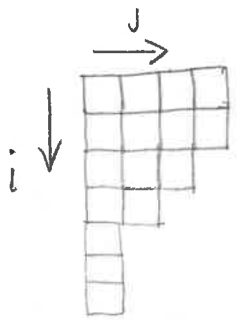
Symmetric functions

1 Partitions

$\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_i \in \mathbb{Z}$, $\lambda_i \geq \lambda_{i+1}$, $\lambda_i = 0$ for all $i > \ell(\lambda)$ for some $\ell(\lambda)$, the length of λ

$|\lambda| = \lambda_1 + \lambda_2 + \cdots$. If $|\lambda| = n$, $\lambda \vdash n$ (λ a partition of n)

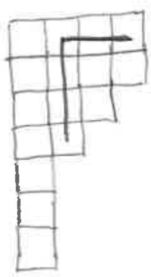
Eg $\lambda = (4,4,3,2,1,1,1,0,\dots) = (4,4,3,2,1,1,1)$, $\ell(\lambda) = 7$, $|\lambda| = 16$



Young diagram of λ (aka Ferrers diagram)

$s = (i,j) \in \lambda$ if $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$

$h(s) = h(i,j) = \lambda_i + \lambda_j + j - i + 1$, the hook length of s.



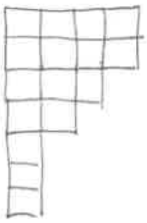
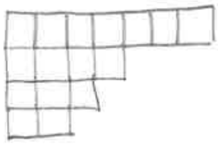
,

10	6	4	2
9	5	3	1
7	3	1	
5	1		
3			
2			
1			

$h(1,2) = 6$

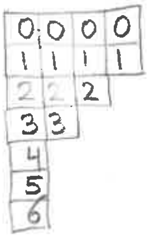
λ' the <u>conjugate</u> of λ

Eg $\lambda = (4,4,3,2,1,1,1)$, $\lambda' = (7,4,3,2)$

λ  , λ' 

$m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ <u>multiplicity</u> of parts of size i

$$n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i = \sum_{i \geq 1} \binom{\lambda'_i}{2}$$



If $\lambda_i \geq \mu_i$ for all i, $\mu \subseteq \lambda$: μ is <u>contained</u> in λ

Skew shape/diagram: $\lambda - \mu$ (or λ/μ) for $\mu \subseteq \lambda$.

Eg $\lambda = (4,4,3,2,1,1,1)$, $\mu = (3,3,1,1,1)$

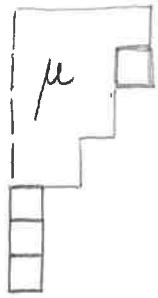
λ/μ 

If λ/μ has at most one square in each column:
<u>horizontal strip</u>

If λ/μ has at most one square in each row:
<u>vertical strip</u>

Eg $\lambda = (4,4,3,2,1,1,1)$, $\mu = (4,3,3,2)$

λ/μ is a vertical strip



If $\mu \subseteq \lambda$ and $(\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots)$ then λ & μ are said to be interlacing, denoted as $\lambda \succ \mu$

<u>Lemma</u> Let $\mu \subseteq \lambda$. Then $\lambda \succ \mu$ iff λ/μ is a horizontal strip.

<u>Pf</u> Note that for $(i,j) \in \lambda$ we have

$(i,j) \in \lambda/\mu \iff \mu_i < j \leq \lambda_i$.

Moreover if $(i,j) \in \lambda/\mu$ then also $(i, \lambda_i) \in \lambda/\mu$.

Now let $(i,j), (i+1,j) \in \lambda/\mu \Rightarrow \mu_i < j \leq \lambda_i$ & $\mu_{i+1} < j \leq \lambda_{i+1}$

(i.e $\begin{array}{c|c} & j \\ i & \square \\ i+1 & \square \end{array} \in \lambda/\mu$)

This is incompatible with $\lambda_{i+1} \leq \mu_i$ since this would imply $\mu_{i+1} < j \leq \lambda_{i+1} \leq \mu_i < j \leq \lambda_i$, i.e., that $j<j$.

Hence $\lambda > \mu$ implies that λ/μ is a horizontal strip.

Conversely, if $(i+1,j) \in \lambda/\mu$ then $(i+1, \lambda_{i+1}) \in \lambda/\mu$.

If λ/μ is a horizontal strip this implies $(i, \lambda_{i+1}) \notin \lambda/\mu$

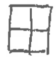
$\Rightarrow \mu_i \geq \lambda_{i+1} \Rightarrow \lambda > \mu$.

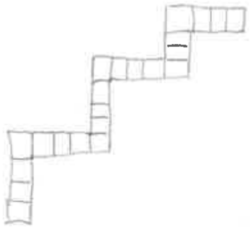
(by $\mu_i \geq j$ with $j = \lambda_{i+1}$)

$\square$

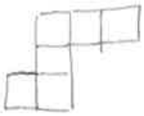
Let $\lambda, \mu \vdash n$. Then the partial order defined by $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots \mu_i$ for all i, is called <u>dominance order</u>.

For all $n \leq 5$ dominance order is a total order, but not for any $n \geq 6$.

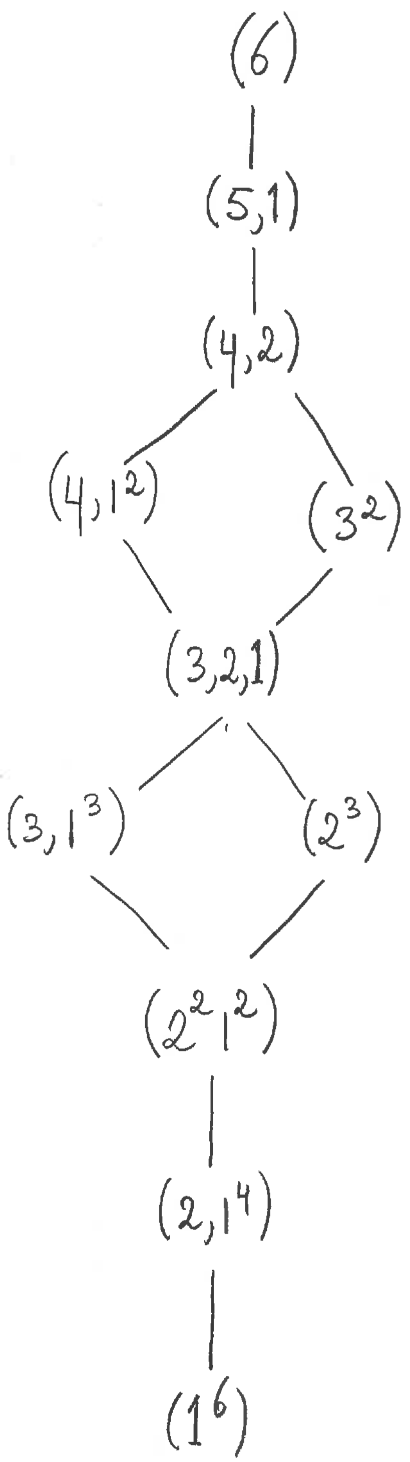
A skew diagram λ/μ is called a <u>border strip</u> (or ribbon) if λ/μ is connected and contains no 2×2 square .

Eg 

The height of a border strip is the number of rows minus one.

Eg height () $= 3-1=2$

Eg.



(2) The ring of symmetric functions

Let S_n be the symmetric group on n letters.

Then the ring of symmetric functions in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}$ is defined as

$$\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{S_n}$$

Λ_n is graded by degree: $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$

where $\Lambda_n^k = \{ f \in \Lambda_n : \deg f = k \} \cup \{0\}$

Let $\ell(\lambda) \leqslant n$. Then the monomial symmetric function $m_\lambda(x_1, \dots, x_n)$ is defined as

$$m_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n / S_n^\lambda} w(x^\lambda),$$

where $x^\lambda := x_1^{\lambda_1} \dots x_n^{\lambda_n}$ and S_n^λ is the stabilizer of λ in S_n (ie $m_\lambda(x_1, \dots, x_n) = \sum_{\substack{\text{distinct} \\ \text{perm. } \alpha \text{ of } \lambda}} x^\alpha$)

$\{m_\lambda\}_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}}$ is a $\mathbb{Z}$-basis of Λ_n^k

(Λ_n^k is a free $\mathbb{Z}$-module of rank $p(k,n)$, the # of partitions of k of length at most n)

The stability property

$$m_\lambda(x_1, \dots, x_{n-1}, 0) = \begin{cases} m_\lambda(x_1, \dots, x_{n-1}) & \text{if } \ell(\lambda) \leq n-1 \\ 0 & \text{if } \ell(\lambda) = n \end{cases}$$

may be used to define the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \dots$.

For $m \geq n$ define $\rho_{m,n} : \Lambda_m \to \Lambda_n$

$$m_\lambda(x_1, .., x_m) \mapsto \begin{cases} m_\lambda(x_1, .., x_n) & \ell(\lambda) \leq n \\ 0 & \text{otherwise.} \end{cases}$$

$\rho_{m,n}$ is a surjective ring homomorphism with kernel

$$\ker \rho_{m,n} = \operatorname{Span}_{\mathbb{Z}} \{m_\lambda(x_1, .., x_m)\}_{n < \ell(\lambda) \leq m}$$

Note that for $m \geq l \geq n$, $\rho_{m,n} = \rho_{l n} \circ \rho_{m l}$ and that $\rho_{n,n}$ is the identity map on Λ_n.

