A q-RIOUS POSITIVITY

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ABSTRACT. The q-binomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{i=1}^{m} (1 - q^{n-m+i})/(1 - q^i)$, for integers $0 \le m \le n$, are known to be polynomials with non-negative integer coefficients. This readily follows from the q-binomial theorem, or the many combinatorial interpretations of $\begin{bmatrix} n \\ m \end{bmatrix}$. In this note we conjecture an arithmetically motivated generalisation of the non-negativity property for products of ratios of q-factorials that happen to be polynomials.

The fact that the binomial coefficients

$$(1) \qquad \binom{n}{m} = \frac{n!}{(n-m)!\, m!}$$

are integers easily follows from the following *arithmetic* argument. The order in which a prime p enters $n!$ is given by

$$(2) \qquad \mathrm{ord}_p\, n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots,$$

where $\lfloor \cdot \rfloor$ is the integer-part function. Setting $x = (n-m)/p^k$ and $y = m/p^k$ in the inequality

$$\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \ge 0,$$

and summing k over the positive integers, we see that

$$\mathrm{ord}_p \binom{n}{m} \ge 0 \quad \text{for any prime } p.$$

This obviously implies that $\binom{n}{m} \in \mathbb{Z}$.

A standard way to establish integrality purely *combinatorially* amounts to interpreting the factorial ratio in (1) as coefficients in the expansion

$$(1 + t)^n = \sum_{m=0}^{n} \binom{n}{m} t^m,$$

that is, as the number of m-element subsets of an n-set.

The arithmetic argument can be extended to more general factorial ratios. For example, the inequality [12, Division 8, Problems 8 and 136]

$$\lfloor 2x \rfloor + \lfloor 2y \rfloor - \lfloor x \rfloor - \lfloor x + y \rfloor - \lfloor y \rfloor \ge 0$$

implies that

$$(3) \qquad A_{n,m} := \frac{(2n)!\,(2m)!}{n!\,(n+m)!\,m!} \in \mathbb{Z}.$$

E. Catalan noted this integrality as early as 1874 [5]. In a much more recent study [8, § 6], I. Gessel named the $A_{n,m}$ the *super Catalan numbers*. He stated several formulae for these numbers including

$$(4) \qquad A_{n,m} = \sum_{k=-\infty}^{\infty} (-1)^k \binom{2n}{n+k}\binom{2m}{m+k}.$$

This identity, attributed to K. von Szily (1894) [14], clearly implies the integrality claimed in (3) but, of course, obscures that $A_{n,m} \geq 0$.

Let $\boldsymbol{a} = (a_1, \ldots, a_r)$ and $\boldsymbol{b} = (b_1, \ldots, b_s)$ be tuples of positive integers subject to the condition

$$(5) \qquad \sum_{i=1}^{r} \lfloor a_i x \rfloor - \sum_{j=1}^{s} \lfloor b_j x \rfloor \geq 0 \qquad \text{for } x \geq 0.$$

In his work on the distribution of primes (cf. [3]) P. Chebyshev considered the ratios

$$(6) \qquad D_n(\boldsymbol{a}, \boldsymbol{b}) := \frac{(a_1 n)! \cdots (a_r n)!}{(b_1 n)! \cdots (b_s n)!}.$$

In view of (2), condition (5) is necessary and sufficient for $D_n(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{Z}$ for all positive n — a fact known in the literature as *Landau's criterion* [10]. Unlike the special cases of binomial coefficients and super Catalan numbers, there seems to be no non-arithmetical approach available in the literature to demonstrate that $D_n(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{Z}$ more generally.

It is worth mentioning that the Chebyshev–Landau factorial ratios appear quite naturally in several deep mathematical problems including, for example, the Riemann hypothesis [3] and arithmetic properties of mirror maps [6]. We refer the interested reader to [3] which, among other things, contains a full classification of pairs of tuples $(\boldsymbol{a}, \boldsymbol{b})$ satisfying (5) for $s \leq r+1$ and $\sum_{i=1}^{r} a_i = \sum_{j=1}^{s} b_j$. (The latter "balancing" condition in fact follows from (5) if $x \geq 0$ is replaced by $x \in \mathbb{R}$; cf. [3, Lemma 3.4].)

