NONSYMMETRIC INTERPOLATION MACDONALD POLYNOMIALS AND $\mathfrak{gl}_n$ BASIC HYPERGEOMETRIC SERIES

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Abstract. The Knop–Sahi interpolation Macdonald polynomials are inhomogeneous and nonsymmetric generalisations of the well-known Macdonald polynomials. In this paper we apply the interpolation Macdonald polynomials to study a new type of basic hypergeometric series of type $\mathfrak{gl}_n$. Our main results include a new q-binomial theorem, new q-Gauss sum, and several transformation formulae for $\mathfrak{gl}_n$ series.

Part I. Interpolation Macdonald polynomials

1. Introduction

The Newton interpolation polynomials

$$N_k(x) = (x - x_1) \cdots (x - x_k) \tag{1.1}$$

were used by Newton in his now famous expansion

$$N(x) = \sum_{i=0}^{k} N(x_1) \partial_1 \partial_2 \cdots \partial_i N_i(x).$$

Here $N(x)$ is an arbitrary polynomial of degree k and ∂_i (operators in this paper act on the *left*) is a Newton divided difference operator

$$f(x_1, x_2, \dots)\partial_i = \frac{f(\dots, x_{i+1}, x_i, \dots) - f(\dots, x_i, x_{i+1}, \dots)}{x_{i+1} - x_i}.$$

Various multivariable generalisations of the Newton interpolation polynomials exist in the literature, such as the Schubert polynomials [12] and several types of Macdonald interpolation polynomials [10, 11, 23, 24, 25, 29, 31]. In this paper we are interested in the latter, providing generalisations of (1.1) when the interpolation points $x_1, \dots, x_k$ form a geometric progression

$$(x_1, x_2, x_3, \dots) = (1, q, q^2, \dots).$$

Then

$$N_k(x) = (x - 1)(x - q) \cdots (x - q^{k-1}), \tag{1.2}$$

and three equivalent characterisations may be given as follows.

(1) $N_k(x)$ is the unique monic polynomial of degree k such that $N_k(q^m) = 0$ for $m \in \{0, 1, \dots, k - 1\}$.
(2) $N_k(x)$ is the solution of the recurrence

$$p_{k+1}(x) = q^k(x - 1)p_k(x/q)$$

with initial condition $p_0(x) = 1$.
(3) Up to normalisation $N_k(x)$ is the unique polynomial eigenfunction, with eigenvalue q^{-k}, of the operator

$$\xi = \tau\left(1 - \frac{1}{x}\right) + \frac{1}{x},$$

where $f(x)\tau = f(x/q)$.

Knop [10] and Sahi [29] generalised the Newton interpolation polynomials to a family of nonsymmetric, inhomogeneous polynomials $M_u(x)$, labelled by compositions $u \in \mathbb{N}^n$ and depending on n variables; $x = (x_1, \dots, x_n)$. The polynomials $M_u(x)$, known as the (nonsymmetric) Macdonald interpolation polynomials or (nonsymmetric) vanishing Macdonald polynomials, form a distinguished basis in the ring $\mathbb{Q}(q, t)[x_1, \dots, x_n]$. Remarkably, Knop and Sahi showed that all three characterisations of the Newton interpolation polynomials carry over to the multivariable theory. What appears not to have been observed before, however, is that the Macdonald interpolation polynomials may be employed to build a multivariable theory of basic hypergeometric series of type $\mathfrak{gl}_n$. For example, with $\mathsf{M}_u(x)$ an appropriate normalisation of $M_u(x)$, the following n-dimensional extension of the famous q-binomial theorem holds

$$\sum_u a^{|u|}\mathsf{M}_u(x) = \prod_{i=1}^n \frac{(at^{n-i})_\infty}{(ax_i)_\infty}. \tag{1.3}$$

The $\mathfrak{gl}_n$ basic hypergeometric series studied in the paper are very different to existing multiple basic hypergeometric series, such as those pioneered by Gustafson and Milne [7, 20, 21] or those studied subsequently by a large number of authors, see e.g., [6, 22] and references therein.

In Part I of this paper, comprising of Sections 1–6, we lay the necessary groundwork for studying basic hypergeometric series based on the interpolation Macdonald polynomials. This in itself will involve the study of another type of multivariable basic hypergeometric series involving the function $\mathsf{E}_{u/v}(a,b)$, which is a normalised connection coefficient between the interpolation polynomials $\mathsf{M}_u(ax)$ and $\mathsf{M}_v(bx)$. An example of an identity for the connection coefficients is the multivariable q-Pfaff–Saalschütz sum

$$\sum_v \frac{(a)_v}{(c)_v}\, \mathsf{E}_{u/v}(a,b)\mathsf{E}_{v/w}(b,c) = \frac{(a)_w(b)_u}{(b)_w(c)_u}\, \mathsf{E}_{u/w}(a,c).$$

Upon symmetrisation our identities for the function $\mathsf{E}_{u/v}(a,b)$ generalise multiple series studied by the second author in [26] in work on BC_n-symmetric polynomials.

In Part II, containing Sections 7–10, we define the $\mathfrak{gl}_n$ basic hypergeometric series and prove a number of important results, such as a multiple analogues of the q-binomial and q-Gauss sums for interpolation Macdonald polynomials. By taking the top-homogeneous components of $\mathfrak{gl}_n$ identities we also obtain results for $\mathfrak{sl}_n$ basic hypergeometric functions involving the nonsymmetric (homogeneous) Macdonald polynomials $E_u(x)$.

2. Compositions

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. In this paper the letters u, v, w will be reserved for compositions of length n, i.e., $u = (u_1, \dots, u_n) \in \mathbb{N}^n$. Occasionally, when more than three compositions are required in one equation, we also use $\bar{u}, \bar{v}, \bar{w}$, where u and $\bar{u}$ are understood to be independent. For brevity the trivial composition $(0, \dots, 0) \in \mathbb{N}^n$ will be written simply as 0. The sum of the parts of the composition u is denoted by $|u|$, i.e., $|u| = u_1 + \cdots + u_n$.

A composition is called *dominant* if $u_i \geq u_{i+1}$ for all $1 \leq i \leq n-1$, in other words, if u is a *partition*. As is customary, we often use the Greek letters λ, μ and ν to denote partitions — all of which are in $\mathbb{N}^n$ in this paper (by attaching strings of zeros if necessary).

The symmetric group $\mathfrak{S}_n$ acts on compositions by permuting the parts. The unique partition in the $\mathfrak{S}_n$-orbit of u is denoted by u^+. The staircase partition δ is defined as $\delta = (n-1, n-2, \dots, 1, 0)$, and t^δ is shorthand for $(t^{n-1}, \dots, t, 1)$. More generally we write $a^u b^v = (a^{u_1} b^{v_1}, \dots, a^{u_n} b^{v_n})$ for a, b scalars and u, v compositions.

Given a composition u, we define its *spectral vector* $\langle u \rangle$ by

$$\langle u \rangle := q^u t^{\delta \sigma_u}, \tag{2.1}$$

where $\sigma_u \in \mathfrak{S}_n$ is the unique permutation of minimal length such that $u = u^+ \sigma_u$. Note that $\langle \lambda \rangle = q^\lambda t^\delta$ and $\langle 0 \rangle = t^\delta$. Less formally, $\langle u \rangle$ is the unique permutation

$$(q^{u_1} t^{k_1}, q^{u_2} t^{k_2}, \dots, q^{u_n} t^{k_n})$$

of $q^{u^+}t^\delta$ such that, if $u_i = u_j$ for $i < j$, then $k_i > k_j$. For example, the spectral vector of $u = (2, 4, 2, 0, 1, 2, 1)$ is given by

$$\langle u \rangle = (q^2 t^5, q^4 t^6, q^2 t^4, 1, qt^2, q^2 t^3, qt).$$

The diagram of the composition u is the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq i \leq n$ and $1 \leq j \leq u_i$. We write $v \subseteq u$ if the diagram of v is contained in the diagram of u (i.e., if $v_i \leq u_i$ for all $1 \leq i \leq n$) and $v \subset u$ if $v \subseteq u$ and $v \neq u$. We follow Knop [10] in defining a partial order $\preceq$ on compositions refining the above containment (partial) order. Given two compositions $u, v \in \mathbb{N}^n$, $v \preceq u$ if there exists a permutation $w \in \mathfrak{S}_n$ such that $v_i < u_{w_i}$ if $i < w_i$ and $v_i \leq u_{w_i}$ if $i \geq w_i$. For example $(2, 1, 2) \preceq (1, 3, 2)$ but $(2, 2, 1) \not\preceq (1, 3, 2)$.

For $s = (i, j)$ in the diagram of u, the arm-length $a(s)$, arm-colength $a'(s)$, leg-length $l(s)$ and leg-colength $l'(s)$ are given by [10, 30]

$$a(s) = u_i - j, \qquad a'(s) = j - 1$$

and

$$l(s) = |\{k > i : \ j \leq u_k \leq u_i\}| + |\{k < i : \ j \leq u_k + 1 \leq u_i\}|$$
$$l'(s) = |\{k > i : \ u_k > u_i\}| + |\{k < i : \ u_k \geq u_i\}|.$$

For dominant compositions these last two definitions reduce to the usual leg-length and leg-colength for partitions [18].

Since $l'(s)$ only depends on the row-coordinate of s, we will also write $l'(i)$. We may then use the above definition for empty rows (i.e., rows such that $u_i = 0$) as well. It is easily seen that

$$\langle u \rangle_i = q^{u_i} t^{n-1-l'(i)}. \tag{2.2}$$

Two statistics on compositions frequently used in this paper are

$$n(u) = \sum_{s \in u} l(s) \qquad \text{and} \qquad n'(u) = \sum_{s \in u} a(s) = \sum_{i=1}^{n} \binom{u_i}{2}.$$

Obviously, $n'(u) = n'(u^+)$. Moreover, if λ' denotes the conjugate of the partition λ, then $n'(\lambda) = n(\lambda')$. The main reason for introducing $n'(u)$ is thus to avoid typesetting $n((u^+)')$ in many of our formulae.

Throughout this paper $q \in \mathbb{C}$ is fixed such that $|q| < 1$. Then

$$(b)_\infty = (b; q)_\infty := \prod_{i=0}^{\infty} (1 - bq^i)$$

and

$$(b)_k = (b; q)_k := \frac{(b)_\infty}{(bq^k)_\infty} = \prod_{i=0}^{k-1} (1 - bq^i)$$

are the standard q-shifted factorials [6]. The last equation is extended to compositions u by

$$(b)_u = (b; q, t)_u := \prod_{s \in u} (1 - b\, q^{a'(s)} t^{-l'(s)}). \tag{2.3}$$

Note that this is invariant under permutations of u

$$(b)_u = (b)_{u^+} = \prod_{i=1}^{n} (bt^{1-i}; q)_{u_i^+}.$$

Alternatively, by (2.2), we can write

$$(bt^{n-1})_u = \prod_{i=1}^{n} \frac{(b\langle 0 \rangle_i)_\infty}{(b\langle u \rangle_i)_\infty}. \tag{2.4}$$

We also employ condensed notation for (generalised) q-shifted factorials, setting

$$(a_1, a_2, \ldots, a_N)_u = \prod_{i=1}^{N} (a_i)_u.$$

A special role in the theory of basic hypergeometric series is played by the q-shifted factorial $(q)_k$. In the multivariable theory this role is played not by $(q)_u$, but by

$$c_u' = c_u'(q, t) := \prod_{s \in u} (1 - q^{a(s)+1} t^{l(s)}).$$

Occasionally we also need the related functions

$$c_u = c_u(q, t) := \prod_{s \in u} (1 - q^{a(s)} t^{l(s)+1})$$

and

$$b_u = b_u(q, t) := \frac{c_u}{c_u'}. \tag{2.5}$$

For partitions these are standard in Macdonald polynomial theory, see [18].