This makes $\{(\Lambda_n)_{n \geq 0}, (\rho_{m,n})_{m \geq n \geq 0}\}$ an <u>inverse system</u> of $\mathbb{Z}$-modules.

Now define $\rho^k_{m,n} : \Lambda^k_m \to \Lambda^k_n$ by the restriction of $\rho_{m,n}$ to Λ^k_m ; $\rho^k_{m,n} = \rho_{m,n}\big|_{\Lambda^k_m}$

$\rho^k_{m,n}$ is injective, and hence bijective, if $m \geq n \geq k$

(The maximum length of a partition of size k is k: )

$\Lambda^k := \varprojlim_n \Lambda^k_n$, the inverse limit of the $\mathbb{Z}$-modules Λ^k_n relative to the homomorphisms $\rho^k_{m,n}$:

$f \in \Lambda^k$, $f = (f_0, f_1, f_2, \dots)$

$f_n \in \Lambda^k_n$ for all n, such that $f_m(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{m-n}) = f_n(x_1, \dots, x_n)$ $(m \geq n)$.

Hence $\rho_n^k : \Lambda^k \to \Lambda_n^k$ is an isomorphism for $n \geq k$.
$f \mapsto f_n$.

This implies Λ^k is a free $\mathbb{Z}$-module with basis $\{m_\lambda\}_{\lambda \vdash k}$ where m_λ is defined by $\rho_n^k(m_\lambda) = m_\lambda(x_1, \dots, x_n)$, $n \geq k$

$$\Lambda := \bigoplus_{k \geq 0} \Lambda^k$$

$\rho_n := \bigoplus_{k \geq 0} \rho_n^k : \Lambda \to \Lambda_n$ is an isomorphism for degrees $\leq n$

Elements of Λ are finite linear combinations of the m_λ, unlike the elements of $\hat{\Lambda} = \varprojlim_n \Lambda_n$ which allows for elements of unbounded degree.

Example:

$$m_0 = 1$$

$$m_1 = x_1 + x_2 + \cdots$$

$$m_{(2)} = x_1^2 + x_2^2 + \cdots$$

$$m_{(1^2)} = \sum_{1 \leq i < j} x_i x_j$$

$$m_{(3)} = x_1^3 + x_2^3 + \cdots$$

$$m_{(2,1)} = \prod_{\substack{i,j \geq 1 \\ i \neq j}} x_i^2 x_j$$

$$m_{(1^3)} = \prod_{1 \leq i < j < k} x_i x_j x_k$$

Besides the monomial symmetric functions there are several other important bases of Λ.

For $r \geqslant 0$ the rth <u>elementary symmetric function</u> e_r is defined as

$$e_r = m_{(1^r)} = \sum_{1 \leq i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Clearly $$\sum_{r \geqslant 0} e_r z^r = \sum_{I \subseteq \mathbb{Z}} \prod_{i \in I} z x_i = \prod_{i \geq 1} \left(\sum_{k=0}^{1} (z x_i)^k \right)$$
$$= \prod_{i \geq 1} (1 + z x_i)$$

<u>Lemma</u>(*) $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$ (and $\Lambda_n = \mathbb{Z}[e_1, \dots, e_n]$)

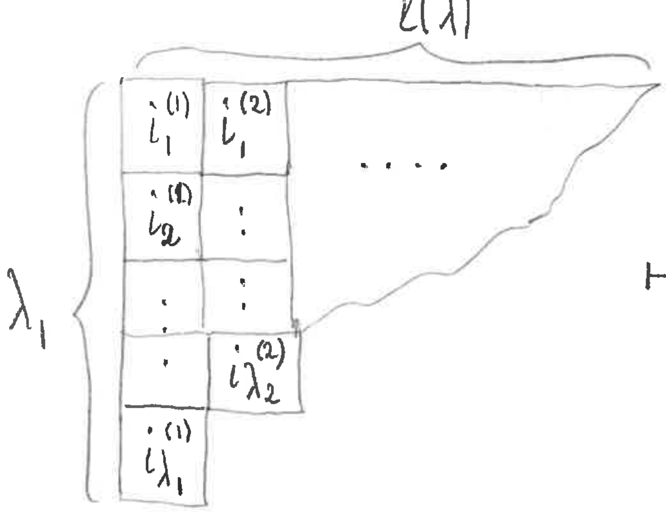
Pf Define $e_\lambda := \prod_{i \geq 1} e_{\lambda_i}$ (recall that $e_0 = 1$)

Each monomial contributing to e_r can be written as a column-strict tableau of shape (1^r):

$$\begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \\ \hline \\ \hline i_r \\ \hline \end{array} \longmapsto x_{i_1} x_{i_2} \cdots x_{i_r}$$

(*) fundamental theorem of symmetric functions.

Hence each monomial contributing to e_λ can be written as a column-strict tableau of shape λ':



$$\mapsto \prod_{k\geq 1} \dots x_{i_1^{(k)}} \dots x_{i_{\lambda_k}^{(k)}}$$

Eg $\quad e_{(2,1,1)} = e_2 e_1^2$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3\\\cline{1-1}\end{array} + \dots + \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 3\\\cline{1-1}\end{array} + \text{etc.}$$

In mth row of of each tableau all entries must be greater or equal to m.

$$x_1^{\lambda_1'} x_2^{\lambda_2'} \dots$$
$$\updownarrow$$

$\Rightarrow \# m$'s is at most λ_m' :

$$\begin{array}{|c|c|c|c|c|c|l}\hline 1&1&1&1&1&1&\lambda_1'\\\hline 2&2&2&2&2&\multicolumn{1}{c}{}&\lambda_2'\\\cline{1-5} 3&3&3&\multicolumn{1}{c}{}&&&\lambda_3'\\\cline{1-3} \vdots \end{array}$$

$$\Rightarrow e_\lambda = m_{\lambda'} + \sum_{\mu < \lambda'} c_{\lambda\mu}\, m_\mu$$

$\Rightarrow \{e_{\lambda'}\}_{\lambda \in P}$ forms a $\mathbb{Z}$-basis of Λ $\qquad \square$

For $r \geq 0$ the rth complete symmetric function h_r is defined as

$$h_r = \sum_{\lambda \vdash r} m_\lambda = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Clearly, $\sigma_z(x_1, x_2, \ldots) := \sum_{r \geq 0} h_r z^r = \prod_{i \geq 1} \left(\sum_{k \geq 0} (z x_i)^k \right)$

$$= \prod_{i \geq 1} \frac{1}{1 - z x_i}$$

Comparing this with $\sum_{r \geq 0} e_r z^r = \prod_{i \geq 1} (1 + z x_i)$ shows that

$$\left(\sum_{r \geq 0} e_r (-z)^r \right) \sigma_z = 1$$

Hence $$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = \delta_{n,0} \qquad (*)$$

We shall see later using plethystic notation that e_r & h_r are really two sides of the same coin.

We can define an involution $\omega : \Lambda \to \Lambda$ by
$\omega(e_r) = h_r$.

That this indeed an involution follows from (*):

$$\sum_{r=0}^{n} (-1)^r h_r \, \omega(h_{n-r}) = \delta_{n,0} \underset{r \to n-r}{\Rightarrow} \sum_{r=0}^{n} (-1)^r \omega(h_r) \, h_{n-r} = \delta_{n,0}$$

so that $\omega(h_r) = e_r$.

Consequently $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$ & $\Lambda_n = \mathbb{Z}[h_1, \dots, h_n]$

<u>Remark</u> In Λ_n, $e_r = 0$ for $r > n$ but $h_r \neq 0$ for $r > n$. However, the $h_1, h_2, \dots$ are no longer algebraically independent. Eg, in Λ_2, $h_3 = 2h_2h_1 - h_1^3 = 2h_{(2,1)} - h_{(1^3)}$

For $r \geq 1$ the rth <u>power sum</u> p_r is defined as

$$p_r = m_{(r)} = \sum_{i \geq 1} x_i^r$$

Then $\psi_z(x_1, x_2, \dots) := \sum_{r \geq 1} \frac{p_r z^r}{r} = $ p.l.o.

$$= \sum_{r\geq 1} \sum_{i\geq 1} \frac{(zx_i)^r}{r} = -\sum_{i\geq 1} \log(1-zx_i)$$
$$= \log \sigma_z$$

In other words, $\sigma_z = e^{\psi_z}$ and $\sigma_z' = \psi_z' \sigma_z$

This implies Newton's relations

$$n h_n = \sum_{r=1}^{n} p_r h_{n-r} \qquad (*)$$

Pf. $\times$ by z^{n-1} & $\sum_{n\geq 1}$ $\Rightarrow$ $\sigma_z' = \sum_{r\geq 1} \sum_{n\geq r} p_r h_{n-r} z^{n-1}$

$$= \sum_{r\geq 1} p_r z^{r-1} \sum_{n\geq 0} h_n z^n = \psi_z' \sigma_z . \qquad \square$$

An immediate consequence of $(*)$ is that

$$\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \dots]$$

but the power sums do not form a $\mathbb{Z}$-basis of Λ:

$$h_2 = \frac{1}{2}(p_1^2 + p_2)$$

Set $p_0 := 1$ and define $p_\lambda := \prod_{i \geq 1} p_{\lambda_i}$ &

$z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ (see tutorial question 2)

__Lemma__ $\sigma_z = \sum_\lambda \frac{p_\lambda z^{|\lambda|}}{z_\lambda}$ i.e., $h_r = \sum_{\lambda \vdash r} \frac{p_\lambda}{z_\lambda}$

__Pf__ $\sigma_z = e^{\psi_z} = e^{\sum_{r \geq 1} \frac{p_r z^r}{r}} = \prod_{r \geq 1} e^{\frac{p_r z^r}{r}}$

$$= \prod_{r \geq 1} \sum_{m_r} \frac{\left(\frac{p_r z^r}{r}\right)^{m_r}}{m_r!} \underset{\substack{\uparrow \\ \lambda := (1^{m_1} 2^{m_2} \ldots) \\ \text{so that } |\lambda| = \sum_{r \geq 1} r m_r}}{=} \sum_\lambda \frac{p_\lambda z^{|\lambda|}}{z_\lambda}$$

$\square$

(3) Plethystic or λ-ring notation

The ring Λ may be viewed as a free λ-ring in a single variable. Without formally defining λ-rings we briefly discuss some convenient notation stemming from this point of view.