The above integrality has an interesting q-counterpart [11]. It follows immediately from the definition of the q-factorial,

$$[n]! = [n]_q! = \prod_{i=1}^{n} \frac{1-q^i}{1-q},$$

that $[n]!$ is a polynomial whose irreducible factors over $\mathbb{Q}$ are cyclotomic polynomials $\Phi_\ell(q) \in \mathbb{Z}[q]$ (cf. [18, § 1]). Moreover,

$$\mathrm{ord}_{\Phi_\ell(q)}[n]! = \left\lfloor \frac{n}{\ell} \right\rfloor \quad \text{for all } \ell = 2, 3, 4, \ldots.$$

We may thus conclude that

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[n-m]!\,[m]!}, \qquad A_{n,m}(q) := \frac{[2n]!\,[2m]!}{[n]!\,[n+m]!\,[m]!},$$

and

$$(7) \qquad\qquad D_n(\boldsymbol{a},\boldsymbol{b};q) := \frac{[a_1 n]! \cdots [a_r n]!}{[b_1 n]! \cdots [b_s n]!}$$

subject to (5) are all polynomials in $\mathbb{Z}[q]$. For this reason the *q*-binomial coefficients are often referred to by their alternative name of *Gaussian polynomials*.

Another well-known fact about the Gaussian polynomials is the *non-negativity* of their coefficients. In fact, since each of the coefficients c_i in $\begin{bmatrix} n+m \\ m \end{bmatrix} = c_0 + \cdots + c_{nm}q^{nm}$ is strictly positive[1], it is customary to refer to them as *positive polynomials*. Following tradition, we relax the term positivity to simply refer to any polynomial with non-negative coefficients. Hence we say that $1 + q^2$ is a positive polynomial even though the linear term has vanishing coefficient.

The only known proofs of the positivity of the *q*-binomial coefficients are essentially all combinatorial. For example, the *q*-binomial theorem [7, Eq. (II.4)]

$$\prod_{i=0}^{n-1}(1+tq^i) = \sum_{m=0}^{n} q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} t^m,$$

implies positivity and, more specifically, the combinatorial interpretation

$$q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{\substack{I \subseteq \{0,\dots,n-1\} \\ |I|=m}} q^{\sum_{i \in I} i}.$$

In view of the preceding discussion the following conjecture arises naturally.

Conjecture 1. *Let* $\boldsymbol{a} = (a_1,\dots,a_r)$ *and* $\boldsymbol{b} = (b_1,\dots,b_s)$ *satisfy* (5). *Then the polynomial*

$$(8) \qquad\qquad D(\boldsymbol{a},\boldsymbol{b};q) := \frac{[a_1]! \cdots [a_r]!}{[b_1]! \cdots [b_s]!}$$

is positive.

Replacing all $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\boldsymbol{a}n$ and $\boldsymbol{b}n$ for a positive integer n, we see that the conjecture is equivalent to the claim that the polynomials $D_n(\boldsymbol{a},\boldsymbol{b};q)$ defined in (7) are positive for all positive integers n.

The conjecture is trivially true whenever the right-hand side in (8) can be represented as a product of *q*-binomial coefficients. To provide some further evidence, we show the validity of the conjecture for the *q*-super Catalan numbers

$$A_{n,m}(q) = D((2n,2m),(n,n+m,m);q)$$

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[1]One may in fact show that $c_0 = 1$ and that the c_i are symmetric and unimodal; $c_i = c_{nm-i}$ and $c_i \le c_{i+1}$ for $0 \le i \le \lfloor nm/2 - 1 \rfloor$. See [13] and [17] for nice surveys on the unimodality.

as well as for

(9) $B_{n,m}(q) := D((2n, m), (n, 2m, n - m); q) = \dfrac{[2n]!\,[m]!}{[n]!\,[2m]!\,[n-m]!} \in \mathbb{Z}[q], \quad n \geq m.$

We note that, as shown in [3], the q-binomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}$ together with $A_{n,m}(q)$ and $B_{n,m}(q)$ exhaust the space of 2-parameter solutions to (5) with $s = r + 1$.

Proposition 2. *The q-super Catalan numbers $A_{n,m}(q)$ are positive polynomials for all $n, m \geq 0$.*

Proof. Our proof rests on a q-analogue of Gessel's formula [8, Eq. (32)]

$$A_{n,n+p} = \sum_{k=0}^{\lfloor p/2 \rfloor} 2^{p-2k} \binom{p}{2k} A_{n,k} \qquad (p \geq 0),$$

given in (11) below.

