambda P_{\lambda/\nu}(x;q,t) Q_{\lambda/\tau}(y;q,t) \\ = \prod_{i,j\geqslant 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} \sum_\lambda P_{\tau/\lambda}(x;q,t) Q_{\nu/\lambda}(y;q,t). \tag{2.28}$$

Applying this with $(x,\tau) \mapsto (Tx,\nu)$ and multiplying both sides by $T^{|\nu|}$, it follows from the homogeneity of the Macdonald polynomials that

$$f(x,y) = \prod_{i,j\geqslant 1} \frac{(tTx_iy_j;q)_\infty}{(Tx_iy_j;q)_\infty} \sum_{\lambda,\nu} T^{|\nu|} P_{\nu/\lambda}(Tx;q,t) Q_{\nu/\lambda}(y;q,t).$$

By the simultaneous variable change $(\lambda,\nu) \mapsto (\nu,\lambda)$ this yields

$$f(x,y) = f(Tx,y) \prod_{i,j\geqslant 1} \frac{(tTx_iy_j;q)_\infty}{(Tx_iy_j;q)_\infty}$$

and thus

$$f(x,y) = f(0,y) \prod_{i,j\geqslant 1} \frac{(tTx_iy_j;q,T)_\infty}{(Tx_iy_j;q,T)_\infty}.$$

By $P_{\lambda/\nu}(0;q,t)=\delta_{\lambda\nu}$ and $Q_{\lambda/\lambda}(y;q,t)=1$ we finally get

$$f(0,y)=\sum_{\lambda}T^{|\lambda|}=\frac{1}{(T;T)_\infty},$$

and the claim follows. □

Proof of Theorem 2.2. If we take Proposition 2.3, replace $\nu\mapsto\mu$, and then make the plethystic substitutions $x\mapsto(a-d)/(1-t)$ and $y\mapsto(c-b)/(1-t)$, we get

$$\sum_{\lambda,\mu}T^{|\lambda|}P_{\lambda/\mu}\Big(\Big[\frac{a-d}{1-t}\Big];q,t\Big)Q_{\lambda/\mu}\Big(\Big[\frac{c-b}{1-t}\Big];q,t\Big)=\frac{1}{(T;T)_\infty}\cdot\frac{(abT,cdT;q,t,T)_\infty}{(acT,bdT;q,t,T)_\infty}.$$

Here the product on the right follows from [37, p. 310]

$$\prod_{i,j\geqslant 1}\frac{(tTx_iy_j;q)_\infty}{(Tx_iy_j;q)_\infty}=\exp\bigg(\sum_{r\geqslant 1}\frac{T^r}{r}\cdot\frac{1-t^r}{1-q^r}\,p_r(x)p_r(y)\bigg),\tag{2.29}$$

equation (2.9) and[3]

$$\exp\bigg(\sum_{r\geqslant 1}\frac{\mp T^r}{r(1-q^r)(1-t^r)}\bigg)=(T;q,t)_\infty^{\pm 1}.$$

Using both equations in (2.10) (which also hold with P replaced by Q) gives

$$\begin{aligned}\sum_{\lambda,\mu,\nu,\tau}&T^{|\lambda|}P_{\lambda/\nu}\Big(\Big[\frac{a}{1-t}\Big];q,t\Big)Q_{\lambda/\tau}\Big(\Big[\frac{-b}{1-t}\Big];q,t\Big)\\&\times P_{\nu/\mu}\Big(\Big[\frac{-d}{1-t}\Big];q,t\Big)Q_{\tau/\mu}\Big(\Big[\frac{c}{1-t}\Big];q,t\Big)=\frac{1}{(T;T)_\infty}\cdot\frac{(abT,cdT;q,t,T)_\infty}{(acT,bdT;q,t,T)_\infty}.\end{aligned}$$

Transforming the sum over μ by the Cauchy identity (2.28) with $(\lambda,\nu,\tau)\mapsto(\mu,\tau,\nu)$, $x\mapsto -d/(1-t)$ and $y\mapsto c/(1-t)$ leads to

$$\begin{aligned}\sum_{\lambda,\mu,\nu,\tau}&T^{|\lambda|}P_{\lambda/\nu}\Big(\Big[\frac{a}{1-t}\Big];q,t\Big)Q_{\lambda/\tau}\Big(\Big[\frac{-b}{1-t}\Big];q,t\Big)\\&\times Q_{\mu/\nu}\Big(\Big[\frac{c}{1-t}\Big];q,t\Big)P_{\mu/\tau}\Big(\Big[\frac{-d}{1-t}\Big];q,t\Big)=\frac{1}{(T;T)_\infty}\cdot\frac{(abT,cd;q,t,T)_\infty}{(acT,bdT;q,t,T)_\infty},\end{aligned}$$

where this time we have used (2.29) with $T=1$. By the duality (2.13) this is also

$$\begin{aligned}\sum_{\lambda,\mu,\nu,\tau}&(-1)^{|\lambda|+|\mu|}T^{|\lambda|}P_{\lambda/\nu}\Big(\Big[\frac{a}{1-t}\Big];q,t\Big)P_{\lambda'/\tau'}\Big(\Big[\frac{b}{1-q}\Big];t,q\Big)\\&\times Q_{\mu/\nu}\Big(\Big[\frac{c}{1-t}\Big];q,t\Big)Q_{\mu'/\tau'}\Big(\Big[\frac{d}{1-q}\Big];t,q\Big)=\frac{1}{(T;T)_\infty}\cdot\frac{(abT,cd;q,t,T)_\infty}{(acT,bdT;q,t,T)_\infty}.\end{aligned}$$

Replacing $(b,d;\tau)\mapsto(-b,-d;\tau')$, and using (2.5) and (2.11b) completes the proof of (2.26). □

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[3]This may be stated more simply as $\mathrm{Exp}\big(\mp T/(1-q)(1-t)\big)=(T;q,t)_\infty^{\pm}$.