3. Interpolation Macdonald polynomials

Let $x = (x_1, \ldots, x_n)$. The definition of the interpolation Macdonald polynomial $M_u(x) = M_u(x; q, t)$ is deceptively simple [10, 11, 13, 29, 31]. It is the unique polynomial of degree $|u|$ such that

$$M_u(\langle v \rangle) = 0 \quad \text{for } |v| \le |u|,\ u \ne v \tag{3.1}$$

and such that the coefficient of x^u is $q^{-n'(u)}$. Note that in the one-variable case

$$M_u(x) = q^{-\binom{u}{2}} N_u(x),$$

with $N_u(x)$ the Newton polynomial (1.2). Comparing the above definition with those of Knop (denoted E_u in [10]) and Sahi (denoted G_u in [31]) we find that

$$M_u(x) = q^{-n'(u)}t^{(n-1)|u|}F_u(xt^{1-n}), \qquad F = E, G.$$

One of the key results in the theory is that the polynomials M_u can be computed recursively in much the same way as the Newton interpolation polynomials. Let $s_i \in \mathfrak{S}_n$ be the elementary transposition interchanging the variables x_i and x_{i+1}. Then the operator T_i (acting on Laurent polynomials in x) is defined as the unique operator that commutes with functions symmetric in x_i and x_{i+1}, such that

$$1T_i = t \qquad \text{and} \qquad x_{i+1}T_i = x_i.$$

More explicitly,

$$T_i = t + (s_i - 1)\frac{tx_{i+1} - x_i}{x_{i+1} - x_i}.$$

It may readily be verified (see e.g., [14, 16]) that the T_i for $1 \le i \le n-1$ satisfy the defining relations of the Hecke algebra of the symmetric group

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},$$

$$T_iT_j = T_jT_i \quad \text{for } |i - j| \ne 1,$$

$$(T_i - t)(T_i + 1) = 0.$$

To describe the recursion we also require the operator τ

$$f(x)\tau := f(x_n/q, x_1, \ldots, x_{n-1})$$

and the raising operator ϕ

$$f(x)\phi := f(x\tau)(x_n - 1).$$

According to [9, Theorems 4.1 & 4.2] and [29, Corollary 4.4 & Theorem 4.5] the interpolation Macdonald polynomials can now be computed as follows:

$$M_0(x) = 1,$$

$$M_{(u_2,\ldots,u_{n-1},u_1+1)}(x) = M_u(x)\phi$$

and

$$M_{us_i}(x) = M_u(x)\left(T_i + \frac{t-1}{\langle u\rangle_{i+1}/\langle u\rangle_i - 1}\right) \qquad \text{for } u_i < u_{i+1}.$$

For a more general view of this recursive construction in terms of Yang–Baxter graphs we refer the reader to [11].

A third description of the M_u requires Knop's generalised Cherednik operators Ξ_i [9, 10]

$$\Xi_i := t^{1-n} T_{i-1} \cdots T_1 \tau (x_n - 1) T_{n-1} \cdots T_i \frac{1}{x_i} + \frac{1}{x_i}$$

$$= t^{1-i} T_{i-1} \cdots T_1 \tau \left(1 - \frac{1}{x_n} \right) T_{n-1}^{-1} \cdots T_i^{-1} + \frac{1}{x_i}$$

for $1 \leq i \leq n$. The Ξ_i are mutually commuting and the interpolation Macdonald polynomials may be shown to be simultaneous eigenfunctions of the Ξ_i. Specifically, [10, Theorem 3.6]

$$M_u(x)\Xi_i = M_u(x)/\langle u \rangle_i.$$

Observe that the T_i (and hence their inverses) and τ are degree preserving operators. The top-homogeneous degree $M^t(x)$ of $M_u(x)$ thus satisfies

$$M_u^t(x)Y_i^{-1} = M_u^t(x)/\langle u \rangle_i,$$

where

$$Y_i^{-1} = t^{1-i} T_{i-1} \cdots T_1 \tau T_{n-1}^{-1} \cdots T_i^{-1}.$$

Since the Y_i are precisely the Cherednik operators [4], which have the nonsymmetric Macdonald polynomials $E_u(x) = E(x; q, t)$ as simultaneous eigenfunctions with eigenvalues $\langle u \rangle_i$, it follows the top-homogeneous component of $M_u(x)$ is given by $E_u(x)$ [10, Theorem 3.9]. To be more precise, since the coefficient of x^u in $E_u(x)$ is 1, it follows that

$$E_u(x) = q^{n'(u)} \lim_{a \to 0} a^{|u|} M_u(x/a). \tag{3.2}$$

The $\mathfrak{gl}_n$ basic hypergeometric series studied in Part II of this paper contain the Macdonald interpolation polynomials as key-ingredient. In developing the theory we also frequently need specific knowledge about $E_u(x)$. Both these functions almost exclusively occur in combination with c'_u, and it will be convenient to define the normalised Macdonald polynomials

$$\mathsf{M}_u(x) := q^{n'(u)} t^{n(u)} \frac{M_u(x)}{c'_u} \tag{3.3}$$

and

$$\mathsf{E}_u(x) := t^{n(u)} \frac{E_u(x)}{c'_u}.$$

Note that in the one-variable case

$$\mathsf{M}_u(x) = x^u \frac{(1/x)_u}{(q)_u} \qquad \text{and} \qquad \mathsf{E}_u(x) = \frac{x^u}{(q)_u}, \quad u \in \mathbb{N}. \tag{3.4}$$

Also, from (3.2),

$$\mathsf{E}_u(x) = \lim_{a \to 0} a^{|u|} \mathsf{M}_u(x/a). \tag{3.5}$$

We use repeatedly (and implicitly) in subsequent sections that $\{\mathsf{M}_u(x) : |u| \le k\}$ (resp. $\{\mathsf{E}_u(x) : |u| = k\}$) forms a $\mathbb{Q}(q, t)$-basis in the space of polynomials of degree $\le k$ (exactly k) in n variables.

On several occasions we also need the symmetric analogues of M_u and E_u, denoted MS_λ and P_λ, respectively. P_λ is of course the Macdonald polynomial as originally introduced by Macdonald [17, 18]. Using the same normalisation as before, i.e.,

$$\mathsf{MS}_\lambda(x) = q^{n'(\lambda)} t^{n(\lambda)} \frac{MS_\lambda(x)}{c'_\lambda} \qquad \text{and} \qquad \mathsf{P}_\lambda(x) = t^{n(\lambda)} \frac{P_\lambda(x)}{c'_\lambda},$$

the symmetric polynomials are given by

$$\mathsf{MS}_\lambda(x) := \sum_{u^+ = \lambda} \mathsf{M}_u(x) \qquad \text{and} \qquad \mathsf{P}_\lambda(x) := \sum_{u^+ = \lambda} \mathsf{E}_u(x). \tag{3.6}$$

(This is not the standard way to define the symmetric polynomials but provides the most useful description for our purposes.)

A final result about the interpolation polynomials needed subsequently is Sahi's principal specialisation formula [31, Theorem 1.1]. Most convenient will be to normalise Sahi's formula by Cherednik's principal specialisation formula for $\mathsf{E}_u(x)$ [5, Main Theorem]. Then

$$\mathsf{M}_u(z\langle 0 \rangle) = (1/z)_u \mathsf{E}_u(z\langle 0 \rangle). \tag{3.7}$$

For later reference we state the $z \to 0$ limit of this separately

$$\mathsf{M}_u(0) = \tau_u \mathsf{E}_u(\langle 0 \rangle), \tag{3.8}$$

where $\mathsf{M}_u(0)$ is shorthand for $\mathsf{M}_u(0, \ldots, 0)$ and $\tau_u = \tau_u(q, t) = \tau_u^{-1}(1/q, 1/t)$ is defined as

$$\tau_u := (-1)^{|u|} q^{n'(u)} t^{-n(u^+)}. \tag{3.9}$$

4. The Okounkov and Sahi binomial formulas

Another important ingredient in the $\mathfrak{gl}_n$ basic hypergeometric series are Sahi's [31] generalised q-binomial coefficients

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}_{q,t} := \frac{\mathsf{M}_v(\langle u \rangle)}{\mathsf{M}_v(\langle v \rangle)}. \tag{4.1}$$

Note that, since $\mathsf{M}_0(x) = 1$,

$$\begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ u \end{bmatrix} = 1.$$

From (3.1) it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \text{if } |u| \le |v|, \, u \ne v. \tag{4.2}$$

A Theorem of Knop [10, Theorem 4.5] states that

$$\begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \text{if } v \npreceq u, \tag{4.3}$$

which in particular implies that

$$\begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \text{if } v^+ \nsubseteq u^+. \tag{4.4}$$

Using (1.2) it follows that in the one-variable case (4.1) reduces to the definition of the classical q-binomial coefficients

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{(q^{u-v+1})_v}{(q)_v}, \quad u, v \in \mathbb{N}, \tag{4.5}$$

known to be polynomials in q with positive integer coefficients. For general n and generic u and v the generalised q-binomial coefficients are, however, rational functions in q and t. For example, if u and

$$u^{(k)} := (u_1, \ldots, u_{k-1}, u_k + 1, u_{k+1}, \ldots, u_n)$$

are both dominant, then [13]

$$\begin{bmatrix} u^{(k)} \\ u \end{bmatrix} = \frac{1 - q^{u_k+1} t^{n-k}}{1 - q} \prod_{i=1}^{k-1} \frac{1 - q^{u_k - u_i} t^{i-k-1}}{1 - q^{u_k - u_i} t^{i-k}} \prod_{i=k+1}^{n} \frac{1 - q^{u_k - u_i + 1} t^{i-k-1}}{1 - q^{u_k - u_i + 1} t^{i-k}}.$$

Since [13, page 14]

$$\mathsf{M}_u(\langle u \rangle) = \tau_u \, t^{(n-1)|u|}$$

definition (4.1) can also be written as

$$\mathsf{M}_v(\langle u \rangle) = \tau_v \, t^{(n-1)|v|} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{4.6}$$

In [31] Sahi proved a binomial formula for the interpolation Macdonald polynomials $\mathsf{M}_u(x)$ that will be of importance later. In fact, we will be needing slightly more than what may be found in Sahi's paper, and for clarity's sake it is best to first recall the analogous results obtained by Okounkov [23] for the symmetric interpolation polynomials $\mathsf{MS}_\lambda(x)$.

Before we can state Okounkov's theorem we need the symmetric analogue of the generalised q, t-binomial coefficients (4.1), introduced independently by Lassalle and Okounkov [15, 23]

$$\binom{\lambda}{\mu} = \binom{\lambda}{\mu}_{q,t} := \frac{\mathsf{MS}_\mu(\langle \lambda \rangle)}{\mathsf{MS}_\mu(\langle \mu \rangle)}.$$

(Lassalle's definition is quite different but may in fact be shown to be equivalent to the above.) By symmetrisation it easily follows that for any composition u in the $\mathfrak{S}_n$-orbit of λ (i.e., any u such that $u^+ = \lambda$) [3]

$$\binom{\lambda}{\mu} = \sum_{v^+=\mu} \begin{bmatrix} u \\ v \end{bmatrix}.$$

For later comparison with the nonsymmetric case it will also be useful to define

$$\mathsf{MS}'_\lambda(x) = \mathsf{MS}'_\lambda(x; q, t) := \tau_\lambda^{-1} (q^{-1} t^{n-1})^{|\lambda|} \mathsf{MS}_\lambda(t^{1-n} x; q^{-1}, t^{-1}). \tag{4.7}$$

Theorem 4.1 (Okounkov's binomial formula). *For λ a partition of at most* *n parts*

$$\mathsf{MS}_\lambda(ax) = \sum_\mu a^{|\mu|} \binom{\lambda}{\mu}_{q^{-1},t^{-1}} \frac{\mathsf{MS}_\lambda(a\langle 0\rangle)}{\mathsf{MS}_\mu(a\langle 0\rangle)} \mathsf{MS}'_\mu(x).$$

Okounkov's theorem has a certain inversion symmetry as follows. Upon replacing $(a, x, q, t) \mapsto (1/a, axt^{1-n}, 1/q, 1/t)$ (note that $\langle 0\rangle|_{(q,t)\mapsto(1/q,1/t)} = t^{1-n}\langle 0\rangle$) and using (4.7) as well as [23, page 537]

$$\mathsf{MS}_\mu(a^{-1}t^{1-n}\langle 0\rangle; q^{-1}, t^{-1}) = (a^{-1}qt^{1-n})^{|\mu|}\mathsf{MS}_\mu(a\langle 0\rangle),$$

one finds

$$a^{|\lambda|}\mathsf{MS}'_\lambda(x) = \sum_\mu \frac{\tau_\mu}{\tau_\lambda} \binom{\lambda}{\mu} \frac{\mathsf{MS}_\lambda(a\langle 0\rangle)}{\mathsf{MS}_\mu(a\langle 0\rangle)} \mathsf{MS}_\mu(ax). \tag{4.8}$$

Replacing $(\lambda, \mu) \mapsto (\mu, \nu)$, multiplying by $\binom{\lambda}{\mu}_{q^{-1},t^{-1}}\mathsf{MS}_\lambda(a\langle 0\rangle)/\mathsf{MS}_\mu(a\langle 0\rangle)$ and then summing over μ, results in

$$\mathsf{MS}_\lambda(ax) = \sum_{\mu,\nu} \frac{\tau_\nu}{\tau_\mu} \binom{\lambda}{\mu}_{q^{-1},t^{-1}} \binom{\mu}{\nu} \frac{\mathsf{MS}_\lambda(a\langle 0\rangle)}{\mathsf{MS}_\nu(a\langle 0\rangle)} \mathsf{MS}_\nu(ax),$$

where the sum over μ on the left has been performed using Okounkov's binomial formula. Equating coefficients of $\mathsf{MS}_\lambda(ax)$ (and replacing $(q,t) \mapsto (1/q, 1/t)$) results in the orthogonality relation [23, page 540]

$$\sum_\mu \frac{\tau_\mu}{\tau_\lambda} \binom{\lambda}{\mu} \binom{\mu}{\nu}_{q^{-1},t^{-1}} = \delta_{\lambda\nu}. \tag{4.9}$$

An alternative viewpoint is that Theorem 4.1 may be inverted using (4.9) but that, up to a change of parameters, this inverted form is equivalent to the original one. As we shall see shortly, this inversion symmetry is absent in Sahi's nonsymmetric analogue of Theorem 4.1.

