A GENERALIZATION OF THE q-SAALSCHÜTZ SUM AND THE BURGE TRANSFORM

ANNE SCHILLING AND S. OLE WARNAAR

Abstract. A generalization of the q-(Pfaff)–Saalschütz summation formula is proved. This implies a generalization of the Burge transform, resulting in an additional dimension of the "Burge tree". Limiting cases of our summation formula imply the (higher-level) Bailey lemma, provide a new decomposition of the q-multinomial coefficients, and can be used to prove the Lepowsky and Primc formula for the $\mathrm{A}_1^{(1)}$ string functions.

1. Introduction

One of the most important summation formulas for basic hypergeometric functions is Jackson's q-analogue of a ${}_3F_2$ summation formula of Pfaff and Saalschütz. Employing standard notation (see e.g., Gasper and Rahman [13]) this q-(Pfaff)–Saalschütz sum is written as

(1.1)
$$
{}_3\phi_2\Big[\genfrac{}{}{0pt}{}{a,b,q^{-n}}{c,abq^{1-n}/c};q,q\Big]:=\sum_{k=0}^{n}\frac{(a)_k(b)_k(q^{-n})_k\,q^k}{(q)_k(c)_k(abq^{1-n}/c)_k}=\frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n}
$$

for $n\in\mathbb{Z}_+$. Here $(a)_n$ is the q-shifted factorial, defined for all integers n by

$$
(a;q)_\infty=(a)_\infty=\prod_{k=0}^{\infty}(1-aq^k)\qquad\text{and}\qquad(a;q)_n=(a)_n=\frac{(a)_\infty}{(aq^n)_\infty}.
$$

Defining the q-binomial coefficient as

$$
\begin{bmatrix}m+n\\ m\end{bmatrix}=\begin{cases}\dfrac{(q)_{m+n}}{(q)_m(q)_n} & \text{for } m,n\geq 0\\ 0 & \text{otherwise,}\end{cases}\tag{1.2}
$$

1991 *Mathematics Subject Classification.* Primary 33D15, 05A30, 05A10.

Key words and phrases. q-Saalschütz sum, Burge transform, Bailey lemma, q-multinomial coefficients, $\mathrm{A}_1^{(1)}$ string functions.

the q-Saalschütz sum is often written as the following summation formula [17, 9, 1]

(1.3)
$$\sum_{i=0}^{M} q^{i(i+\ell)} \begin{bmatrix} L_1+L_2+M-i \\ M-i \end{bmatrix} \begin{bmatrix} L_1 \\ i+\ell \end{bmatrix} \begin{bmatrix} L_2 \\ i \end{bmatrix} = \begin{bmatrix} L_1+M \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\ell \\ M \end{bmatrix},$$

valid for all $L_1, L_2, M, \ell \in \mathbb{Z}$ except when $-L_1 \le -\ell \le L_2 < 0 \le M$ or $-L_2 \le \ell \le L_1 < 0 \le M+\ell$. (In these cases the left-hand side is zero whereas the right-hand side is not.)

In this paper we generalize the representation (1.3) of the q-Saalschütz sum to a summation formula which transforms an N-fold sum over a product of $N+2$ q-binomials to an $(N-1)$-fold sum over a product of $N+1$ q-binomials as stated in Theorem 2.1 of the next section. This generalized q-Saalschütz sum contains many important special cases and can be applied in connection with the Burge transform, the Bailey lemma, q-multinomial coefficients and level-N $\mathrm{A}_1^{(1)}$ string functions as summarized below.

(1) In Ref. [7] Burge used equation (1.3) to establish a transformation on generating functions of (restricted) partition pairs. This "Burge transform", which generalizes a special case of the Bailey lemma, can be used to derive a tree of identities for doubly bounded Virasoro characters [7, 12]. Our generalization of (1.3) adds a further dimension to the Burge tree as discussed in Section 3.

(2) Letting b tend to infinity in (1.1) yields the q-Chu–Vandermonde summation [13, Eq. (II.7)]. The q-binomial version of this is obtained by letting M tend to infinity in (1.3), resulting in

$$\sum_{i=0}^{L_2} q^{i(i+\ell)} \begin{bmatrix} L_1 \\ i+\ell \end{bmatrix} \begin{bmatrix} L_2 \\ i \end{bmatrix} = \begin{bmatrix} L_1+L_2 \\ L_1-\ell \end{bmatrix} \tag{1.4}$$

for $L_1, L_2, M, \ell \in \mathbb{Z}$ except when $-L_1 \le -\ell \le L_2 < 0$ or $-L_2 \le \ell \le L_1 < 0$. This identity can be viewed as a decomposition of the q-binomial and is easily understood combinatorially using the notion of the Durfee rectangle of a partition.

The q-binomials have been generalized to q-trinomials in Ref. [3], and more generally to q-multinomials in Refs. [2, 8, 18, 22, 27]. Our generalized q-Saalschütz sum implies a generalized q-Chu–Vandermonde sum which provides a new decomposition formula for q-multinomials in terms of q-binomials (see Section 4.1).

(3) When L_1 and L_2 tend to infinity in (1.3) we are left with

$$\sum_{i=0}^{M} \frac{q^{i(i+\ell)}}{(q)_{M-i}} \frac{1}{(q)_i(q)_{i+\ell}} = \frac{1}{(q)_M(q)_{M+\ell}}. \tag{1.5}$$

Let $\{\gamma\}_{L\geq 0}$ and $\{\delta\}_{L\geq 0}$ be sequences that satisfy

$$\gamma_L = \sum_{r=L}^{\infty} \frac{\delta_r}{(q)_{r-L}(aq)_{r+L}}. \tag{1.6}$$

Then the pair (γ, δ) is called a conjugate Bailey pair relative to a [5, 24]. Replacing $M \to M-L$ and $\ell \to \ell + 2L$ in equation (1.5) implies the conjugate Bailey pair

$$\gamma_L = \frac{a^L q^{L^2}}{(q)_{M-L}(aq)_{M+L}}, \qquad \delta_L = \frac{a^L q^{L^2}}{(q)_{M-L}},$$

with $a = q^\ell$.

A limit of our generalized q-Saalschütz sum yields (a special case of) the higher-level generalization of this conjugate Bailey pair of Refs. [23, 24]. For details see Section 4.2. This paper thus provides a new proof of the higher-level Bailey lemma of [23, 24] for a special choice of one of the parameters.

(4) Finally, letting L_1, L_2 and M all tend to infinity in (1.3) yields the well-known Durfee rectangle identity

$$\sum_{i=0}^{\infty} \frac{q^{i(i+\ell)}}{(q)_i(q)_{i+\ell}} = \frac{1}{(q)_\infty}.$$

This formula has many interpretations. Here we only mention that the right-hand side can be identified with the level-1 $\mathrm{A}_1^{(1)}$ string function. Combined with the spinon formula of the string function of Refs. [4, 6, 20, 21, 25], the analogous limit of our generalized q-Saalschütz sum yields the fermionic expression for the string function due to Lepowsky and Primc [19] (see Section 4.3).

2. A generalized q-Saalschütz identity

The next theorem states the main result of this paper and provides a generalization of the q-Saalschütz summation formula (1.3). Let C be the Cartan matrix of A_{N-1} (i.e., $C_{ij} = 2\delta_{i,j} - \delta_{|i-j|,1}$ for $i,j = 1,\ldots,N-1$ where $\delta_{i,j}$ is the Kronecker delta symbol) and let $\mathcal{I} = 2I - C$ be the corresponding incidence matrix where I is the identity matrix. Furthermore, let $\boldsymbol{e}_i$, $i = 1,\ldots,N-1$ be the standard unit vectors in $\mathbb{Z}^{N-1}$, $(\boldsymbol{e}_i)_j = \delta_{i,j}$, and denote $\boldsymbol{n}C^{-1}\boldsymbol{n} = \sum_{i,j=1}^{N-1} n_i C^{-1}_{ij} n_j$ and $\boldsymbol{e}_i C^{-1}\boldsymbol{n} = (C^{-1}\boldsymbol{n})_i$ for $\boldsymbol{n} \in \mathbb{Z}^{N-1}$.