3. Proof of Theorem 1.3

Instead of giving a direct proof of the q,t-Nekrasov–Okounkov formula we will first study a rational function $f_{n,m}$ which, as follows from Proposition 3.3 below, may be viewed as a rational function analogue of the sum side of (1.4). As it turns out, almost all of our steps towards proving Theorem 1.3 can be carried out at the level of $f_{n,m}$.

Let

$$\begin{aligned} f_{n,m}(u,T;q,t) &:= (-u)^{nm} q^{n\binom{m+1}{2}} t^{m\binom{n}{2}} \\ &\quad\times \sum_{\lambda,\mu\subset(m^n)} T^{|\lambda|}(-uq^m t^{n-1})^{-|\lambda|-|\mu|} P_\lambda(t^{\delta_n};q,t)P_{\lambda'}(q^{\delta_m};t,q) \\ &\qquad\qquad \times P_\mu(q^\lambda t^{\delta_n};q,t)P_{\mu'}(t^{\lambda'}q^{\delta_m};t,q). \end{aligned} \tag{3.1}$$

An obvious symmetry of $f_{n,m}$ is

$$f_{n,m}(u,T;q,t) = f_{m,n}(uq/t,T;t,q).$$

Not as apparent are the following two additional symmetries.

Lemma 3.1. *We have*

$$f_{n,m}(u,T;q,t) = T^{nm} f_{n,m}(u/T,1/T;q,t) \tag{3.2a}$$

$$\qquad\qquad = f_{n,m}(tT/uq,T;q,t). \tag{3.2b}$$

Proof. By the duality (2.8) we get

$$\begin{aligned} f_{n,m}(u,T;q,t) &= (-u)^{nm} q^{n\binom{m+1}{2}} t^{m\binom{n}{2}} \\ &\quad\times \sum_{\lambda,\mu\subset(m^n)} T^{|\lambda|}(-uq^m t^{n-1})^{-|\lambda|-|\mu|} P_\mu(t^{\delta_n};q,t)P_{\mu'}(q^{\delta_m};t,q) \\ &\qquad\qquad \times P_\lambda(q^\mu t^{\delta_n};q,t)P_{\lambda'}(t^{\mu'}q^{\delta_m};t,q). \end{aligned}$$

Renaming the summation index λ as μ and vice versa yields

$$\begin{aligned} f_{n,m}(u,T;q,t) &= (-u)^{nm} q^{n\binom{m+1}{2}} t^{m\binom{n}{2}} \\ &\quad\times \sum_{\lambda,\mu\subset(m^n)} T^{|\mu|}(-uq^m t^{n-1})^{-|\lambda|-|\mu|} P_\lambda(t^{\delta_n};q,t)P_{\lambda'}(q^{\delta_m};t,q) \\ &\qquad\qquad \times P_\mu(q^\lambda t^{\delta_n};q,t)P_{\mu'}(t^{\lambda'}q^{\delta_m};t,q). \end{aligned}$$

Comparing this with (3.1) implies (3.2a).

Next we replace the sum over μ in (3.1) by a sum over its complement with respect to (m^n), denoted by $\tilde{\mu} = (m-\mu_n,\dots,m-\mu_1)$. Recalling that (see e.g., [2, Equation (4.3)])

$$P_{\tilde{\mu}}(x_1,\dots,x_n;q,t) = (x_1\cdots x_n)^m P_\mu(x_1^{-1},\dots,x_n^{-1};q,t),$$

this yields

$$\begin{aligned} f_{n,m}(u,T;q,t) &= \sum_{\lambda,\mu\subset(m^n)} (-tT/u)^{|\lambda|}(-uq^m t^{n-1})^{|\mu|} P_\lambda(t^{\delta_n};q,t)P_{\lambda'}(q^{\delta_m};t,q) \\ &\qquad\qquad \times P_\mu(q^{-\lambda} t^{-\delta_n};q,t)P_{\mu'}(t^{-\lambda'}q^{-\delta_m};t,q). \end{aligned}$$

Since $P_\lambda(t^{\delta_n};q,t)=P_\lambda(t^{\rho_n};q,t)$ and $P_\mu(q^{-\lambda}t^{-\delta_n};q,t)=t^{(1-n)|\mu|}P_\mu(q^{-\lambda}t^{\rho_n};q,t)$, this can be further transformed into

$$f_{n,m}(u,T;q,t)=\sum_{\lambda,\mu\subset(m^n)}(-tT/u)^{|\lambda|}(-uq)^{|\mu|}P_\lambda(t^{\rho_n};q,t)P_{\lambda'}(q^{\rho_m};t,q) \times P_\mu(q^{-\lambda}t^{\rho_n};q,t)P_{\mu'}(t^{-\lambda'}q^{\rho_m};t,q). \tag{3.3}$$

Applying the symmetry $P_\lambda(x;q,t)=P_\lambda(x;1/q,1/t)$ (see [37, p. 324]) to (2.8) and then replacing (q,t) by $(1/q,1/t)$ we obtain

$$P_\lambda(t^{\rho_n};q,t)P_\mu(q^{-\lambda}t^{\rho_n};q,t)=P_\mu(t^{\rho_n};q,t)P_\lambda(q^{-\mu}t^{\rho_n};q,t).$$

Using this in (3.3) and then again swapping λ and μ, we find

$$f_{n,m}(u,T;q,t)=\sum_{\lambda,\mu\subset(m^n)}(-tT/u)^{|\mu|}(-uq)^{|\lambda|}P_\lambda(t^{\rho_n};q,t)P_{\lambda'}(q^{\rho_m};t,q) \times P_\mu(q^{-\lambda}t^{\rho_n};q,t)P_{\mu'}(t^{-\lambda'}q^{\rho_m};t,q).$$