Since $f \in \Lambda$ is symmetric it is natural to think of symmetric functions as operators acting on sets, like $\{x_1, x_2, ..\}$, which we will call <u>alphabets</u>

Instead of the usual set notation, we adopt additive notation, writing

$$X = \{x_1, x_2, \dots\} = x_1 + x_2 + \cdots$$

To avoid confusion, when such notation is used for symmetric functions, <u>plethystic brackets</u> $[\cdot]$ are used; $f(X) = f(x_1, x_2, ..) = f[X] = f[x_1 + x_2 + \cdots]$

The idea is now to allow for more complicated alphabets, not all of which are necessarily countable. (When an alphabet X is countable, we can write $X = \sum_{x \in X} x$).

First we simply consider $X + Y := \sum_{x \in X} x + \sum_{y \in Y} y$ for countable alphabets (set union if X & Y are disjoint). Obviously, by the definition of p_r,

$$p_r[X+Y] = p_r[X] + p_r[Y],$$

and for example $p_r[\underbrace{X + \cdots + X}_{n \text{ times}}] = p_r[nX] = n p_r[X]$

For arbitrary alphabets (we are yet to construct example of non-countable alphabets) we use the same definition of $X+Y$ (or p_r acting on $X+Y$)

$$p_r[X+Y] := p_r[X] + p_r[Y].$$

Given X, Y we now form $X-Y$ as the alphabet such that

$$\mathrm{Pr}\left[X \underset{\text{plethystic-sign}}{-} Y\right] = \mathrm{Pr}[X] \underset{\text{normal-sign}}{-} \mathrm{Pr}[Y], \quad r \geq 1$$

as well as XY $\left(= \left(\sum_{x \in X} x\right)\left(\sum_{y \in Y} y\right)\right.$ in the countable case i.e. Cartesian product$\Big)$

$$\mathrm{Pr}[XY] = \mathrm{Pr}[X]\,\mathrm{Pr}[Y].$$

Note that we can manipulate alphabets as if they are ordinary elements of a commutative ring

$$\mathrm{Pr}\left[(X-Y)+Y\right] = \mathrm{Pr}\left[X+(Y-Y)\right] = \mathrm{Pr}[X]$$

$$\mathrm{Pr}\left[X(Y-Z)\right] = \mathrm{Pr}\left[XY-XZ\right] \quad \text{etc.}$$

We have addition & multiplication but only a special case of division

$$\mathrm{Pr}\left[\frac{X}{1-q}\right] := \frac{\mathrm{Pr}[X]}{1-q^r} = \mathrm{Pr}[X]\,\mathrm{Pr}\left[1+q+q^2+\dots\right]$$

$$= \mathrm{Pr}\left[X\left(1+q+q^2+\dots\right)\right] \qquad r \geq 1$$

Note that $p_r\left[\frac{X(1-q)}{1-q}\right] \overset{①}{=} p_r[X]$

$$\overset{②}{=} p_r\left[\frac{X}{1-q}\right] p_r[1-q]$$

$$= \frac{p_r[X]}{1-q^r}(1-q^r) = p_r[X]$$

(In other words $1-q$ & $1+q+q^2+\dots$ are units).

Letters in an alphabet should not be confused with ordinary "scalars" or what are sometimes referred to as binomial variables. For example if z is a single letter alphabet then

$$p_r[zX] = z^r p_r[X]. \quad \text{But } p_r[\overset{\text{binomial variable}}{\overset{\downarrow}{n}}X] = n\, p_r[X]$$

(More generally, for ξ a binomial variable (e.g $\xi \in \mathbb{R}$)

$$p_r[\xi X] := \xi\, p_r[X]).$$

Sometimes it is also convenient to use an ordinary minus sign in plethystic notation, so we can represent the set of variables $\{-x_1, -x_2, \dots\} =: \varepsilon X$

Then $p_r[\varepsilon X] = (-1)^r p_r[X]$, so that

$$p_r[-\varepsilon X] = (-1)^{r-1} p_r[X] \qquad (*)$$

Lemma • $e_r[X] = (-1)^r h_r[-X]$

• $\omega : \Lambda \to \Lambda$ corresponds to the plethystic substitution $X \mapsto -\varepsilon X$.

Remark More generally, plethystic substitutions correspond to ring homomorphism, e.g.,

$$\begin{cases} \Lambda \to \Lambda \\ X \mapsto -\varepsilon X \end{cases}, \quad \begin{matrix} \Lambda \to \Lambda \otimes \Lambda \\ X \to X \pm Y \end{matrix}, \quad \begin{matrix} \Lambda \to \Lambda \otimes \mathbb{Q}(q) \\ X \to \dfrac{X}{1-q} \end{matrix}$$

Pf • $\sigma_z[-X] = e^{\Psi_z[-X]} \underset{\substack{\uparrow \\ p_r[-X] = -p_r[X]}}{=} e^{-\Psi_z[X]} = \dfrac{1}{\sigma_z[X]}$

$$= \sum_{r \geq 0} e_r[X](-z)^r \Rightarrow h_r[-X] = (-1)^r e_r[X]$$

• $\omega(\sigma_z[X]) = \sum_{r \geq 0} e_r[X] z^r = \sum_{r \geq 0} h_r[-X](-z)^r = \sigma_{-z}[-X]$

Hence, since $\psi_z = \log \sigma_z$,

$$\omega(\psi_z[X]) = \psi_{-z}[-X] \quad \text{, i.e.,}$$

$$\omega(p_r[X]) = (-1)^r p_r[-X] = p_r[-\varepsilon X] \qquad \square$$

<u>Corollary</u> $\omega(p_r) = (-1)^{r-1} p_r$

<u>Pf</u> This immediately follows from (*).

<u>Remark</u> Plethystic substitutions are closely related to the structure of Λ as a <u>self-dual</u>, cocommutative, graded <u>Hopf algebra</u>. The <u>comultiplication</u> $\mu : \Lambda \to \Lambda \otimes \Lambda$ can be realised as $f[X] \mapsto f[X+Y]$, the <u>multiplication</u> $m : \Lambda \otimes \Lambda \to \Lambda$ as $f[X]g[Y] \mapsto f[X]g[X]$ and the <u>antipode</u> $S : \Lambda \to \Lambda$ as $f[X] \mapsto f[-X]$.

We will explore this more in exercise 5.

4 The Hall scalar product

Definition The Cauchy product of two alphabets X & Y is given by $\sigma_1[XY]$

Lemma
$$\sigma_1[XY] = \sum_\lambda \frac{1}{z_\lambda} p_\lambda[X]p_\lambda[Y] \qquad (1)$$
$$= \sum_\lambda h_\lambda[X]\, m_\lambda[Y] \qquad (2)$$

Pf
$$\sigma_1[XY] = \sum_\lambda \frac{p_\lambda[XY]}{z_\lambda} \underset{\substack{\uparrow \\ p_r[XY]=p_r[X]p_r[Y]}}{=} \sum_\lambda \frac{p_\lambda[X]p_\lambda[Y]}{z_\lambda}$$

which gives (1). For (2) it suffices to consider $Y = y_1 + \cdots + y_n$. Then

$$\sigma_1[XY] \underset{\substack{\uparrow \\ XY=\sum_i Xy_i \ \& \ \sigma_z[A+B]=\sigma_z[A]\sigma_z[B]}}{=} \prod_{i=1}^n \sigma_{y_i}[X] = \prod_{i=1}^n \left(\sum_{r_i \geq 0} h_{r_i}[X]\, y_i^{r_i} \right)$$

$$\underset{\substack{\uparrow \\ \alpha=(r_1,\dots,r_n) \\ \text{composition}}}{=} \sum_\alpha \left(\prod_{i=1}^n h_{\alpha_i}[X] \right) y^\alpha \qquad (y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n})$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} h_\lambda[X] \sum_{w\in S_n/S_n^\lambda} w(y^\lambda)$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} h_\lambda[X]\, m_\lambda[Y] \qquad \square$$

<u>Definition</u> (Hall scalar product on Λ)

$\langle\, ,\, \rangle : \Lambda \times \Lambda \to \mathbb{Z}$

$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$

<u>Proposition</u> Let (a_λ) & (b_λ) be bases of Λ. Then

$\langle a_\lambda, b_\mu \rangle = \delta_{\lambda\mu}$ (1) iff $\sum_\lambda a_\lambda[X]\, b_\lambda[X] = \sigma_1[XY]$

<u>Remark</u> • (a_λ) & (b_λ) as above are referred to as dual bases w.r.t. the Hall scalar product

• The lin. operator $f^\perp : \Lambda \to \Lambda$ for $f \in \Lambda$ is defined as $\langle f^\perp g, h \rangle = \langle g, fh \rangle$ and referred to as the adjoint of multiplication by f.