Before we can state this result we need to introduce the nonsymmetric analogue of the function $\mathsf{MS}'_\lambda(x)$ as follows. Extend definition (2.1) to all integral vectors $I \in \mathbb{Z}^n$ by $\langle I\rangle := q^I t^{\delta \sigma_I}$, where $\sigma_I \in \mathfrak{S}_n$ is the unique permutation of minimal length such that $I = I^+\sigma_u$. Here I^+ is a dominant integral vector, i.e., $I_i^+ \geq I_{i+1}^+$ for all i. Then $\mathsf{M}'_u(x) = \mathsf{M}'_u(x; q, t)$ is the unique polynomial such that the top-homogeneous terms (i.e., the terms of degree $|u|$) of $\mathsf{M}'_u(x)$ and $\mathsf{M}_u(x)$ coincide (and are thus given by $\mathsf{E}_u(x)$), and such that

$$\mathsf{M}'_u(\langle -v\rangle) = 0 \qquad \text{for } |v| < |u|.$$

We remark that, generally, $\mathsf{M}'_u(\langle -v\rangle) \neq 0$ if $|v| = |u|$. Unlike the symmetric case, $\mathsf{M}'_u(x)$ does not simply follow from $\mathsf{M}_u(x)$ by the map $(q,t) \mapsto (1/q, 1/t)$.

Theorem 4.2 (Sahi's binomial theorem). *For* $u \in \mathbb{N}^n$

$$\mathsf{M}_u(ax) = \sum_v a^{|v|} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1}, t^{-1}} \frac{\mathsf{M}_u(a\langle 0 \rangle)}{\mathsf{M}_v(a\langle 0 \rangle)} \mathsf{M}'_v(x).$$

Replacing $x \mapsto x/a$ and letting a tend to zero using

$$\lim_{a \to 0} a^{|u|} \mathsf{M}'_u(x/a) = \mathsf{E}_u(x)$$

and (3.8) expresses $\mathsf{M}_u(x)$ in terms of $\mathsf{E}_v(x)$.

Corollary 4.3. *For* $u \in \mathbb{N}^n$

$$\mathsf{M}_u(x) = \sum_v \frac{\tau_u}{\tau_v} \frac{\mathsf{E}_u(\langle 0 \rangle)}{\mathsf{E}_v(\langle 0 \rangle)} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1}, t^{-1}} \mathsf{E}_v(x). \tag{4.10}$$

In the absence of a non-symmetric analogue of (4.7) Sahi's result lacks the inversion symmetry of Theorem 4.1, and by only minor changes of the proof given in [31] one can also obtain the nonsymmetric version of (4.8).

Theorem 4.4. *For* $u \in \mathbb{N}^n$

$$a^{|u|} \mathsf{M}'_u(x) = \sum_v \frac{\tau_v}{\tau_u} \begin{bmatrix} u \\ v \end{bmatrix} \frac{\mathsf{M}_u(a\langle 0 \rangle)}{\mathsf{M}_v(a\langle 0 \rangle)} \mathsf{M}_v(ax).$$

The key to the proof is the following dual version of Sahi's reciprocity theorem [31, Theorem 1.2]: there exists a unique polynomial $\mathsf{R}_v(x)$ of degree not exceeding $|v|$ with coefficients in $\mathbb{Q}(q, t, a)$, such that

$$\mathsf{R}_v(\langle u \rangle) = \tau_u \, a^{|u|} \frac{\mathsf{M}'_u(\langle v \rangle/a)}{\mathsf{M}_u(a\langle 0 \rangle)}$$

for all $u \in \mathbb{N}^n$. From this one can compute the coefficients c_{uv} in

$$\tau_u \, a^{|u|} \frac{\mathsf{M}'_u(x/a)}{\mathsf{M}_u(a\langle 0 \rangle)} = \sum_v c_{uv} \mathsf{M}_v(x),$$

see [31] for more details.

Of course, the consistency of the last two theorems dictates that

$$\sum_v \frac{\tau_v}{\tau_u} \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}_{q^{-1}, t^{-1}} = \delta_{uw}. \tag{4.11}$$

Theorem 4.4 and equation (4.11) each imply the inverse of (4.10).

Corollary 4.5. *For* $u \in \mathbb{N}^n$

$$\mathsf{E}_u(x) = \sum_v \frac{\mathsf{E}_u(\langle 0 \rangle)}{\mathsf{E}_v(\langle 0 \rangle)} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_v(x). \tag{4.12}$$

5. One-variable basic hypergeometric series

Suppressing the q-dependence, the standard notation for single-variable basic hypergeometric series is [6]

$$
{}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};x\right] := \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r)_k}{(q,b_1,\ldots,b_s)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r+1} x^k. \tag{5.1}
$$

A series is called *terminating* if only a finite number of terms contribute to the sum, for example if $a_r = q^{-m}$ with m a nonnegative integer.

Nearly all important summation and transformation formulas concern ${}_{r+1}\phi_r$ series. A ${}_{r+1}\phi_r$ series such that

$$
a_1 \cdots a_{r+1}x = b_1 \cdots b_r, \tag{5.2a}
$$
$$
x = q \tag{5.2b}
$$

is called *balanced*. For reasons that will become clear in Part II we somewhat relax this terminology and refer to a ${}_{r+1}\phi_r$ series as balanced if (5.2a) holds but not necessarily (5.2b). (On page 70 of [6] a series satisfying (5.2a) is referred to as *a series of type II*, but we prefer a somewhat more descriptive adjective.)

In the next section and in Part II we will generalise most of the simple summation and transformation formulas for ${}_{r+1}\phi_r$ series, and for later reference we list a number of one-variable series below. The reader may find proofs of all of the identities in the book by Gasper and Rahman [6].

q-Binomial theorem. The q-binomial theorem [6, Equation (III.3)] is one of the most famous identities for basic hypergeometric series, discovered around 1850 by a number of mathematicians including Cauchy and Heine,

$$
{}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix};x\right] = \frac{(ax)_\infty}{(x)_\infty}, \qquad |x| < 1. \tag{5.3}
$$

q-Gauss sum. The q-Gauss sum is a generalisation of the q-binomial theorem due to Heine [6, Equation (II.8)]

$$
{}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};\frac{c}{ab}\right] = \frac{(c/a,c/b)_\infty}{(c,c/ab)_\infty}, \qquad |c/ab| < 1. \tag{5.4}
$$

In our terminology, the series on the left is balanced. To obtain the q-binomial theorem it suffices to replace $b \mapsto c/ax$ and take the $c \to 0$ limit.

q-Chu–Vandermonde sums. The first q-Chu–Vandermonde sum is simply the terminating version of the q-Gauss sum [6, Equation (II.7)]

$$
{}_2\phi_1\left[\begin{matrix} a,q^{-k} \\ c \end{matrix};\frac{cq^k}{a}\right] = \frac{(c/a)_k}{(c)_k}. \tag{5.5}
$$

By replacing $q \mapsto 1/q$ or by reversing the order of summation a second q-Chu–Vandermonde sum follows [6, Equation (II.6)]

$$
{}_2\phi_1\left[\begin{matrix} a, q^{-k} \\ c \end{matrix}; q\right] = \frac{(c/a)_k}{(c)_k}\, a^k.
\tag{5.6}
$$

The binomial formulas of Okounkov and Sahi are easily seen to be generalisations of the q-Chu–Vandermonde sums. For example, Theorem 4.2 for $n = 1$ is (5.6) with $(a, c, k) \mapsto (x, 1/a, u)$ and Theorem 4.4 for $n = 1$ is (5.5) with $(a, c, k) \mapsto (1/ax, 1/a, u)$.

q-Pfaff–Saalschütz sum. At the top of the summations considered in this paper is the q-Pfaff–Saalschütz sum, first derived by Jackson and generalising the q-Gauss sum [6, Equation (II.12)],

$$
{}_3\phi_2\left[\begin{matrix} a, b, q^{-k} \\ c, abq^{1-k}/c \end{matrix}; q\right] = \frac{(c/a, c/b)_k}{(c, c/ab)_k}.
\tag{5.7}
$$

This sum is balanced in the traditional sense and yields the q-Gauss in the large k limit.

Heine's transformations. There are three Heine transformations for ${}_2\phi_1$ series. Of interest in this paper are only two of these. The first one is [6, Equation (III.2)]

$$
{}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \frac{(c/a, ax)_\infty}{(c, x)_\infty}\, {}_2\phi_1\left[\begin{matrix} a, abx/c \\ ax \end{matrix}; \frac{c}{a}\right]
\tag{5.8}
$$

for $\max\{|x|, |c/a|\} < 1$, and simplifies to the q-Gauss sum (5.4) when the balancing condition $x = c/ab$ is imposed. The second one is Heine's q-analogue of Euler's ${}_2F_1$ transformation [6, Equation (III.3)]

$$
{}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \frac{(abx/c)_\infty}{(c)_\infty}\, {}_2\phi_1\left[\begin{matrix} c/a, c/b \\ c \end{matrix}; \frac{abx}{c}\right]
\tag{5.9}
$$

for $\max\{|x|, |abx/c|\} < 1$, and simplifies to the q-binomial theorem (5.3) when $c = b$.

q-Kummer–Thomae–Whipple formula. Sears' q-analogue of the Kummer–Thomae—Whipple formula for ${}_2F_1$ series is given by [6, Equation (III.9)]

$$
{}_3\phi_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}; \frac{f}{a}\right] = \frac{(e/a, f)_\infty}{(f/a, e)_\infty}\, {}_3\phi_2\left[\begin{matrix} a, d/b, d/c \\ d, f \end{matrix}; \frac{e}{a}\right],
\tag{5.10}
$$

where $f = de/bc$ and $\max\{|e/a|, |f/a|\} < 1$. Note that both ${}_3\phi_2$ series are balanced, and that the limits $c, d \to \infty$ (such that $de/abc = x$) and $a, e \to 0$ (such that $de/abc = x$) yield the Heine transformations (5.8) and (5.9) respectively.

Sears' ${}_4\phi_3$ transformation. The most general result in our list is Sears' ${}_4\phi_3$ transformation formula [6, Equation (III.15)]

$$
{}_4\phi_3\left[\begin{matrix} a, b, c, q^{-k} \\ d, e, f \end{matrix}; q\right] = \frac{(e/a, f/a)_k}{(e, f)_k}\, a^n\, {}_4\phi_3\left[\begin{matrix} a, d/b, d/c, q^{-k} \\ d, aq^{1-n}/e, aq^{1-n}/f \end{matrix}; q\right]
\tag{5.11}
$$

for $abc = defq^{k-1}$. In the large k limit this yields (5.10) whereas for $d = c$ it simplifies to the q-Pfaff–Saalschütz sum (5.7).

6. The function $\mathsf{E}_{u/v}(a, b)$

A final ingredient needed in our study of $\mathfrak{gl}_n$ basic hypergeometric series are certain normalised connection coefficients between interpolation Macdonald polynomials.

Definition 6.1. *For $u, v \in \mathbb{N}^n$ let the connection coefficients $c_{uv}(a, b) = c_{uv}(a, b; q, t)$ be given by*

$$\mathsf{M}_u(ax) = \sum_v c_{uv}(a, b)\mathsf{M}_v(bx). \tag{6.1}$$

Then

$$\mathsf{E}_{u/v}(a, b) = \mathsf{E}_{u/v}(a, b; q, t) := \frac{\mathsf{E}_v(\langle 0 \rangle /a)}{\mathsf{E}_u(\langle 0 \rangle /b)}\, c_{uv}(a, b). \tag{6.2}$$

From the definition of the coefficients $c_{uv}(a, b)$ it immediately follows that

$$c_{uv}(a, b) = c_{uv}(ac, bc).$$

Hence

$$\mathsf{E}_{u/v}(ac, bc) = c^{|u|-|v|}\mathsf{E}_{u/v}(a, b). \tag{6.3}$$

Two other easy consequence of the definition are

$$\lim_{a \to b} \mathsf{E}_{u/v}(a, b) = \delta_{uv} \tag{6.4}$$

and the orthogonality relation

$$\sum_v \mathsf{E}_{u/v}(a, b)\mathsf{E}_{v/w}(b, a) = \delta_{uw}. \tag{6.5}$$

By (3.4) and (5.5) with $(a, c, k) \mapsto (1/bx, q^{1-u}a/b, u)$ it can be shown that

$$\mathsf{E}_{u/v}(1, b) = (b)_{u-v} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{for } u, v \in \mathbb{N}. \tag{6.6}$$

In general such a simple factorisation does not hold, although some of the features of (6.6) lift to $n > 1$.