Theorem 2.1. *Let $\sigma = 0, 1$ and let $N, \ell, M, L_1 + \frac{\ell+\sigma}{2}, L_2 + \frac{\ell+\sigma}{2}$ be integers such that $\ell + \sigma N$ is even, $N \geq 1$ and $L_1, L_2 \geq 0$. Then*

$$\begin{aligned}
(2.1)\quad & \sum_{i=0}^{M} q^{i(i+\ell)/N} \begin{bmatrix} L_1+L_2+M-i \\ M-i \end{bmatrix} \\
& \times \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1} \\ \frac{2i+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{n} \end{bmatrix} \begin{bmatrix} L_1+\frac{1}{2}m_1 \\ i+\ell \end{bmatrix} \begin{bmatrix} L_2+\frac{1}{2}m_1 \\ i \end{bmatrix} \\
& = \sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1} \\ \frac{\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}} q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}} \begin{bmatrix} \boldsymbol{\mu}+\boldsymbol{\eta} \\ \boldsymbol{\eta} \end{bmatrix} \begin{bmatrix} L_1+\frac{1}{2}(M+\mu_1) \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+\frac{1}{2}(M+\ell+\mu_{N-1}) \\ M \end{bmatrix}
\end{aligned}$$

with

$$\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(\mathcal{I}\boldsymbol{m} + (2i+\ell)\boldsymbol{e}_1) \tag{2.2}$$

and

$$\boldsymbol{\mu}+\boldsymbol{\eta} = \frac{1}{2}(\mathcal{I}\boldsymbol{\mu} + (M+\ell)\boldsymbol{e}_1 + M\boldsymbol{e}_{N-1}). \tag{2.3}$$

The vector $\boldsymbol{m} \in \mathbb{Z}^{N-1}$ on the left-hand side is determined by the (summation) variable $\boldsymbol{n}$ through the $(\boldsymbol{m}, \boldsymbol{n})$-system (2.2). Similarly $\boldsymbol{\mu} \in \mathbb{Z}^{N-1}$ is determined by (2.3). Also, $\begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{n} \end{bmatrix} = \prod_{j=1}^{N-1} \begin{bmatrix} m_j+n_j \\ n_j \end{bmatrix}$ and similarly for $\begin{bmatrix} \boldsymbol{\mu}+\boldsymbol{\eta} \\ \boldsymbol{\eta} \end{bmatrix}$. We further note that the nature of the solutions of (2.2) depends on the parity of N. When N is odd one must have

$$\begin{aligned}
m_1 \equiv m_3 \equiv \cdots \equiv m_{N-2} \equiv 0 \pmod 2, \\
m_2 \equiv m_4 \equiv \cdots \equiv m_{N-1} \equiv \ell \pmod 2
\end{aligned}$$

whereas for N even one finds

$$\begin{aligned}
& m_1 \equiv m_3 \equiv \cdots \equiv m_{N-1} \pmod 2, \\
(2.4)\qquad & m_2 \equiv m_4 \equiv \cdots \equiv m_{N-2} \equiv \ell \equiv 0 \pmod 2.
\end{aligned}$$

This implies that m_1 is even for N odd so that L_1, L_2 must be integers. This indeed follows from (since N is odd) $0 \equiv \ell + \sigma N \equiv \ell + \sigma \pmod 2$. When N is even the partity of m_1 is not fixed and there is the freedom to choose m_1 even corresponding to $\sigma = 0$ or m_1 odd corresponding to $\sigma = 1$. (Since for N even $0 \equiv \ell + \sigma N \equiv \ell \pmod 2$, ℓ is even in accordance with (2.4) and hence, since L_1, L_2 must be integers when m_1 even and half an odd integer when m_1 odd, it thus follows from

$L_i + (\ell+\sigma)/2 \in \mathbb{Z}$ that σ has the same parity as m_1.) A similar analysis of the solutions of the $(\boldsymbol{\mu},\boldsymbol{\eta})$-system (2.3) can be carried out. The restrictions on the sums over $\boldsymbol{n}$ and $\boldsymbol{\eta}$ ensure that the components of $\boldsymbol{m}$ and $\boldsymbol{\mu}$ are integer and have the parity as discussed above.

Equation (2.1) yields a summation formula for every $N \geq 1$. When $N = 1$ the sums over $\boldsymbol{n}$ and $\boldsymbol{\eta}$ drop out; on the left-hand side $m_1 = 0$ and on the right-hand side one needs to interpret $\mu_1 = M$ and $\mu_0 = M + \ell$. Then (2.1) indeed reduces to (1.3) for $N = 1$.

Proof of Theorem 2.1. Note that both sides of (2.1) are zero unless $M + \ell \geq 0$ and $M \geq 0$. Furthermore, denoting the identity (2.1) by $I(L_1, L_2, M, \ell)$, it enjoys the symmetry $I(L_1, L_2, M, \ell) = I(L_2, L_1, M + \ell, -\ell)$. Hence we may assume $\ell \geq 0$ and $M \geq 0$ in the proof below.

Throughout the proof we use modified q-binomials defined as

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{(q^{n+1})_m}{(q)_m} \qquad \text{for } m \in \mathbb{Z}_+,\ n \in \mathbb{Z}, \tag{2.5}$$

and zero otherwise. Note that $\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right]$ is zero if $n < 0$ unless $m + n < 0$. Let us now show that on both sides of (2.1) the q-binomials (1.2) can be replaced by the modified q-binomials. Since $M, \ell, L_1, L_2 \geq 0$ we find from (2.2) and (2.3) that $m_i + n_i \geq 0$ and $\mu_i + \eta_i \geq 0$ if $m_i, \mu_i \geq 0$ so that $\left[\begin{smallmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{n} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \boldsymbol{\mu}+\boldsymbol{\eta} \\ \boldsymbol{\eta} \end{smallmatrix}\right]$ in (2.1) can be replaced by the modified q-binomials $\left[\begin{smallmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{m} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \boldsymbol{\mu}+\boldsymbol{\eta} \\ \boldsymbol{\mu} \end{smallmatrix}\right]$, respectively. The other q-binomials can be turned into modified q-binomials since the top entries are nonnegative by the conditions on the parameters.

The proof of (2.1) makes frequent use of the following identity which is a corollary of Sears' transformation formula for a balanced ${}_4\phi_3$ series [13, Eq. (III.15)]

$$\begin{aligned} \sum_{i\in\mathbb{Z}} q^{i(i-a+e+g)} \left[\begin{smallmatrix} i+a \\ a \end{smallmatrix}\right] \left[\begin{smallmatrix} b-i \\ c-i \end{smallmatrix}\right] \left[\begin{smallmatrix} d \\ i+e \end{smallmatrix}\right] \left[\begin{smallmatrix} f \\ i+g \end{smallmatrix}\right] \\ = \sum_{i\in\mathbb{Z}} q^{i(i-a+e+g)} \left[\begin{smallmatrix} a-g \\ a-g-i \end{smallmatrix}\right] \left[\begin{smallmatrix} b-d+e \\ c-i \end{smallmatrix}\right] \left[\begin{smallmatrix} c+d-i \\ c+e \end{smallmatrix}\right] \left[\begin{smallmatrix} i+f \\ i+g \end{smallmatrix}\right], \end{aligned} \tag{2.6}$$

where $a, b, c, d, e, f, g \in \mathbb{Z}$ and the condition $a + b = c + d + f$ applies. Since we need the Sears transform (2.6) with negative entries in the q-binomials it is essential that definition (2.5) is used here. (The above formula is not correct for all $a, \ldots, g \in \mathbb{Z}$ with the use of (1.2)).

