q-Trinomial identities

S. Ole Warnaar[a)]
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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We obtain connection coefficients between q-binomial and q-trinomial coefficients. Using these, one can transform q-binomial identities into q-trinomial identities and back again. To demonstrate the usefulness of this procedure we rederive some known trinomial identities related to partition theory and prove many of the conjectures of Berkovich, McCoy and Pearce, which have recently arisen in their study of the $\phi_{2,1}$ and $\phi_{1,5}$ perturbations of minimal conformal field theory. © *1999 American Institute of Physics.* [S0022-2488(99)01105-6]

I. INTRODUCTION

The q-binomial coefficients can be defined by the q-analog of Newton's binomial expansion,

$$(1+x)(1+qx)\cdots(1+q^{n-1}x)=\sum_{a=0}^{n} x^a q^{a(a-1)/2}\begin{bmatrix} n \\ a \end{bmatrix}. \tag{1}$$

An explicit expression for the q-binomial coefficients is given by

$$\begin{bmatrix} n \\ a \end{bmatrix}_q=\begin{bmatrix} n \\ a \end{bmatrix}=\begin{cases} \dfrac{(q)_n}{(q)_a(q)_{n-a}} & \text{for } 0\leqslant a\leqslant n, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$(q)_n=\prod_{j=1}^{n}(1-q^j), \quad n\geqslant 1 \quad \text{and} \quad (q)_0=1.$$

q-Binomials play an essential role in combinatorics, partition theory, and statistical mechanics; see, e.g., Refs. 1–4, and one of MacMahon's famous results is that $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right]$ is the generating function of partitions with no more than m parts, no part exceeding n. Less well understood are the q-trinomial coefficients, defined as q-analogs of the numbers appearing in the generalized Pascal triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 3 & 2 & 1 & & \\ & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ . & . & . & . & . & . & . & . & . \end{array} . \tag{2}$$

Andrews, Baxter, and Forrester[5,6] were the first to encounter q-trinomial coefficients, and in Ref. 6 Andrews and Baxter defined

[a)]Electronic mail: warnaar@wins.uva.nl

$$\begin{bmatrix} L,b;q \\ a \end{bmatrix}_2 = \begin{bmatrix} L,b \\ a \end{bmatrix}_2 = \sum_{k\geqslant 0} q^{k(k+b)} \begin{bmatrix} L \\ k \end{bmatrix} \begin{bmatrix} L-k \\ k+a \end{bmatrix} \tag{3}$$

and

$$T_n(L,a;q) = T_n(L,a) = q^{(L-a)(L+a-n)/2} \begin{bmatrix} L,a-n;q^{-1} \\ a \end{bmatrix}_2 . \tag{4}$$

The q-trinomial T_n can be expressed explicitly as

$$T_n(L,a) = \sum_{\substack{r=0 \\ L-a-r \text{ even}}}^{L-|a|} \frac{q^{r(r-n)/2}(q)_L}{(q)_{(L-a-r)/2}(q)_{(L+a-r)/2}(q)_r} . \tag{5}$$

Clearly, the q-trinomial coefficients are nonzero for $a=-L,-L+1,...,L$ only and satisfy the symmetries

$$\begin{bmatrix} L,b;q \\ a \end{bmatrix}_2 = q^{a(a-b)} \begin{bmatrix} L,b-2a \\ -a \end{bmatrix}_2 \quad \text{and} \quad T_n(L,a) = T_n(L,-a).$$

To see that (3) indeed defines q-analogs of the trinomial coefficients, set $q=1$ and twice apply the binomial formula to find that

$$\sum_{a=-L}^{L} x^a \begin{bmatrix} L,b;1 \\ a \end{bmatrix}_2 = (1+x+x^{-1})^L,$$

in accordance with (2). The only further properties of q-trinomials needed in this paper are the limiting formulas[6]

$$\lim_{\substack{L\to\infty \\ L-a \text{ even}}} T_0(L,a) = \frac{(-q^{1/2})_\infty + (q^{1/2})_\infty}{2(q)_\infty}, \tag{6}$$

$$\lim_{\substack{L\to\infty \\ L-a \text{ odd}}} T_0(L,a) = \frac{(-q^{1/2})_\infty - (q^{1/2})_\infty}{2(q)_\infty}, \tag{7}$$

and

$$\lim_{L\to\infty} \begin{bmatrix} L,a \\ a \end{bmatrix}_2 = \frac{1}{(q)_\infty} . \tag{8}$$

Finally, we introduce the abbreviation

$$\begin{bmatrix} L,a \\ a \end{bmatrix}_2 = \begin{bmatrix} L \\ a \end{bmatrix}_2 .$$

Since their discovery about a decade ago, q-trinomials have found numerous applications in, again, combinatorics, partition theory, and statistical mechanics.[5–23] Among the most striking results is a q-trinomial proof of Schur's partition theorem and Capparelli's (then) conjecture,[9] a q-trinomial proof of the Göllnitz–Gordon partition theorem[7] and their Andrews–Bressoud generalizations,[13,16] the proof of an E_8 Rogers–Ramanujan-type identity,[10] and a trinomial analog of Bailey's lemma.[19]

Most of the above-cited papers contain q-trinomial identities. Upon close inspection of many of these identities, one is struck by their similarity with well-known q-binomial identities. This strongly suggests that many q-trinomial identities can be simply viewed as corollaries of q-binomial identities. In an earlier paper[23] we made a first, only partially successful, attempt to relate q-trinomial identities to q-binomial identities, showing that each Bailey pair (which implies a q-binomial identity) implies a trinomial Bailey pair (which implies a q-trinomial identity). The problem with the idea of Ref. 23 is that it applies to q-trinomial identities in which the parameter a in (3) and (4) takes even values only. Therefore, q-trinomial identities in which a takes arbitrary integer values remained irreducible to q-binomial identities.

In this paper we intend to deal with this problem, and in the next section connection coefficients between q-binomial and q-trinomial coefficients are obtained. Using these coefficients and the idea of Ref. 23, many q-trinomial identities are derived from known q-binomial identities. In Sect. III, several q-trinomial identities related to partitions are obtained and in Sec. IV general classes of q-trinomial identities are proved, including many of the recent conjectures of Berkovich, McCoy, and Pearce.[21] To make contact with the recently discovered trinomial analog of Bailey's lemma, our results are finally formulated in the language of Bailey pairs in Sec. V. In the Appendix some necessary formulas for q-binomial coefficients are collected.

II. CONNECTION COEFFICIENTS

To relate q-binomials and q-trinomials, we consider the simple problem of finding the coefficients $C_{L,k}$ and $C'_{L,k}$, such that

$$T_0(L,a)=\sum_{k=0}^{L} C_{L,k}(a)\begin{bmatrix}2k\\k-a\end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix}2L\\L-a\end{bmatrix}=\sum_{k=0}^{L} C'_{L,k}(a)T_0(k,a). \tag{10}$$

Of course, the two equations imply that

$$\sum_{k=M}^{L} C_{L,k}(a)C'_{k,M}(a)=\delta_{L,M}. \tag{11}$$

The answer to the above connection coefficient problem is given by the following lemma.