Lemma 6.1. *For $u, v \in \mathbb{N}^n$*

$$\mathsf{E}_{u/v}(1, 0) = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{6.7a}$$

$$\mathsf{E}_{u/v}(0, 1) = \frac{\tau_u}{\tau_v} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1}, t^{-1}} \tag{6.7b}$$

and

$$\mathsf{E}_{u/0}(1, b) = (b)_u. \tag{6.7c}$$

The connection coefficient $\mathsf{E}_{u/v}(a, b)$ thus interpolates between generalised q-shifted factorials and generalised q-binomial coefficients.

Proof. From (6.1) and (6.2) with $(a, b) \mapsto (1/a, 1)$

$$a^{|u|} \mathsf{M}_u(x/a) = \sum_v \frac{\mathsf{E}_u(\langle 0 \rangle)}{\mathsf{E}_v(\langle 0 \rangle)} \mathsf{E}_{u/v}(1, a) \mathsf{M}_v(x),$$

where we have also used (6.3) to replace $\mathsf{E}_{u/v}(1/a, 1)$ by $a^{|v|-|u|} \mathsf{E}_{u/v}(1, a)$. By (3.5) we can take the $a \to 0$ limit, giving

$$\mathsf{E}_u(x) = \sum_v \frac{\mathsf{E}_u(\langle 0 \rangle)}{\mathsf{E}_v(\langle 0 \rangle)} \mathsf{E}_{u/v}(1, 0) \mathsf{M}_v(x).$$

Comparing this with (4.12) completes the proof of (6.7a).

From (6.1) and (6.2) with $(a, b) \mapsto (1, 1/a)$

$$\mathsf{M}_u(x) = \sum_v a^{|v|} \frac{\mathsf{E}_u(\langle 0 \rangle)}{\mathsf{E}_v(\langle 0 \rangle)} \mathsf{E}_{u/v}(a, 1) \mathsf{M}_v(x/a),$$

where we have also used (6.3) to replace $\mathsf{E}_{u/v}(1, 1/a)$ by $a^{|v|-|u|} \mathsf{E}_{u/v}(a, 1)$. Taking the $a \to 0$ limit using (3.5) yields

$$\mathsf{M}_u(x) = \sum_v \frac{\mathsf{E}_u(\langle 0 \rangle)}{\mathsf{E}_v(\langle 0 \rangle)} \mathsf{E}_{u/v}(0, 1) \mathsf{E}_v(x).$$

Comparing this with (4.10) results in (6.7b).

If we take $x = \langle 0 \rangle / ab$ in (6.1) and use the principal specialisation formula (3.7) we get

$$(b)_u = \sum_v \mathsf{E}_{u/v}(a, b)(a)_v. \tag{6.8}$$

Letting a tend to 0 using $\lim_{a \to 0}(a)_v = \delta_{v,0}$ results in (6.7c).

The identity (6.8) is the $w = 0$ instance of our next result.

Proposition 6.2 (Multiple q-Chu–Vandermonde sum I). *For $u, w \in \mathbb{N}^n$*

$$\sum_v \mathsf{E}_{u/v}(a, b) \mathsf{E}_{v/w}(1, a) = \mathsf{E}_{u/w}(1, b). \tag{6.9}$$

For $n = 1$ this is the q-Chu–Vandermonde sum (5.5) with (c, k) replaced by $(q^{w-u+1}a/b, u - w)$. Corollary 6.5 below shows that we may restrict the sum over v in (6.9) to $w \preceq v \preceq u$.

Needed in our proof of the $\mathfrak{gl}_n$ analogue of the q-Gauss sum in Section 9 is the following special case of (6.9) obtained by substituting $b \to ab$, applying (6.3) and letting $b \to 0$ using 6.7a.

Corollary 6.3. *For $u, w \in \mathbb{N}^n$*

$$\sum_v a^{|u|-|v|} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_{v/w}(1, a) = \begin{bmatrix} u \\ w \end{bmatrix}.$$

Proof of Proposition 6.2. Double use of (6.1) gives

$$\sum_v c_{uv}(a,b)c_{vw}(b,c) = c_{uw}(a,c).$$

Hence, by (6.2),

$$\sum_v \left(\frac{a}{c}\right)^{|v|} \mathsf{E}_{u/v}(a,b)\mathsf{E}_{v/w}(b,c) = \left(\frac{b}{c}\right)^{|u|}\left(\frac{a}{b}\right)^{|w|}\mathsf{E}_{u/w}(a,c).$$

Scaling $(a,b,c) \mapsto (ac, bc, abc)$ and applying (6.3) we obtain the desired result.

Proposition 6.4. *For $u, w \in \mathbb{N}^n$*

$$\mathsf{E}_{u/w}(a,b) = \sum_v \frac{\tau_w}{\tau_v} \frac{(b)_u(a)_v}{(b)_v(a)_w} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1},t^{-1}} \begin{bmatrix} v \\ w \end{bmatrix} \tag{6.10a}$$

$$= \sum_v a^{|v|-|w|}b^{|u|-|v|} \frac{\tau_u}{\tau_v} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1},t^{-1}} \begin{bmatrix} v \\ w \end{bmatrix}. \tag{6.10b}$$

These results not only give explicit expressions for the function $\mathsf{E}_{u/w}(a,b)$ but also show that it is polynomial in a and b, a fact that will be used later in our proof of Proposition 6.9. By (4.3) it also implies the following simple corollary.

Corollary 6.5. *If $v \not\preceq u$ then $\mathsf{E}_{u/v}(a,b) = 0$.*

Proof of Proposition 6.4. By (3.7), (3.8) and (6.2), and by substituting (a,b) by $(1/b, 1/a)$ using

$$\mathsf{E}_{u/v}(b^{-1}, a^{-1}) = (ab)^{|v|-|u|}\mathsf{E}_{u/v}(a,b),$$

the first claim can be stated in equivalent form as

$$c_{uw}(a,b) = \sum_v \frac{\tau_w}{\tau_v} \left(\frac{a}{b}\right)^{|v|} \frac{\mathsf{M}_u(a\langle 0\rangle)\mathsf{M}_v(b\langle 0\rangle)}{\mathsf{M}_v(a\langle 0\rangle)\mathsf{M}_w(b\langle 0\rangle)} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1},t^{-1}} \begin{bmatrix} v \\ w \end{bmatrix}.$$

If we multiply this by $\mathsf{M}_w(bx)$ and sum over w then

$$\sum_w c_{uw}(a,b)\mathsf{M}_w(bx) = \sum_{v,w} \frac{\tau_w}{\tau_v} \left(\frac{a}{b}\right)^{|v|} \frac{\mathsf{M}_u(a\langle 0\rangle)\mathsf{M}_v(b\langle 0\rangle)}{\mathsf{M}_v(a\langle 0\rangle)\mathsf{M}_w(b\langle 0\rangle)} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1},t^{-1}} \begin{bmatrix} v \\ w \end{bmatrix}\mathsf{M}_w(bx)$$

$$= \sum_v a^{|v|} \frac{\mathsf{M}_u(a\langle 0\rangle)}{\mathsf{M}_v(a\langle 0\rangle)} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1},t^{-1}} \mathsf{M}_v'(x)$$

$$= \mathsf{M}_u(ax).$$

Here the second line follows from Theorem 4.4 and the third line from Theorem 4.2.

The second claim follows from the first by substituting $(a,b) \mapsto (ac, bc)$ and then multiplying both sides by $c^{|w|-|u|}$. Using (6.3) the large c limit can now be taken leading to (6.10b).

Our next result generalises the q-Pfaff–Saalschütz sum (5.7), and will be applied in Section 10 to prove a $\mathfrak{gl}_n$ version of the q-Kummer–Thomae–Whipple formula (5.10).

Theorem 6.6 (Multiple q-Pfaff–Saalschütz sum). *For $u, w \in \mathbb{N}^n$*

$$\sum_v \frac{(a)_v}{(c)_v} \, \mathsf{E}_{u/v}(a,b) \mathsf{E}_{v/w}(b,c) = \frac{(a)_w (b)_u}{(b)_w (c)_u} \, \mathsf{E}_{u/w}(a,c). \tag{6.11}$$

When $n = 1$ this is (5.7) with $(a,b,c,k) \mapsto (aq^w, c/b, cq^w, u - w)$, and for $b \to \infty$ this is (recall (6.7b)) (6.10a) with $b \mapsto c$. Again the sum in the above may be restricted to $w \preceq v \preceq u$.

Proof. Twice using (6.10a) we find that

$$\sum_v \frac{(a)_v (c)_w}{(b)_u (d)_v} \, \mathsf{E}_{u/v}(a,b) \mathsf{E}_{v/w}(c,d)$$

$$= \sum_{v, \bar{v}, \bar{w}} \frac{\tau_v \tau_w}{\tau_{\bar{v}} \tau_{\bar{w}}} \frac{(a)_{\bar{v}}}{(b)_{\bar{v}}} \frac{(c)_{\bar{w}}}{(d)_{\bar{w}}} \begin{bmatrix} u \\ \bar{v} \end{bmatrix}_{q^{-1}, t^{-1}} \begin{bmatrix} \bar{v} \\ v \end{bmatrix} \begin{bmatrix} v \\ \bar{w} \end{bmatrix}_{q^{-1}, t^{-1}} \begin{bmatrix} \bar{w} \\ w \end{bmatrix}.$$

Summing over v using (4.11), then performing the trivial sum over $\bar{w}$, and finally replacing $\bar{v}$ by v leads to

$$\sum_v \frac{(a)_v (c)_w}{(b)_u (d)_v} \, \mathsf{E}_{u/v}(a,b) \mathsf{E}_{v/w}(c,d) = \sum_v \frac{\tau_w}{\tau_v} \frac{(a,c)_v}{(b,d)_v} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1}, t^{-1}} \begin{bmatrix} v \\ w \end{bmatrix}.$$

The right-hand side is invariant under the maps $a \leftrightarrow c$ or $b \leftrightarrow d$. Hence,

$$\sum_v \frac{(a)_v (c)_w}{(b)_u (d)_v} \, \mathsf{E}_{u/v}(a,b) \mathsf{E}_{v/w}(c,d) = \sum_v \frac{(c)_v (a)_w}{(b)_u (d)_v} \, \mathsf{E}_{u/v}(c,b) \mathsf{E}_{v/w}(a,d) \tag{6.12a}$$

$$= \sum_v \frac{(c)_v (a)_w}{(d)_u (b)_v} \, \mathsf{E}_{u/v}(c,d) \mathsf{E}_{v/w}(a,b). \tag{6.12b}$$

Setting $b = c$ in (6.12a) using (6.4) completes the proof. $\quad\square$

Several further multiple q-Chu–Vandermonde sums follow as limiting cases of Theorem 6.6.

Corollary 6.7 (Multiple q-Chu–Vandermonde sums II–IV). *For $u, w \in \mathbb{N}^n$*

$$\sum_v \mathsf{E}_{u/v}(1,a) \mathsf{E}_{v/w}(a,b) = \mathsf{E}_{u/w}(1,b) \tag{6.13a}$$

$$\sum_v \frac{\tau_v}{\tau_w} \frac{(b)_u}{(b)_v} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_{v/w}(a,b) = \frac{(a)_u}{(a)_w} \begin{bmatrix} u \\ w \end{bmatrix} \tag{6.13b}$$

and

$$\sum_v b^{|v|-|w|} \frac{(a)_v}{(a)_w} \, \mathsf{E}_{u/v}(a,b) \begin{bmatrix} v \\ w \end{bmatrix} = a^{|u|-|w|} \frac{(b)_u}{(b)_w} \begin{bmatrix} u \\ w \end{bmatrix}. \tag{6.13c}$$

For $n = 1$ (6.13a) reduces to (5.6) with $(a,c,k) \mapsto (b/a, q^{w-u+1}/a, u - w)$, (6.13b) to (5.5) with $(a,c,k) \mapsto (aq^w, q^{w-u+1}a/b, u - w)$ and (6.13c) to (5.6)

with $(a, c, k) \mapsto (b/a, bq^w, u - w)$. We also remark that if we replace $a \mapsto 1/b$ in Theorem 4.4, then specialise $x = a\langle 0 \rangle$ using (3.7) and (6.7c), we obtain

$$\sum_v \frac{\tau_v}{\tau_u} \frac{(b)_u}{(b)_v} \, \mathsf{E}_{u/0}(a, b) \begin{bmatrix} u \\ v \end{bmatrix} = \frac{\mathsf{M}'_u(a\langle 0 \rangle)}{\mathsf{E}_u(\langle 0 \rangle)}.$$

Comparing this with the $w = 0$ case of (6.13b) yields the principal specialisation formula

$$\mathsf{M}'_u(a\langle 0 \rangle) = \tau_u^{-1}(a)_u \mathsf{E}_u(\langle 0 \rangle).$$

Proof of Corollary 6.7. Equation (6.13a) follows by first rescaling the parameters in (6.11) as $(a, b, c) \mapsto (c, ac, bc)$ and then taking the $c \to 0$ limit. Equation (6.13b) is (6.11) in the limit $(a, b, c) \to (\infty, a, b)$, whereas (6.13c) corresponds to the $c \to 0$ limit of (6.11).