We start by shifting $\boldsymbol{n} \to \boldsymbol{n} + i\boldsymbol{e}_1$, followed by $i \to i - n_1$. This transforms the left-hand side of (2.1) into

$$\sum_{i,\boldsymbol{n}} q^{(i-n_1)(i-n_1-m_1+\ell)+\boldsymbol{n}C^{-1}\boldsymbol{n}} \\ \times \begin{bmatrix} L_1+L_2+M+n_1-i \\ M+n_1-i \end{bmatrix} \begin{bmatrix} L_1+\frac{1}{2}m_1 \\ i+\ell-n_1 \end{bmatrix} \begin{bmatrix} L_2+\frac{1}{2}m_1 \\ i-n_1 \end{bmatrix} \begin{bmatrix} m_1+i \\ m_1 \end{bmatrix} \prod_{\alpha=2}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix}$$

where the sum over $\boldsymbol{n}$ is restricted by

$$\frac{\ell+\sigma N}{2N} + (C^{-1}\boldsymbol{n})_1 \in \mathbb{Z} \tag{2.7}$$

and the $(\boldsymbol{m},\boldsymbol{n})$-system is given by

$$\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(\mathcal{I}\boldsymbol{m} + \ell\boldsymbol{e}_1). \tag{2.8}$$

Since the $(\boldsymbol{m},\boldsymbol{n})$-system has become i-independent, only the first four q-binomials depend on the summation variable i. Hence we may apply (2.6) with $a = m_1$, $b = L_1 + L_2 + M + n_1$, $c = M + n_1$, $d = L_1 + \frac{1}{2}m_1$, $e = \ell - n_1$, $f = L_2 + \frac{1}{2}m_1$ and $g = -n_1$ to obtain

$$\sum_{i,\boldsymbol{n}} q^{(i-n_1)(i-n_1-m_1+\ell)+\boldsymbol{n}C^{-1}\boldsymbol{n}} \\ \times \begin{bmatrix} L_1+M+n_1+\frac{1}{2}m_1-i \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\ell-\frac{1}{2}m_1 \\ M+n_1-i \end{bmatrix} \begin{bmatrix} L_2+\frac{1}{2}m_1+i \\ i-n_1 \end{bmatrix} \begin{bmatrix} m_1+n_1 \\ m_1+n_1-i \end{bmatrix} \prod_{\alpha=2}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix}.$$

Shifting $\boldsymbol{n} \to \boldsymbol{n} + i(2\boldsymbol{e}_1 - \boldsymbol{e}_2)$ and $\boldsymbol{m} \to \boldsymbol{m} - 2i\boldsymbol{e}_1$, which leaves the $(\boldsymbol{m},\boldsymbol{n})$-system (2.8) and the restriction (2.7) on the summation over $\boldsymbol{n}$ invariant, yields

$$\sum_{i,\boldsymbol{n}} q^{(i+\frac{m_2-m_1}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} L_1+M+\frac{1}{2}m_1+n_1 \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\ell-\frac{1}{2}m_1+i \\ M+n_1+i \end{bmatrix} \\ \times \begin{bmatrix} L_2+\frac{1}{2}m_1 \\ -i-n_1 \end{bmatrix} \begin{bmatrix} m_1+n_1 \\ m_1+n_1-i \end{bmatrix} \begin{bmatrix} m_2+n_2-i \\ m_2 \end{bmatrix} \prod_{\alpha=3}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix},$$

where we have used the $(\boldsymbol{m},\boldsymbol{n})$-system to simplify the exponent of q. Shifting $i \to n_2 - i$ one can apply (2.6) with $a = m_2$, $b = L_2 + M + \ell - \frac{1}{2}m_1 + n_2$, $c = M + n_1 + n_2$, $d = L_2 + \frac{1}{2}m_1$, $e = -n_1 - n_2$, $f = m_1 + n_1$ and $g = m_1 + n_1 - n_2$, observing that

$$c + d + f - a - b = 2m_1 + 2n_1 - m_2 - \ell = 0$$

thanks to (2.8). This yields

$$\sum_{i,\boldsymbol{n}} q^{(i-\frac{m_3-m_2}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} L_1+M+\frac{1}{2}m_1+n_1 \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\frac{1}{2}m_1+n_1+n_2-i \\ M \end{bmatrix}$$
$$\times \begin{bmatrix} m_2+n_2-m_1-n_1 \\ m_2+n_2-m_1-n_1-i \end{bmatrix} \begin{bmatrix} M+\ell-m_1-n_1 \\ M+n_1+n_2-i \end{bmatrix} \begin{bmatrix} m_1+n_1+i \\ m_1+n_1-n_2+i \end{bmatrix} \prod_{\alpha=3}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix}.$$

Shifting $\boldsymbol{n} \to \boldsymbol{n} + i(\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3)$ and $\boldsymbol{m} \to \boldsymbol{m} - 2i(\boldsymbol{e}_1 + \boldsymbol{e}_2)$, which again leaves the $(\boldsymbol{m}, \boldsymbol{n})$-system (2.8) and the restriction (2.7) on the sum over $\boldsymbol{n}$ unchanged, leads to

$$\sum_{i,\boldsymbol{n}} q^{(i+\frac{m_3-m_2}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} L_1+M+n_1+\frac{1}{2}m_1 \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\frac{1}{2}m_1+n_1+n_2 \\ M \end{bmatrix}$$
$$\times \begin{bmatrix} m_2+n_2-m_1-n_1 \\ m_2+n_2-m_1-n_1-i \end{bmatrix} \begin{bmatrix} M+\ell-m_1-n_1+i \\ M+n_1+n_2+i \end{bmatrix} \begin{bmatrix} m_1+n_1 \\ m_1+n_1-n_2-i \end{bmatrix} \begin{bmatrix} m_3+n_3-i \\ m_3 \end{bmatrix} \prod_{\alpha=4}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix}. \tag{2.9}$$

We now need the following lemma.

Lemma 2.2. *For $p = 3, \ldots, N$, let*

$$f_p = \sum_{i,\boldsymbol{n}} q^{(i+\frac{m_p-m_{p-1}}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} L_1+M+\frac{1}{2}m_1+n_1 \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\frac{1}{2}m_1+n_1+n_2 \\ M \end{bmatrix}$$
$$\times \Bigl(\prod_{\alpha=1}^{p-3} \begin{bmatrix} M+\sum_{\beta=1}^{\alpha+2} n_\beta+\sum_{\beta=1}^{\alpha}(-1)^{\alpha-\beta}(m_\beta+n_\beta) \\ M+\sum_{\beta=1}^{\alpha} n_\beta+\sum_{\beta=1}^{\alpha}(-1)^{\alpha-\beta}(m_\beta+n_\beta) \end{bmatrix}\Bigr) \Bigl(\prod_{\alpha=p+1}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix}\Bigr)$$
$$\times \begin{bmatrix} \sum_{\alpha=1}^{p-1}(-1)^{p-\alpha-1}(m_\alpha+n_\alpha) \\ \sum_{\alpha=1}^{p-1}(-1)^{p-\alpha-1}(m_\alpha+n_\alpha)-i \end{bmatrix} \begin{bmatrix} M+\ell-m_1-\sum_{\alpha=1}^{p-2} n_\alpha+i \\ M+\sum_{\alpha=1}^{p-1} n_\alpha+i \end{bmatrix}$$
$$\times \begin{bmatrix} \sum_{\alpha=1}^{p-2}(-1)^{p-\alpha}(m_\alpha+n_\alpha) \\ \sum_{\alpha=1}^{p-2}(-1)^{p-\alpha}(m_\alpha+n_\alpha)-n_{p-1}-i \end{bmatrix} \begin{bmatrix} m_p+n_p-i \\ m_p \end{bmatrix},$$

with $(\boldsymbol{m}, \boldsymbol{n})$-system (2.8) and $m_N = n_N = 0$. Then $f_p = f_{p+1}$ for $3 \le p < N$.