Lemma II.1: For $C_{L,k}$ and $C'_{L,k}$ as above,

$$C_{L,k}(a)=(-1)^{L-k}q^{\binom{L-k}{2}+(a^2-L^2)/2}\begin{bmatrix}L\\k\end{bmatrix}, \tag{12}$$

$$C'_{L,k}(a)=q^{(k^2-a^2)/2}\begin{bmatrix}L\\k\end{bmatrix}. \tag{13}$$

Proof: Substitution of the expression for $C'_{L,k}$ into the right-hand side of (10) and using Eq. (5) for T_0 gives

$$\sum_{k=0}^{L} C'_{L,k}(a)T_0(k,a)=\sum_{k=0}^{L}\sum_{\substack{r=0\\k-a-r\ \text{even}}}^{k-|a|}\frac{q^{(k^2-a^2+r^2)/2}(q)_L}{(q)_{L-k}(q)_{(k-a-r)/2}(q)_{(k+a-r)/2}(q)_r}.$$

To proceed, we introduce new summation variables i,j defined by $k=i+j+a$ and $r=i-j$, and apply the q-Chu–Vandermonde sum, i.e.,

$$\sum_{k=0}^{L} C'_{L,k}(a)T_0(k,a)=\sum_{i=0}^{L}\sum_{j=0}^{i} q^{i(i+a)+j(j+a)}\begin{bmatrix}L\\i\end{bmatrix}\begin{bmatrix}i\\j\end{bmatrix}\begin{bmatrix}L-i\\j+a\end{bmatrix}\overset{\text{by (A1)}}{=}\sum_{i=0}^{L} q^{i(i+a)}\begin{bmatrix}L\\i\end{bmatrix}\begin{bmatrix}L\\i+a\end{bmatrix}$$

$$\overset{\text{by (A1)}}{=}\begin{bmatrix}2L\\L-a\end{bmatrix}.$$

This settles (13), and to prove (12) we show that (11) holds. Taking the left-hand side of (11) and substituting the claim of the lemma, we find

$$\sum_{k=M}^{L} C_{L,k}(a)C'_{k,M}(a)=\sum_{k=M}^{L}(-1)^{L-k}q^{\binom{L-k}{2}+(M^2-L^2)/2}\begin{bmatrix}L\\k\end{bmatrix}\begin{bmatrix}k\\M\end{bmatrix}$$

$$=q^{(M^2-L^2)/2}\begin{bmatrix}L\\M\end{bmatrix}\sum_{k=0}^{L-M}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}L-M\\k\end{bmatrix}=\delta_{L,M},$$

where in the last step we have used (1) with $x=-1$. □

We note that a proof of (12) that does not rely on (13) is implied by Eqs. (2.12) and (2.35) of Ref. 6.

The analogous result involving T_1 instead of T_0 can be stated as follows. Define $D_{L,k}$ and $D'_{L,k}$ by

$$T_1(L,a)=\sum_{k=0}^{L} D_{L,k}(a)\begin{bmatrix}2k\\k-a\end{bmatrix}$$

and

$$\begin{bmatrix}2L\\L-a\end{bmatrix}=\sum_{k=0}^{L} D'_{L,k}(a)T_1(k,a). \tag{14}$$

Lemma II.2: For $D_{L,k}$ and $D'_{L,k}$ as above,

$$D_{L,k}(a)=(-1)^{L-k}q^{\binom{L-k}{2}+\binom{a}{2}-\binom{L}{2}}\frac{1+q^a}{1+q^k}\begin{bmatrix}L\\k\end{bmatrix}, \tag{15}$$

$$D'_{L,k}(a)=q^{\binom{k}{2}-\binom{a}{2}}\frac{1+q^L}{1+q^a}\begin{bmatrix}L\\k\end{bmatrix}. \tag{16}$$

Proof: Following the proof of Lemma II.1 with T_0 replaced by T_1, one finds after application of the q-Chu–Vandermonde sum (A1), that the right-hand side of (14) is equal to

$$\frac{1+q^L}{1+q^a}\sum_{i=0}^{L} q^{i(i+a-1)}\begin{bmatrix}L\\i\end{bmatrix}\begin{bmatrix}L\\i+a\end{bmatrix}.$$

Before (A1) can again be applied, the recurrence (A5) is needed to rewrite this as

$$\frac{1+q^L}{1+q^a}\left\{\sum_{i=0}^{L} q^{i(i+a-1)}\begin{bmatrix}L\\i\end{bmatrix}\begin{bmatrix}L-1\\i+a-1\end{bmatrix}+q^a\sum_{i=0}^{L} q^{i(i+a)}\begin{bmatrix}L\\i\end{bmatrix}\begin{bmatrix}L-1\\i+a\end{bmatrix}\right\}.$$

Using (A1) and combining terms gives $\left[\begin{smallmatrix}2L\\L-a\end{smallmatrix}\right]$. To prove (15) it again suffices to consider $\sum_{k=M}^{L} D_{L,k}(a)D'_{k,M}(a)$. After substituting the results for D and D' and replacing $k\to L-k$, one finds that this becomes $\delta_{L,M}$ after using (1) with $x=-1$. □

To conclude this section, we note that the representations (3) and (5) for the q-trinomial coefficients can also be written as a relation between q-trinomials and q-binomials. That is,

$$T_n(L,2a)=\sum_{k\geqslant 0} q^{(L-2k)(L-2k-n)/2}\begin{bmatrix} L \\ 2k \end{bmatrix}\begin{bmatrix} 2k \\ k-a \end{bmatrix} \tag{17}$$

and

$$\begin{bmatrix} L,b \\ 2a \end{bmatrix}_2=\sum_{k\geqslant 0} q^{(k-a)(k-a+b)}\begin{bmatrix} L \\ 2k \end{bmatrix}\begin{bmatrix} 2k \\ k-a \end{bmatrix}. \tag{18}$$

These results, which, unlike the previous transformations are not invertible, will be needed later.

III. SIMPLE EXAMPLES FROM PARTITION THEORY

Before proving general series of q-trinomial identities using the results of the previous section, we treat some simple examples related to partition identities first.

The first example concerns the following result of Andrews[7] (see also Ref. 21). It is well known[24] that the (first) Rogers–Ramanujan identity can be obtained as a limiting case of the polynomial identity,

$$\sum_{n\geqslant 0} q^{n^2}\begin{bmatrix} L-n \\ n \end{bmatrix}=\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2}\begin{bmatrix} L \\ (L-5j)/2 \end{bmatrix}. \tag{19}$$

Here the polynomials appearing on either side are known to be the generating function of partitions with the difference between parts of at least two and the largest part not exceeding $L-1$.[4,25] In Ref. 7, Andrews remarks that it is ''most surprising and intriguing'' that the following also holds:

$$\sum_{n\geqslant 0} q^{n^2}\begin{bmatrix} L-n \\ n \end{bmatrix}=\sum_{j=-\infty}^{\infty}\left\{ q^{j(10j+1)}\begin{bmatrix} L \\ 5j \end{bmatrix}_2 - q^{(2j+1)(5j+2)}\begin{bmatrix} L \\ 5j+2 \end{bmatrix}_2\right\}. \tag{20}$$

We now show that (20) is a corollary of (19), or for those who prefer to decrease instead of increase complexity, that (19) is a corollary of (20). Replacing $q\to 1/q$ in (9) and (12), using (4) and (A7), we find that [see also Ref. 6, Eqs. (2.12) and (2.35)]

$$\begin{bmatrix} L \\ a \end{bmatrix}_2=\sum_{k=0}^{L} (-1)^{L-k} q^{(L-k)(L+k+1)/2}\begin{bmatrix} L \\ k \end{bmatrix}\begin{bmatrix} 2k \\ k-a \end{bmatrix}.$$