Comparing the above with (3.3) yields (3.2b). □

Next we compute $f_{n,m}$ in two different ways. First, using the homogeneity of the Macdonald polynomials and the dual Cauchy identity [37, p. 329]

$$\sum_\mu T^{|\mu|}P_\mu(x;q,t)P_{\mu'}(y;t,q)=\prod_{i,j\geqslant 1}(1+Tx_iy_j), \tag{3.4}$$

we can perform the sum over μ in (3.1). Also using

$$\prod_{i=1}^n\prod_{j=1}^m\big(1-u^{-1}q^{\lambda_i-j}t^{\lambda'_j-i+1}\big)=(-u)^{-nm}q^{-n\binom{m+1}{2}}t^{-m\binom{n}{2}}(q^mt^n)^{|\lambda|} \times\prod_{i=1}^n\prod_{j=1}^m\big(1-uq^{j-\lambda_i}t^{i-\lambda'_j-1}\big),$$

this gives

$$f_{n,m}(u,T;q,t)=\sum_{\lambda\subset(m^n)}(-tT/u)^{|\lambda|}P_\lambda(t^{\delta_n};q,t)P_{\lambda'}(q^{\delta_m};t,q) \times\prod_{i=1}^n\prod_{j=1}^m\big(1-uq^{j-\lambda_i}t^{i-\lambda'_j-1}\big). \tag{3.5}$$

Before we proceed we remark that if $u=t$ then the summand contains the factor

$$\prod_{i=1}^n\prod_{j=1}^m\big(1-q^{j-\lambda_i}t^{i-\lambda'_j}\big).$$

This vanishes for all $\lambda\subset(m^n)$ with the exception of $\lambda=0$. Similarly, if $u=1/q$ then the summand contains the factor

$$\prod_{i=1}^n\prod_{j=1}^m\big(1-q^{j-\lambda_i-1}t^{i-\lambda'_j-1}\big),$$

which vanishes unless $\lambda=(m^n)$. Finally, if we replace T by uT and then let u tend to 0 we are left with

$$\lim_{u\to 0} f_{n,m}(u,uT;q,t)=\sum_{\lambda\subset(m^n)}(-tT)^{|\lambda|}P_\lambda(t^{\delta_n};q,t)P_{\lambda'}(q^{\delta_m};t,q),$$

which can be summed by (3.4). We summarise these three observations in the following lemma, where we have also used that $P_{(m^n)}(x_1,\dots,x_n;q,t)=(x_1\cdots x_n)^m$.

Lemma 3.2. *We have*

$$f_{n,m}(t,T;q,t)=\prod_{i=1}^n\prod_{j=1}^m(1-q^jt^i) \tag{3.6a}$$

$$f_{n,m}(1/q,T;q,t)=T^{mn}\prod_{i=1}^n\prod_{j=1}^m(1-q^jt^i) \tag{3.6b}$$

$$\lim_{u\to 0} f_{n,m}(u,uT;q,t)=\prod_{i=1}^n\prod_{j=1}^m(1-Tq^{j-1}t^i). \tag{3.6c}$$

Returning to (3.5), we use

$$\begin{aligned}\prod_{i=1}^n\prod_{j=1}^m\big(1-uq^{j-\lambda_i}t^{i-\lambda'_j-1}\big)&=(-u)^{|\lambda|}q^{-n(\lambda')}t^{-n(\lambda)-|\lambda|}\\&\times\prod_{i=1}^n\prod_{j=1}^m\big(1-uq^jt^{i-1}\big)\prod_{s\in\lambda}\frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-uq^{m-a'(s)}t^{l'(s)})(1-uq^{a'(s)+1}t^{n-l'(s)-1})}\end{aligned}$$

together with the principal specialisation formula (2.7) to obtain the following.

Proposition 3.3. *The rational function $f_{n,m}$ can be expressed as*

$$\begin{aligned}f_{n,m}(u,T;q,t)&=\prod_{i=1}^n\prod_{j=1}^m\big(1-uq^jt^{i-1}\big)\\&\times\sum_{\lambda\subset(m^n)}T^{|\lambda|}\prod_{s\in\lambda}\bigg(\frac{(1-q^{a'(s)}t^{n-l'(s)})(1-q^{m-a'(s)}t^{l'(s)})}{(1-uq^{a'(s)+1}t^{n-l'(s)-1})(1-uq^{m-a'(s)}t^{l'(s)})}\\&\qquad\qquad\times\frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}\bigg).\end{aligned}$$

As an immediate application of the proposition we have that $f(u,T;q,t)$ defined in (1.7) is given by

$$f(u,T;q,t)=\lim_{n,m\to\infty}f_{n,m}(u,T;q,t).$$

For our second computation of $f_{n,m}$ we start with the representation given in (3.3) and twice apply the finite form of Proposition 2.1 given by (2.25). Then

$$\begin{aligned}f_{n,m}(u,T;q,t)&=\sum_{\lambda,\mu,\nu,\tau\subset(m^n)}(-tT/u)^{|\lambda|}(-uq)^{|\mu|}q^{-|\tau|}t^{-|\nu|}\\&\times\frac{(t^n;q,t)_\lambda(t^n;q,t)_\mu}{(t^n;q,t)_{\tau'}\,b_{\tau'}(q,t)}\cdot\frac{(q^m;t,q)_{\lambda'}(q^m;t,q)_{\mu'}}{(q^m;t,q)_{\nu'}\,b_{\nu'}(t,q)}\\&\times Q_{\lambda/\nu}(t^\rho;q,t)Q_{\lambda'/\tau}(q^\rho;t,q)Q_{\mu/\nu}(t^\rho;q,t)Q_{\mu'/\tau}(q^\rho;t,q).\end{aligned}$$

Since we do not know of a suitable finite analogue of (2.26), we next let n and m tend to infinity, and use (2.3) as well as the homogeneity of the Macdonald polynomials. This yields

$$f(u,T;q,t)=\sum_{\lambda,\mu,\nu,\tau\subset(m^n)} T^{|\lambda|}b_\nu(q,t)b_\tau(t,q)Q_{\lambda/\nu}(-t^{\rho+1}/u;q,t)Q_{\lambda'/\tau}(q^\rho;t,q) \\ \times Q_{\mu/\nu}(-ut^\rho;q,t)Q_{\mu'/\tau}(q^{\rho+1};t,q).$$

By (2.26) with $(a,b,c,d)=(-t/u,1,-u,q)$ the sum evaluates in closed form as

$$f(u,T;q,t)=\frac{1}{(T;T)_\infty}\cdot\frac{(uq,u^{-1}tT;q,t,T)_\infty}{(qT,tT;q,t,T)_\infty} = \frac{(uq,u^{-1}tT;q,t,T)_\infty}{(T,qtT;q,t,T)_\infty}. \tag{3.7}$$

Equating this with (1.7) and dividing both sides by $(uq;q,t)_\infty$ results in (1.4).