Proof. Change $i \to n_p - i$ and apply (2.6) with $a = m_p$, $b = M + \ell - m_1 - \sum_{\alpha=1}^{p-2} n_\alpha + n_p$, $c = M + \sum_{\alpha=1}^{p} n_\alpha$, $d = \sum_{\alpha=1}^{p-2}(-1)^{p-\alpha}(m_\alpha + n_\alpha)$, $e = d - n_{p-1} - n_p$, $f = \sum_{\alpha=1}^{p-1}(-1)^{p-\alpha-1}(m_\alpha + n_\alpha)$ and $g = f - n_p$, observing that

$$c + d + f - a - b = 2\sum_{\alpha=1}^{p-1} n_\alpha + m_1 + m_{p-1} - m_p - \ell = 0$$

by summing up the first $p-1$ components of the $(\boldsymbol{m},\boldsymbol{n})$-system (2.8). This leads to

(2.10)

$$\begin{aligned}
f_p = \sum_{i,\boldsymbol{n}} & q^{(n_p-i+\frac{m_p-m_{p-1}}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n}} \\
& \times \begin{bmatrix} L_1+M+\frac{1}{2}m_1+n_1 \\ M+\ell \end{bmatrix} \begin{bmatrix} L_2+M+\frac{1}{2}m_1+n_1+n_2 \\ M \end{bmatrix} \\
& \times \Bigl(\prod_{\alpha=1}^{p-3} \begin{bmatrix} M+\sum_{\beta=1}^{\alpha+2} n_\beta+\sum_{\beta=1}^{\alpha}(-1)^{\alpha-\beta}(m_\beta+n_\beta) \\ M+\sum_{\beta=1}^{\alpha} n_\beta+\sum_{\beta=1}^{\alpha}(-1)^{\alpha-\beta}(m_\beta+n_\beta) \end{bmatrix}\Bigr)\Bigl(\prod_{\alpha=p+1}^{N-1} \begin{bmatrix} m_\alpha+n_\alpha \\ m_\alpha \end{bmatrix}\Bigr) \\
& \times \begin{bmatrix} \sum_{\alpha=1}^{p}(-1)^{p-\alpha}(m_\alpha+n_\alpha) \\ \sum_{\alpha=1}^{p}(-1)^{p-\alpha}(m_\alpha+n_\alpha)-i \end{bmatrix} \begin{bmatrix} M+\ell-m_1-\sum_{\alpha=1}^{p-1} n_\alpha \\ M+\sum_{\alpha=1}^{p} n_\alpha-i \end{bmatrix} \\
& \times \begin{bmatrix} M+\sum_{\alpha=1}^{p} n_\alpha+\sum_{\alpha=1}^{p-2}(-1)^{p-\alpha}(m_\alpha+n_\alpha)-i \\ M+\sum_{\alpha=1}^{p-2} n_\alpha+\sum_{\alpha=1}^{p-2}(-1)^{p-\alpha}(m_\alpha+n_\alpha) \end{bmatrix} \begin{bmatrix} \sum_{\alpha=1}^{p-1}(-1)^{p-\alpha-1}(m_\alpha+n_\alpha)+i \\ \sum_{\alpha=1}^{p-1}(-1)^{p-\alpha-1}(m_\alpha+n_\alpha)-n_p+i \end{bmatrix}.
\end{aligned}$$

We now carry out the transformations $\boldsymbol{n} \to \boldsymbol{n}+i(\boldsymbol{e}_1+\boldsymbol{e}_p-\boldsymbol{e}_{p+1})$ and $\boldsymbol{m} \to \boldsymbol{m}-2i(\boldsymbol{e}_1+\boldsymbol{e}_2+\cdots+\boldsymbol{e}_p)$, which leave the $(\boldsymbol{m},\boldsymbol{n})$-system unchanged. (Here $\boldsymbol{e}_N := 0$.) Using $\boldsymbol{n}C^{-1}(\boldsymbol{e}_1+\boldsymbol{e}_p-\boldsymbol{e}_{p+1}) = \sum_{\alpha=1}^{p} n_\alpha$ and $(\boldsymbol{e}_1+\boldsymbol{e}_p-\boldsymbol{e}_{p+1})C^{-1}(\boldsymbol{e}_1+\boldsymbol{e}_p-\boldsymbol{e}_{p+1}) = 2$, as well as the $(\boldsymbol{m},\boldsymbol{n})$-system, yields

$$\begin{aligned}
(n_p-i+\frac{m_p-m_{p-1}}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n} \to \\
(n_p+\frac{m_p-m_{p-1}}{2})^2-(i-\frac{m_1-\ell}{2})^2+2i(i+\sum_{\alpha=1}^{p} n_\alpha)+\boldsymbol{n}C^{-1}\boldsymbol{n} \\
= (i+\frac{m_{p+1}-m_p}{2})^2-(\frac{m_1-\ell}{2})^2+\boldsymbol{n}C^{-1}\boldsymbol{n}
\end{aligned}$$

transforming (2.10) into f_{p+1} as desired. □

Equation (2.9) corresponds to f_3 and we can thus use the above lemma to replace it with f_N. Since $m_N = 0$, the last q-binomial in f_N is 1 and we can perform the sum over i using the q-Saalschütz sum, which is the special case $a = 0$ of the Sears transformation (2.6). (When $a = 0$, the only nonvanishing term on the right-hand side of (2.6) corresponds to $i = -g$.) Specifically, we take f_N, replace i by $-i$ and apply (2.6) with the same choice of parameters as in the proof of

Lemma 2.2 but with $p=N$, $n_N=0$ and $a=m_N=0$. Then we get

(2.11)
$$\sum_{\boldsymbol{n}} q^{(\frac{m_{N-1}}{2})^2-(\frac{m_1-\ell}{2})^2+(n_{N-1}-\sum_{\alpha=1}^{N-2}(-1)^{N-\alpha}(m_\alpha+n_\alpha))(\sum_{\alpha=1}^{N-1}(-1)^{N-\alpha-1}(m_\alpha+n_\alpha))}$$
$$\times q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\Bigl(\prod_{\alpha=1}^{N-3}\left[{M+\sum_{\beta=1}^{\alpha+2}n_\beta+\sum_{\beta=1}^{\alpha}(-1)^{\alpha-\beta}(m_\beta+n_\beta) \atop M+\sum_{\beta=1}^{\alpha}n_\beta+\sum_{\beta=1}^{\alpha}(-1)^{\alpha-\beta}(m_\beta+n_\beta)}\right]\Bigr)$$
$$\times\left[{L_1+M+\frac{1}{2}m_1+n_1 \atop M+\ell}\right]\left[{M+\ell-m_1-\sum_{\alpha=1}^{N-2}n_\alpha \atop M+\ell-m_1-\sum_{\alpha=1}^{N-1}n_\alpha-\sum_{\alpha=1}^{N-1}(-1)^{N-\alpha-1}(m_\alpha+n_\alpha)}\right]$$
$$\times\left[{L_2+M+\frac{1}{2}m_1+n_1+n_2 \atop M}\right]\left[{M+\ell-m_1-\sum_{\alpha=1}^{N-1}n_\alpha \atop M+\ell-m_1-\sum_{\alpha=1}^{N-2}n_\alpha-\sum_{\alpha=1}^{N-2}(-1)^{N-\alpha}(m_\alpha+n_\alpha)}\right].$$