If we thus take (19) with L replaced by $2k$, multiply by $(-1)^{L-k}q^{(L-k)(L+k+1)/2}\left[\begin{smallmatrix} L \\ k \end{smallmatrix}\right]$ and sum over k, we arrive at

$$\sum_{k\geqslant 0}\sum_{n\geqslant 0} (-1)^{L-k} q^{(L-k)(L+k+1)/2+n^2}\begin{bmatrix} L \\ k \end{bmatrix}\begin{bmatrix} 2k-n \\ n \end{bmatrix}$$
$$=\sum_{j=-\infty}^{\infty}\left\{ q^{j(10j+1)}\begin{bmatrix} L \\ 5j \end{bmatrix}_2 - q^{(2j+1)(5j+1)}\begin{bmatrix} L \\ 5j+2 \end{bmatrix}_2\right\}.$$

To simplify the left-hand side, we set $k=L-m+n$ followed by $n\to m-n$ to get

$$\sum_{m\geqslant 0} q^{m^2}\sum_{n\geqslant 0} (-1)^n q^{\binom{n}{2}+n(L-2m+1)}\begin{bmatrix} L \\ n \end{bmatrix}\begin{bmatrix} 2L-m-n \\ m-n \end{bmatrix}=\sum_{m\geqslant 0} q^{m^2}\begin{bmatrix} L-m \\ m \end{bmatrix},$$

where the sum over n has been performed using the q-Chu–Vandermonde summation (A3). As remarked before, one can equally well take the reverse route and starting from (20), using Lemma II.1, one readily obtains (19). We leave this to the reader.

Our second example concerns the following identity of Slater[26] related to the (first) Göllnitz–Gordon partition identity:[27,28]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8j+1})(1-q^{8j+4})(1-q^{8j+7})}. \tag{21}$$

A polynomial identity that implies this equation is given by[13,16]

$$\sum_{m,n\geqslant 0} q^{(m^2+n^2)/2} \begin{bmatrix} L-m \\ n \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j/2} \{T_0(L,4j)+T_0(L,4j+1)\}. \tag{22}$$

It was observed in Ref. 7 that for fixed L the polynomial appearing on the right-hand side with q replaced by q^2 is the generating function of partitions $\lambda=(\lambda_1,\lambda_2,\ldots)$ with $\lambda_i-\lambda_{i+1}\geqslant 2$ for λ_i odd, $\lambda_i-\lambda_{i+1}\geqslant 3$ for λ_i even, and with the largest part not exceeding $2L-1$. To see that (22) indeed implies (21), let L tend to infinity using (6), (7), and (A6). Hence,

$$\sum_{m,n\geqslant 0} \frac{q^{(m^2+n^2)/2}}{(q)_n} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{(-q^{1/2})_\infty}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j/2}.$$

Using Jacobi's triple product identity [Eq. (2.2.10) of Ref. 1] and Eq. (1) with $x=q^{1/2}$ gives

$$\sum_{n\geqslant 0} \frac{q^{n^2/2}(-q^{1/2})_n}{(q)_n} = \frac{(-q^{1/2})_\infty (q^{3/2};q^4)_\infty (q^{5/2};q^4)_\infty (q^4;q^4)_\infty}{(q)_\infty}.$$

Letting $q\to q^2$ and cleaning up the right-hand side finally yields (21).

The companion q-binomial identity of (22) is given by the following identity of Refs. 29 and 30:

$$\begin{aligned}\sum_{\substack{m_1,m_2\geqslant 0 \\ m_1+m_2 \text{ even}}} & q^{(m_1^2+m_2^2)/4} \begin{bmatrix} L+\frac{1}{2}(m_1-m_2) \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(m_1+m_2) \\ m_2 \end{bmatrix} \\ &= \sum_{j=-\infty}^{\infty} (-1)^j \left\{ q^{j(20j+1)/2} \begin{bmatrix} 2L \\ L-4j \end{bmatrix} + q^{(4j+1)(5j+1)/2} \begin{bmatrix} 2L \\ L-4j-1 \end{bmatrix} \right\}.\end{aligned}$$

To prove this we replace L by k, multiply by $q^{-a^2/2}C_{L,k}(a)$ as given by (12), and sum over k using (9). The resulting equation is

$$\begin{aligned}\sum_{j=-\infty}^{\infty} & (-1)^j q^{2j^2+j/2} \{T_0(L,4j)+T_0(L,4j+1)\} \\ &= \sum_{\substack{m_1,m_2\geqslant 0 \\ m_1+m_2 \text{ even}}} q^{(m_1^2+m_2^2-2L^2)/4} \begin{bmatrix} \frac{1}{2}(m_1+m_2) \\ m_2 \end{bmatrix} \sum_{k=0}^{L} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} L \\ k \end{bmatrix} \begin{bmatrix} L-k+\frac{1}{2}(m_1-m_2) \\ m_1 \end{bmatrix} \\ &\overset{\text{by (A2)}}{=} \sum_{\substack{m_1,m_2\geqslant 0 \\ m_1+m_2 \text{ even}}} q^{((m_1-L)^2+(m_2-L)^2)/4} \begin{bmatrix} \frac{1}{2}(m_1+m_2) \\ m_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(m_1-m_2) \\ m_1-L \end{bmatrix}.\end{aligned}$$

Making the variable change $m_1\to L+n-m$ and $m_2\to L-n-m$, we find Eq. (22).

IV. *q*-TRINOMIAL IDENTITIES

After the previous examples, we now derive general classes of q-trinomial identities, as stated in Propositions IV.1–IV.5 below. The setup will be as follows. First we describe a family of q-binomial identities for bounded analogs of Virasoro characters, based on continued fraction expansions. We then transform these identities into q-trinomial identities, by either using (9) or (18). Many of the q-trinomial identities available in the literature are contained in Propositions IV.1–IV.5 or can be derived in a completely analogous fashion.

A. *q*-binomial identities for bounded Virasoro characters

Using the inclusion–exclusion construction of Feigin and Fuchs,[31] the (normalized) characters of the Virasoro algebra of central charge $c=1-6(p'-p)^2/pp'$, with p,p' integers such that $1<p<p'$ and $\gcd(p,p')=1$, are given by[32,33]

$$\chi_{r,s}^{(p,p')}(q)=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\{q^{j(pp'j+p'r-ps)}-q^{(pj+r)(p'j+s)}\}.$$

Here $r=1,...,p-1$ and $s=1,...,p'-1$ label the highest weight representations.

For simplicity we only deal with the ''vacuum'' character, determined by $|p'r-ps|=1$. The following polynomial analogs of the vacuum Virasoro characters have arisen in the context of statistical mechanics[34,35] and partition theory,[36]

$$B_L(p,p';q)=\sum_{j=-\infty}^{\infty}\left\{q^{j(pp'j+1)}\begin{bmatrix}2L\\L-p'j\end{bmatrix}-q^{(pj+r)(p'j+s)}\begin{bmatrix}2L\\L-p'j-s\end{bmatrix}\right\}. \tag{23}$$

The polynomials $B_L(p,p')$ are known to be related to the minimal conformal field theory $M(p,p')$ perturbed by the operator $\phi_{1,3}$.