To conclude this section we make some final remarks about $f_{n,m}$ and why it is an interesting function in its own right. First of all, from (3.1) we have $f_{n,m}(u,T;q,t)\in\mathbb{Q}(q,t)[u,u^{-1},T]$. A stronger result appears to hold as follows.

Conjecture 3.4. *The function $f_{n,m}(u,T;q,t)$ lies in $\mathbb{Z}[q,t,u,u^{-1},T]$.*

For $n=1$ (or $m=1$) this is easily seen to be true. For example, taking $n=1$ in Proposition 3.3 yields

$$f_{1,m}(u,T;q,t)=\sum_{k=0}^{m} T^k \begin{bmatrix} m\\ k\end{bmatrix}_q (t/u;q)_k(uq;q)_{m-k}, \tag{3.8}$$

where $\begin{bmatrix} m\\ k\end{bmatrix}_q$ is the classical q-binomial coefficient (see e.g., [14, Equation (I.39)]). Since the summand lies in $\mathbb{Z}[q,t,u,u^{-1},T]$, so does $f_{1,m}(u,T;q,t)$. For $n,m>1$, however, the conjectured polynomiality is much deeper.

The function $f_{n,m}$ may be viewed as a generalised basic hypergeometric series, and some of the symmetries and evaluations proved in this section are generalisations of well-known results for such series. Defining [45, p. 68]

$$C^-_\lambda(z;q,t):=\prod_{s\in\lambda}\big(1-zq^{a(s)}t^{l(s)}\big),$$

we may rewrite Proposition 3.3 as

$$f_{n,m}(u,T;q,t)=\prod_{i=1}^{n}\prod_{j=1}^{m}\big(1-uq^jt^{i-1}\big) \\ \times\sum_{\lambda\subset(m^n)}\frac{C^-_\lambda(uq;q,t)C^-_\lambda(t/u;q,t)}{C^-_\lambda(q;q,t)C^-_\lambda(t;q,t)}\cdot\frac{(t^n;q,t)_\lambda(q^{-m};q,t)_\lambda}{(uqt^{n-1};q,t)_\lambda(q^{-m}/u;q,t)_\lambda}\Big(\frac{T}{u}\Big)^{|\lambda|}.$$

For $n=1$ (see also (3.8)) this simplifies to the ${}_2\phi_1$ basic hypergeometric series

$$f_{1,m}(u,T;q,t)=(uq;q)_m\,{}_2\phi_1\left[\begin{matrix} t/u,q^{-m}\\ q^{-m}/u\end{matrix};q,\frac{T}{u}\right], \tag{3.9}$$

where [14]

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\right]:=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k}\,z^k.$$

The symmetry (3.2a) can thus be viewed as a generalisation of Heine's well-known transformation formula [14, Equation (III.2)]

$$ {}_2\phi_1\left[\begin{matrix} aw, q^{-m} \\ wq^{-m} \end{matrix}; q, z\right] = \Big(\frac{z}{w}\Big)^m \frac{(q/z;q)_m}{(q/w;q)_m}\, {}_2\phi_1\left[\begin{matrix} az, q^{-m} \\ zq^{-m} \end{matrix}; q, w\right]. $$

Similarly, (3.2b) generalises

$$ {}_2\phi_1\left[\begin{matrix} q/w, q^{-m} \\ aq^{-m}/w \end{matrix}; q, z\right] = \frac{(zq/a;q)_m}{(wq/a;q)_m}\, {}_2\phi_1\left[\begin{matrix} q/z, q^{-m} \\ aq^{-m}/z \end{matrix}; q, w\right], $$

which is a limiting case of the ${}_3\phi_2$ transformation formula [14, Equation (III.11)]. Also, by (3.2b), the evaluation (3.6a) is equivalent to

$$ f_{n,m}(T/q, T; q, t) = \prod_{i=1}^{n}\prod_{j=1}^{m}(1-q^j t^i). \tag{3.10} $$

Comparing this with the $u = T/q$ case of (3.9) shows that (3.10) can be viewed as generalisations of the q-Chu–Vandermonde summation [14, Equation (II.6)]

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-m} \\ bq^{-m} \end{matrix}; q, q\right] = \frac{(aq/b;q)_m}{(q/b;q)_m}. $$

Finally we note that if we let m tend to infinity in (3.9) we can sum the resulting ${}_1\phi_0$ series by the q-binomial theorem [14, Equation (II.3)]. Hence

$$ \lim_{m\to\infty} f_{1,m}(u, T; q, t) = \frac{(uq, tT/u;q)_\infty}{(T;q)_\infty}. \tag{3.11} $$

Unfortunately, $\lim_{m\to\infty} f_{n,m}(u,T;q,t)$ for finite $n>1$, which interpolates between (3.7) and (3.11), does not admit a simple factorised expression.

4. Special cases of the q,t-Nekrasov–Okounkov formula

The Nekrasov–Okounkov formula (1.5) contains many classical identities as special cases. For $\mu = 0$ it yields Euler's formula for the generating function of partitions. For $z = 2$ only the staircase partitions δ_n for $n \geqslant 1$ contribute to the sum and (1.5) simplifies to Jacobi's identity for the third power of the Dedekind eta function $\eta(\tau)$. More generally, for $z = p$ with p a positive integer, (1.5) it is related to Macdonald's expansion [36, pp. 134 and 135] for the (p^2-1)th power of $\eta(\tau)$. In a different vein (see e.g., [18]), by setting $z^2 = -x/T$, taking the $T \to 0$ limit, and then extracting coefficients of x^n, the Nekrasov–Okounkov formula simplifies to

$$ \sum_{\lambda\vdash n}\prod_{s\in\lambda}\frac{1}{h(s)^2} = \frac{1}{n!}, \tag{4.1} $$

which is a well-known identity related to the Robinson–Schensted–Knuth correspondence [30, 46, 48], the Frame–Robinson–Thrall formula [10] and the Plancherel measure on partitions [3].