Proposition 6.8 (Multiple Sears transformation). *For $u, w \in \mathbb{N}^n$*

$$\sum_v \frac{(aq/b, aq/c)_u (d, e)_v}{(aq/b, aq/c)_v (d, e)_w} \, \mathsf{E}_{u/v}(1, aq/de) \mathsf{E}_{v/w}(aq/de, a^2 q^2/bcde)$$

$$= \sum_v \frac{(aq/b, aq/d)_u (c, e)_v}{(aq/b, aq/d)_v (c, e)_w} \, \mathsf{E}_{u/v}(1, aq/ce) \mathsf{E}_{v/w}(aq/ce, a^2 q^2/bcde). \quad (6.14)$$

When $n = 1$ this is (5.11) with

$$(a, b, c, d, e, f, k) \mapsto (eq^w, dq^w, aq/bc, aq^{w+1}/b, aq^{w+1}/c, q^{w-u}de/a, u - w).$$

Later we shall also encounter the limiting case of (6.14) obtained by taking the $e \to \infty$ limit and relabelling the remaining variables

$$\sum_v \tau_v \frac{(d, e)_u (b)_v}{(d, e)_v (b)_w} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_{v/w}(a, ac) = \sum_v \tau_v \frac{(d, a)_u (d/c)_v}{(d, a)_v (d/c)_w} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_{v/w}(e, ac), \quad (6.15)$$

where $abc = de$.

Proof of Proposition 6.8. First we observe that thanks to (6.3) the q-Pfaff–Saal-schütz sum (6.11) can also be written with an additional parameter d as

$$\sum_v \frac{(a)_v}{(c)_v} \, \mathsf{E}_{u/v}(ad, bd) \mathsf{E}_{v/w}(bd, cd) = \frac{(a)_w (b)_u}{(b)_w (c)_u} \, \mathsf{E}_{u/w}(ad, cd). \quad (6.16)$$

Making the substitutions $(a, b, c, d, u, v) \mapsto (c, d, aq/b, aq/cde, v, \bar{v})$ this leads to

$$\mathsf{E}_{v/w}(aq/de, a^2 q^2/bcde)$$

$$= \frac{(d)_w (aq/b)_v}{(c)_w (d)_v} \sum_{\bar{v}} \frac{(c)_{\bar{v}}}{(aq/b)_{\bar{v}}} \, \mathsf{E}_{v/\bar{v}}(aq/de, aq/ce) \mathsf{E}_{\bar{v}/w}(aq/ce, a^2 q^2/bcde).$$

Inserting the above expansion and interchanging the order of the v and $\bar{v}$ sums we find

$$\text{LHS}(6.14) = \sum_{\bar{v}} \frac{(aq/b, aq/c)_u}{(c,e)_w} \frac{(c)_{\bar{v}}}{(aq/b)_{\bar{v}}} \mathsf{E}_{\bar{v}/w}(aq/ce, a^2q^2/bcde)$$

$$\times \sum_{v} \frac{(e)_v}{(aq/c)_v} \mathsf{E}_{u/v}(1, aq/de)\mathsf{E}_{v/\bar{v}}(aq/de, aq/ce).$$

The sum over v can now be performed by (6.16) with

$$(a, b, c, d) \mapsto (e, aq/d, aq/c, 1/e)$$

resulting in the right-hand side of the multiple Sears transform.

Proposition 6.9 (Duality, type I). *For $k \in \mathbb{N}$ and $u, w \in \mathbb{N}^n$*

$$\sum_{|v|=k} \mathsf{E}_{u/v}(a,b)\mathsf{E}_{v/w}(c,d) = \sum_{|v|=|u|+|w|-k} \mathsf{E}_{u/v}(c,d)\mathsf{E}_{v/w}(a,b)$$

and

$$\sum_{|v|=k} \begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} = \sum_{|v|=|u|+|w|-k} \begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}.$$

Proof. Replace $(a, b, c, d) \to (ae, be, ce, de)$ in (6.12b), use (6.3), and let e tend to 0. Then

$$\sum_{v} \mathsf{E}_{u/v}(a,b)\mathsf{E}_{v/w}(c,d) = \sum_{v} \mathsf{E}_{u/v}(c,d)\mathsf{E}_{v/w}(a,b).$$

From (6.10b) it follows that $\mathsf{E}_{u/v}(a,b)$, when viewed as a polynomial in a and b, is homogeneous of degree $|u| - |v|$. If we therefore read off the term of degree $k + |w|$ in c and d in the above transformation the first claim follows.

The second identity follows by taking $b = d = 0$ and using (6.3) and (6.7a).

Proposition 6.10 (Duality, type II). *For $u, v \in \mathbb{N}^n$*

$$\mathsf{E}_{u/v}(a,b) = \frac{\tau_u}{\tau_v} \mathsf{E}_{u/v}(b, a; q^{-1}, t^{-1}). \tag{6.17}$$

Together with (6.5) this implies a generalisation of (4.11).

Proof. If we take the $b \to 0$ limit in (6.11) using (6.3) and Lemma 6.1, and replace $c \mapsto b$, we obtain

$$\mathsf{E}_{u/w}(a,b) = \frac{a^{|u|}}{b^{|w|}} \frac{(b)_u}{(a)_w} \sum_{v} \frac{\tau_v}{\tau_w} \left(\frac{b}{a}\right)^{|v|} \frac{(a)_v}{(b)_v} \begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}_{q^{-1}, t^{-1}}.$$

If we compare this with (6.10a), use (6.3) and

$$(a)_u = \tau_u\, a^{|u|}(a^{-1}; q^{-1}, t^{-1})_u,$$

the claim follows.

Let $\mathsf{g}_{vw}^u = \mathsf{g}_{vw}^u(q,t)$ be the structure constants of the normalised nonsymmetric Macdonald polynomials

$$\mathsf{E}_v(x)\mathsf{E}_w(x) = \sum_{u} \mathsf{g}_{vw}^u \mathsf{E}_u(x). \tag{6.18}$$

Proposition 6.11. *For $u, v \in \mathbb{N}^n$*

$$\mathsf{E}_{u/v}(a, b) = a^{|u|-|v|} \sum_w (b/a)_w \, \mathsf{g}^u_{vw}. \tag{6.19}$$

For $(a, v) = (1, 0)$ this is (6.7c) since $\mathsf{g}^u_{0,w} = \delta_{uw}$. With some considerable pain (6.19) can be proved using the q-Pfaff–Saalschütz sum (6.11), but since it follows as an easy corollary of Theorem 9.1 we omit a proof here.

We conclude this section with several remarks about the function $\mathsf{E}_{u/v}(a, b)$ and its symmetric counterpart. For this purpose we first introduce λ-ring notation [12], which also plays a crucial role in our proof of the $\mathfrak{gl}_n$ q-binomial theorem in the next section.

For $\mathbb{A}$ an alphabet (i.e., countable set with elements referred to as letters) let $|\mathbb{A}|$ be its cardinality. If we adopt the usual additive notation for alphabets, that is, $\mathbb{A} = \sum_{a \in \mathbb{A}} a$, then the disjoint union and Cartesian product of $\mathbb{A}$ and $\mathbb{B}$ may be written as

$$\mathbb{A} + \mathbb{B} = \sum_{a \in \mathbb{A}} a + \sum_{b \in \mathbb{B}} b \qquad \text{and} \qquad \mathbb{A}\mathbb{B} = \sum_{a \in \mathbb{A}} \sum_{b \in \mathbb{B}} ab.$$

For f a symmetric function, we define

$$f[\mathbb{A}] = f(a_1, a_2, \dots) \quad \text{for } \mathbb{A} = a_1 + a_2 + \cdots,$$

where $[\,,\,]$ are referred to as plethystic brackets. Let

$$\sigma_z[\mathbb{A}] = \prod_{a \in \mathbb{A}} \frac{1}{1 - za}.$$

Then the complete symmetric function $S_k[\mathbb{A}]$ is defined by its generating function as

$$\sigma_z[\mathbb{A}] = \sum_{k=0}^{\infty} z^k S_k[\mathbb{A}]. \tag{6.20}$$

More generally we define the complete symmetric function (and thus any symmetric function) of the difference of two alphabets as

$$\sigma_z[\mathbb{A} - \mathbb{B}] = \frac{\prod_{b \in \mathbb{B}}(1 - zb)}{\prod_{a \in \mathbb{A}}(1 - za)}. \tag{6.21}$$

Hence, if $1/(1 - q)$ denotes the infinite alphabet $1 + q + q^2 + \cdots$,

$$\sigma_z\left[\frac{\mathbb{A} - \mathbb{B}}{1 - q}\right] = \frac{\prod_{b \in \mathbb{B}}(zb)_\infty}{\prod_{a \in \mathbb{A}}(za)_\infty}. \tag{6.22}$$

Below we need (6.22) with $q \mapsto t$ and $\mathbb{A} = a$, $\mathbb{B} = b$ single-letter alphabets. More specifically, we consider the function

$$Q_{\lambda/\mu}\left[\frac{a - b}{1 - t}\right],$$

where $Q_{\lambda/\mu}(x) = Q_{\lambda/\mu}(x; q, t)$ is a skew Macdonald polynomial defined by

$$Q_\lambda[\mathbb{A} + \mathbb{B}] = \sum_\mu Q_{\lambda/\mu}[\mathbb{A}] Q_\mu[\mathbb{B}]$$

and $Q_\lambda(x) = b_\lambda P_\lambda(x)$. If $f_{\mu\nu}^\lambda = f_{\mu\nu}^\lambda(q, t)$ are the q, t-Littlewood–Richardson coefficients

$$P_\mu(x) P_\nu(x) = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda(x),$$

then, by [18, page 344],

$$Q_{\lambda/\mu}(x) = \sum_\nu f_{\mu\nu}^\lambda Q_\nu(x),$$

where b_λ is defined in (2.5). Combining this with [18, page 338]

$$Q_\lambda\left[\frac{1-a}{1-t}\right] = t^{n(\lambda)} \frac{(a)_\lambda}{c_\lambda'},$$

it follows that

$$Q_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] = a^{|\lambda|-|\mu|} \sum_\nu t^{n(\nu)} \frac{(b/a)_\nu}{c_\nu'} f_{\mu\nu}^\lambda, \qquad (6.23)$$

where we have also used the homogeneity of $Q_{\lambda/\mu}(x)$ and the fact that $f_{\mu\nu}^\lambda = 0$ unless $|\lambda| = |\mu| + |\nu|$. If we now define

$$\mathsf{f}_{\mu\nu}^\lambda := t^{n(\mu)+n(\nu)-n(\lambda)} \frac{c_\lambda'}{c_\mu' c_\nu'} f_{\mu\nu}^\lambda,$$

and

$$\mathsf{Q}_{\lambda/\mu}(x) := t^{n(\mu)-n(\lambda)} \frac{c_\lambda'}{c_\mu'} Q_{\lambda/\mu}(x),$$

then (compare with (6.18))

$$\mathsf{P}_\mu(x) \mathsf{P}_\nu(x) = \sum_\lambda \mathsf{f}_{\mu\nu}^\lambda \mathsf{P}_\lambda(x),$$

and

$$\mathsf{Q}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] = a^{|\lambda|-|\mu|} \sum_\nu (b/a)_\nu \, \mathsf{f}_{\mu\nu}^\lambda. \qquad (6.24)$$

Comparing (6.24) with (6.19) we are led to conclude that $\mathsf{E}_{u/v}(a, b)$ is the non-symmetric analogue of $\mathsf{Q}_{\lambda/\mu}[(a - b)/(1 - t)]$. Indeed, from (3.6) it follows that

$$\mathsf{Q}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] = \sum_{v^+ = \mu} \mathsf{E}_{u/v}(a, b),$$

where $u^+ = \lambda$. This in turn implies that the multiple q-Saalschütz sum (6.11) and Sears transformation (6.14) are the nonsymmetric analogues of [26, Corollaries 4.9 & 4.8] respectively.