Recently, very different, so-called fermionic representations for the above polynomials have been obtained by Berkovich, McCoy and Schilling using continued fractions.[29,30] Assume $p<p'<2p$, $\gcd(p,p')=1$ and define non-negative integers n and $\nu_0,\dots,\nu_n$ by the continued fraction expansion

$$\frac{p}{p'-p}=\nu_0+\cfrac{1}{\nu_1+\cfrac{1}{\nu_2+\cdots+\cfrac{1}{\nu_n+2}}}=[\nu_0,\dots,\nu_{n-1},\nu_n+2].$$

Using n and ν_j, set

$$t_m=\sum_{j=0}^{m-1}\nu_j,\quad 1\leqslant m\leqslant n\quad\text{and}\quad d=\sum_{j=0}^{n}\nu_j. \tag{24}$$

The t_m and d are used to define a fractional incidence matrix $\mathcal{I}$ and a fractional Cartan-type matrix $2B=2I-\mathcal{I}$ (with I the d by d unit matrix) as follows:

$$\mathcal{I}_{i,j}=\begin{cases}\delta_{i,j+1}+\delta_{i,j-1}, & \text{for } 1\leqslant i<d,\quad i\neq t_m,\\ \delta_{i,j+1}+\delta_{i,j}-\delta_{i,j-1}, & \text{for } i=t_m,\quad 1\leqslant m\leqslant n-\delta_{\nu_n,0},\\ \delta_{i,j+1}+\delta_{\nu_n,0}\delta_{i,j}, & \text{for } i=d.\end{cases} \tag{25}$$

When $p'=p+1$, the incidence matrix $\mathcal{I}$ has components $\mathcal{I}_{i,j}=\delta_{|i-j|,1}$ $(i,j=1,...,p-2)$, so that $2B$ corresponds to the Cartan matrix of the Lie algebra A_{p-3}. When $p=2k-1$ and $p'=2k+1$ the matrix $\mathcal{I}$ has components $\mathcal{I}_{i,j}=\delta_{|i-j|,1}+\delta_{i,j}\delta_{i,k-1}$ $(i,j=1,...,k-1)$, so that $2B$ corresponds to the Cartan-type matrix of the tadpole graph of $k-1$ nodes.

Using the above definition, the fermionic representation for the bounded Virasoro characters with $p<p'<2p$ can be given as

$$F_L(p,p';q)=\sum_{m\in 2\mathbb{Z}^d} q^{mBm/2}\prod_{j=1}^{d}\begin{bmatrix} L\,\delta_{j,1}+\frac{1}{2}(\mathcal{I}m)_j \\ m_j\end{bmatrix}. \tag{26}$$

Here we use the notations $vMw=\Sigma_{j,k}v_jM_{j,k}w_k$, $(Mv)_j=\Sigma_k M_{j,k}v_k$ and $(vM)_j=\Sigma_k v_k M_{k,j}$. These conventions are important since, generally, $M(=\mathcal{I},B)$ is not a symmetric matrix. The general form (26) for $F_L(p,p')$ can be found in Refs. 29 and 30 (see also Ref. 37). The important special cases $(p,p')=(p,p+1)$ and $(2k-1,2k+1)$ were proven prior to this is in Refs. 38, 39 and Ref. 40, respectively.

The expression for $F_L(p,p';q)$ with $p'>2p$ follows from the duality transformation

$$F_L(p,p';1/q)=q^{-L^2}F_L(p'-p,p';q). \tag{27}$$

To obtain fermionic character formulas for $\chi_{r,s}^{(p,p')}(q)$ with $|p'r-ps|=1$, one simply lets L tend to infinity in (26).

Before we proceed to use the identity,

$$F_L(p,p';q)=B_L(p,p';q), \tag{28}$$

to derive trinomial identities, let us comment on the convention of writing $2B$ for a Cartan-type matrix in the above formulas. This has its origin in the work of Ref. 41, where, in more general situations, the matrix B has a (nontrivial) tensor product structure, $B=b_1\otimes b_2$. In the identities of this section the matrix b_1 is simply the inverse of the A_1 Cartan matrix, $(b_1)=(\frac{1}{2})$. In Sec. IV D, however, we indeed encounter a different situation, b_1 being the (still trivial) Cartan-type matrix of the tadpole graph with just a single node, so that $b_1=(1)$.

B. *q*-trinomial identities I

We start with the q-binomial identity (28) for $p<p'<2p$, assuming that $d\geqslant 2$. Applying Eq. (9), with $C_{L,k}$ given by (12), we find

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\{q^{(p'(2p-p')j+2)j/2}T_0(L,p'j)-q^{((2p-p')j+2r-s)(p'j+s)/2}T_0(L,p'j+s)\}\\
&=\sum_{k=0}^{L}(-1)^{L-k}q^{\binom{L-k}{2}-L^2/2}\begin{bmatrix}L\\k\end{bmatrix}F_k(p,p';q)\\
&=\sum_{m\in 2\mathbb{Z}^d}q^{(mBm-L^2)/2}\left(\prod_{j=2}^{d}\begin{bmatrix}\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}\right)\sum_{k=0}^{L}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}L\\k\end{bmatrix}\begin{bmatrix}L-k+\frac{1}{2}(\mathcal{I}m)_1\\ m_1\end{bmatrix}\\
&\overset{\text{by (A2)}}{=} q^{L^2/2}\sum_{m\in 2\mathbb{Z}^d}q^{mBm/2-L(Bm)_1}\begin{bmatrix}\frac{1}{2}(\mathcal{I}m)_1\\ m_1-L\end{bmatrix}\prod_{j=2}^{d}\begin{bmatrix}\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}\\
&=q^{L^2\mathcal{I}_{1,1}/2}\sum_{m+Le_1\in 2\mathbb{Z}^d}q^{mBm/2+L(mB-Bm)_1/2}\prod_{j=1}^{d}\begin{bmatrix}\frac{1}{2}L\mathcal{I}_{j,1}+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix},
\end{aligned}$$

with e_j $(j=1,...,d)$ the standard unit vectors in $\mathbb{Z}^d$. We now have to distinguish two cases according to whether $\nu_0=1$ (so that $3p/2<p'<2p$) or $\nu_0>1$ (so that $p<p'\leqslant 3p/2$). In the latter case $\mathcal{I}_{1,j}=\mathcal{I}_{j,1}=\delta_{1,j-1}$, and we obtain the following polynomial identities.

Proposition IV.1: For integers p,p′ with $p<p'\leqslant 3p/2$ *and* $\gcd(p,p')=1$, *let integers* $1\leqslant r<p$ *and* $1\leqslant s<p'$ *be fixed by* $|p'r-ps|=1$ *and let* $\mathcal{I}$ *and B be defined by (24) and (25). Then the following polynomial identity holds for* $L\in\mathbb{Z}$:

$$\sum_{m+Le_1\in 2\mathbb{Z}^d} q^{mBm/2}\prod_{j=1}^{d}\begin{bmatrix}\frac{1}{2}L\delta_{j,2}+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}$$

$$=\sum_{j=-\infty}^{\infty}\{q^{(p'(2p-p')j+2)j/2}T_0(L,p'j)-q^{((2p-p')j+2r-s)(p'j+s)/2}T_0(L,p'j+s)\}.$$

The admissible pairs $(p,p')=(3,4)$ and $(p,p')=(2,3)$ have been neglected in our derivation due to the assumption that $d\geqslant 2$. These two cases can be treated in a similar fashion, and when $(p,p')=(3,4)$ the left-hand side is 1 for L even and 0 for L odd. When $(p,p')=(2,3)$, in which case $F_L(2,3;q)=1$, the left-hand side becomes $\delta_{L,0}$. All of the identities of Proposition IV.1 have been derived before, and for $p'=p+1$ they were first found by Schilling.[42,14] The more general case can be found in Ref. 22.

Next we treat the case $\nu_0=1$. When this occurs $\mathcal{I}_{1,j}=\delta_{j,1}-\delta_{1,j-1}$ and $\mathcal{I}_{j,1}=\delta_{j,1}+\delta_{1,j-1}$, and we obtain the following polynomial identities.