Pf We have $a_\lambda = \sum_\nu c_{\lambda\nu} h_\nu$, $b_\mu = \sum_\omega d_{\omega\mu} m_\omega$ so that

$\langle a_\lambda, b_\mu \rangle = \sum_\nu c_{\lambda\nu} d_{\nu\mu}$. Hence (1) is equivalent to

$$\sum_\nu c_{\lambda\nu} d_{\nu\mu} = \delta_{\lambda\mu} \Leftrightarrow \sum_\lambda d_{\mu\lambda} c_{\lambda\nu} = \delta_{\mu\nu}$$

Also

$$\sum_\lambda a_\lambda[X] b_\lambda[Y] \overset{①}{=} \sum_{\lambda,\mu,\nu} d_{\mu\lambda} c_{\lambda\nu} h_\nu[X] m_\mu[Y]$$

$$\overset{②}{=} \sigma_1[XY]$$

$$= \sum_\mu h_\mu[X] m_\mu[Y]$$

Hence (2) is also equivalent to $\sum_\lambda d_{\mu\lambda} c_{\lambda\nu} = \delta_{\mu\nu}$ □

5 Schur functions

Let $\ell(\lambda)\leqslant n$. Then the Schur function $s_\lambda(x_1,..,x_n)$ is defined as

$$s_\lambda(x_1,\dots,x_n) := \frac{\det\limits_{1\leqslant i,j\leqslant n}\left(x_i^{\lambda_j+n-j}\right)}{\prod\limits_{1\leqslant i<j\leqslant n}(x_i-x_j)} = \frac{\sum\limits_{w\in S_n}\operatorname{sgn}(w)\, w\left(x^{\lambda+\delta}\right)}{\prod\limits_{i<j}(x_i-x_j)}$$

where $\delta=(n-1,..,2,1,0)$, $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and

$$\prod_{1\leqslant i<j\leqslant n}(x_i-x_j) = \det_{1\leqslant i,j\leqslant n}\left(x_i^{n-j}\right) \quad \left(=\Delta(x_1,\dots,x_n)\right)$$

are the Vandermonde product and determinant respectively.

<u>Lemma</u> The Schur function s_λ is a symmetric polynomial of degree $|\lambda|$ and $\{s_\lambda(x_1,\dots,x_n)\}_{\substack{\ell(\lambda)\leqslant n\\ \lambda\vdash k}}$ forms a $\mathbb{Z}$-basis of Λ_n^k.

Pf Symmetry is obvious since both the numerator and denominator are skew symmetric polynomials. Since the numerator vanishes if $x_i = x_j$ for some $1 \leq i < j \leq n$ polynomiality is also clear.

The degree claim is also obvious.

Now, since $\left\{ \det_{1 \leq i,j \leq n} \left(x_i^{\mu_j} \right) \right\}_{\substack{\ell(\mu) \leq n \\ \mu \text{ strict}}}$ is a basis of the free $\mathbb{Z}$-module $\overset{A_n}{\vee}$ of skew-symmetric polynomials in $x_1, \ldots, x_n$ (We are anti-symmetrising x^μ which vanishes unless μ is strict). But a strict partition μ of length at most n can be represented as $\mu = \delta + \lambda$. Since

$$\begin{aligned} \varphi : \Lambda_n &\to A_n \\ f &\mapsto \Delta f \end{aligned}$$

is an isomorphism, $\{ s_\lambda \}_{\ell(\lambda) \leq n}$ is a $\mathbb{Z}$-basis for Λ_n

$\square$

Remark The same determinant definition may be used to define the Schur functions for arbitrary (weak) compositions $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then, if $\alpha = w(\lambda + \delta) - \delta$ for some partition λ and $w \in S_n$, then $s_\alpha = \operatorname{sgn}(w) s_\lambda$. Otherwise $s_\alpha = 0$

<u>Lemma</u> The Schur functions are stable:

$$s_\lambda(x_1,\dots,x_{n-1},0) = \begin{cases} s_\lambda(x_1,\dots,x_{n-1}) & \text{if } \ell(\lambda)\leqslant n-1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Pf</u>

$$s_\lambda(x_1,\dots,x_n) \underset{①}{\overset{\lambda_n=0}{=}} \frac{\det\limits_{1\leqslant i,j\leqslant n-1}\left(x_i^{\lambda_j+n-j}\right)}{\prod\limits_{1\leqslant i<j\leqslant n-1}(x_i-x_j)\prod\limits_{i=1}^{n} x_i}$$

$$= \frac{\det\limits_{1\leqslant i,j\leqslant n-1}\left(x_i^{\lambda_j+(n-1)-j}\right)}{\prod\limits_{1\leqslant i,j\leqslant n-1}(x_i-x_j)}$$

$$\underset{②}{\overset{\lambda_n>0}{=}} 0$$

□

It thus makes sense to define $s_\lambda(x_1,\dots,x_n)=0$ if $\ell(\lambda)>n$. Moreover, $s_\lambda(x_1,x_2,\dots)$ is well-defined and $\{s_\lambda\}$ forms a $\mathbb{Z}$-basis of Λ.

<u>Remark</u> It may be shown that the Schur functions on n letters are the characters of the polynomial representations of $GL(n, \mathbb{C})$ as well as related to the characters of S_n, see exercise 9. The occurence of both $GL(n,\mathbb{C})$ & S_n can be understood through Schur-Weyl duality.

A semistandard Young tableau of shape λ and content / filling / weight α on n letters is a filling of the Young diagram of λ with the numbers $1, 2, .., n$ such that row are weakly increasing from left to right and strictly increasing from top to bottom, and such that there are α_i boxes filled with i. (Hence $|\lambda| = |\alpha|$)

E.g
$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 4 & 6 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{l} \lambda = (5,4,2,2,1) \\ \alpha = (2,2,1,4,2,3) \end{array}$$

Note that a Young tableau T of shape λ can alternatively be represented by a sequence of interlacing partitions:

$$0 = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)} = \lambda$$

where the skew shape $\lambda^{(i)}/\lambda^{(i-1)}$ represents those boxes of T filled with i, i.e., $|\lambda^{(i)}/\lambda^{(i-1)}| = \alpha_i$

Eg
$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 4 & 6 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad 0 \prec (2) \prec (3,1) \prec (3,2) \prec (5,3,1) \prec (5,3,2,1) \prec (5,4,2,2,1)$$

Let $SSYT(\lambda,\alpha) = \{T : \text{shape}(T) = \lambda,\ \text{content}(T) = \alpha\}$

<u>Theorem</u>
$$s_\lambda = \sum_T x^T = \sum_{T \in SSYT(\lambda,\cdot)} x^{\text{content}(T)}$$

where, if $x = (x_1, \ldots, x_n)$, all T are on n letters.

Eg $s_{(2,1)}(x_1,x_2,x_3) = m_{(2,1)}(x_1,x_2,x_3) + 2\, m_{(1^3)}(x_1,x_2,x_3)$

$$\begin{array}{cccccccc}
\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 2&2\\ \hline 3\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1&3\\ \hline 3\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 2&3\\ \hline 3\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array} \\
x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_2^2x_3 & x_1x_3^2 & x_2x_3^2 & \multicolumn{2}{c}{\underbrace{\qquad\qquad}_{2x_1x_2x_3}}
\end{array}$$

<u>Remark</u> Since s_λ is symmetric, the theorem implies $|SSYT(\lambda,\alpha)| = |SSYT(\lambda, w(\alpha))|$, $w \in S_n$.
This number is known as the Koska number $K_{\lambda\alpha}$.

More generally, in the representation theory of semi-simple Lie algebras

$$\mathrm{char}_{V(\lambda)} = \sum_{\mu\in\mathfrak{h}^*} \mathrm{mult}(\mu)\, e^{\mu} = \sum_{\mu\in\mathfrak{h}^*} K_{\lambda\mu}\, e^{\mu}$$

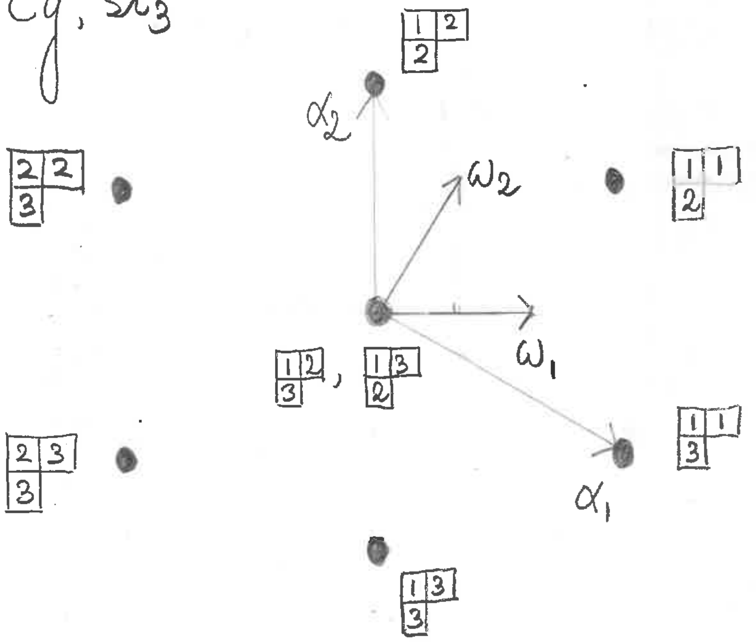
where $V(\lambda)$ is an irreducible $\mathfrak{g}$-module of highest weight λ, μ is an arbitrary weight and $K_{\lambda\mu} = \mathrm{mult}(\mu)$ is the dimension of the weight space indexed by μ in the weight space decomposition of $V(\lambda)$.