All that remains to be done is to clean up the above expression. Introduce a new variable $\boldsymbol{\eta}\in\mathbb{Z}^{N-1}$ through its components as follows

$$\begin{aligned}
\eta_i &= n_{2i}+n_{2i+1} && \text{for } i=1,\dots,\lfloor N/2\rfloor-1\\
\eta_{N-i} &= n_{2i+1}+n_{2i+2} && \text{for } i=1,\dots,\lfloor (N-1)/2\rfloor-1\\
\eta_{\lfloor (N+1)/2\rfloor} &= \sum_{\alpha=1}^{N-2}(-1)^{N-\alpha}(m_\alpha+n_\alpha)-n_{N-1}\\
\eta_{\lfloor (N+1)/2\rfloor\pm 1} &= \sum_{\alpha=1}^{N-1}(-1)^{N-\alpha-1}(m_\alpha+n_\alpha)+n_{N-1}
\end{aligned}$$

for N even/odd. Also define $\boldsymbol{\mu}$ through the $(\boldsymbol{\mu},\boldsymbol{\eta})$-system (2.3) Eliminating $\boldsymbol{m}$ and $\boldsymbol{n}$ from (2.11) in favour of $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$, we finally get the right-hand side of (2.1). We also note that $(C^{-1}\boldsymbol{\eta})_1=\sum_{i=1}^{N-1}(N-i)\eta_i/N$ yields $(-n_1+\sum_{i=2}^{N-1}(N-i)n_i)/N=(C^{-1}\boldsymbol{n})_1-n_1$ so that the restriction (2.7) on the sum over $\boldsymbol{n}$ translates into the restriction

$$\frac{\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}$$

for the sum over $\boldsymbol{\eta}$ as it should.

□

3. The Burge transform

Perhaps the most interesting application of our generalized q-Saalschütz sum (2.1) arises when it is combined with the Burge transform [7, 12]. The Burge transform is a generalization of (a special case) of the Bailey lemma and can be utilized to derive an infinite tree (a Burge tree) of polynomial identities from a single initial identity. In this section we show that each element of a Burge tree can be transformed using (2.1) to yield an additional infinite series of polynomial identities.

In his study of restricted partition pairs Burge considered the polynomial

$$
\begin{aligned}
(3.1)\quad & X^{(p,p')}_{r,s}(M_1,L_1,M_2,L_2) \\
&= \sum_{j=-\infty}^{\infty} \Big\{ q^{j(pp'j+p'(M_{12}+r)-ps)} \begin{bmatrix} M_1+L_1-(p'-p)j \\ M_1+pj \end{bmatrix} \begin{bmatrix} M_2+L_2+(p'-p)j \\ M_2-pj \end{bmatrix} \\
&\qquad - q^{(pj+M_{12}+r)(p'j+s)} \begin{bmatrix} M_1+L_1-(p'-p)j+r-s \\ M_1+pj+r \end{bmatrix} \begin{bmatrix} M_2+L_2+(p'-p)j-r+s \\ M_2-pj-r \end{bmatrix} \Big\},
\end{aligned}
$$

with $M_{12} = M_1 - M_2$, and proved that it is the generating function of pairs of partitions (λ, μ) such that

$$0 \le \lambda_1 \le \cdots \le \lambda_{M_1} \le L_1, \qquad 0 \le \mu_1 \le \cdots \le \mu_{M_2} \le L_2,$$

and

$$\lambda_i - \mu_{i-r+1} \ge 1 - s, \qquad \mu_i - \lambda_{i-p+r+1} \ge 1 - p' + s.$$

Here the integers p, p', r, s are restricted to $p, p' \ge 1$, $0 \le r + M_{12} \le p$ and $0 \le s - L_{12} \le p'$, with $L_{12} = L_1 - L_2$. There are four exceptional cases, $r = 0$, $r = p$, $r = -M_{12}$ and $r = p - M_{12}$ that demand the additional conditions $\mu_1 \le s - 1$, $\lambda_1 \le p' - s - 1$, $\lambda_{M_2} \ge L_1 - s + 1$ and $\mu_{M_1} \ge L_2 - p' + s + 1$, respectively [14, 12].

The important observation made in [7] is that

$$
\begin{aligned}
(3.2)\quad & X^{(p,p+p')}_{r,r+s}(M_1,L_1,M_2,L_2) \\
&= \sum_{i\in\mathbb{Z}} q^{i(i+M_{12})} \begin{bmatrix} L_1+L_2+M_2-i \\ M_2-i \end{bmatrix} X^{(p,p')}_{r,s}(i+M_{12}, L_1-i, i, L_2-M_{12}-i)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.3)\quad & X^{(p',p+p')}_{s-M_{12},r+s+L_{12}}(M_1,L_1,M_2,L_2) \\
&= \sum_{i\in\mathbb{Z}} q^{i(i+M_{12})} \begin{bmatrix} L_1+L_2+M_2-i \\ M_2-i \end{bmatrix} X^{(p,p')}_{r,s}(L_1-i, i+M_{12}, L_2-M_{12}-i, i)
\end{aligned}
$$

where the second equation follows from the first by exploiting the symmetry

$$(3.4)\qquad X^{(p,p')}_{r,s}(M_1,L_1,M_2,L_2) = X^{(p',p)}_{s-L_{12},r+M_{12}}(L_1,M_1,L_2,M_2).$$

The proof of the Burge transform follows from the q-Saalschütz formula (1.3). In [7, 12] the defining equation (3.1) is substituted into (3.2), then the sums over i and j are interchanged, followed by the variable change $i \to i + pj$ and $i \to i + pj + r$ in the terms corresponding to the second and third line of (3.1), respectively (referred to as the positive and negative terms below). Then the q-Saalschütz sum is used with $L_1 \to L_1+M_{12}-(p'-p)j$, $L_2 \to L_2-M_{12}+(p'-p)j$, $M \to M_2-pj$ and $\ell \to M_{12}+2pj$ for the positive terms and $L_1 \to L_1+M_{12}-(p'-p)j+r-s$, $L_2 \to L_2-M_{12}+(p'-p)j-r+s$, $M \to M_2-pj-r$ and $\ell \to M_{12}+2pj+2r$ for the negative terms. This gives the left-hand side of (3.2). However, we note that it needs to be verified that the summation (1.3) has not been employed when the variables therein lie in the ranges given just below (1.3). This means that

$$\begin{aligned}(3.5)\quad & -L_1 - M_{12} + (p'-p)j - r + s \le -M_{12} - 2pj - 2r \\ & \qquad \le L_2 - M_{12} + (p'-p)j - r + s < 0 \le M_2 - pj - r\end{aligned}$$

and

$$\begin{aligned}(3.6)\quad & -L_2 + M_{12} - (p'-p)j + r - s \le M_{12} + 2pj + 2r \\ & \qquad \le L_1 + M_{12} - (p'-p)j + r - s < 0 \le M_1 + pj + r\end{aligned}$$

and the corresponding inequalities obtained by setting $r = s = 0$ should not hold for any $j \in \mathbb{Z}$. Eliminating j gives several conditions on the parameters in (3.2). In particular (3.5) can only hold if

$$2pj > -M_{12} - 2r \quad \text{and} \quad 2p'j < M_{12} + L_{12} - 2s.$$

Similarly, (3.6) can only hold if

$$2pj < -M_{12} - 2r \quad \text{and} \quad 2p'j > M_{12} + L_{12} - 2s.$$

If, for example, $M_{12} = L_{12} = 0$ these conditions cannot be satisfied for any j recalling that $0 \le r \le p$ and $0 \le s \le p'$. Hence, setting

$$X_{r,s}^{(p,p')}(M, L, M, L) = X_{r,s}^{(p,p')}(M, L),$$

the symmetric version of the Burge transform (3.2)

$$(3.7)\qquad X_{r,r+s}^{(p,p+p')}(M, L) = \sum_{i=0}^{M} q^{i^2} \begin{bmatrix} 2L + M - i \\ 2L \end{bmatrix} X_{r,s}^{(p,p')}(i, L - i)$$

always holds. By the same arguments one can show that the symmetric form of (3.3)

$$X_{s,r+s}^{(p',p+p')}(M, L) = \sum_{i=0}^{M} q^{i^2} \begin{bmatrix} 2L + M - i \\ 2L \end{bmatrix} X_{r,s}^{(p,p')}(L - i, i)$$

is true for arbitrary M and L.