Proposition IV.2: For integers p,p′ with $3p/2<p'<2p$ *and* $\gcd(p,p')=1$ *let integers* $1\leqslant r<p$ *and* $1\leqslant s<p'$ *be fixed by* $|p'r-ps|=1$ *and let* $\mathcal{I}$ *and B defined by (24) and (25). Then the following polynomial identity holds for* $L\in\mathbb{Z}$:

$$\sum_{m+Le_1\in 2\mathbb{Z}^d} q^{L(L-2m_2)/4+mBm/2}\prod_{j=1}^{d}\begin{bmatrix}\frac{1}{2}L(\delta_{j,1}+\delta_{j,2})+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}$$

$$=\sum_{j=-\infty}^{\infty}\{q^{(p'(2p-p')j+2)j/2}T_0(L,p'j)-q^{((2p-p')j+2r-s)(p'j+s)/2}T_0(L,p'j+s)\}.$$

The case $(p,p')=(3,5)$ has again escaped a proper derivation, but has, in fact, been treated previously, corresponding to identity (20) with q replaced by $1/q$. Apart from this special case due to Andrews,[7] the identities of Proposition IV.2 have been proved by Berkovich, McCoy, and Orrick[13,16] for $(p,p')=(2\nu+1,4\nu)$ and were conjectured for general p and p' by Berkovich, McCoy, and Pearce [Eq. (8.8) of Ref. 21].

C. *q*-trinomial identities II

Our starting point for deriving q-trinomial identities is again Eq. (28), but this time we rely on (18). This implies that (28) with L replaced by k, multiplied by $q^{k^2}\left[\begin{smallmatrix}L\\2k\end{smallmatrix}\right]$, and summed over k yields

$$\sum_{k\geqslant 0} q^{k^2}\begin{bmatrix}L\\2k\end{bmatrix}F_k(p,p';q)=\sum_{j=-\infty}^{\infty}\left\{q^{j(p'(p+p')j+1)}\begin{bmatrix}L\\2p'j\end{bmatrix}_2-q^{(p'j+s)((p+p')j+r+s)}\begin{bmatrix}L\\2p'j+2s\end{bmatrix}_2\right\}. \tag{29}$$

To transform this into explicit polynomial identities we need to distinguish between $p<p'<2p$ and $p'>2p$.

First, assume that $p<p'<2p$. After substituting expression (26) for F_L, the left side of (29) is

$$\sum_{k\geqslant 0}\sum_{m\in 2\mathbb{Z}^d} q^{k^2+mBm/4}\begin{bmatrix}L\\2k\end{bmatrix}\prod_{j=1}^{d}\begin{bmatrix}k\delta_{j,1}+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}.$$

By the q-Chu–Vandermonde summation (A1), with $L\to L-k+m_1/2$, $a\to k-m_1/2$, and $b\to -k-m_1/2$, this can be rewritten as

$$\sum_{i,k\geqslant 0}\sum_{m\in 2\mathbb{Z}^d} q^{i(i-k-m_1/2)+k^2+mBm/4}\begin{bmatrix} L-k+\frac{1}{2}m_1\\ i\end{bmatrix}\begin{bmatrix} k-\frac{1}{2}m_1\\ 2k-i\end{bmatrix}\prod_{j=1}^{d}\begin{bmatrix} k\,\delta_{j,1}+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}.$$

Replacing $m_j\rightarrow m_{j+2}$, $j=1,...,d$, followed by $k\rightarrow(m_1+m_2)/2$ and $i\rightarrow m_1$ yields

$$\sum{}' q^{(3m_1^2+m_2^2-2m_1m_3)/4+\Sigma_{j,k=1}^d m_{j+2}B_{j,k}m_{k+2}/4}$$
$$\times\begin{bmatrix} L-\frac{1}{2}(m_1+m_2-m_3)\\ m_1\end{bmatrix}\begin{bmatrix} \frac{1}{2}(m_1+m_2-m_3)\\ m_2\end{bmatrix}\prod_{j=1}^{d}\begin{bmatrix} \frac{1}{2}(m_1+m_2)\,\delta_{j,1}+\frac{1}{2}\Sigma_{k=1}^d\mathcal{I}_{j,k}m_{k+2}\\ m_j+2\end{bmatrix},$$

where the primed sum denotes a sum over $m\in\mathbb{Z}^{d+2}$ such that m_1+m_2 and $m_3,\ldots,m_{d+2}$ are all even.

Now define a new incidence matrix $\mathcal{I}'$ and Cartan-type matrix $2B'=2I-\mathcal{I}'$ of dimension $d'=d+1$ by replacing the continued fraction expansion $[\nu_0,\ldots\nu_n+2]$ by $[1,\nu_0,\ldots,\nu_n+2]$, so that $\mathcal{I}'$ becomes the incidence matrix corresponding to the continued fraction expansion of p'/p. Also define $\mathcal{I}''$ and $2B''=2I-\mathcal{I}''$ of dimension $d''=d+2$ as

$$\mathcal{I}''_{i,j}=\begin{cases}-\delta_{i,1}\delta_{j,1}+\delta_{i,2}+\delta_{i,3}-\delta_{j,2}+\delta_{j,3}, & \text{for } i=1 \text{ or } j=1,\\ \mathcal{I}'_{i-1,j-1}, & \text{for } i,j=2,...,d+2.\end{cases}\tag{30}$$

Then the above sequence of transformations implies the following proposition.

Proposition IV.3: For integers p,p′ with p<p′<2 and gcd(p,p′)=1 let integers 1⩽r<p and 1⩽s<p′ be fixed by |p′r−ps|=1 and let $\mathcal{I}''$ and B″ be defined by (30). Then the following polynomial identity holds for L∈ℤ:

$$\sum{}' q^{mB''m/4}\prod_{j=1}^{d''}\begin{bmatrix} L\,\delta_{j,1}+\frac{1}{2}(\mathcal{I}''m)_j\\ m_j\end{bmatrix}$$
$$=\sum_{j=-\infty}^{\infty}\left\{q^{j(p'(p+p')j+1)}\begin{bmatrix} L\\ 2p'j\end{bmatrix}_2-q^{(p'j+s)((p+p')j+r+s)}\begin{bmatrix} L\\ 2p'j+2s\end{bmatrix}_2\right\}.$$

The identities of Proposition IV.3 are the $n=0$ case of the conjectured equation (8.11) [which contains the $n=0$ instances of (6.19) and (8.3)] of Ref. 21, and are related to the $\phi_{2,1}$ perturbation of the minimal conformal field theory $M(p',p+p')$.

When $p'>2p$ we replace $p\rightarrow p'-p$ in (29) and use the duality property (27). Hence

$$\sum_{k\geqslant 0} q^{2k^2}\begin{bmatrix} L\\ 2k\end{bmatrix}F_k(p,p';q^{-1})$$
$$=\sum_{j=-\infty}^{\infty}\left\{q^{j(p'(2p'-p)j+1)}\begin{bmatrix} L\\ 2p'j\end{bmatrix}_2-q^{(p'j+s)((2p'-p)j+r+s)}\begin{bmatrix} L\\ 2p'j+2s\end{bmatrix}_2\right\}.\tag{31}$$

Observe that the transformation carried out above implies $p<p'<2p$ and $|p'(r-s)+ps|=1$.