Let n and p be non-negative integers. Twice applying the q-Chu–Vandermonde sum [7, Eq. (II.7)] in the form

$$\begin{bmatrix} a+b \\ c \end{bmatrix} = \sum_{k=0}^{\infty} q^{k(b-c+k)} \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ c-k \end{bmatrix}$$

yields

(10)
$$\begin{bmatrix} 2n+2p \\ p \end{bmatrix} = \sum_{j=0}^{\infty} q^{j(n+j)} \begin{bmatrix} n+p \\ j \end{bmatrix} \begin{bmatrix} n+p \\ p-j \end{bmatrix}$$
$$= \sum_{j=0}^{\infty} q^{j(n+j)} \begin{bmatrix} n+p \\ j \end{bmatrix} \sum_{k=0}^{\infty} q^{k(n+k)} \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} n+p-j \\ p-j-k \end{bmatrix}.$$

Multiplying this by $[2n]!\,[p]!/([n]!\,[n+p]!)$ implies the recurrence

(11)
$$A_{n,n+p}(q) = \sum_{k=0}^{\lfloor p/2 \rfloor} A_{n,k}(q) \sum_{j=k}^{p-k} q^{k(n+k)+j(n+j)} \begin{bmatrix} p \\ 2k \end{bmatrix} \begin{bmatrix} p-2k \\ j-k \end{bmatrix}$$

for $p \geq 0$. Together with the initial conditions $A_{n,n}(q) = A_{n,0}(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}$, the symmetry $A_{n,m}(q) = A_{m,n}(q)$ and the positivity of q-binomial coefficients, formula (11) implies the desired positivity of $A_{n,m}(q)$. $\qquad \square$

Another positivity result related to $A_{n,m}(q)$ may be found in [9]. Before stating this result we remark that by taking $(a, b, c) \mapsto (1, \infty, q^{-m})$ in the very-well poised ${}_6\phi_5$ summation [7, Eq. (II.21)] a q-analogue of von Szily's identity (4) arises. Namely,

$$A_{n,m}(q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}+k^2} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \begin{bmatrix} 2m \\ m+k \end{bmatrix}.$$

After the substitution $q \mapsto 1/q$ this may also be written as

(12)
$$A_{n,m}(q) = q^{-nm} \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \begin{bmatrix} 2m \\ m+k \end{bmatrix}.$$

Now a special case of [9, Theorem 4.7] amounts to the following claim. For r, s positive integers and n, m non-negative integers, the functions $R_{n,m;r,s}(q)$ defined by

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}^r \begin{bmatrix} 2m \\ m+k \end{bmatrix}^s = A_{n,m}(q) R_{n,m;r,s}(q)$$

are positive polynomials. Of course, from (12) it follows that $R_{n,m;1,1}(q) = q^{nm}$. In view of the positivity of $A_{m,n}(q)$ the positivity of the right-hand side should come as no surprise, since, intuitively, raising the value of r and/or s should result in a "more positive (or less-negative) polynomial". The fact that the right-side factors, with $A_{m,n}(q)$ as one of its factors, is much more remarkable. We also note that in general it is hard to prove the positivity of alternating sum expressions of the form given above. For example, showing that

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}+4k^2} \begin{bmatrix} 2n \\ n+3k \end{bmatrix}$$

is positive is key to proving the longstanding Borwein conjecture, see [1, 2, 4, 15, 16].

Proposition 3. *The $B_{n,m}(q)$ defined in (9) are positive polynomials for all $n \geq m \geq 0$.*

Proof. This time we simply multiply (10) by $[n]! \, [2n+p]!/([2n]! \, [n+p]!)$ to get

$$(13) \qquad B_{n+p,n}(q) = \sum_{k=0}^{\lfloor p/2 \rfloor} B_{n+k,n}(q) \sum_{j=k}^{p-k} q^{k(n+k)+j(n+j)} \begin{bmatrix} 2n+p \\ 2n+2k \end{bmatrix} \begin{bmatrix} p-2k \\ j-k \end{bmatrix}.$$

Since $B_{n,n}(q) = 1$, the result follows from (13) by induction on p. $\qquad \square$

Note that $[m]! = 1 + (m-1)q + O(q^2)$ for $m \geq 1$ implying that

$$D_n(\boldsymbol{a}, \boldsymbol{b}; q) = 1 + (s-r)q + O(q^2) \quad \text{for } n \geq 1,$$

where $s - r > 0$. With a little work one can also check that the coefficient of q^2 in the q-expansion of $D_n(\boldsymbol{a}, \boldsymbol{b}; q)$ is positive, assuming (5); verifying the non-negativity of coefficients of terms beyond q^2 quickly becomes unfeasible. To provide additional support for Conjecture 1 we have computed the polynomials $D_n(\boldsymbol{a}, \boldsymbol{b}; q)$ for all n up to 20 for the 52 choices for $\boldsymbol{a}$ and $\boldsymbol{b}$ listed in [3, Table 2]. Since $s > r$ for each of these, and since the large n limit of $D_n(\boldsymbol{a}, \boldsymbol{b})$ is given by the positive power series $\prod_{i \geq 1} (1 - q^i)^{r-s}$, one would expect potential counter examples to occur for "small" values of n. However, our computation resulted in polynomials with non-negative coefficients only.

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