Some of the above-mentioned special cases have nice generalisations to the Macdonald polynomial or the $t = q$ (i.e., Schur) level. For example, if we replace u by $-u/qT$ in (1.4), then let T tend to 0 and finally extract coefficients of u^n we obtain a q,t-analogue of (4.1)

$$ \sum_{\lambda\vdash n}\frac{q^{n(\lambda')}t^{n(\lambda)}}{c_\lambda(q,t)c'_\lambda(q,t)} = [u^n](-u;q,t)_\infty = e_n\bigg(\bigg[\frac{1}{(1-q)(1-t)}\bigg]\bigg), $$

where $e_n(x)$ is the n-th elementary symmetric function and $1/(1-q)(1-t)$ is plethystic notation for the cartesian product of the alphabets $\{1,q,q^2,\dots\}$ and $\{1,t,t^2,\dots\}$. As an identity this is not actually new — it for example follows by specialising $x=t^\rho$, $y=q^\rho$, $T=u$ in the dual Cauchy identity (3.4) and using the large large-n limit of (2.7), see also [11] — but the point is that it is contained in (1.4).

Another interesting special case corresponds to $u=q^{-p}$ for p a positive integer. Then the summand of (1.4) contains the factor

$$\prod_{s\in\lambda}\big(1-q^{a(s)-p+1}t^{l(s)}\big),$$

which vanishes unless λ is a partition such that $\lambda_i-\lambda_{i+1}\leqslant p-1$ for all $1\leqslant i\leqslant l(\lambda)$. In other words, consecutive parts should differ by at most $p-1$ and also the smallest part has size at most $p-1$. If we denote this set of partitions by D_p (for example, $D_1=\{0\}$, $D_2=\{\lambda:\lambda' \text{ is strict}\}$, and the number of partitions in D_p of length l is p^l-p^{l-1}), then

$$\sum_{\lambda\in D_p} T^{|\lambda|}\prod_{s\in\lambda}\frac{(1-q^{a(s)-p+1}t^{l(s)})(1-q^{a(s)+p}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}=\prod_{i=1}^{p-1}\frac{(q^{i-p}T;t,T)_\infty}{(q^itT;t,T)_\infty}. \tag{4.2}$$

A much stronger restriction results if we take $q=t$ in (4.2). Then partitions with hook-lengths equal to p vanish. Partitions with no such hook-lengths are known as p-cores and play an important role in the modular representation theory of the symmetric group, see e.g., [40, 46]. Thus, with C_p denoting the set of p-cores,

$$\begin{aligned}\sum_{\lambda\in C_p} T^{|\lambda|}\prod_{h\in\mathscr{H}(\lambda)}&\frac{(1-t^{h-p})(1-t^{h+p})}{(1-t^h)^2}\\ &=(T;T)_\infty^{p-1}\prod_{1\leqslant i<j\leqslant p}(t^{j-i}T,t^{i-j}T;T)_\infty.\end{aligned} \tag{4.3}$$

The set of 2-cores is given by $C_2=\{\delta_n:n\geqslant 1\}$, and for $p=2$ we thus recover the Jacobi triple product identity [14]

$$\sum_{n\geqslant 1}(-1)^nT^{\binom{n}{2}}\frac{t^n-t^{1-n}}{1-t}=(T,tT,t^{-1}T;T)_\infty.$$

More generally, (4.3) is the Macdonald identity for the affine root system $\mathrm{A}_{p-1}^{(1)}$ [36] specialised as

$$\mathrm{e}^{-\alpha_0}\mapsto Tt^{1-p},\quad \mathrm{e}^{-\alpha_1},\dots,\mathrm{e}^{-\alpha_{p-1}}\mapsto t,$$

where $\alpha_0,\dots,\alpha_{p-1}$ are the simple roots. This can be seen using a well-known parametrisation of p-cores due to Klyachko [27] and "Bijection 2" from the work of Garvan, Kim and Stanton [13]. For more details we also refer to [9, 18, 19]. Identity (4.2) should thus be regarded as a generalisation of the Jacobi triple product identity and the specialised Macdonald identity of type A.

After completion of an earlier version of this paper, Amer Iqbal informed us of his joint work with Kozçaz and Shabbir [24] on the refined topological vertex. This is defined as the rational function

$$C_{\lambda\mu\nu}(t,q) := q^{n(\mu')+n(\nu')+\frac{1}{2}(|\lambda|+|\mu|+|\nu|)}t^{-n(\mu)}c_\nu(q,t)^{-1} \times \sum_\eta t^{-|\eta|}s_{\lambda'/\eta}\big(t^\rho q^{-\nu}\big)s_{\mu/\eta}\big(q^\rho t^{-\nu'}\big),$$

(where $s_{\lambda/\mu}$ is a skew Schur function) and reduces to the ordinary topological vertex [1, 44] for $t=q$. In their paper Iqbal *et al.* use geometric considerations as a heuristic to generate identities for the refined topological vertex. This in turn leads to numerous q,t-hook-length formulas, see [24, Section 6]. As remarked in their paper, these identities are not rigorously proved, but checked up to some fixed order in the parameters using a computer. Their Example 3, arising from a 5-dimensional $U(1)$ gauge theory is, up to a renaming of the variables, precisely our (1.4).[4]

Macdonald polynomials can also be applied to deal with the other identities from [24], and below we discuss in detail [24, Example 4] arising from a 5-dimensional supersymmetric $U(1)$ gauge theory with two hypermultiplets.