Substituting expression (26) for F_L and using (A7), the left side of (31) yields

$$\sum_{k\geqslant 0}\sum_{m\in 2\mathbb{Z}^d} q^{k(2k-m_1)+mBm/4}\begin{bmatrix} L\\ 2k\end{bmatrix}\prod_{j=1}^{d}\begin{bmatrix} k\,\delta_{j,1}+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}.$$

By the q-Chu–Vandermonde summation (A1), with $L\rightarrow L-m_1/2$, $a\rightarrow m_1/2$, $b\rightarrow m_1/2-2k$, this can be rewritten as

$$\sum_{i,k\geqslant 0}\sum_{m\in 2\mathbb{Z}^d} q^{i(i-2k+m_1/2)+k(2k-m_1)+mBm/4}\begin{bmatrix} L-\frac{1}{2}m_1\\ i\end{bmatrix}\begin{bmatrix}\frac{1}{2}m_1\\ 2k-1\end{bmatrix}\prod_{j=1}^{d}\begin{bmatrix} k\delta_{j,1}+\frac{1}{2}(\mathcal{I}m)_j\\ m_j\end{bmatrix}.$$

Replacing $m_j\to m_{j+2}$, $j=1,...,d$, followed by $t\to m_1+m_2$ and $i\to m_1$, gives

$$\sum{}' q^{(m_1^2+m_2^2-m_2m_3)/2+\Sigma_{j,k=1}^d m_{j+2}B_{j,k}m_{k+2}/4}\begin{bmatrix} L-\frac{1}{2}m_3\\ m_1\end{bmatrix}$$
$$\times\begin{bmatrix}\frac{1}{2}m_3\\ m_2\end{bmatrix}\prod_{j=1}^{d}\begin{bmatrix}\frac{1}{2}(m_1+m_2)\delta_{j,1}+\frac{1}{2}\Sigma_{k=1}^d\mathcal{I}_{j,k}m_{k+2}\\ m_{j+2}\end{bmatrix},$$

where the primed sum again denotes a sum over $m\in\mathbb{Z}^{d+2}$ such that m_1+m_2 and $m_3,\ldots,m_{d+2}$ are all even.

Now define a new incidence matrix $\mathcal{I}$ and Cartan-type matrix $2B'=2I-\mathcal{I}$ of dimension $d'=d+1$ by replacing the continued fraction expansion $[\nu_0,\ldots,\nu_n+2]$ by $[\nu_0+1,\nu_1,\ldots,\nu_n+2]$, so that $\mathcal{I}'$ becomes the incidence matrix corresponding to the continued fraction expansion of $p'/(p'-p)$. Also define $\mathcal{I}''$ and $2B''=2I-\mathcal{I}''$ of dimension $d''=d+2$ as

$$\mathcal{I}''_{i,j}=\begin{cases}\delta_{i,3}-\delta_{j,3}, & \text{for } i=1 \text{ or } j=1,\\ \mathcal{I}'_{i-1,j-1}, & \text{for } i,j=2,...,d+2.\end{cases} \tag{32}$$

Then the above sequence of transformations implies the following proposition.

Proposition IV.4: For integers p,p′ with $p<p'<3p/2$ *and* $gcd(p,p')=1$ *let integers* $1\leqslant r<p$ *and* $1\leqslant s<p'$ *be fixed by* $|p'(r-s)r+ps|=1$ *and let* $\mathcal{I}$ *and B be defined by (32). Then the following polynomial identity holds for* $L\in\mathbb{Z}$:

$$\sum{}' q^{mB''m/4}\prod_{j=1}^{d''}\begin{bmatrix} L\delta_{j,1}+\frac{1}{2}(\mathcal{I}''m)_j\\ m_j\end{bmatrix}$$
$$=\sum_{j=-\infty}^{\infty}\left\{q^{j(p'(2p'-p)j+1)}\begin{bmatrix} L\\ 2p'j\end{bmatrix}_2-q^{(p'j+s)((2p'-p)j+r+s)}\begin{bmatrix} L\\ 2p'j+2s\end{bmatrix}_2\right\}.$$

The identities of Proposition IV. 4, which are related to the $\phi_{2,1}$ perturbation of the conformal field theory $M(p',2p'-p)$, were conjectured in Ref. 21 [as Eq. (6.9)]. For $p=p'-1$ a proof using recurrences was recently given in Ref. 20.

D. *q*-trinomial identities III

There are, of course, many more q-trinomial identities that can be derived using the techniques of the previous sections. Our final application is to show that in some cases a bit more ingenuity is required to arrive at the desired result. The identities we set out to prove here were again conjectured by Berkovich, McCoy, and Pearce [Eq. (9.4) of Ref. 21] and are interesting, as they contain the (polynomial) Rogers–Ramanujan identity (20) as the simplest case. It also provides an example for which the matrix $B=b_1\otimes b_2$ (in the proposition below denoted as C_n) of Sec. IV A has $b_1=(1)$ and not $(\frac{1}{2})$.

Proposition IV.5: For $n\geqslant 1$, *let* C_n *be the Cartan matrix of* A_n. *Then for all* $L\in\mathbb{Z}$,

$$\sum_{m\in\mathbb{Z}^n} q^{mC_nm/2}\prod_{j=1}^{n}\begin{bmatrix} L\delta_{j,1}+m_j-(C_nm)_j\\ m_j\end{bmatrix}$$
$$=\sum_{j=-\infty}^{\infty}\left\{q^{((n+3)(n+4)j+2)j/2}\begin{bmatrix} L\\ (n+4)_j\end{bmatrix}_2-q^{((n+3)j+2)((n+4)j+2)/2}\begin{bmatrix} L\\ (n+4)j+2\end{bmatrix}_2\right\}. \tag{33}$$

Letting L tend to infinity using (8) and (A6), this yields the following Virasoro-character identities.

Corollary IV.1: For $n\geqslant 1$ and $|q|<1$,

$$\sum_{m\in\mathbb{Z}^n}\frac{q^{mC_nm/2}}{(q)_{m_1}}\prod_{j=2}^{n}\begin{bmatrix}m_j-(C_nm)_j\\ m_j\end{bmatrix}=\begin{cases}\chi_{1,2}^{((n+3)/2,n+4)}(q), & n\ \text{odd},\\ \chi_{1,2}^{((n+4)/2,n+3)}(q), & n\ \text{even}.\end{cases}\tag{34}$$

In Ref. 21, the identities (33) and (34) were associated with the $\phi_{2,1}$ perturbation of the conformal field theories $M((n+4)/2,n+3)$ when n is odd and the $\phi_{1,5}$ perturbation of $M((n+3)/2,n+4)$ when n is even.

Proof: The corollary betrays a hidden parity dependence of (33), which also plays a role in the proof. Treating n being odd first, we set $n=2k-1$. The left-hand side of (33) then reads as

$$\sum_{m\in\mathbb{Z}^{2k-1}}q^{mC_{2k-1}m/2}\prod_{j=1}^{2k-1}\begin{bmatrix}\frac{1}{2}L\,\delta_{j,1}+m_{j-1}-m_j+m_{j+1}\\ m_j\end{bmatrix},\tag{35}$$

with the convention that $m_0=L/2$ and $m_{2k}=0$. We eliminate the variables m_{2j-1}, $j=1,...,k$ in favor of new variables $M_1,\ldots,M_k$, defined as

$$m_{2j-1}=m_{2j-2}-\tfrac{1}{2}(M_j-M_{j+1}),$$

where $M_{k+1}=0$. If after this replacement we relabel m_{2j} to m_j for $j=1,...,k$ (so that $m_k=0$), expression (35) becomes

$$\begin{aligned}&\sum_{M+Le_1\in 2\mathbb{Z}^k}q^{(L(L-2M_1)+M_1^2+\Sigma_{i,j=2}^{k}M_i(C_{k-1})_{i,j}M_j)/4}\\ &\times\sum_{m_1,\ldots,m_{k-1}}q^{\Sigma_{j=1}^{k-1}(M_{j+1}-m_j)(m_{j-1}-m_j-(M_j-M_{j+2})/2)}\\ &\times\begin{bmatrix}m_0+m_1+\frac{1}{2}(M_1-M_2)\\ m_0-\frac{1}{2}(M_1-M_2)\end{bmatrix}\prod_{j=1}^{k-1}\begin{bmatrix}m_{j-1}-\frac{1}{2}(M_j-M_{j+2})\\ m_j\end{bmatrix}\begin{bmatrix}m_{j+1}+\frac{1}{2}(M_{j+1}-M_{j+2})\\ M_j-\frac{1}{2}(M_{j+1}-M_{j+2})\end{bmatrix}.\end{aligned}\tag{36}$$