We also note that, since $K_{\lambda\alpha} = K_{\lambda w(\alpha)}$ we may write

$$s_\lambda = \sum_{\mu} K_{\lambda\mu}\, m_\mu$$

summed over partitions μ such that $|\mu| = |\lambda|$.

Eg. sl_3



$$V(\omega_1+\omega_2)$$

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = (e_1 \wedge e_2) \otimes e_1 \quad ; \quad f_2 \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = f_2(e_1 \wedge e_2) \otimes e_1 \oplus (e_1 \wedge e_2) \otimes \underbrace{f_2 e_1}_{=0}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (e_1 \wedge e_3) \otimes e_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\| $$

$$f_1 \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = \overbrace{f_1(e_1 \wedge e_2)}^{0} \otimes e_1 \oplus (e_1 \wedge e_2) \otimes f_1 e_1$$

$$= (e_1 \wedge e_2) \otimes e_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$f_2 \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = f_2(e_1 \wedge e_2) \otimes e_2 \oplus (e_1 \wedge e_2) \otimes f_2 e_2 = (e_1 \wedge e_3) \otimes e_2 \oplus (e_1 \wedge e_2) \otimes e_3$$

$$= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

<u>Pf</u> By the correspondence between semistandard tableaux and sequences of interlacing partitions,

$$\sum_T x^T = \sum_{T \in SSYT(\lambda,\cdot)} x^{\mathrm{content}(T)}$$

$$= \sum_{0=\lambda^{(0)} \prec \dots \prec \lambda^{(n)} = \lambda} \prod_{i=1}^{n} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}$$

$$\underset{\mu := \lambda^{(n-1)}}{\overset{\uparrow}{=}} \sum_{\mu \prec \lambda} x_n^{|\lambda/\mu|} \sum_{0=\lambda^{(0)} \prec \dots \prec \lambda^{(n-1)} = \mu} \prod_{i=1}^{n-1} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}$$

$$= \sum_{\mu \prec \lambda} x_n^{|\lambda/\mu|} S_\mu(x_1, \dots, x_{n-1})$$

It thus suffices to prove the <u>branching rule</u>

$$S_\lambda(x_1, \dots, x_n) = \sum_{\mu \prec \lambda} x_n^{|\lambda/\mu|} S_\mu(x_1, \dots, x_{n-1})$$

(Clearly both descriptions of the Schur functions satisfy the same initial condition $S_\lambda[\underset{\text{empty alphabet}}{\overset{\uparrow}{0}}] = \delta_{\lambda 0}$

By homogeneity it suffices to prove

$$s_\lambda(x_1,\dots,x_{n-1},1) = \sum_{\mu\prec\lambda} s_\mu(x_1,\dots,x_{n-1})$$

(so that $s_\lambda[X+1] = \sum_{\mu\prec\lambda} s_\mu[X]$)

From the determinantal definition of s_λ:

$$s_\lambda(x_1,\dots,x_{n-1},1) = \frac{\det\begin{pmatrix} x_i^{\lambda_j+n-j} & i<n \\ 1 & i=n \end{pmatrix}}{\Delta(x_1,\dots,x_{n-1})\prod_{i=1}^{n-1}(x_i-1)}$$

subtract last row from row i & divide by (x_i-1) $\rightarrow$ $= \det\begin{pmatrix} \sum_{k=0}^{\lambda_j+n-j-1} x_i^k & i<n \\ 1 & i=n \end{pmatrix} / \Delta_{n-1}$

subtract column 2 from col 1
" C3 from C2
etc
$\rightarrow$ $= \det\begin{pmatrix} \sum_{k=\lambda_{j+1}+n-j-1}^{\lambda_j+n-j-1} x_i^k & i<n \\ \delta_{jn} & i=n \end{pmatrix} / \Delta_{n-1}$

$$= \det_{1\leq i,j\leq n-1}\left(\sum_{\mu_j=\lambda_{j+1}}^{\lambda_j} x_i^{\mu_j+n-j-1}\right) / \Delta_{n-1}$$

multilinearity $\rightarrow$ $= \sum_{\mu\prec\lambda} \underbrace{\det_{1\leq i,j\leq n-1}\left(x_i^{\mu_i+(n-1)-j}\right) / \Delta_{n-1}}_{s_\mu(x_1,\dots,x_{n-1})}$ $\square$

The branching rule can also be written as

$$s_\lambda(x_1,\dots,x_n) = \sum_\mu s_{\lambda/\mu}(x_n)\, s_\mu(x_1,\dots,x_{n-1})$$

where the <u>skew Schur function</u> is defined by

$$s_\lambda[X+Y] = \sum_\mu s_{\lambda/\mu}[X]\, s_\mu[Y]$$

Clearly, $s_{\lambda/\mu}(z) = \begin{cases} z^{|\lambda/\mu|} & \text{if } \lambda > \mu \\ 0 & \text{otherwise} \end{cases}$

and, more generally, for $X = x_1 + x_2 + \cdots + x_n$

$$s_{\lambda/\mu}[X] = \prod_{i=1}^{n} s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i) \quad , \quad \lambda^{(n)} := \lambda,\ \lambda^{(0)} := \mu$$

Hence also

$$s_{\lambda/\mu} = \sum_{SSYT(\lambda/\mu,\,\cdot\,)} x^T$$

Remark We have

$$s_\lambda[X+u+v] \overset{①}{=} \sum_\mu s_{\lambda/\mu}(u)\, s_\mu[X+v]$$

$$= \sum_{\mu,\nu} s_{\lambda/\mu}(u)\, s_{\mu/\nu}(v)\, s_\nu[X]$$

$$\overset{②}{=} \sum_\mu s_{\lambda/\mu}(v)\, s_\mu[X+u]$$

$$= \sum_{\mu,\nu} s_{\lambda/\mu}(v)\, s_{\mu/\nu}(u)\, s_\nu[X]$$

and thus $\displaystyle\sum_\mu s_{\lambda/\mu}(u)s_{\mu/\nu}(v) = \sum_\mu s_{\lambda/\mu}(v)s_{\mu/\nu}(u)$.

Defining the 'transfer matrix' $T(u)$ with $(T(u))_{\lambda,\mu} = s_{\lambda/\mu}(u)$, we see that

$T(u)\,T(v) = T(v)\,T(u)$, a hallmark of quantum integrability.

Theorem (Jacobi-Trudi identity)

Let λ be a partition of length at most k. Then

$$s_\lambda = \det_{1\leq i,j\leq k}\left(h_{\lambda_i+j-i}\right) \qquad (h_r := 0 \text{ for } r<0)$$

Remark • More generally $s_{\lambda/\mu} = \det_{1\leq i,j\leq k}\left(h_{\lambda_i-\mu_j+j-i}\right)$

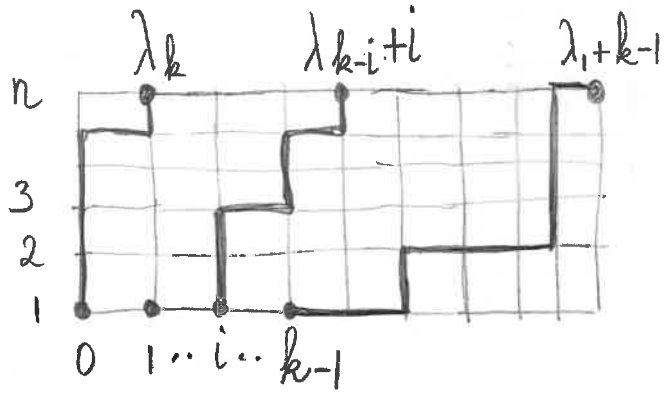
Proof We sketch the proof, which is essentially an application of a special case of the Lindström-Gessel-Viennot lemma.

Wlog we may assume $\ell(\lambda)=k$.

Recall $s_\lambda = \sum_{T\in SSYT(\lambda,\cdot)} x^{\text{content}(T)}$

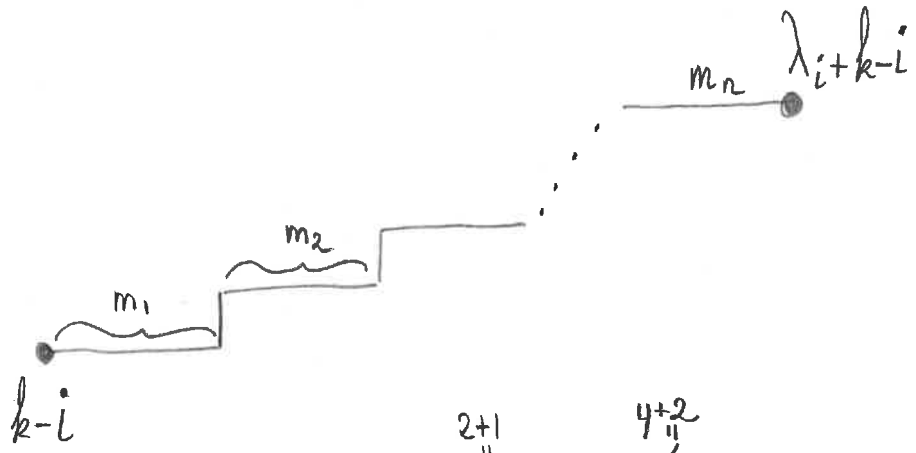
Each T in SSYT is in bijection with a set of nonintersecting lattice paths in a rectangular grid as follows.