By iterating the two Burge transformations, starting with an appropriate initial identity for $X_{r,s}^{(p,p')}$, one can derive an infinite tree of polynomial identities. This was mentioned in [7] and explicitly carried out in [12]. To illustrate this we follow [12] and use the trivial result

$$X_{0,1}^{(1,2)}(M,L) = \delta_{L,0} \tag{3.8}$$

to derive the Burge tree

$$\begin{tikzcd}
 & & & & X_{0,1}^{(1,2)} \arrow[dll] \arrow[drr] & & & \\
 & & X_{0,1}^{(1,3)} \arrow[dl] \arrow[dr] & & & & X_{1,1}^{(2,3)} \arrow[dl] \arrow[dr] & \\
 & X_{0,1}^{(1,4)} \arrow[dl] \arrow[dr] & & X_{1,1}^{(3,4)} & & X_{1,2}^{(2,5)} & & X_{1,2}^{(3,5)} \\
X_{0,1}^{1,5} & & X_{1,1}^{4,5} & & & & &
\end{tikzcd}$$

where a node with label $X_{r,s}^{(p,p')}$ corresponds to a polynomial identity for $X_{r,s}^{(p,p')}(M,L)$. (Actually, in Ref. [12] an extension of the Burge tree was constructed by exploiting various symmetries of $X_{r,s}^{(p,p')}$.) Explicitly some of the identities in the above tree are [7, 12],

$$X_{0,1}^{(1,3)}(M,L) = q^{L^2} \begin{bmatrix} L+M \\ 2L \end{bmatrix} \tag{3.9}$$

$$X_{1,1}^{(2,3)}(M,L) = \begin{bmatrix} 2L+M \\ 2L \end{bmatrix} \tag{3.10}$$

$$X_{1,1}^{(3,4)}(M,L) = \sum_{\substack{m=0 \\ m \text{ even}}}^{L} q^{\frac{1}{2}m^2} \begin{bmatrix} 2L+M-\frac{1}{2}m \\ 2L \end{bmatrix} \begin{bmatrix} L \\ m \end{bmatrix} \tag{3.11}$$

$$X_{1,2}^{(2,5)}(M,L) = \sum_{n=0}^{L} q^{n^2} \begin{bmatrix} 2L+M-n \\ 2L \end{bmatrix} \begin{bmatrix} 2L-n \\ n \end{bmatrix}. \tag{3.12}$$

Equation (3.10) is a doubly bounded version of the Euler identity, equation (3.11) is a doubly bounded analogue of the vacuum-character identity of the Ising model

$$\sum_{\substack{m=0\\ m\text{ even}}}^{\infty}\frac{q^{\frac{1}{2}m^2}}{(q)_m}=\frac{1}{2}\Big\{(-q^{1/2})_\infty+(q^{1/2})_\infty\Big\}$$
$$=\prod_{j=1}^{\infty}\frac{(1+q^{8j-3})(1+q^{8j-5})(1-q^{8j})}{1-q^{2j}}$$

and (3.12) is a doubly bounded version of the (first) Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\prod_{j=1}^{\infty}\frac{1}{(1-q^{5j-1})(1-q^{5j-4})}.$$

To see how (2.1) transforms an identity in the Burge tree, let us first introduce a generalization of the polynomial $X^{(p,p')}_{r,s}(M_1,L_1,M_2,L_2)$ as follows. Let N be a positive integer, $\sigma=0,1$ and let $M_1,M_2,L_1+\frac{M_{12}+\sigma}{2},L_2+\frac{M_{12}+\sigma}{2}$ be integers such that $M_{12}+\sigma N$ is even. Also assume that $(p'-p)/N\in\mathbb{Z}_+$ and $(r-s)/N\in\mathbb{Z}$, for r,s integers. Then

(3.13)

$$\begin{aligned}
&X^{(p,p'),N}_{r,s,\sigma}(M_1,L_1,M_2,L_2)\\
&\quad=\sum_{j=-\infty}^{\infty}q^{\frac{j}{N}(pp'j+p'(M_{12}+r)-ps)}\sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{M_{12}+2pj+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}}q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}}\begin{bmatrix}\boldsymbol{\eta}+\boldsymbol{\mu}\\ \boldsymbol{\eta}\end{bmatrix}\\
&\qquad\times\begin{bmatrix}M_1+L_1-(p'-p)j/N-\frac{1}{2}(M_2-pj-\mu_1)\\ M_1+pj\end{bmatrix}\\
&\qquad\qquad\times\begin{bmatrix}M_2+L_2+(p'-p)j/N-\frac{1}{2}(M_1+pj-\mu_{N-1})\\ M_2-pj\end{bmatrix}\\
&\quad-\sum_{j=-\infty}^{\infty}q^{\frac{1}{N}(pj+M_{12}+r)(p'j+s)}\sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{M_{12}+2pj+2r+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}}q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}}\begin{bmatrix}\boldsymbol{\eta}+\boldsymbol{\mu}\\ \boldsymbol{\eta}\end{bmatrix}\\
&\qquad\times\begin{bmatrix}M_1+L_1-((p'-p)j-r+s)/N-\frac{1}{2}(M_2-pj-r-\mu_1)\\ M_1+pj+r\end{bmatrix}\\
&\qquad\qquad\times\begin{bmatrix}M_2+L_2+((p'-p)j-r+s)/N-\frac{1}{2}(M_1+pj+r-\mu_{N-1})\\ M_2-pj-r\end{bmatrix},
\end{aligned}$$

with $(\boldsymbol{\mu},\boldsymbol{\eta})$-systems

$$\boldsymbol{\mu}+\boldsymbol{\eta}=\frac{1}{2}(\mathcal{I}\boldsymbol{\mu}+(M_1+pj)\boldsymbol{e}_1+(M_2-pj)\boldsymbol{e}_{N-1})$$

for the first term of the right-hand side and

$$\boldsymbol{\mu}+\boldsymbol{\eta}=\frac{1}{2}(\mathcal{I}\boldsymbol{\mu}+(M_1+pj+r)\boldsymbol{e}_1+(M_2-pj-r)\boldsymbol{e}_{N-1})$$

for the second term of the right-hand side. In Section 4.1 we will show that in the limit when M_1, M_2 tend to infinity for fixed M_{12} the above polynomials become proportional to the one-dimensional configuration sums of solvable lattice models of Date et al. [10, 11], which are bounded analogues of level-N $\mathrm{A}_1^{(1)}$ branching functions.

Using (2.1), it follows that

$$\begin{aligned}(3.14)\quad & X^{(p,p+Np'),N}_{r,r+Ns,\sigma}(M_1,L_1,M_2,L_2)\\ &=\sum_{i\in\mathbb{Z}} q^{i(i+M_{12})/N}\begin{bmatrix}L_1+L_2+M_2-i\\ M_2-i\end{bmatrix}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\ \frac{2i+M_{12}+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}\\ &\qquad\times X^{(p,p')}_{r,s}(i+M_{12},L_1-i+\frac{1}{2}m_1,i,L_2-M_{12}-i+\frac{1}{2}m_1),\end{aligned}$$

where on the right-hand side we assume the $(\boldsymbol{m},\boldsymbol{n})$-system

$$\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(\mathcal{I}\boldsymbol{m}+(2i+M_{12})\boldsymbol{e}_1). \tag{3.15}$$

Because of the conditions $L_1, L_2 \geq 0$ in (2.1), a sufficiency condition for the above transformation to hold is

$$\begin{aligned}&\left\lfloor\frac{L_1+M_{12}(N-1)/(2N)-s-r/N}{p'+p/N}\right\rfloor\leq\left\lfloor\frac{L_1+M_{12}+r-s}{p'-p}\right\rfloor\\ &\left\lfloor\frac{L_2-M_{12}(N-1)/(2N)+s+r/N}{p'+p/N}\right\rfloor\leq\left\lfloor\frac{L_2-M_{12}-r+s}{p'-p}\right\rfloor\end{aligned}\tag{3.16}$$

together with the inequalities obtained by setting $r=s=0$, where we assumed that $p'>p$. (The kernel of $X^{(p,p+Np'),N}_{r,r+Ns,\sigma}$ and of $X^{(p,p')}_{r,s}$ on either side of (3.14) is zero unless the summation variable j lies in certain ranges. The above conditions make sure that in these ranges of j the conditions $L_1, L_2 \geq 0$ of Theorem 2.1 apply).