This allows for successive summation over $m_{k-1},\ldots,m_1$ by the q-Saalschütz sum (A4). When summing over m_j, we take (A4) with $L\to m_{j-1}-(M_j-M_{j+2})/2$, $a\to(M_{j+1}-M_{j+2})/2$, $b\to-(M_{j+1}+M_{j+2})/2$, $c\to(M_j-M_{j+1})/2$ (for $j\geqslant 2$), and $c\to m_0+(M_1-M_2)/2$ (for $j=1$). As a result, (36) collapses into

$$\sum_{M+Le_1\in 2\mathbb{Z}^k}q^{L(L-2M_1)/4+MBM/2}\prod_{j=1}^{k}\begin{bmatrix}\frac{1}{2}L(\delta_{j,1}+\delta_{j,2})+\frac{1}{2}(\mathcal{I}M)_j\\ M_j\end{bmatrix},\tag{37}$$

with matrices $\mathcal{I}$ and $2B=2I-\mathcal{I}$ defined in Eqs. (24) and (25) corresponding to the continued fraction expansion of $(k+2)/(k+1)=[1,k-1]$, i.e.,

$$\mathcal{I}_{i,j}=\begin{cases}\delta_{i,1}\delta_{j,1}+\delta_{i,2}-\delta_{j,2}, & \text{for } i=1 \text{ or } j=1,\\ \delta_{i,j-1}+\delta_{i,j+1}, & \text{for } i,j=2,...,k.\end{cases}$$

The last part of the proof consists of the observation that the identity obtained by equating (37) with the right-hand side of (33) (with $n=2k-1$) is nothing but the identity of Proposition IV.2 with $(p,p')=(k+2,2k+3)$ after letting $q\to 1/q$. This is readily seen using (4) and (A7).

Next, we deal with n being even, setting $n=2k$. The left-hand side of (33) then is

$$\sum_{m\in\mathbb{Z}^{2k}} q^{mC_{2k}m/2}\prod_{j=1}^{2k}\begin{bmatrix} L\delta_{j,1}+m_{j-1}-m_j+m_{j+1}\\ m_j\end{bmatrix}, \tag{38}$$

where $m_0=m_{2k+1}=0$. We eliminate the variables m_{2j}, $j=1,...,k$, introducing new variables $m_0,\ldots,M_{k-1}$ by

$$m_{2j}=m_{2j-1}-\tfrac{1}{2}(M_{j-1}-M_j),$$

where $M_k=0$. After this replacement we shift $m_{2j-1}\to m_j$ for $j=1,...,$ k so that expression (38) becomes

$$\sum_{\substack{M_0,\ldots,M_{k-1}\\ M_j\ \text{even}}} q^{(M_0^2+\Sigma_{i,j=1}^{k-1}M_i(C_{k-1})_{i,j}M_j)/4}$$
$$\times\sum_{m_1,\ldots,m_k} q^{m_1(m_1-(M_0+M_1)/2)+\Sigma_{j=2}^{k}(M_{j-1}-m_j)(m_{j-1}-m_j-(M_{j-2}-M_j)/2)}$$
$$\times\begin{bmatrix} L-\frac{1}{2}(M_0-M_1)\\ m_1\end{bmatrix}\left(\prod_{j=2}^{k}\begin{bmatrix} m_{j-1}-\frac{1}{2}(M_{j-2}-M_j)\\ m_j\end{bmatrix}\right)\prod_{j=1}^{k}\begin{bmatrix} m_{j+1}+\frac{1}{2}(M_{j-1}-M_j)\\ m_j-\frac{1}{2}(M_{j-1}-M_j)\end{bmatrix}.$$

We now sum over $m_k,\ldots,m_3$ by successive application of the q-Saalschütz sum (A4). When summing over m_j we take (A4) with $L\to m_{j-1}-(M_{j-2}-M_j)/2$, $a\to(M_{j-1}-M_j)/2$, $b\to -(M_{j-1}+M_j)/2$, and $c\to(M_{j-2}-M_{j-1})/2$. The final sum over m_1 follows from (A1) with $L\to L-(M_0-M_1)/2$, $a\to(M_0-M_1)/2$, and $b\to-(M_0+M_1)/2$. Setting $M_0\to 2i$, the resulting expression is

$$\sum_{i\geqslant 0} q^{i^2}\begin{bmatrix} L\\ 2i\end{bmatrix}\sum_{M\in 2\mathbb{Z}^{k-1}} q^{MC_{k-1}M/4}\prod_{j=1}^{k-1}\begin{bmatrix} i\delta_{j,1}+M_j-\frac{1}{2}(C_{k-1}M)_j\\ M_j\end{bmatrix}.$$

Equating this with the right-hand side of (33) for $n=2k$, we recognize identity (29) with $(p,p')=(k+1,k+2)$. □

V. THE TRINOMIAL BAILEY LEMMA

In this final section of our paper we formulate some of our results in the language of Bailey pairs. As we will see, the connection coefficients obtained in Sec. II provide a very elementary proof of the trinomial analog of Bailey's lemma recently obtained by Andrews and Berkovich.[19]

First, some definitions are needed. In subsequent formulas, $T_n(L,a)/(q)_L$ will be abbreviated to $Q_n(L,a)$.

Definition V.1: A pair of sequences $\alpha=\{\alpha_L\}_{L\geqslant 0}$ *and* $\beta=\{\beta_L\}_{L\geqslant 0}$ *that satisfies*

$$\beta_L=\sum_{r=0}^{L}\frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}},$$

forms a (binomial) Bailey pair relative to a.

Definition V.2: A pair of sequences $A=\{A_L\}_{L\geqslant 0}$ *and* $B=\{B_L\}_{L\geqslant 0}$ *that satisfies*

$$B_L=\sum_{r=0}^{L} Q_n(L,r)A_r,$$

forms a trinomial Bailey pair relative to n.

The Bailey lemma[43] and trinomial Bailey lemma[19] can now be stated as the following summation formulas.