Up to trivial factors the generalised q-binomial coefficients arise as evaluations of the interpolation Macdonald polynomials $M_u(x)$, see (4.1) or (4.6). It is thus natural to ask for generalisations of the $M_u(x)$ that yield the $E_{u/v}(a, b)$ upon evaluation. In fact, in the symmetric theory such functions have already been constructed in [27]. Specifically, $\mathfrak{S}_n$-invariant *rational functions* $f_\mu(x; a, b; q, t) = f_\mu(x; a, b)$ were defined satisfying the following three conditions. (In comparison with [27] the roles of a and b have been interchanged.) (1) The function

$$f_\mu(x; a, b) \prod_{i=1}^n (qt^{n-1}a/x_i)_{\mu_1}$$

is holomorphic in $\mathbb{C}^*$, (2)

$$f_\mu(b\langle\lambda\rangle; a, b) = 0 \quad \text{unless } \mu \subseteq \lambda,$$

and (3)

$$\lim_{x \to a\langle\lambda\rangle} f_\mu(x; a, b) \prod_{i=1}^n (qt^{n-1}a/x_i)_{\lambda_1} = 0 \quad \text{unless } \lambda \subseteq \mu.$$

Up to normalisation this uniquely fixes the functions $f_\mu(x; a, b)$. Moreover,

$$Q_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] = f_\mu(b\langle\lambda\rangle; a, b)\, d_\lambda e_\mu,$$

where d_λ and e_μ are simple factors. In the nonsymmetric theory similar rational interpolation functions can be defined, which generalise the M_u and which yield the $E_{u/v}(a, b)$ upon evaluation. We hope to report on these functions in a future publication.

Part II. $\mathfrak{gl}_n$ Basic Hypergeometric Series

Finally everything is in place to develop the theory of $\mathfrak{gl}_n$ basic hypergeometric series based on interpolation Macdonald polynomials.

7. Introduction

The multiple-series identities of the previous section involving the connection coefficients $E_{u/v}(a, b)$ generalise all of the terminating identities listed in Section 5. The identities for $\mathfrak{gl}_n$ basic hypergeometric series proved in the next few sections will generalise the non-terminating identities in the list, except for (5.9).

Below two different types of $\mathfrak{gl}_n$ basic hypergeometric series will be considered. At the top level are series of the form

$$\sum_u \frac{(a_1, \ldots, a_{r-1})_u}{(b_1, \ldots, b_{r-1}, b_r t^{n-1})_u} E_{u/v}(c, d) M_u(x), \tag{7.1}$$

where $u, v \in \mathbb{N}^n$, and where the parameters satisfy the $\mathfrak{gl}_n$ *balancing condition*

$$a_1 \cdots a_{r-1}d = b_1 \cdots b_r. \tag{7.2}$$

From (3.4) and (6.6) it follows that (7.1) for $n = 1$ simplifies to

$$x^v \frac{(a_1, \ldots, a_{r-1}, 1/x)_v}{(q, b_1, \ldots, b_r)_v} \, {}_{r+1}\phi_r \left[\begin{array}{c} a_1 q^v, \ldots, a_{r-1} q^v, q^v/x, d/c \\ b_1 q^v, \ldots, b_r q^v \end{array} ; cx \right]$$

which for $v = 0$ is simply

$${}_{r+1}\phi_r \left[\begin{array}{c} a_1, \ldots, a_{r-1}, 1/x, d/c \\ b_1, \ldots, b_r \end{array} ; cx \right].$$

Note that the $\mathfrak{gl}_n$ balancing condition (7.2) implies that the ${}_{r+1}\phi_r$ series are balanced in the sense of (5.2a).

At a lower level are $\mathfrak{gl}_n$ series of the type

$$\sum_u c^{|u|} \frac{(a_1, \ldots, a_{r-1})_u}{(b_1, \ldots, b_{r-1})_u} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x), \tag{7.3}$$

obtained from (7.1) by taking $d = b_r = 0$, so that no balancing condition holds.

In this paper we will consider the series (7.1) and (7.3) (and some related series discussed below) as formal power series. However, for $\max\{|cx_1|, \ldots, |cx_n|\}$ sufficiently small the $\mathfrak{gl}_n$ series may also be viewed as $\mathfrak{gl}_n$ functions.

Typically, if a classical summation or transformation formula for nonterminating basic hypergeometric series is balanced (like the q-Gauss sum or the q-Kummer–Thomae–Whipple transformation) it admits a $\mathfrak{gl}_n$ generalisation involving series of the type (7.1). If, however, the parameters in the identity can be chosen freely (like in the q-binomial theorem or Heine's ${}_2\phi_1$ transformations) it, at best, admits a $\mathfrak{gl}_n$ generalisation involving series of the type (7.3).

Replacing $(b_r, c, d, x) \mapsto (b_r e, ce, de, x/e)$ in (7.1), and using (3.5) and (6.3), it follows that

$$\lim_{c \to 0} e^{|v|} \sum_u \frac{(a_1, \ldots, a_{r-1})_u}{(b_1, \ldots, b_{r-1}, b_r e t^{n-1})_u} \mathsf{E}_{u/v}(ec, ed) \mathsf{M}_u(x/e)$$

$$= \sum_u \frac{(a_1, \ldots, a_{r-1})_u}{(b_1, \ldots, b_{r-1})_u} \mathsf{E}_{u/v}(c, d) \mathsf{E}_u(x).$$

We will refer to the series on the right (or special cases thereof) as $\mathfrak{sl}_n$ basic hypergeometric series. Obviously, by the above limiting procedure, any $\mathfrak{gl}_n$ series identity implies a corresponding $\mathfrak{sl}_n$ series identity.

Replacing $(b_r, c, d, u, v) \mapsto (b_r t^{1-n}, c t^{1-n}, d t^{1-n}, v, w)$ in (7.1), multiplying by $t^{(n-1)|w|}$ and specialising $x = \langle u \rangle$ we obtain the terminating series

$$\sum_v \tau_v \frac{(a_1, \ldots, a_{r-1})_v}{(b_1, \ldots, b_r)_v} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_{v/w}(c, d)$$

subject to the balancing condition (7.2). The q-Chu–Vandermonde sum (6.13b) and the transformation (6.15) provide example of the above series. We shall find in the next few sections that both (6.13b) and (6.15) arise by the above type of specialisation from $\mathfrak{gl}_n$ basic hypergeometric series identities.

8. A $\mathfrak{gl}_n$ q-binomial theorem

Our first main result is a generalisation of the celebrated q-binomial theorem (5.3). An equivalent way of stating this theorem follows from the substitutions $(a, x) \mapsto (q^v/x, ax)$, where v is a nonnegative integer. Then, after a shift in the summation index,

$$\sum_{u=0}^{\infty}(ax)^u \begin{bmatrix} u \\ v \end{bmatrix} \frac{(1/x)_u}{(q)_u} = (ax)^v \frac{(1/x)_v}{(q)_v} \frac{(aq^v)_\infty}{(ax)_\infty},$$

where $\begin{bmatrix} u \\ v \end{bmatrix}$ is the q-binomial coefficient (4.5). Recalling (3.4), this may also be stated as

$$\sum_{u=0}^{\infty} a^u \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x) = a^v \mathsf{M}_v(x) \frac{(aq^v)_\infty}{(ax)_\infty},$$

for $v \in \mathbb{N}$. The following theorem provides a $\mathfrak{gl}_n$ analogue of the type (7.3).

Theorem 8.1 ($\mathfrak{gl}_n$ q-binomial theorem I). *Let*

$$\mathbb{X} = x_1 + \cdots + x_n \qquad and \qquad \mathbb{V} = \langle v \rangle_1 + \cdots + \langle v \rangle_n, \tag{8.1}$$

where $v \in \mathbb{N}^n$. Then

$$\sum_u a^{|u|} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x) = a^{|v|} \mathsf{M}_v(x) \prod_{i=1}^n \frac{(a\langle v \rangle_i)_\infty}{(ax_i)_\infty} \tag{8.2}$$

$$= a^{|v|} \mathsf{M}_v(x)\, \sigma_a \begin{bmatrix} \dfrac{\mathbb{X} - \mathbb{V}}{1 - q} \end{bmatrix}.$$

Note that the product in the first expression on the right can be written in several different forms since

$$\prod_{i=1}^n (a\langle v \rangle_i)_\infty = \prod_{i=1}^n (aq^{v_i^+} t^{n-i})_\infty = \frac{1}{(at^{n-1})_v} \prod_{i=1}^n (at^{n-i})_\infty. \tag{8.3}$$

We also note that unlike the case $n = 1$, where v is readily eliminated from the identity by shifting the summation index, (8.2) depends nontrivially on $v \in \mathbb{N}^n$ when $n > 1$.

Before giving a proof we state four simple corollaries of the $\mathfrak{gl}_n$ q-binomial theorem.

Corollary 8.2 ($\mathfrak{gl}_n$ q-binomial theorem II). *We have*

$$\sum_u a^{|u|} \mathsf{M}_u(x) = \prod_{i=1}^n \frac{(at^{n-i})_\infty}{(ax_i)_\infty}.$$

This is of course nothing but the $v = 0$ case of Theorem 8.1 (and corresponds to (1.3) of the introduction). In view of (3.4) it is, however, closer to the standard formulation of the q-binomial theorem (5.3).

A well-known finite form of the q-binomial theorem gives the expansion of the q-shifted factorial $(a)_u$ as a power series in a [1, Theorem 3.3]

$$(a)_u = \sum_{v=0}^u (-a)^v q^{\binom{v}{2}} \begin{bmatrix} u \\ v \end{bmatrix}, \qquad u \in \mathbb{N}.$$

Theorem 8.1 extends this to a q-shifted factorial indexed by compositions.

Corollary 8.3. *For $u \in \mathbb{N}^n$*

$$(a)_u = \sum_v \tau_v \, a^{|v|} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{8.4}$$

Proof. Take Theorem 8.1 with $(x, a, u, v) \mapsto (\langle u \rangle, at^{1-n}, v, w)$. Using (2.4) and (4.6) yields

$$\sum_v \tau_v \, a^{|v|} \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \tau_w \, a^{|w|} \frac{(a)_u}{(a)_w} \begin{bmatrix} u \\ w \end{bmatrix}. \tag{8.5}$$

The identity (8.4) is the special case $w = 0$.

The sum (8.5) also corresponds to the $b \to 0$ limit of the q-Chu–Vandermonde (6.13b). We stress however that the above proof is independent of the results of Section 6 as the proof of Theorem 8.1 given below only uses vanishing properties of the Macdonald interpolation polynomials.

Corollary 8.4 ($\mathfrak{sl}_n$ **Euler sum**)**.** *For $v \in \mathbb{N}^n$*

$$\sum_u \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_u(x) = \mathsf{E}_v(x) \prod_{i=1}^n \frac{1}{(x_i)_\infty}. \tag{8.6}$$

For $n = 1$ and $v = 0$ (or $u \mapsto u + v$) this is Euler's q-exponential sum [6, Equation (II.1)]

$$\sum_{u=0}^{\infty} \frac{x^u}{(q)_u} = \frac{1}{(x)_\infty}.$$

For general n it can also be found in [3, Equation (3.30)].

Proof. Replace $x \mapsto x/a$ in (8.2) and let a tend to zero using (3.5).

Our final corollary contains a generalisation of the $_1\phi_1$ summation [6, Equation (II.5)].

Corollary 8.5. *For $v \in \mathbb{N}^n$*

$$\sum_u \frac{\tau_u \, a^{|u|}}{(at^{n-1})_u} \begin{bmatrix} u \\ v \end{bmatrix}_{q^{-1}, t^{-1}} \mathsf{M}_u(x) = \tau_v \, a^{|v|} \mathsf{M}_v(x) \prod_{i=1}^n \frac{(ax_i)_\infty}{(at^{n-i})_\infty}.$$

Proof. This results by inverting (8.2) using (4.11).

Proof of Theorem 8.1. By (6.20) the right-hand side of (8.2) can be expanded in terms of complete symmetric functions as

$$\mathsf{M}_v(x) \sum_{k=0}^{\infty} a^{k+|v|} S_k \begin{bmatrix} \mathbb{X} - \mathbb{V} \\ 1 - q \end{bmatrix}.$$

Comparing this with the left-hand side of (8.2) and equating coefficients of a^k, we are left to prove that

$$\sum_{|u|=k+|v|} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x) = \mathsf{M}_v(x) S_k \begin{bmatrix} \mathbb{X} - \mathbb{V} \\ 1 - q \end{bmatrix}.$$

Both sides are polynomials in x of degree $k + |v|$ so that it suffices to check that the above is true for $x = \langle w \rangle$ where w is any compositions such that $|w| \le k + |v|$. In other words, introducing $\mathbb{W} = \langle w \rangle_1 + \cdots + \langle w \rangle_n$ and using (4.6), we need to show that

$$t^{(n-1)k} \sum_{|u|=k+|v|} \tau_u \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \tau_v \begin{bmatrix} w \\ v \end{bmatrix} S_k \left[\frac{\mathbb{W} - \mathbb{V}}{1 - q} \right]$$

holds for all w such that $|w| \le k + |v|$.