Using the symmetry (3.4) one also finds

$$\begin{aligned}(3.17)\quad & X^{(p',Np+p'),N}_{s-M_{12},N(r+L_{12}+M_{12})+s-M_{12},\sigma}(M_1,L_1,M_2,L_2)\\ &=\sum_{i\in\mathbb{Z}} q^{i(i+M_{12})/N}\begin{bmatrix}L_1+L_2+M_2-i\\ M_2-i\end{bmatrix}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\ \frac{2i+M_{12}+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}\\ &\qquad\times X^{(p,p')}_{r,s}(L_1-i+\frac{1}{2}m_1,i+M_{12},L_2-M_{12}-i+\frac{1}{2}m_1,i),\end{aligned}$$

where again (3.15) holds. This time a sufficient condition is that

$$
\begin{aligned}
\left\lfloor \frac{L_2 - M_{12}(N-1)/(2N) - s - r/N}{p' + p/N} \right\rfloor &\leq \left\lfloor \frac{L_1 + M_{12} + r - s}{p' - p} \right\rfloor \\
\left\lfloor \frac{L_1 + M_{12}(N-1)/(2N) + s + r/N}{p' + p/N} \right\rfloor &\leq \left\lfloor \frac{L_2 - M_{12} - r + s}{p' - p} \right\rfloor
\end{aligned} \tag{3.18}
$$

holds, as well as the inequalities obtained by setting $r = s = 0$, where again $p' > p$.

Again we consider the simpler case when $M_{12} = L_{12} = 0$. Setting

$$X^{(p,p'),N}_{r,s,\sigma}(M,L,M,L) = X^{(p,p'),N}_{r,s,\sigma}(M,L),$$

the generalized Burge transformations (3.14) and (3.17) simplify to

$$
\begin{aligned}
X^{(p,p+Np'),N}_{r,r+Ns,\sigma}(M,L) &= \sum_{i=0}^{M} q^{i^2/N} \begin{bmatrix} 2L+M-i \\ 2L \end{bmatrix} \\
&\times \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1} \\ \frac{2i+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{n} \end{bmatrix} X^{(p,p')}_{r,s}(i, L-i+\frac{1}{2}m_1)
\end{aligned} \tag{3.19}
$$

and

$$
\begin{aligned}
X^{(p',Np+p'),N}_{s,Nr+s,\sigma}(M,L) &= \sum_{i=0}^{M} q^{i^2/N} \begin{bmatrix} 2L+M-i \\ 2L \end{bmatrix} \\
&\times \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1} \\ \frac{2i+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{n} \end{bmatrix} X^{(p,p')}_{r,s}(L-i+\frac{1}{2}m_1, i)
\end{aligned} \tag{3.20}
$$

both with $(\boldsymbol{m},\boldsymbol{n})$-system

$$\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(\mathcal{I}\boldsymbol{m} + 2i\boldsymbol{e}_1). \tag{3.21}$$

The sufficiency conditions (3.16) and (3.18) (and their $r = s = 0$ counterparts) reduce to the single condition

$$\left\lfloor \frac{L+s+r/N}{p'+p/N} \right\rfloor \leq \left\lfloor \frac{L-r+s}{p'-p} \right\rfloor. \tag{3.22}$$

To end this section let us give some simple examples of our extensions to the Burge transform, by finding the generalizations of equations (3.9)–(3.12) to arbitrary N. First, applying (3.19) to (3.8) yields

$$X^{(1,2N+1),N}_{0,N,\sigma}(M,L) = q^{L^2} \sum_{\boldsymbol{m}\in\mathbb{Z}^{N-1}} q^{\frac{1}{4}\boldsymbol{m}\mathcal{T}\boldsymbol{m}} \begin{bmatrix} L+M-\frac{1}{2}m_1 \\ 2L \end{bmatrix} \begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{m} \end{bmatrix},$$

with $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(\mathcal{I}_T\boldsymbol{m}+2L\boldsymbol{e}_1)$ and $(\mathcal{I}_T)_{i,j}=\delta_{|i-j|,1}+\delta_{i,j}\delta_{i,1}$ the incidence matrix of the tadpole graph with $N-1$ nodes, and $T=2I-\mathcal{I}_T$ the corresponding Cartan-like matrix. When N is odd $\sigma=0$, $L\in\mathbb{Z}$ and $\boldsymbol{m}\in 2\mathbb{Z}^{N-1}$. When N is even $m_{2i+1}\equiv 2L\equiv\sigma \pmod 2$ and $m_{2i}\equiv 0 \pmod 2$. The sufficiency condition (3.22) is satisfied. Next applying (3.20) to (3.8) yields

$$X^{(2,N+2),N}_{1,1,\sigma}(M,L)=\begin{bmatrix}2L+M\\2L\end{bmatrix}\delta_{\sigma,0}$$

which, for $\sigma=0$, is a doubly bounded version of the Euler identity for the level-N string functions of type $\mathrm{A}^{(1)}_1$. Our third example follows after inserting (3.9) into (3.20),

$$X^{(3,N+3),N}_{1,1,\sigma}(M,L)=\sum_{\boldsymbol{m}\in\mathbb{Z}^N}q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}}\begin{bmatrix}2L+M-\frac{1}{2}m_1\\2L\end{bmatrix}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\\boldsymbol{m}\end{bmatrix},$$

with $(\boldsymbol{m},\boldsymbol{n})$-system $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(\mathcal{I}\boldsymbol{m}+2L\boldsymbol{e}_1)\in\mathbb{Z}^N$, where $\mathcal{I}$ now is the incidence matrix of the A_N Dynkin diagram. When N is odd $\sigma=0$, $L\in\mathbb{Z}$ and $\boldsymbol{m}\in 2\mathbb{Z}^{N-1}$ and when N is even $m_{2i}\equiv 2L\equiv\sigma \pmod 2$ and $m_{2i+1}\equiv 0 \pmod 2$. These identities are bounded analogues of identities for level-N $\mathrm{A}^{(1)}_1$ branching functions isomorphic to unitary minimal Virasoro characters. Finally we use (3.19) and (3.10) to find

$$X^{(2,3N+2),N}_{1,N+1,\sigma}(M,L)=\\\sum_{i=0}^{M}q^{i^2/N}\begin{bmatrix}2L+M-i\\2L\end{bmatrix}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\\frac{2i+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\\boldsymbol{n}\end{bmatrix}\begin{bmatrix}2L-i+m_1\\i\end{bmatrix}$$

where (3.21) holds. As remarked before, for $N=1$ ($\sigma=0$) this is a doubly bounded version of the (first) Rogers–Ramanujan identity. For $N=2$ it becomes

$$X^{(2,8),2}_{1,3,\sigma}(M,L)=\sum_{i=0}^{M}\sum_{\substack{n=0\\n+i+\sigma\text{ even}}}^{i}q^{(i^2+n^2)/2}\begin{bmatrix}2L+M-i\\2L\end{bmatrix}\begin{bmatrix}i\\n\end{bmatrix}\begin{bmatrix}2L-n\\i\end{bmatrix}$$

which can be recognized as a doubly bounded version of

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}=\prod_{j=1}^{\infty}\frac{1}{(1-q^{8j-1})(1-q^{8j-4})(1-q^{8j-7})}$$

due to Slater [26] and related to the (first) Göllnitz–Gordon partition identity [15, 16],

4. Special limits of Theorem 2.1

4.1. **q-Multinomial coefficients.** In Refs. [2, 8, 18, 22, 27] q-multinomial coefficients were introduced as q-analogues of the coefficients in the expansion

$$(1+x+x^2+\cdots+x^N)^L=\sum_{a=-\frac{NL}{2}}^{\frac{NL}{2}}\binom{L}{a}_N x^{a+\frac{NL}{2}},$$

for $L\in\mathbb{Z}_+$. The q-multinomial coefficients are the generating function of a wide class of combinatorial objects: (i) unrestricted lattice paths related to the RSOS lattice models of Date et al. with H-function statistic [10, 11], (ii) Durfee dissection partitions [27] and (iii) tabloids of shape (N^L) and content $(1^a 2^{NL-a})$ with the statistic "value" [8], et cetera.

