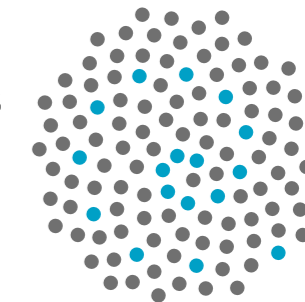


The Rogers – Ramanujan Identities

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The identities

The Rogers–Ramanujan identities

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

and

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

are two of the most remarkable and important results in the theory of q -series. Although originally discovered by L.J. Rogers in 1894 [3], they shot to fame through the work of the Indian genius S. Ramanujan.

Ramanujan included the following immediate consequence of the Rogers–Ramanujan identities in his famous letter to G.H. Hardy,

$$\frac{1}{1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \dots}}}} = \left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2} \right) e^{2\pi/5},$$

who wrote (about this and a similar such continued fraction):



“...defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class.”

Leonard James Rogers
(1862 – 1933)
b. Oxford, England.

Partitions

Another simple consequence of the Rogers–Ramanujan identities involves partitions of the integers [1]. If we write the positive integer 9 as the ordered sum of smaller positive integers (called parts), like

$$9 = 3 + 2 + 2 + 1 + 1$$

then this sum is called a partition of 9. The total number of partitions of 9 is 30 and the number of partitions of n denoted $p(n)$ grows exponentially with n :

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

The Rogers–Ramanujan identities imply two theorems for partitions of n subject to some simple restrictions. One of these may be stated as follows:

The number of partitions of n such that all parts differ by at least 2 is equal to the number of partitions of n such that all parts are congruent to 1 or 4 modulo 5.

For example, if we take $n = 9$, then out of the 30 unrestricted partitions only 5 are of the first type:

$$9, \quad 8 + 1, \quad 7 + 2, \quad 6 + 3, \\ 5 + 3 + 1$$

and five are of the second type:

$$9, \quad 6 + 1 + 1 + 1, \quad 4 + 4 + 1, \\ 4 + 1 + 1 + 1 + 1 + 1, \\ 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

Representation theory

The Rogers–Ramanujan identities are also related to the representation theory of the special linear Lie algebra \mathfrak{sl}_2 . It thus seems a natural question to ask for a generalization to the more general Lie algebra \mathfrak{sl}_n . This turns out to be a surprisingly difficult problem and more than a hundred years after Rogers' initial discovery the only other case that has been settled corresponds to \mathfrak{sl}_3 .

Using ideas and techniques from algebraic combinatorics, the following \mathfrak{sl}_3 analogue of the first Rogers–Ramanujan identity may be proved [2, 4]:

$$\sum_{\substack{n_1, n_2=0 \\ n_2 \leq 2n_1}}^{\infty} \frac{q^{n_1^2 - n_1 n_2 + n_2^2} (1-q) \cdots (1-q^{2n_1})}{(1-q) \cdots (1-q^{n_2}) (1-q) \cdots (1-q^{n_2}) (1-q) \cdots (1-q^{2n_1 - n_2})} \\ = \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})^2 (1-q^{7n-3}) (1-q^{7n-4}) (1-q^{7n-6})^2}.$$

Although the right-hand side of this identity can again be interpreted combinatorially in terms of partitions, it remains an open problem to also understand the left-hand side from a partition theoretic point of view.



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