Since $|u| = k + |v|$ it follows from (4.2) that the left-hand side vanishes if $|w| < k + |v|$. If on the other hand $|w| = k + |v|$ then, again by (4.2), only the term $u = w$ contributes to the sum. The previous equation can thus also be stated as

$$\begin{bmatrix} w \\ v \end{bmatrix} S_k \left[\frac{\mathbb{W} - \mathbb{V}}{1 - q} \right] = \frac{\tau_w}{\tau_v} t^{(n-1)k} \chi(|w| = k + |v|) \begin{bmatrix} w \\ v \end{bmatrix}, \tag{8.7}$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

By (4.4) both sides are identically zero if $v^+ \not\subseteq w^+$. It is thus sufficient to show that

$$S_k \left[\frac{\mathbb{W} - \mathbb{V}}{1 - q} \right] = \frac{\tau_w}{\tau_v} t^{(n-1)k} \chi(|w| = k + |v|)$$

for $v^+ \subseteq w^+$ such that $|w| \le k + |v|$. Assuming these conditions we find

$$\frac{\mathbb{W} - \mathbb{V}}{1 - q} = -\mathbb{Y},$$

where $\mathbb{Y} = \mathbb{Y}_1 + \cdots + \mathbb{Y}_n$, with

$$\mathbb{Y}_i = t^{n-i} \frac{q^{v_i^+} - q^{w_i^+}}{1 - q} = t^{n-i} q^{v_i^+} + t^{n-i} q^{v_i^+ + 1} + \cdots + t^{n-i} q^{w_i^+ - 1}. \tag{8.8}$$

The crucial observation is that

$$|\mathbb{Y}| < k \qquad \text{if } |w| < k + |v|, \tag{8.9a}$$
$$|\mathbb{Y}| = k \qquad \text{if } |w| = k + |v|. \tag{8.9b}$$

From (6.21) it follows that

$$\sigma_z[-\mathbb{Y}] = \prod_{y \in \mathbb{Y}} (1 - yz)$$

so that

$$S_k[-\mathbb{Y}] = 0 \qquad\qquad\qquad \text{if } |\mathbb{Y}| < k \tag{8.10a}$$
$$S_k[-\mathbb{Y}] = (-1)^k \prod_{y \in \mathbb{Y}} y \qquad \text{if } |\mathbb{Y}| = k. \tag{8.10b}$$

(More generally $S_k[-\mathbb{Y}] = (-1)^k e_k[\mathbb{Y}]$, with $e_k[\mathbb{Y}]$ the elementary symmetric function). Equations (8.9a) and (8.10a) imply that $S_k[-\mathbb{Y}] = 0$ for $|w| < k + |v|$, in accordance with (8.7). Furthermore (8.9b) and (8.10b) imply that for $|w| = k + |v|$

$$S_k[-\mathbb{Y}] = (-1)^k \prod_{y \in \mathbb{Y}} y = \frac{\tau_w}{\tau_v} t^{(n-1)k}$$

again in agreement with (8.7).

9. A $\mathfrak{gl}_n$ q-Gauss sum

To generalise the q-Gauss sum we substitute (a, b, c, k) by $(q^v/x, b/a, bq^v, u)$ in (5.4), shift the summation index and recall (3.4). Hence

$$\sum_{u=0}^{\infty} a^{u-v} \frac{(b/a)_{u-v}}{(b)_u} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x) = \frac{\mathsf{M}_v(x)}{(a)_v} \frac{(a, bx)_{\infty}}{(b, ax)_{\infty}}.$$

By (6.3) and (6.6) this can also be written as

$$\sum_{u=0}^{\infty} \frac{(b)_v}{(b)_u} \mathsf{E}_{u/v}(a, b) \mathsf{M}_u(x) = \mathsf{M}_v(x) \frac{(aq^v, bx)_{\infty}}{(bq^v, ax)_{\infty}},$$

for $v \in \mathbb{N}$.

Theorem 9.1 ($\mathfrak{gl}_n$ q-Gauss sum I). *With the notation of Theorem 8.1*

$$\sum_u \frac{(bt^{n-1})_v}{(bt^{n-1})_u} \mathsf{E}_{u/v}(a, b) \mathsf{M}_u(x) = \mathsf{M}_v(x) \prod_{i=1}^{n} \frac{(a\langle v \rangle_i, bx_i)_{\infty}}{(b\langle v \rangle_i, ax_i)_{\infty}} \tag{9.1}$$

$$= \mathsf{M}_v(x) \, \sigma_1 \left[\frac{a-b}{1-q} \left(\mathbb{X} - \mathbb{V} \right) \right].$$

Note that the series on the left satisfies the $\mathfrak{gl}_n$ balancing condition (7.2). By (6.3) and (6.7a) the $\mathfrak{gl}_n$ q-Gauss sum simplifies to the $\mathfrak{gl}_n$ q-binomial theorem (8.2) when b tends to zero. We also remark that the $\mathfrak{gl}_n$ q-Gauss sum may be viewed as a nonterminating analogue of the q-Chu–Vandermonde sum (6.13b). Specifically, taking $(u, v, x) \mapsto (v, w, \langle u \rangle)$ in (9.1), using (4.6) and (8.3), and finally scaling $(a, b) \mapsto (at^{1-n}, bt^{1-n})$ using (6.3) yields (6.13b).

Proof of Theorem 9.1. If we take Corollary 6.3, multiply both sides by $b^{|u|}\mathsf{M}_u(x)$ and sum over u, we obtain

$$\sum_v a^{-|v|} \mathsf{E}_{v/w}(1, a) \sum_u (ab)^u \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x) = \sum_u b^{|u|} \begin{bmatrix} u \\ w \end{bmatrix} \mathsf{M}_u(x).$$

Both the u-sums can be performed by the $\mathfrak{gl}_n$ q-binomial theorem (8.2) so that

$$\sum_v b^{|v|} \mathsf{E}_{v/w}(1, a) \mathsf{M}_v(x) \prod_{i=1}^{n} \frac{(ab\langle v \rangle_i)_{\infty}}{(abx_i)_{\infty}} = b^{|w|} \mathsf{M}_w(x) \prod_{i=1}^{n} \frac{(b\langle w \rangle_i)_{\infty}}{(bx_i)_{\infty}}.$$

By (2.4) and (6.3) this can be rewritten as

$$\sum_v \frac{(abt^{n-1})_w}{(abt^{n-1})_v} \mathsf{E}_{v/w}(b, ab) \mathsf{M}_v(x) = \mathsf{M}_w(x) \prod_{i=1}^{n} \frac{(b\langle w \rangle_i, abx_i)_{\infty}}{(ab\langle w \rangle_i, bx_i)_{\infty}},$$

which is (9.1) with $(a, b, u, v) \mapsto (b, ab, v, w)$.

Again we give a number of simple corollaries.

Corollary 9.2 ($\mathfrak{gl}_n$ q-Gauss sum II). *We have*

$$\sum_u a^{|u|} \frac{(b)_u}{(abt^{n-1})_u} \mathsf{M}_u(x) = \prod_{i=1}^n \frac{(at^{n-i}, abx_i)_\infty}{(abt^{n-i}, ax_i)_\infty}. \qquad (9.2)$$

Proof. By (6.3), (2.4) and (8.3) this follows from the $(v, b) \mapsto (0, ab)$ case of (9.1).

Corollary 9.3 ($\mathfrak{sl}_n$ q-binomial theorem I). *We have*

$$\sum_u \mathsf{E}_{u/v}(a, b)\mathsf{E}_u(x) = \mathsf{E}_v(x) \prod_{i=1}^n \frac{(bx_i)_\infty}{(ax_i)_\infty} = \mathsf{E}_v(x)\,\sigma_1\left[\frac{a-b}{1-q}\,\mathbb{X}\right]. \qquad (9.3)$$

For $n = 1$ and $v = 0$ this is (5.3) with $(a, x) \mapsto (b/a, ax)$.

Proof. In (9.1) we scale $(a, b, x) \mapsto (ac, bc, x/c)$. By (6.3) this results in

$$\sum_u c^{|u|} \frac{(bct^{n-1})_v}{(bct^{n-1})_u} \mathsf{E}_{u/v}(a, b)\mathsf{M}_u(x/c) = c^{|v|}\mathsf{M}_v(x/c) \prod_{i=1}^n \frac{(ac\langle v\rangle_i, bx_i)_\infty}{(bc\langle v\rangle_i, ax_i)_\infty}.$$

Taking the $c \to 0$ limit using (3.5) yields (9.3).

For later reference we also state the $(a, b, v) \mapsto (1, a, 0)$ case of (9.3).

Corollary 9.4 ($\mathfrak{sl}_n$ q-binomial theorem II). *We have*

$$\sum_u (a)_u \mathsf{E}_u(x) = \prod_{i=1}^n \frac{(ax_i)_\infty}{(x_i)_\infty}. \qquad (9.4)$$

This is the nonsymmetric analogue of the Kaneko–Macdonald q-binomial theorem for symmetric Macdonald polynomials [8, 19]

$$\sum_\lambda (a)_\lambda \mathsf{P}_\lambda(x) = \prod_{i=1}^n \frac{(ax_i)_\infty}{(x_i)_\infty}. \qquad (9.5)$$

It is in fact easily shown that (9.4) and (9.5) are equivalent:

$$\sum_\lambda (a)_\lambda \mathsf{P}_\lambda(x) = \sum_\lambda \sum_{u^+ = \lambda} (a)_{u^+} \mathsf{E}_u(x) = \sum_u (a)_u \mathsf{E}_u(x).$$

Corollary 9.5. *Proposition 6.11 is true.*

Proof. If we multiply (6.19) by $\mathsf{E}_u(x)$ and sum over u using (6.18) we obtain

$$\sum_u \mathsf{E}_{u/v}(a, b)\mathsf{E}_u(x) = \mathsf{E}_v(x) \sum_w (b/a)_w \mathsf{E}_u(ax) = \mathsf{E}_v(x/a) \prod_{i=1}^n \frac{(bx_i)_\infty}{(ax_i)_\infty}.$$

Since this is (9.3) the proof is complete.

Let

$$\mathsf{P}_{\lambda/\mu}(x) := t^{n(\lambda)-n(\mu)} \frac{c'_\mu}{c'_\lambda} P_{\lambda/\mu}(x),$$

where $P_{\lambda/\mu}(x)$ is a skew Macdonald polynomial (i.e., $P_{\lambda/\mu}(x) = Q_{\lambda/\mu}(x) b_\mu/b_\lambda$).
For later reference we note that

$$\mathsf{P}_{\lambda/\mu}(x) = t^{2n(\lambda)-2n(\mu)} \frac{c_\mu c'_\mu}{c_\lambda c'_\lambda} \mathsf{Q}_{\lambda/\mu}(x) \tag{9.6a}$$

$$= \frac{(t^n)_\mu}{(t^n)_\lambda} \frac{\mathsf{P}_\lambda(\langle 0\rangle)}{\mathsf{P}_\mu(\langle 0\rangle)} \mathsf{Q}_{\lambda/\mu}(x). \tag{9.6b}$$

Further assume the balancing condition $b_0 b_1 v_0 v_1 w_0 w_1 = qt$. Then the following skew Cauchy-type identity is implied by [28, Corollary 3.8]:

$$\frac{1}{Z} \sum_\lambda q^{|\lambda|} \frac{(v_0, v_1)_\lambda}{(q/w_0, q/w_1)_\lambda} \mathsf{P}_{\lambda/\mu}\left[\frac{1-b_0}{1-t}\right] \mathsf{Q}_{\lambda/\nu}\left[\frac{1-b_1}{1-t}\right]$$

$$= \frac{(q/b_0)^{|\mu|}(v_0, v_1)_\mu}{(q/b_0 w_0, q/b_0 w_1)_\mu} \frac{(q/b_1)^{|\nu|}(v_0, v_1)_\nu}{(q/b_1 w_0, q/b_1 w_1)_\nu}$$

$$\times \sum_\lambda (b_0 b_1/q)^{|\lambda|} \frac{(q/b_0 b_1 w_0, q/b_0 b_1 w_1)_\lambda}{(v_0, v_1)_\lambda} \mathsf{P}_{\nu/\lambda}\left[\frac{1-b_0}{1-t}\right] \mathsf{Q}_{\mu/\lambda}\left[\frac{1-b_1}{1-t}\right],$$

provided the sum on the left terminates. The normalisation Z is given by the sum on the left for $\mu = \nu = 0$. Choosing

$$(b_0, b_1, v_0, v_1, w_0, w_1) \mapsto (b/a, d/c, t^n, q^{-k}, qt^{1-n}/bd, q^k ac)$$

with k a nonnegative integer and taking the large k limit yields

$$\frac{1}{Z'} \sum_\lambda \frac{(t^n)_\lambda}{(bdt^{n-1})_\lambda} \mathsf{P}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] \mathsf{Q}_{\lambda/\nu}\left[\frac{c-d}{1-t}\right]$$

$$= \frac{(t^n)_\mu (t^n)_\nu}{(adt^{n-1})_\mu (bct^{n-1})_\nu} \sum_\lambda \frac{(act^{n-1})_\lambda}{(t^n)_\lambda} \mathsf{P}_{\nu/\lambda}\left[\frac{a-b}{1-t}\right] \mathsf{Q}_{\mu/\lambda}\left[\frac{c-d}{1-t}\right], \quad (9.7)$$

where Z' again denotes the sum on the left for $\mu = \nu = 0$. Explicitly,

$$Z' = \sum_\lambda \frac{(b/a, d/c)_\lambda}{(bdt^{n-1})_\lambda} \mathsf{P}_\lambda(ac\langle 0\rangle) = \prod_{i=1}^n \frac{(adt^{n-i}, bct^{n-i})_\infty}{(act^{n-i}, bdt^{n-i})_\infty},$$

where the second equality follows from Kaneko's q-Gauss sum for Macdonald polynomials [8, Proposition 5.4]. Consequently, (9.7) may also be written as

$$\sum_\lambda \mathsf{P}_{\lambda/\mu}\left[\frac{a-b}{1-t}\right] \mathsf{Q}_{\lambda/\nu}\left[\frac{c-d}{1-t}\right] \prod_{i=1}^n \frac{(bd\langle\lambda\rangle_i)_\infty}{(t\langle\lambda\rangle_i)_\infty}$$

$$= \left(\prod_{i=1}^n \frac{(ad\langle\mu\rangle_i, bc\langle\nu\rangle_i)_\infty}{(t\langle\mu\rangle_i, t\langle\nu\rangle_i)_\infty}\right) \sum_\lambda \mathsf{P}_{\nu/\lambda}\left[\frac{a-b}{1-t}\right] \mathsf{Q}_{\mu/\lambda}\left[\frac{c-d}{1-t}\right] \prod_{i=1}^n \frac{(t\langle\lambda\rangle_i)_\infty}{(ac\langle\lambda\rangle_i)_\infty}.$$