Proposition 4.1. *We have*

$$\sum_{\mu,\nu}\frac{(-u)^{|\mu|}(-v)^{|\nu|}q^{n(\mu')+n(\nu')}t^{n(\mu)+n(\nu)}}{c_\mu(q,t)c'_\mu(q,t)c_\nu(q,t)c'_\nu(q,t)}\prod_{i,j\geqslant 1}\big(1-wq^{i-\nu_j}t^{j-\mu'_i}\big) \tag{4.4a}$$

$$= \sum_\lambda \frac{(-wqt)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}}{c_\lambda(q,t)c'_\lambda(q,t)}\prod_{i,j\geqslant 1}\big(1-uq^{i-1}t^{j-\lambda'_i-1}\big)\big(1-vq^{i-\lambda_j-1}t^{j-1}\big) \tag{4.4b}$$

$$= \frac{(u,v,wqt,uvw;q,t)_\infty}{(uwq,vwt;q,t)_\infty}. \tag{4.4c}$$

By applying the 'flop transition' to this theory, see [24, p. 450], Iqbal *et al.* also obtained the following companion identity.

Proposition 4.2. *We have*

$$\sum_\lambda \frac{w^{|\lambda|}t^{2n(\lambda)}}{c_\lambda(q,t)c'_\lambda(q,t)}\prod_{i,j\geqslant 1}\big(1-uq^{i-1}t^{j-\lambda'_i}\big)\big(1-vq^{i-1}t^{j-\lambda'_i}\big) = \frac{(ut,vt,uw,vw;q,t)_\infty}{(w,uvw;q,t)_\infty}. \tag{4.5}$$

For reasons that will become clear later, we first prove the second proposition.

Proof of Proposition 4.2. By (2.1) the claim may also be stated as

$$\sum_\lambda \frac{w^{|\lambda|}t^{2n(\lambda)}(u,v;q,t)_\lambda}{c_\lambda(q,t)c'_\lambda(q,t)} = \prod_{i,j\geqslant 1}\frac{(uw,vw;q,t)_\infty}{(w,uvw;q,t)_\infty}, \tag{4.6}$$

where $(a_1,\dots,a_k;q,t)_\lambda := (a_1;q,t)_\lambda\cdots(a_k;q,t)_\lambda$. The shortest proof of this is to start with the Cauchy identity (2.6) and carry out the plethystic substitutions $x\mapsto (w-uw)/(1-t)$ and $y\mapsto(1-v)/(1-t)$. By [37, p. 338]

$$P_\lambda\Big(\Big[\frac{a-b}{1-t}\Big];q,t\Big) = a^{|\lambda|}\frac{t^{n(\lambda)}(b/a;q,t)_\lambda}{c_\lambda(q,t)},$$

[4] In the subsequent paper [25] Iqbal *et al.* prove this for $t=q$ using the cyclic symmetry of the ordinary topological vertex.

and the simple relation $Q_\lambda = b_\lambda P_\lambda$ (see (2.5)) the identity (4.6) immediately follows.

It is in fact not hard to show that (4.6) and hence also (4.5) admit a bounded analogue in which λ is summed over partitions of length at most n. To this end we recall the symmetric rational function $R_\lambda(x;b;q,t)$ defined by the branching formula [32],

$$R_\lambda(x_1,\dots,x_n;b;q,t) = \sum_{\mu\subset\lambda} \frac{(bx_n/t;q,t)_\mu}{(bx_n;q,t)_\lambda} P_{\lambda/\mu}(x_n;q,t) R_\mu(x_1,\dots,x_{n-1};b;q,t)$$

and initial condition $R_\lambda(-;b;q,t) = \delta_{\lambda,0}$. Note that $R_\lambda(x;0;q,t) = P_\lambda(x;q,t)$ and $R_{(k)}(x_1;b;q,t) = x^k/(bx;q)_k$. According to [32, Corollary 5.4], the function $R_\lambda(x;b;q,t)$ admits the following $\mathfrak{sl}_n$ analogue of the classical q-Gauss sum:

$$\sum_\lambda t^{n(\lambda)} \Big(\frac{c}{ab}\Big)^{|\lambda|} \frac{(a,b;q,t)_\lambda}{c'_\lambda(q,t)} R_\lambda(x;c,q,t) = \prod_{i=1}^n \frac{(cx_i/a, cx_i/b;q)_\infty}{(cx_i, cx_i/ab;q)_\infty}.$$

Specialising $x = t^{\delta_n}$, using [32, Proposition 4.4]

$$R_\lambda(t^{\delta_n};b;q,t) = \frac{t^{n(\lambda)}(t^n;q,t)_\lambda}{(bt^{n-1};q,t)_\lambda c_\lambda(q,t)},$$

and finally replacing $(a,b,c) \mapsto (u,v,uvw)$, yields

$$\sum_\lambda \frac{w^{|\lambda|} t^{2n(\lambda)} (t^n,u,v;q,t)_\lambda}{(uvwt^{n-1};q,t)_\lambda\, c_\lambda(q,t) c'_\lambda(q,t)} = \prod_{i=1}^n \frac{(uwt^{i-1}, vwt^{i-1};q)_\infty}{(wt^{i-1}, uvwt^{i-1};q)_\infty},$$

where we note that $(t^n;q,t)_\lambda = 0$ unless $l(\lambda) \leqslant n$. In the large-n limit this gives (4.6). □

Proof of Proposition 4.1. In the following we denote the double sum in (4.4a) by LHS. Replacing $\mu \mapsto \mu'$ and using (2.2) as well as

$$P_\lambda(t^\rho;q,t) = \frac{t^{n(\lambda)}}{c_\lambda(q,t)} \tag{4.7}$$

(this is the large-n limit of (2.7)), we get

$$\mathrm{LHS} = \sum_{\mu,\nu} \frac{(-u)^{|\mu|}(-v)^{|\nu|} q^{n(\nu')} t^{n(\mu')} P_\mu(q^\rho;t,q) P_\nu(t^\rho;q,t)}{c'_\mu(t,q) c'_\nu(q,t)} \prod_{i,j\geqslant 1} \big(1 - wq^{i-\nu_j} t^{j-\mu_i}\big).$$

In order to decouple the sums over μ and ν we apply the dual Cauchy identity (3.4) with $(x,y,T) \mapsto (q^{-\nu}t^\rho, q^\rho t^{-\mu}, -wqt)$. Then

$$\mathrm{LHS} = \sum_{\lambda,\mu,\nu} (-wqt)^{|\lambda|} (-u)^{|\mu|} (-v)^{|\nu|} q^{n(\nu')} t^{n(\mu')} \\ \times \frac{P_\mu(q^\rho;t,q) P_{\lambda'}(q^\rho t^{-\mu};t,q) P_\nu(t^\rho;q,t) P_\lambda(q^{-\nu}t^\rho;q,t)}{c'_\mu(t,q) c'_\nu(q,t)}.$$