If $T \in SSYT(\lambda, \alpha)$, $\lambda = (\lambda_1, \dots, \lambda_k)$
$\alpha = (\alpha_1, \dots, \alpha_n)$

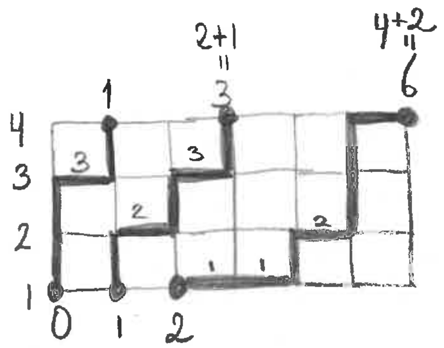


$\longleftrightarrow \quad T$

where, if the ith row of T has entries $1^{m_1} 2^{m_2} \dots n^{m_n}$ then the ith path from the right is



E.g. $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$

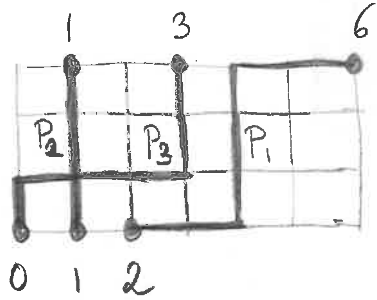


$= P$

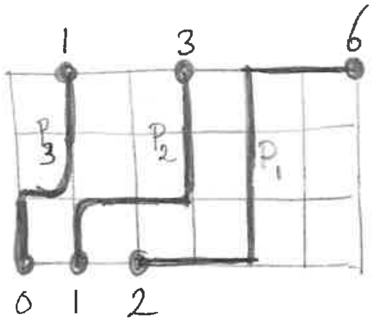
$$\text{weight}(P) = \prod_i x_i^{\#\text{ horizontal steps at height } i} := x^P$$

Enlarge the set of paths by considering all paths from $(0,1,\dots,k)$ to $(\lambda_k, \lambda_{k-1}+1, \dots, \lambda_1+k-1)$

E.g.



$\sigma = (1,3,2)$



$\sigma = (1,2,3)$

By assigning the sign $\mathrm{sgn}(\sigma)$ to each set of paths according to the permutation σ encoding the arrival order, and by noting the intersecting sets of paths can be paired according to a flip in the first crossing (so that pairs have opposite sign) only nonintersecting sets of path contribute.

Hence

$$S_\lambda(x_1,\dots,x_n) = \sum_{P \text{ nonintersecting}} x^P$$

$$= \sum_{\text{all } P} \operatorname{sgn}(P)\, x^P$$

$$= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\text{all } P \text{ from} \\ k-\sigma_i \mapsto \lambda_i+k-i \\ 1\le i\le k}} x^P$$

$$= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} h_{(\lambda_i+\cancel{k}-i)-\cancel{k}-\sigma_i}(x_1,\dots,x_n)$$

$$= \det_{1\le i,j\le k}\left(h_{\lambda_i+j-i}(x_1,\dots,x_n)\right)$$

$\square$

Theorem (Cauchy identity)

$$\sum_\lambda s_\lambda[X]\, s_\lambda[Y] = \sigma_1[XY]$$

Pf There is a beautiful proof using RSK, but we have no time for that. Instead we will use the Jacobi-Trudi identity.

Let $X = \sum_{i=1}^n x_i$ and $Y = \sum_{i=1}^n y_i$.

Then $\sigma_1[XY] = \sum_\alpha h_\alpha[X]\, y^\alpha$; $h_\alpha := h_{\alpha_1} \cdots h_{\alpha_n}$

Vandermonde determinant $\quad \alpha = (\alpha_1, \dots, \alpha_n)$ a (weak) composition

$$\sum_{w \in S_n} sgn(w)\, w(y^\alpha) = \Delta(Y)$$

$$= \frac{1}{\Delta(Y)} \sum_{w \in S_n} \sum_\alpha sgn(w)\, h_\alpha[X]\, y^{\alpha + w(\delta)}$$

$$\overset{\beta := \alpha + w(\delta) - \delta}{=} \frac{1}{\Delta(Y)} \sum_\beta y^{\beta+\delta} \left(\sum_{w \in S_n} sgn(w)\, h_{\beta+\delta-w(\delta)}[X] \right)$$

$$\overset{JT}{=} \frac{1}{\Delta(Y)} \sum_\beta y^{\beta+\delta}\, s_\beta[X]$$

Since $s_\beta = 0$ unless $\beta = w(\lambda+\delta) - \delta$, we may replace $\sum_\beta \mapsto \sum_\lambda \sum_{w \in S_n}$

$$= \frac{1}{\Delta(Y)} \sum_\lambda s_\lambda[X] \overbrace{\sum_{w \in S_n} sgn(w)\, y^{w(\lambda+\delta)}}^{\Delta(Y) s_\lambda[Y]}$$

□

<u>Corollary</u> $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

<u>Lemma</u> $\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$

<u>Pf</u> Let the <u>Littlewood-Richardson</u> coeffs $c^\lambda_{\mu\nu}$ be defined as $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$

Then $\sum_{\lambda,\mu} s_{\lambda/\mu}[X] \; s_\mu[Y] s_\lambda[Z]$

$$= \sum_\lambda s_\lambda[X+Y] s_\lambda[Z]$$

$$= \sigma_1 [(X+Y)Z]$$

$$= \sigma_1 [XZ] \sigma_1 [YZ]$$

$$= \sum_{\mu,\nu} s_\nu[X] s_\nu[Z] s_\mu[Y] s_\mu[Z]$$

$$= \sum_{\lambda,\mu,\nu} c^\lambda_{\mu\nu} s_\nu[X] s_\mu[Y] s_\lambda[Z]$$

Equating coefficients of $s_\mu[Y]s_\lambda[Z]$ yields

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$$

Finally

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \sum_\omega c^\lambda_{\mu\omega} \langle s_\omega, s_\nu \rangle = c^\lambda_{\mu\nu}$$

and

$$\langle s_\lambda, s_\mu s_\nu \rangle = \sum_\omega c^\omega_{\mu\nu} \langle s_\lambda, s_\omega \rangle = c^\lambda_{\mu\nu} \qquad \square$$

<u>Theorem</u> (Pieri rule)

$$h_r s_\mu = \sum_{\substack{\lambda \supset \mu \\ |\lambda/\mu| = r}} s_\lambda$$

<u>Pf</u> $\sum_\mu \sum_{r \geq 0} z^r h_r[X] s_\mu[X] s_\mu[Y]$

$$= \sigma_z[X]\sigma_1[XY] = \sigma_1[X(Y+z)]$$

$$= \sum_\lambda s_\lambda[X] s_\lambda[Y+z]$$

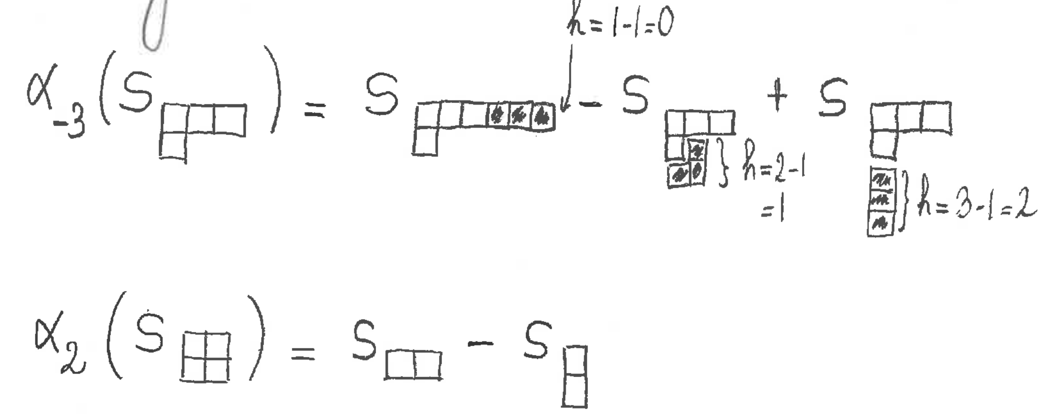
$$\underset{\substack{\uparrow\\ \text{branching rule}}}{=} \sum_{\lambda} s_\lambda[X] \sum_{\mu<\lambda} z^{|\lambda/\mu|} s_\mu[Y]$$

Equating coefficients of $z^r s_\mu[Y]$ yields the Pieri rule. $\square$

<u>Remark</u> The branching and Pieri rule may be regarded as dual. This can be made even more precise using vertex operators.