As our final corollary of the $\mathfrak{gl}_n$ q-Gauss sum we will show that this has a nonsymmetric analogue. In analogy with (9.6b) let

$$\mathsf{F}_{u/v}(a, b) := \frac{(t^n)_v}{(t^n)_u} \frac{\mathsf{E}_u(\langle 0\rangle)}{\mathsf{E}_v(\langle 0\rangle)} \mathsf{E}_{u/v}(a, b)$$

Corollary 9.6. *For $v, w \in \mathbb{N}^n$ and $|ac|, |ad|, |bc|, |bd|$ sufficiently small so that the sum on the left converges*

$$\sum_u \mathsf{F}_{u/v}(a,b)\mathsf{E}_{u/w}(c,d) \prod_{i=1}^n \frac{(bd\langle u\rangle_i)_\infty}{(t\langle u\rangle_i)_\infty}$$

$$= \left(\prod_{i=1}^n \frac{(ad\langle v\rangle_i, bc\langle w\rangle_i)_\infty}{(t\langle v\rangle_i, t\langle w\rangle_i)_\infty} \right) \sum_u \mathsf{F}_{w/u}(a,b)\mathsf{E}_{v/u}(c,d) \prod_{i=1}^n \frac{(t\langle u\rangle_i)_\infty}{(ac\langle u\rangle_i)_\infty}.$$

Proof. Application of (6.1) to both sides of the $\mathfrak{gl}_n$ q-Gauss sum results in

$$\sum_{u,w} \frac{(bt^{n-1})_v}{(bt^{n-1})_u} \mathsf{E}_{u/v}(a,b)c_{uw}(1,c)\mathsf{M}_w(cx) = \sum_u c_{vu}(1,c)\mathsf{M}_u(cx) \prod_{i=1}^n \frac{(a\langle v\rangle_i, bx_i)_\infty}{(b\langle v\rangle_i, ax_i)_\infty}.$$

The right-hand side can again be transformed by the q-Gauss sum (9.1) with $(a,b,x,v) \mapsto (a/c, b/c, cx, w)$ so that

$$\sum_{u,w} \left(\prod_{i=1}^n (b\langle u\rangle_i)_\infty \right) \mathsf{E}_{u/v}(a,b)c_{uw}(1,c)\mathsf{M}_w(cx)$$

$$= \sum_{u,w} \left(\prod_{i=1}^n \frac{(a\langle v\rangle_i, b\langle w\rangle_i/c)_\infty}{(a\langle u\rangle_i/c)_\infty} \right) c_{vu}(1,c)\mathsf{E}_{w/u}(a/c, b/c)\mathsf{M}_w(cx),$$

where we have also used (2.4). Next we equate coefficients of $\mathsf{M}_w(cx)$ and substitute $(a,b,c) \mapsto (ad, bd, d/c)$. After carrying out some simplifications using (6.2) and (6.3) the result follows.

10. A $\mathfrak{gl}_n$ q-Kummer–Thomae–Whipple formula

By the substitutions

$$(a,b,d,e,f) \mapsto (1/x, bq^v, dq^v, eq^v, a)$$

and the use of (6.3) and (6.6), the Kummer–Thomae–Whipple formula (5.10) can be written in the form

$$\sum_{u=0}^\infty \frac{(b)_u}{(d,e)_u} \mathsf{E}_{u/v}(a, ac)\mathsf{M}_u(x) = \frac{(b)_v}{(d/c)_v} \frac{(a, ex)_\infty}{(e, ax)_\infty} \sum_{u=0}^\infty \frac{(d/c)_u}{(d,a)_u} \mathsf{E}_{u/v}(e, ac)\mathsf{M}_u(x).$$

Theorem 10.1 ($\mathfrak{gl}_n$ q-Kummer–Thomae–Whipple formula).   *For $v \in \mathbb{N}^n$ and $abc = de$*

$$\sum_u \frac{(b)_u}{(d, et^{n-1})_u} \mathsf{E}_{u/v}(a, ac)\mathsf{M}_u(x)$$

$$= \frac{(b)_v}{(d/c)_v} \left(\prod_{i=1}^n \frac{(at^{n-i}, ex_i)_\infty}{(et^{n-i}, ax_i)_\infty} \right) \sum_u \frac{(d/c)_u}{(d, at^{n-1})_u} \mathsf{E}_{u/v}(e, ac)\mathsf{M}_u(x).$$

Note that the condition $abc = de$ corresponds to the $\mathfrak{gl}_n$ balancing condition (7.2) and that the transformation may alternatively be written as

$$\sum_u \frac{(b)_u (d, et^{n-1})_v}{(b)_v (d, et^{n-1})_u} \, \mathsf{E}_{u/v}(a, ac) \mathsf{M}_u(x)$$

$$= \left(\prod_{i=1}^{n} \frac{(a\langle v \rangle_i, ex_i)_\infty}{(e\langle v \rangle_i, ax_i)_\infty} \right) \sum_u \frac{(d/c)_u (d, at^{n-1})_v}{(d/c)_v (d, at^{n-1})_u} \, \mathsf{E}_{u/v}(e, ac) \mathsf{M}_u(x),$$

with the product on the right corresponding to $\sigma_1\big[(a - e)(\mathbb{X} - \mathbb{V})/(1 - q)\big]$ in the notation of Theorem 8.1.

The above theorem generalises a number of earlier results. For example, recalling (6.4) and taking the limit $e \to ac$ (so that $d \to b$), and finally replacing $c \mapsto b/a$ we obtain the $\mathfrak{gl}_n$ q-Gauss sum of Theorem 9.1. Furthermore, making the substitutions $(a, e, u, v, x) \mapsto (at^{1-n}, et^{1-n}, v, w, \langle u \rangle)$, and using (2.4), (4.6) and (6.3), we arrive at (6.15).

Proof of Theorem 10.1. If we replace $(a, c, d) \mapsto (d/c, d, ac/d)$ in (6.16) and define $e = abc/d$ we obtain

$$\sum_v \frac{(d/c)_v}{(d)_v} \, \mathsf{E}_{u/v}(a, e) \mathsf{E}_{v/w}(e, ac) = \frac{(d/c)_w (b)_u}{(b)_w (d)_u} \, \mathsf{E}_{u/w}(a, ac). \qquad (10.1)$$

Now denote the left-hand side of Theorem 10.1 with v replaced by w as LHS. By (10.1),

$$\mathrm{LHS} = \sum_u \frac{(b)_u}{(d, et^{n-1})_u} \, \mathsf{E}_{u/w}(a, ac) \mathsf{M}_u(x)$$

$$= \frac{(b)_w}{(d/c)_w} \sum_{u,v} \frac{(d/c)_v}{(d)_v} \frac{1}{(et^{n-1})_u} \, \mathsf{E}_{u/v}(a, e) \mathsf{E}_{v/w}(e, ac) \mathsf{M}_u(x).$$

The sum over u can now be performed by the $\mathfrak{gl}_n$ q-Gauss sum (9.1) with $b \mapsto e$. Hence

$$\mathrm{LHS} = \frac{(b)_w}{(d/c)_w} \left(\prod_{i=1}^{n} \frac{(ex_i)_\infty}{(ax_i)_\infty} \right) \sum_v \left(\prod_{i=1}^{n} \frac{(a\langle v \rangle_i)_\infty}{(e\langle v \rangle_i)_\infty} \right) \frac{(d/c)_v}{(d)_v} \, \mathsf{E}_{v/w}(e, ac) \mathsf{M}_v(x).$$

Finally using (8.3) this yields the right-hand side of the theorem with v replaced by w.

A number of new results follow from the $\mathfrak{gl}_n$ q-Kummer–Thomae–Whipple formula.

Corollary 10.2 ($\mathfrak{gl}_n$ Heine transformation). *For $v \in \mathbb{N}^n$*

$$\sum_u a^{|u|} \frac{(b)_u}{(ct^{n-1})_u} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x)$$

$$= \left(\frac{a}{c} \right)^{|v|} \frac{(b)_v}{(ab/c)_v} \left(\prod_{i=1}^{n} \frac{(at^{n-i}, cx_i)_\infty}{(ct^{n-i}, ax_i)_\infty} \right) \sum_u c^{|u|} \frac{(ab/c)_u}{(at^{n-1})_u} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x).$$

An equivalent way to state the above transformation is

$$\sum_u a^{|u|-|v|} \frac{(b)_u (ct^{n-1})_v}{(b)_v (ct^{n-1})_u} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x)$$
$$= \left(\prod_{i=1}^n \frac{(a\langle v \rangle_i, cx_i)_\infty}{(c\langle v \rangle_i, ax_i)_\infty} \right) \sum_u c^{|u|-|v|} \frac{(ab/c)_u (at^{n-1})_v}{(ab/c)_v (at^{n-1})_u} \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{M}_u(x).$$

For $n = 1$ the $\mathfrak{gl}_n$ Heine transformation simplifies to (5.8) with (a, b, c, x) replaced by $(q^v/x, bq^v, cq^v, ax)$.

Proof. Let $(c, d) \to (0, 0)$ in Theorem 10.1 such that $d/c = ab/e$. By (6.3) and (6.7a) the $\mathfrak{gl}_n$ Heine transformation with $c \mapsto e$ follows.

Corollary 10.3 ($\mathfrak{sl}_n$ q-Euler transformation). *For $v \in \mathbb{N}^n$*

$$\sum_u \frac{(a)_u}{(c)_u} \mathsf{E}_{u/v}(b, c) \mathsf{E}_u(x) = \frac{(a)_v}{(b)_v} \left(\prod_{i=1}^n \frac{(ax_i)_\infty}{(bx_i)_\infty} \right) \sum_u \frac{(b)_u}{(c)_u} \mathsf{E}_{u/v}(a, c) \mathsf{E}_u(x).$$

For $n = 1$ this is (5.9) with $(a, b, c, x) \mapsto (aq^v, c/b, cq^v, bx)$, and for $v = 0$ it is the nonsymmetric analogue of [2, Proposition 3.1]. The $\mathfrak{sl}_n$ Euler transformation is easily seen to be equivalent to the q-Pfaff–Saalschütz sum (6.11). Indeed, if we take the latter, multiply both sides by $\mathsf{E}_u(x)$ and then sum over u using (9.3) we obtain the former. We also note that for $c = 0$ the $\mathfrak{sl}_n$ q-Euler transformation becomes

$$\sum_u b^{|u|}(a)_u \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_u(x) = \left(\frac{b}{a} \right)^{|v|} \frac{(a)_v}{(b)_v} \left(\prod_{i=1}^n \frac{(ax_i)_\infty}{(bx_i)_\infty} \right) \sum_u a^{|u|}(b)_u \begin{bmatrix} u \\ v \end{bmatrix} \mathsf{E}_u(x).$$

When $n = 1$ this is trivial since both sides are summable by the q-binomial theorem (5.3). Curiously, for $n > 1$ it no longer appears possible to explicitly perform the sums in closed form.

Proof. Corollary 10.3 follows from Theorem 10.1 by replacing $(x, e) \mapsto (x/a, ae)$, then taking the limit $a \to 0$ using (3.5), and finally making the substitutions

$$(x, b, c, d, e) \mapsto (bx, a, c/b, c, a/b).$$

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