Lemma V.1. Let (α,β) be a Bailey pair relative to a. Then

$$\sum_{L=0}^{M}\frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L\alpha_L}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{M-L}(aq)_{M+L}}=\sum_{L=0}^{M}\frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L(aq/\rho_1\rho_2)_{M-L}\beta_L}{(aq/\rho_1)_M(aq/\rho_2)_M(q)_{M-L}}.$$

Lemma V.2. Let (A,B) form a trinomial Bailey pair relative to 0. Then

$$\sum_{L=0}^{M}(-1)_L q^{L/2}B_L=(-1)_{M+1}\sum_{L=0}^{M}q^{L/2}A_L\frac{Q_1(M,L)}{1+q^L}. \tag{39}$$

If (A,B) is a trinomial Bailey pair relative to 1, then

$$\sum_{L=0}^{M}(-q^{-1})_L q^L B_L=(-1)_M\sum_{L=0}^{M}A_L\left\{Q_1(M,L)-\frac{Q_1(M-1,L+1)}{1+q^{-L-1}}-\frac{Q_1(M-1,L-1)}{1+q^{L-1}}\right\}.$$

Before we translate the results of Sec. II in the language of Bailey pairs, let us point out that the connection coefficients between q binomials and q trinomials can be applied to yield a very simple proof of the trinomial Bailey lemma. At the heart of the proof of Lemma V.2 is the following identity derived in Ref. 19 by a considerable amount of work,

$$T_0(L,a)=q^{(a-L)/2}\left\{\frac{1+q^L}{1+q^a}T_1(L,a)-\frac{1-q^L}{1+q^a}T_1(L-1,a)\right\}. \tag{40}$$

To see, for example, that this implies (39), we multiply (40) by $q^{L/2}(-1)_L/(q)_L$ and sum over L from a to M. On the right-hand side all but one term cancels, so that

$$\sum_{L=a}^{M}q^{L/2}(-1)_L Q_0(L,a)=\frac{q^{a/2}}{1+q^a}(-1)_{M+1}Q_1(M,a),$$

which obviously implies (39).

By Eqs. (9)–(13), Eq. (40) is proved if we can show its validity when multiplied by $C_{M,L}(a)$ and summed over L. Doing this and using (10), one finds (replacing $L\to k$ and $M\to L$)

$$\begin{aligned}\begin{bmatrix}2L\\L-a\end{bmatrix}&=\sum_{k=0}^{L}q^{\binom{k}{2}-\binom{a}{2}}\begin{bmatrix}L\\k\end{bmatrix}\left\{\frac{1+q^k}{1+q^a}T_1(k,a)-\frac{1-q^k}{1+q^a}T_1(k-1,a)\right\}\\&=\sum_{k=0}^{L}q^{\binom{k}{2}-\binom{a}{2}}\left\{\frac{1+q^k}{1+q^a}\begin{bmatrix}L\\k\end{bmatrix}-q^k\frac{1-q^{k+1}}{1+q^a}\begin{bmatrix}L\\k+1\end{bmatrix}\right\}T_1(k,a)\\&=\sum_{k=0}^{L}q^{\binom{k}{2}-\binom{a}{2}}\frac{1+q^L}{1+q^a}\begin{bmatrix}L\\k\end{bmatrix}T_1(k,a).\end{aligned}$$

But the extremes of this string of equations is nothing but Eq. (14), with $D'_{L,k}(a)$ given by Eq. (16) of Lemma II.2, establishing (40).

We now give a series of lemmas that are all straightforward consequences of the results of Sec. II.

Lemma V.3: Let (α,β) be a Bailey pair relative to 1. Then

$$A_L=q^{-L^2/2}\alpha_L,\quad B_L=\sum_{k=0}^{L}\frac{(-1)^{L-k}q^{\binom{L-k}{2}-L^2/2}(q)_{2k}}{(q)_k(q)_{L-k}}\beta_k$$

is a trinomial Bailey pair relative to 0 and

$$A_L=\frac{q^{-\binom{L}{2}}}{1+q^L}\alpha_L, \quad B_L=\sum_{k=0}^{L}\frac{(-1)^{L-k}q^{\binom{L-k}{2}-\binom{L}{2}}(q)_{2k}}{(1+q^k)(q)_k(q)_{L-k}}\beta_k$$

is a trinomial Bailey pair relative to 1.

The converse statement is as follows.

Lemma V.4: Let $(A(n),B(n))$ be a trinomial Bailey pair relative to n. Then,

$$\alpha_L=q^{L^2/2}A_L(0), \quad \beta_L=\frac{(q)_L}{(q)_{2L}}\sum_{k=0}^{L}\frac{q^{k^2/2}}{(q)_{L-k}}B_k(0)$$

and

$$\alpha_L=q^{\binom{L}{2}}(1+q^L)A_L(1), \quad \beta_L=\frac{(q)_L}{(q)_{2L}}(1+q^L)\sum_{k=0}^{L}\frac{q^{\binom{k}{2}}}{(q)_{L-k}}B_k(1),$$

are Bailey pairs relative to 1.

Lemma V.3 is to be compared with the following result of Ref. 23.

Lemma V.5: Let l be a non-negative integer and (α,β) a Bailey pair relative to $a=q^l$. Then

$$A_L=\begin{cases}\alpha_{(L-l)/2}, & \text{for } L=l,l+2,\ldots,\\ 0, & \text{otherwise};\end{cases}$$

$$B_L=\begin{cases}\displaystyle\sum_{k=0}^{[(L-l)/2]}\frac{q^{(L-l-2k)(L-l-2k-n)/2}}{(q)_l(q)_{L-l-2k}}\beta_k, & for\ L\geqslant l;\\ 0, & \text{otherwise},\end{cases}$$

forms a trinomial Bailey pair relative to n.

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APPENDIX: SOME *q*-BINOMIAL FORMULAS

In this appendix we list some standard q-binomial identities that are repeatedly used in the main text.

The following three formulas all hold for integers a, b, L such that a, $L\geqslant 0$,

$$\sum_{k=0}^{L}q^{k(k+b)}\begin{bmatrix}L\\k\end{bmatrix}\begin{bmatrix}a\\k+b\end{bmatrix}=\begin{bmatrix}a+L\\b+L\end{bmatrix}, \tag{A1}$$

$$\sum_{k=0}^{L}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}L\\k\end{bmatrix}\begin{bmatrix}L+a-k\\b\end{bmatrix}=q^{L(L+a-b)}\begin{bmatrix}a\\b-L\end{bmatrix}, \tag{A2}$$

$$\sum_{k=0}^{L}(-1)^k q^{\binom{k}{2}+k(b-L+1)}\begin{bmatrix}L\\k\end{bmatrix}\begin{bmatrix}L+a-k\\b\end{bmatrix}=\begin{bmatrix}a\\b-L\end{bmatrix}. \tag{A3}$$

The first two equations are specializations of the q-Chu–Vandermonde sum (II.7) of Ref. 3 and the last equation is a specialization of the q-Chu–Vandermonde sum (II.6) of Ref. 3. Identity (A2) is also given in Ref. 1 as Eq. (3.3.10). A useful specialization of the q-Saalschütz sum [(II.12) of Ref. 3] is given by

$$\sum_{k=0}^{L} q^{(a-b-k)(L-k)} \begin{bmatrix} L \\ k \end{bmatrix} \begin{bmatrix} a \\ k+b \end{bmatrix} \begin{bmatrix} k+c \\ a+L \end{bmatrix} = \begin{bmatrix} c \\ b+L \end{bmatrix} \begin{bmatrix} c-b \\ a-b \end{bmatrix}, \tag{A4}$$

true for integers a, b, c, L such that a, c, $L \geqslant 0$. This is Eq. (3.3.11) of Ref. 1. Finally, we list the elementary results:

$$\begin{bmatrix} L \\ a \end{bmatrix} = \begin{bmatrix} L-1 \\ a-1 \end{bmatrix} + q^a \begin{bmatrix} L-1 \\ a \end{bmatrix}, \quad \text{for } L,a \geqslant 0, \quad L+a \neq 0, \tag{A5}$$

$$\lim_{L\to\infty} \begin{bmatrix} L \\ a \end{bmatrix} = \frac{1}{(q)_a} \tag{A6}$$

and

$$\begin{bmatrix} L \\ a \end{bmatrix}_{1/q} = q^{-a(L-a)} \begin{bmatrix} L \\ a \end{bmatrix}_q . \tag{A7}$$

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