For $n \in \mathbb{Z}$ let $\alpha_{-n} : \Lambda \to \Lambda$ be the linear operator which adds a borderstrip/ribbon of size n is all possible ways to a Schur function, where each such strip is weighted by $(-1)^{\text{height}(b.s.)}$

Eg:



Then the α_n satisfy the commutation relations of the <u>Heisenberg algebra</u>:

$$[\alpha_n, \alpha_m] = n\,\delta_{n,-m}$$

E.g. $\alpha_{-2}\alpha_{-1} S_{\ydiagram{1}}$

$$= \alpha_{-2}\left(S_{\ydiagram{2}} + S_{\ydiagram{1,1}}\right)$$

$$= \left(S_{\ydiagram{4}} + \cancel{S_{\ydiagram{2,2}}} - S_{\ydiagram{2,1,1}}\right) + \left(S_{\ydiagram{3,1}} - \cancel{S_{\ydiagram{2,2}}} - S_{\ydiagram{1,1,1,1}}\right)$$

$\alpha_{-1}\alpha_{-2} S_{\ydiagram{1}}$

$$= \alpha_{-1}\left(S_{\ydiagram{3}} - S_{\ydiagram{1,1,1}}\right)$$

$$= S_{\ydiagram{4}} + S_{\ydiagram{3,1}} - S_{\ydiagram{2,1,1}} - S_{\ydiagram{1,1,1,1}}$$

Using the α_n we can define the vertex operators

$$\Gamma_{\pm}(z) = \exp\left(\sum_{n\geq 1} \frac{z^n}{n}\,\alpha_{\pm n}\right)$$

E.g. $\Gamma_-(z) = 1 + \alpha_{-1} z + \frac{z^2}{2}(\alpha_{-2} + \alpha_{-1}^2)$
$$+ \frac{z^3}{6}\left(2\alpha_{-3} + 3\alpha_{-2}\alpha_{-1} + \alpha_{-1}^3\right) + \cdots$$

The vertex operators satisfy the commutation relation

$$\Gamma_+(w)\Gamma_-(z) = \frac{1}{1-zw}\,\Gamma_-(z)\Gamma_+(w)$$

Moreover,

$$\Gamma_-(z)\, S_\mu[X] = \sigma_z[X]\, S_\mu[X] \quad (\text{Pieri})$$

$$\Gamma_+(z)\, S_\mu[X] = S_\mu[X+z] \quad (\text{branching})$$

6) Schur processes in less than 5 minutes

(For more, see Okounkov (2001), Okounkov-Reshetikhin (2003), Borodin-Rains (2006), and many subsequent papers, including works by our magnificent host Leo P.)

Let G be a finite group and consider (for simplicity) the set I_G of irreducible representations over $\mathbb{C}$.

(This set is in 1-1 correspondence with the set of conjugacy classes of G.)

From character theory it immediately follows that

$$\sum_{\rho \in I_G} (\dim \rho)^2 = |G|$$

Hence we can define the <u>Plancherel measure</u> on I_G by

$$\mu(\rho) = \frac{(\dim \rho)^2}{|G|}$$

For example, for $G = S_n$, we can label the irreps by $\lambda \vdash n$ and write

$$\mu(\lambda) = \frac{(f^\lambda)^2}{n!}$$

where $f^\lambda = |SYT(\lambda)|$, $SYT(\lambda)$ the set of standard Young tableaux of shape λ

E.g. $SYT(2,1) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \right\}$.

By the Frame-Robinson-Thrall hook-length formula,

$$f^\lambda = \frac{n!}{\prod_{h \in \mathcal{H}} h}$$

multiset of hook-lengths

E.g. $f^{(2,1)} = \frac{3!}{1 \cdot 1 \cdot 3} = 2$.

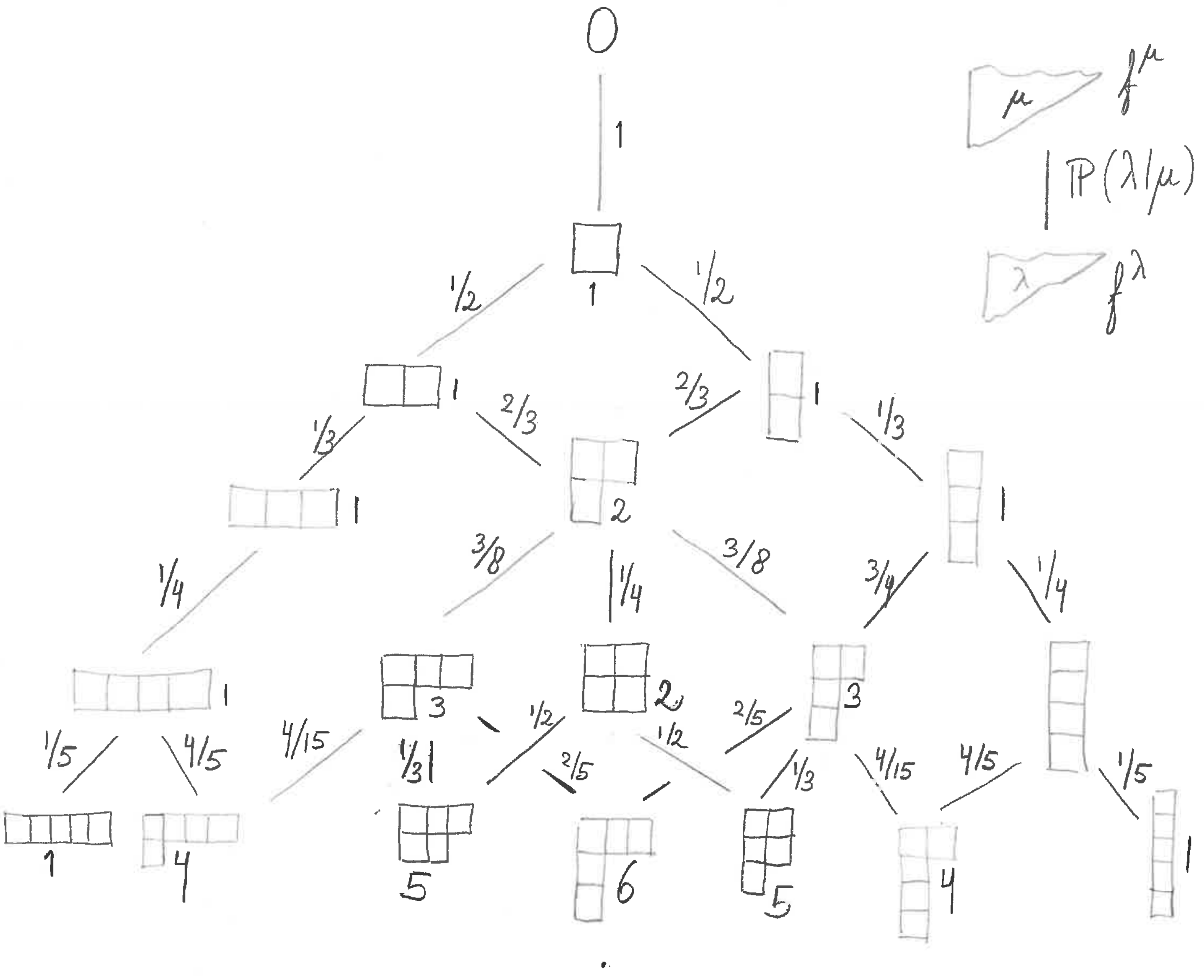
We may obviously view $\mu(\lambda)$ as a measure on the set of partitions of size n.

(This can be turned into a measure on all partition through 'Poissonisation' $\mu_{PPM}(\lambda) := e^{-\xi} \xi^{|\lambda|} \frac{(f^\lambda)^2}{|\lambda|!}$, $\xi > 0$)

Correspondingly, one can define the Plancherel (growth) process as a directed random walk on the Young lattice, with transition probabilities

$(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \dots)$, $|\lambda^{(n)}| = n$

$$\mathbb{P}\left(\lambda^{(n)} = \lambda \mid \lambda^{(n-1)} = \mu\right) = \frac{f^{\lambda}}{n\, f^{\mu}}$$



$\lambda \vdash 5$

Eg. $\mathbb{E}\left(\ell(\lambda)\right) = \frac{67}{24} \approx 2.79 < \frac{20}{7}$ for uniform distribution.