By a double application of the Macdonald–Koornwinder duality (2.8) (with $\rho_n \mapsto \rho$) this can be transformed into

$$\mathrm{LHS} = \sum_{\lambda,\mu,\nu} (-wqt)^{|\lambda|} (-u)^{|\mu|} (-v)^{|\nu|} q^{n(\nu')} t^{n(\mu')} \\ \times \frac{P_{\lambda'}(q^\rho;t,q) P_\mu(q^\rho t^{-\lambda'};t,q) P_\lambda(t^\rho;q,t) P_\nu(q^{-\lambda}t^\rho;q,t)}{c'_\mu(t,q) c'_\nu(q,t)}.$$

Specialising $T=-1$ and $y=q^{\rho}$ in (3.4) and using (4.7), we obtain the following q,t-analogue of Euler's q-exponential sum (see also [33, p. 294]):

$$\sum_{\lambda}\frac{(-1)^{\lambda}q^{n(\lambda')}P_{\lambda}(x;q,t)}{c'_{\lambda}(q,t)}=\prod_{i\geqslant 1}(x_i;q)_{\infty}.$$

This can be used to carry out the sums over μ and ν, resulting in

$$\begin{aligned}\mathrm{LHS}&=\sum_{\lambda}(-wqt)^{|\lambda|}P_{\lambda}(t^{\rho};q,t)P_{\lambda'}(q^{\rho};t,q)\prod_{i\geqslant 1}(uq^{i-1}t^{-\lambda'_i};t)_{\infty}(vq^{-\lambda_i}t^{i-1};q)_{\infty}\\&=\sum_{\lambda}\frac{(-wqt)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}}{c_{\lambda}(q,t)c'_{\lambda}(q,t)}\prod_{i,j\geqslant 1}\big(1-uq^{i-1}t^{j-\lambda'_i-1}\big)\big(1-vq^{i-\lambda_j-1}t^{j-1}\big),\end{aligned}$$

where in the second step we have once again used (4.7) followed by (2.2). This proves the equality between (4.4a) and (4.4b). In fact, the entire proof is now done since the equality of (4.4b) and (4.4c) is equivalent to the identity (4.5) arising from the flop transition. Indeed, by (2.1) the second half of Proposition 4.1 can also be stated as

$$\sum_{\lambda}\frac{(-wqt)^{|\lambda|}q^{n(\lambda')}t^{n(\lambda)}(u/t;q,t)_{\lambda}(v/q;t,q)_{\lambda'}}{c_{\lambda}(q,t)c'_{\lambda}(q,t)}=\frac{(wqt,uvw;q,t)_{\infty}}{(uwq,vwt;q,t)_{\infty}}.$$

Since

$$(z;t,q)_{\lambda'}=(-z)^{|\lambda|}q^{-n(\lambda')}t^{n(\lambda)}(z^{-1};q,t)_{\lambda}$$

this is (4.6) in which (u,v,w) has been replaced by $(u/t,q/v,vwt)$. □

Appendix A.

Jim Bryan suggested an alternative derivation of (1.4) based on the equivariant DMVV formula for the Hilbert scheme of n points in the plane, $(\mathbb{C}^2)^{[n]}$. This formula was first conjectured by Li, Liu and Zhou in [34] and subsequently proved by Waelder [51] as a consequence of the equivariant MacKay correspondence.

Let (u_1,u_2) be the equivariant parameters of the natural torus action on $(\mathbb{C}^2)^{[n]}$, and set $t_1:=\mathrm{e}^{2\pi\mathrm{i}u_1}$ and $t_2:=\mathrm{e}^{2\pi\mathrm{i}u_2}$. Let $\mathrm{Ell}\big((\mathbb{C}^2)^{[n]};u,p,t_1,t_2\big)$ be the equivariant elliptic genus of $(\mathbb{C}^2)^{[n]}$, where $p:=\exp(2\pi\mathrm{i}\tau)$ and $u:=\exp(2\pi\mathrm{i}z)$ for $\tau\in\mathbb{H}$ and $z\in\mathbb{C}$. Treating u,p,t_1 and t_2 as formal variables, the equivariant DMVV formula [51, Theorem 12] expresses the generating function for the elliptic genera as a product:

$$\begin{aligned}\text{(A.1)}\qquad &\sum_{n\geqslant 0}T^n\,\mathrm{Ell}\big((\mathbb{C}^2)^{[n]};u,p,t_1,t_2\big)\\&\qquad=\prod_{m\geqslant 0}\prod_{k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}}\frac{1}{(1-p^mT^ku^{\ell}t_1^{n_1}t_2^{n_2})^{c(km,\ell,n_1,n_2)}}.\end{aligned}$$

The integers $c(m,\ell,n_1,n_2)$ on the right are determined by the equivariant elliptic genus of $\mathbb{C}^2$, given by a simple ratio of Jacobi theta functions:

$$\begin{aligned}\text{(A.2)}\qquad \mathrm{Ell}(\mathbb{C}^2,u,p,t_1,t_2)&=\frac{\theta(ut_1^{-1},u^{-1}t_2;p)}{\theta(t_1^{-1},t_2;p)}\\&=\sum_{m\geqslant 0}\sum_{\ell,n_1,n_2\in\mathbb{Z}}c(m,\ell,n_1,n_2)p^mu^{\ell}t_1^{n_1}t_2^{n_2},\end{aligned}$$

where

$$\theta(u;p) := \sum_{k\in\mathbb{Z}} (-u)^k p^{\binom{k}{2}} = (u, p/u, p; p)_\infty$$

and

$$\theta(u_1,\dots,u_k;p) := \theta(u_1;p)\cdots\theta(u_k;p).$$

In [34] an explicit formula in terms of arm and leg-lengths is obtained for the generating function (over n) of elliptic genera of the framed moduli spaces $M(r,n)$ of torsion-free sheaves on $\mathbb{P}^2$ of rank r and second Chern class n, see [39]. Since $M(1,n)$ coincides with $(\mathbb{C}^2)^{[n]}$ this implies [34, Equation (2.4); $\mu\mapsto\lambda'$]

$$\text{(A.3)}\quad \sum_{n\geqslant 0} T^n \operatorname{Ell}\big((\mathbb{C}^2)^{[n]}; u,p,t_1,t_2\big) \\ = \sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda} \frac{\theta(ut_1^{-a(s)-1}t_2^{l(s)}, u^{-1}t_1^{-a(s)}t_2^{l(s)+1};p)}{\theta(t_1^{-a(s)-1}t_2^{l(s)}, t_1^{-a(s)}t_2^{l(s)+1};p)}.$$

