The Rogers – Ramanujan Identities

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The identities

The Rogers–Ramanujan identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})},$$

are two of the most remarkable and important results in the theory of q-series. Although originally discovered by L.J. Rogers in 1894 [3], they shot to fame through the work of the Indian genius S. Ramanujan.

Ramanujan included the following immediate consequence of the Rogers–Ramanujan identities in his famous letter to G.H. Hardy,

$$\cfrac{1}{1+\cfrac{\mathrm{e}^{-2\pi}}{1+\cfrac{\mathrm{e}^{-4\pi}}{1+\cfrac{\mathrm{e}^{-6\pi}}{1+\cdots}}}}=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)\mathrm{e}^{2\pi/5},$$

who wrote (about this and a similar such continued fraction):

"...defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class."

Leonard James Rogers
(1862 – 1933)
b. Oxford, England.

Partitions

Another simple consequence of the Rogers–Ramanujan identities involves partitions of the integers [1]. If we write the positive integer 9 as the ordered sum of smaller positive integers (called parts), like

$$9=3+2+2+1+1$$

then this sum is called a partition of 9. The total number of partitions of 9 is 30 and the number of partitions of n denoted $p(n)$ grows exponentially with n:

$$p(n)\sim\frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

The Rogers–Ramanujan identities imply two theorems for partitions of n subject to some simple restrictions. One of these may be stated as follows:

> The number of partitions of n such that all parts differ by at least 2 is equal to the number of partitions of n such that all parts are congruent to 1 or 4 modulo 5.

For example, if we take $n=9$, then out of the 30 unrestricted partitions only 5 are of the first type:

$$9,\quad 8+1,\quad 7+2,\quad 6+3,\quad 5+3+1$$

and five are of the second type:

$$9,\quad 6+1+1+1,\quad 4+4+1,\quad 4+1+1+1+1+1,\quad 1+1+1+1+1+1+1+1+1.$$

Representation theory

The Rogers–Ramanujan identities are also related to the representation theory of the special linear Lie algebra $\mathfrak{sl}_2$. It thus seems a natural question to ask for a generalization to the more general Lie algebra $\mathfrak{sl}_n$. This turns out to be a surprisingly difficult problem and more than a hundred years after Rogers' initial discovery the only other case that has been settled corresponds to $\mathfrak{sl}_3$.

Using ideas and techniques from algebraic combinatorics, the following $\mathfrak{sl}_3$ analogue of the first Rogers–Ramanujan identity may be proved [2, 4]:

$$\sum_{\substack{n_1,n_2=0\\ n_2\le 2n_1}}^{\infty}\frac{q^{n_1^2-n_1n_2+n_2^2}(1-q)\cdots(1-q^{2n_1})}{(1-q)\cdots(1-q^{n_2})\,(1-q)\cdots(1-q^{n_2})\,(1-q)\cdots(1-q^{2n_1-n_2})}$$
$$=\prod_{n=1}^{\infty}\frac{1}{(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2}.$$

Although the right-hand side of this identity can again be interpreted combinatorially in terms of partitions, it remains an open problem to also understand the left-hand side from a partition theoretic point of view.

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(1887 – 1920)
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References

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