$\mathbb{E}\left(f^{\lambda}\right) = \frac{149}{30} \approx 4.97 < \frac{26}{7}$ ″

In the Plancherel process, one □ is added to a partition at every time step. More generally, let $\omega = (\omega_1, \omega_2, \dots)$ be a finite or infinite sequence with $\omega_i \in \{\prec, \succ, \prec', \succ'\}$ where $\mu \prec' \lambda$ iff $\mu' \prec \lambda'$ (i.e $\mu \prec' \lambda$ iff λ/μ is a vertical strip) and

$\Lambda = (\underset{0}{\underset{\shortparallel}{\lambda^{(0)}}}, \lambda^{(1)}, \lambda^{(2)}, \dots)$ a sequence of partitions

such that $\lambda^{(i-1)} \omega_i \lambda^{(i)}$

Then the Schur process is the measure on ω-interlaced partitions, such that

$$\mathrm{Prob}(\Lambda) \propto \prod_{i \geq 1} x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|}$$

Many variants are possible, e.g.,

$$\Lambda = (\mu = \lambda^{(-r)}, \lambda^{(1-r)}, \dots, \lambda^{(-1)}, \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(s)} = \lambda)$$

etc

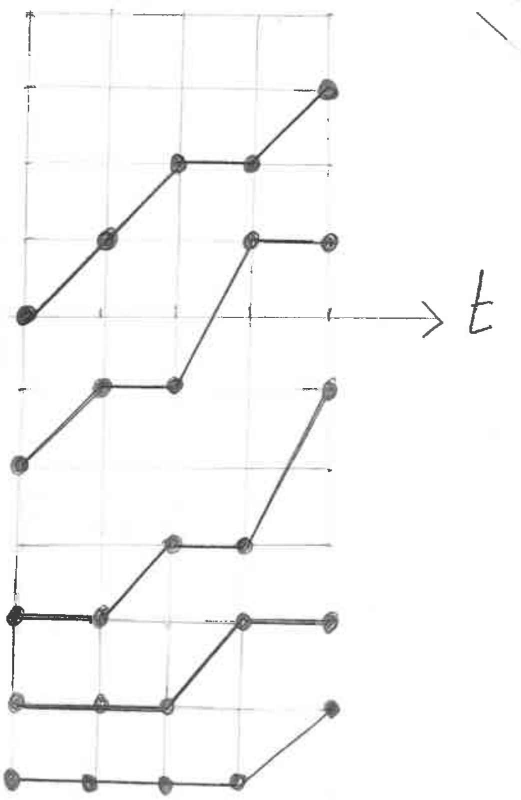
Examples

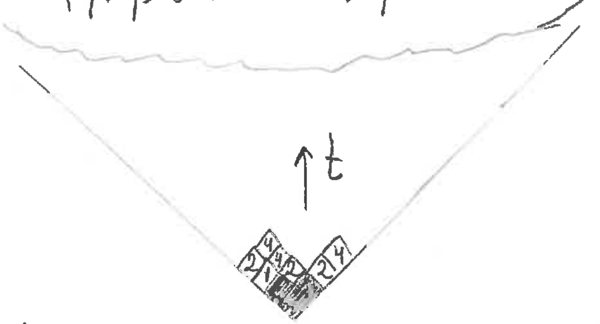
(1) $\omega = \{\prec\}^n$

$$\Lambda = \{\mu = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \dots \prec \lambda^{(n)} = \lambda\}$$

$$\mathrm{Prob}(\Lambda) = \frac{\prod_{i\geq 1} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}}{s_{\lambda/\mu}(x_1,\dots,x_n)}$$

Eg $(2,1) \prec (3,2) \prec (4,2,1) \prec (4,4,1,1) \prec (5,4,3,1,1)$





The point configuration $\mathcal{L}(\Lambda) = \{(t, \lambda_i^{(t)} - i)\}_{\substack{i\geq 1\\ 0\leq t\leq T}}$
$\subseteq \{0,\dots,T\}\times\mathbb{Z}$

(2) $\omega = \{ \underbrace{<, <, \dots, <}_{r} \ \underbrace{>, \dots >, >}_{s} \}$

$$\Lambda = \{ 0 = \lambda^{(-r)} < \dots < \lambda^{(-1)} < \lambda^{(0)} > \lambda^{(1)} > \dots > \lambda^{(s)} = 0 \}$$

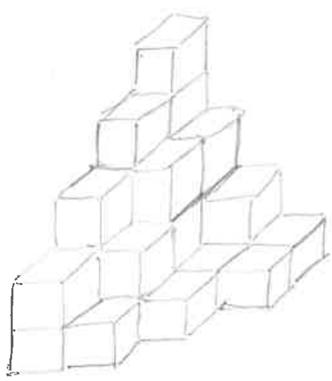
Eg (After Okounkov & Reshetikhin)

$$(2) < (3,1) < (4,2) < (5,3,1) > (3,1) > (2,1) > (1)$$

$\xleftarrow{-t}$ $\prod_i \Gamma_-(y_i)$

$\prod_i \Gamma_-(x_i)$ $\uparrow$ $\quad t \uparrow$

5	3	2	1	$\nwarrow y_1$
4	3	1	1	$\nwarrow y_2$
3	2	1	$\nwarrow y_s$	
2	1	$\nwarrow x_r$		
$\nwarrow x_1$	$\nwarrow x_2$			



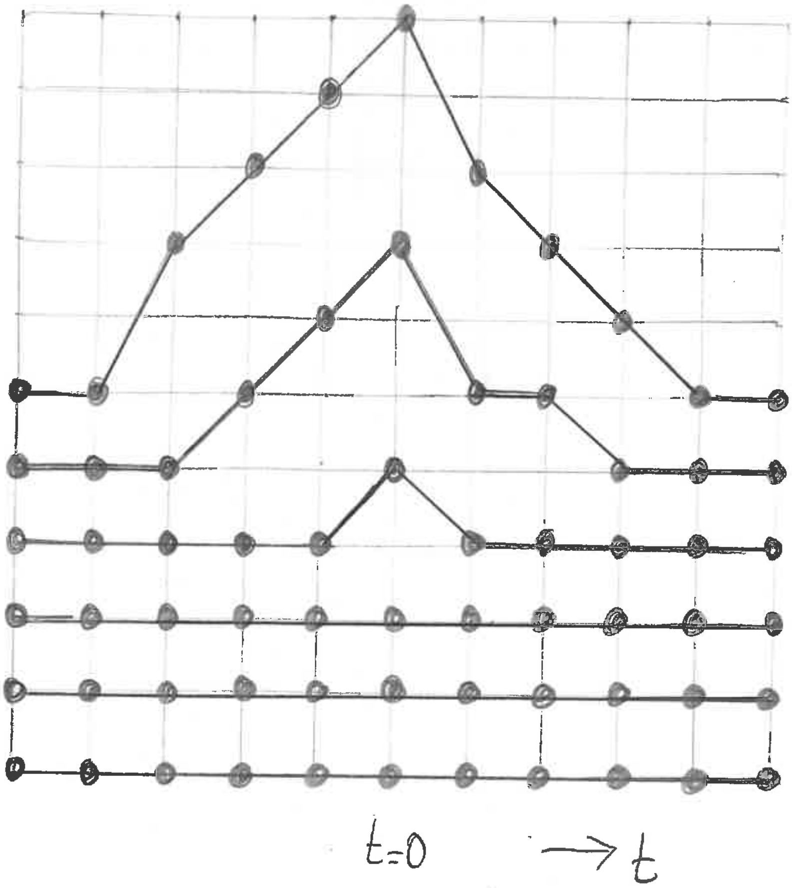
plane partition confined in a "box" $B(r,s,\infty)$.

MacMahon $\displaystyle\sum_{\pi \in B(r,s,\infty)} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1}{1-q^{i+j-1}}$

$$\sum_{\Lambda} \cdots = \sum_{\lambda^{(0)}} S_{\lambda^{(0)}}(x_1, \dots, x_r)\, S_{\lambda^{(0)}}(y_1, \dots, y_s)$$

$$\underset{\substack{\uparrow \\ \text{Cauchy}}}{=} \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1}{1-x_i y_j} \xrightarrow[y_i \mapsto q^{s-i+1/2}]{x_i \mapsto q^{r-i+1/2}} \text{MacMahon}$$

This time we have the point configuration



(3) $\omega = \underbrace{<, <', <, <', \dots, <, <'}_{r \text{ times}} \; \underbrace{>' > \cdots >' > >' >}_{s \text{ times}}$

Eg $\quad 0 < (1) <' (2) < (2,2) \underset{x_{-2}}{<'} (3,3) \underset{x_{-1}}{<} (3,3,2) \underset{x_1}{>'} (2,2,1) \underset{x_2}{>}$

$\quad > 21 >' (1,1) > (1) >' 0$

Pyramid partitions $\quad x_i = x_{-i} = q^{i-1/2}$

$$\sum_{\pi} q^{|\pi|} = \prod_{i \geqslant 1} \frac{(1+q^{2i-1})^{2i-1}}{(1-q^{2n})^{2n}} \qquad \text{(Young)}$$

Theorem (Okounkov-Reshetikhin '03)

The Schur process is a determinantal point process.

$$\mathbb{P}\left(\lambda^{(i_k)} \text{ has a } \bullet \text{ at } y_k \text{ for } 1 \leqslant k \leqslant n\right)$$

$$= \det_{1 \leqslant l, m \leqslant n} K(i_l, k_l; i_m, k_m)$$

$$K(i,k;j,l) = \begin{cases} \left[\dfrac{z^k}{w^l}\right] \dfrac{\phi(z; x_1, \ldots, x_n; \omega_1, \ldots, \omega_n; i)}{\phi(w; x_1, \ldots, x_n; \omega_1, \ldots, \omega_n; j)} \dfrac{(zw)^{1/2}}{z-w} & i \leqslant j \\ -\left[\dfrac{z^k}{w^l}\right] \dfrac{\phi(\quad " \quad ; j)}{\phi(\quad " \quad ; i)} \dfrac{(zw)^{1/2}}{w-z} & i > j \end{cases}$$

$$\phi(z; x_1, \ldots, x_n; \omega_1, \ldots, \omega_n; i)$$

$$= \prod_{\substack{j \leqslant i \\ \omega_j = <}} \sigma_z[x_j] \prod_{\substack{j \leqslant i \\ \omega_j = <'}} \sigma_z^{-1}[-x_j] \prod_{\substack{j > i \\ \omega_j = >}} \sigma_{z^{-1}}^{-1}[x_j] \prod_{\substack{j > i \\ \omega_j = >'}} \sigma_{z^{-1}}[-x_j]$$