Combining (A.1) with (A.3) we can derive an elliptic analogue of the Nekrasov–Okounkov formula as follows. Define a second set of integers $C(m,\ell,n_1,n_2)$ by

$$\text{(A.4)}\quad \frac{(put_1^{-1}, pu^{-1}t_1, put_2^{-1}, pu^{-1}t_2;p)_\infty}{(pt_1^{-1}, pt_1, pt_2^{-1}, pt_2;p)_\infty} \\ = \sum_{m\geqslant 0}\sum_{\ell,n_1,n_2\in\mathbb{Z}} C(m,\ell,n_1,n_2)p^m u^\ell t_1^{n_1} t_2^{n_2}.$$

From the invariance of the left-hand side under the substitutions $(u,t_1,t_2)\mapsto(u,t_2,t_1)$ and $(u,t_1,t_2)\mapsto(u^{-1},t_1^{-1},t_2^{-1})$ it follows that

$$C(m,\ell,n_1,n_2) = C(m,\ell,n_2,n_1) = C(m,-\ell,-n_1,-n_2).$$

By (A.2) and $\theta(u;p) = (1-u)(pu,pu^{-1};p)_\infty$,

$$\begin{aligned}
&\operatorname{Ell}(\mathbb{C}^2, u,p,t_1,t_2) \\
&\quad = \frac{(1-ut_1^{-1})(1-u^{-1}t_2)}{(1-t_1)(1-t_2)}\cdot\frac{(put_1^{-1}, pu^{-1}t_1, put_2^{-1}, pu^{-1}t_2;p)_\infty}{(pt_1^{-1}, pt_1, pt_2^{-1}, pt_2;p)_\infty} \\
&\quad = \frac{(1-ut_1^{-1})(1-u^{-1}t_2)}{(1-t_1^{-1})(1-t_2)} \sum_{m\geqslant 0}\sum_{\ell,n_1,n_2\in\mathbb{Z}} C(m,\ell,n_1,n_2)p^m u^\ell t_1^{n_1}t_2^{n_2} \\
&\quad = \sum_{m\geqslant 0}\sum_{\ell,n_1,n_2\in\mathbb{Z}}\sum_{i,j\geqslant 1} D(m,\ell,n_1+i,n_2-j)p^m u^\ell t_1^{n_1}t_2^{n_2},
\end{aligned}$$

where

$$\begin{aligned}
D(m,\ell,n_1,n_2) := {} & C(m,\ell,n_1-1,n_2+1) + C(m,\ell,n_1,n_2) \\
& - C(m,\ell-1,n_1,n_2+1) - C(m,\ell+1,n_1-1,n_2).
\end{aligned}$$

Comparison with (A.1) yields

$$c(m,\ell,n_1,n_2) = \sum_{i,j\geqslant 1} D(m,\ell,n_1+i,n_2-j).$$

Hence

$$
\begin{aligned}
&\prod_{m\geqslant 0}\prod_{k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}}\frac{1}{(1-p^mT^ku^\ell t_1^{n_1}t_2^{n_2})^{c(km,\ell,n_1,n_2)}}\\
&=\prod_{m\geqslant 0}\prod_{i,j,k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}}\frac{1}{(1-p^mT^ku^\ell t_1^{n_1}t_2^{n_2})^{D(km,\ell,n_1+i,n_2-j)}}\\
&=\prod_{m\geqslant 0}\prod_{i,j,k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}}\bigg(\frac{(1-p^mT^ku^{\ell+1}t_1^{n_1-i}t_2^{n_2+j-1})}{(1-p^mT^ku^{\ell}t_1^{n_1-i+1}t_2^{n_2+j-1})}\\
&\qquad\qquad\times\frac{(1-p^mT^ku^{\ell-1}t_1^{n_1-i+1}t_2^{n_2+j})}{(1-p^mT^ku^{\ell}t_1^{n_1-i}t_2^{n_2+j})}\bigg)^{C(km,\ell,n_1,n_2)}.
\end{aligned}
$$

Equating the right-hand sides of (A.1) and (A.3), using the above rewriting of the former, and finally replacing $(t_1,t_2)\mapsto(q^{-1},t)$ yields

$$
\begin{aligned}
&\sum_\lambda T^{|\lambda|}\prod_{s\in\lambda}\frac{\theta(uq^{a(s)+1}t^{l(s)},u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)},q^{a(s)}t^{l(s)+1};p)}\\
&\quad=\prod_{m\geqslant 0}\prod_{i,j,k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}}\bigg(\frac{(1-p^mT^ku^{\ell+1}q^{i-n_1}t^{j+n_2-1})}{(1-p^mT^ku^{\ell}q^{i-n_1-1}t^{j+n_2-1})}\\
&\qquad\qquad\times\frac{(1-p^mT^ku^{\ell-1}q^{i-n_1-1}t^{j+n_2})}{(1-p^mT^ku^{\ell}q^{i-n_1}t^{j+n_2})}\bigg)^{C(km,\ell,n_1,n_2)}.
\end{aligned}
$$

Since the left-hand side of (A.4) trivialises to 1 when the elliptic nome p tends to 0,

$$C(0,\ell,n_1,n_2)=\delta_{\ell,0}\delta_{n_1,0}\delta_{n_2,0}.$$

In the $p\to 0$ limit the above result thus simplifies to (1.4).

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