Here we need the following explicit representation for the q-multinomials [22]

(4.1)
$$T_n^{(N)}(L,a)=\sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{L}{2}+\frac{a}{N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}}\frac{q^{\boldsymbol{\eta}C^{-1}(\boldsymbol{\eta}-\boldsymbol{e}_n)}(q)_L}{(q)_{\frac{L}{2}-\frac{a}{N}-(C^{-1}\boldsymbol{\eta})_1}(q)_{\frac{L}{2}+\frac{a}{N}-(C^{-1}\boldsymbol{\eta})_{N-1}}(q)_{\boldsymbol{\eta}}},$$

where $L\in\mathbb{Z}_+$, $2a\in\{-NL,-NL+2,\dots,NL\}$ and $n\in\{0,1,\dots,N-1\}$. Repeated use of Newton's binomial expansion shows that

$$\lim_{q\to 1}T_n^{(N)}(L,a)=\binom{L}{a}_N$$

so that $T_n^{(N)}(L,a)$ is indeed a q-analogue of the multinomial coefficient.

Theorem 2.1 provides a new representation of the q-multinomials when $n=0$. To see this we let M tend to infinity in (2.1) resulting in

$$\sum_{i=0}^{\infty}q^{i(i+\ell)/N}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\ \frac{2i+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}}q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}\begin{bmatrix}L_1+\frac{1}{2}m_1\\ i+\ell\end{bmatrix}\begin{bmatrix}L_2+\frac{1}{2}m_1\\ i\end{bmatrix}$$
$$=\sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}}\frac{q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}}(q)_{L_1+L_2}}{(q)_{L_1-\frac{\ell}{2N}-\frac{\ell}{2}-(C^{-1}\boldsymbol{\eta})_1}(q)_{L_2+\frac{\ell}{2N}+\frac{\ell}{2}-(C^{-1}\boldsymbol{\eta})_{N-1}}(q)_{\boldsymbol{\eta}}}.$$

If we now set $L_1 = \frac{1}{2}(L+\ell)$ and $L_2 = \frac{1}{2}(L-\ell)$ (so that $\sigma \equiv L \pmod 2$) and compare with the right-hand side of (4.1), we find that

$$(4.2)\quad T_0^{(N)}(L,\ell/2) = \sum_{i=0}^{\infty} q^{i(i+\ell)/N} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1} \\ \frac{L}{2}+\frac{2i+\ell}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix} \begin{bmatrix}\frac{1}{2}(L+\ell+m_1)\\ i+\ell\end{bmatrix} \begin{bmatrix}\frac{1}{2}(L-\ell+m_1)\\ i\end{bmatrix},$$

with $\boldsymbol{m}$ given by (2.2).

When $N=1$ the above decomposition of the q-multinomial coefficients reduces to the q-Chu–Vandermonde sum (1.4) and a combinatorial interpretation is easily given as follows. The q-binomial $\begin{bmatrix}m+n\\ m\end{bmatrix}$ is the generating function of partitions that fit in a box of dimension m times n. Hence the summand on the left-hand side of (1.4) is the generating function of partitions that fit in a box of dimension $L_1-\ell$ times $L_2+\ell$ which have a Durfee rectangle of size i by $i+\ell$ (maximal rectangle of the Ferrers graph that has a horizontal excess of ℓ nodes). Summing over i removes the Durfee rectangle restriction resulting in the right-hand side. It seems an interesting problem to also explain the q-multinomial decomposition (4.2) combinatorially.

There is a corresponding formula for $1\le n<N-1$ which, however, is less appealing (and which we will not prove here)

$$\begin{aligned} T_n^{(N)}&(L,(n-\ell)/2) - q^{(\ell+1)/N} T_n^{(N)}(L,(n+\ell+2)/2) \\ &= \sum_{i=0}^{\infty} q^{i(i+\ell)/N} \sum_{\substack{n\in\mathbb{Z}^{N-1} \\ \frac{L}{2}+\frac{2i+\ell-n}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_{N-n})} \begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix} \\ &\quad\times \left(\begin{bmatrix}\frac{1}{2}(L+\ell+m_1\\ i+\ell\end{bmatrix} \begin{bmatrix}\frac{1}{2}(L-\ell+m_1)\\ i\end{bmatrix} - \begin{bmatrix}\frac{1}{2}(L+\ell+2+m_1)\\ i+\ell+1\end{bmatrix} \begin{bmatrix}\frac{1}{2}(L-\ell-2+m_1)\\ i-1\end{bmatrix} \right) \end{aligned}$$

with

$$\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(\mathcal{I}\boldsymbol{m} + (2i+\ell)\boldsymbol{e}_1 + \boldsymbol{e}_{N-n}).$$

Although this identity has the structure $f(L,\ell)-q^{(\ell+1)/N}f(L,-\ell-2) = g(L,\ell)-q^{(\ell+1)/N}g(L,-\ell-2)$, it is not true that $f(L,\ell)=g(L,\ell)$.

To conclude our discussion of the q-multinomial coefficients, let us point out that the polynomials defined in equation (3.13) are related to one-dimensional configuration sums of lattice models of Date et al [10, 11]. Let $L\in\mathbb{Z}$ and choose

$$L_1 = \frac{1}{2}\Big(L - M_{12} - \frac{r-s}{N}\Big) \qquad L_2 = \frac{1}{2}\Big(L + M_{12} + \frac{r-s}{N}\Big)$$

so that $\sigma = 0, 1$ is fixed by the condition that $L - (r-s)/N + \sigma$ is even. Then

$$\begin{aligned}
&\lim_{\substack{M_1,M_2\to\infty\\ M_{12}\text{ fixed}}} (q)_{2L} X^{(p,p'),N}_{r,s,\sigma}(M_1,L_1,M_2,L_2)\\
&\qquad = \sum_{j=-\infty}^{\infty} \Big\{ q^{\frac{j}{N}(pp'j+p'(M_{12}+r)-ps)} T_0^{(N)}\big(L, \tfrac{1}{2}(r+M_{12}-s)+p'j\big)\\
&\qquad\qquad - q^{\frac{1}{N}(pj+M_{12}+r)(p'j+s)} T_0^{(N)}\big(L, \tfrac{1}{2}(r+M_{12}+s)+p'j\big)\Big\},
\end{aligned}$$

which, for $p' = p+N$, is proportional to the configuration sums of the models of Date et al. in the representation obtained in [22, Eq. (3.15)].

4.2. **Bailey's lemma.** In this section we show that the limit $L_1, L_2 \to \infty$ of Theorem 2.1 gives rise to the higher-level Bailey lemma (or more precisely the higher-level conjugate Bailey pairs) of Refs. [23, 24].

Bailey's lemma [5] is an elegant tool to prove q-series identities such as the famous Rogers–Ramanujan identities. Let $\alpha = \{\alpha_L\}_{L\geq 0}, \beta = \{\beta_L\}_{L\geq 0}$ be a pair of sequences that satisfies

$$\beta_L = \sum_{i=0}^{L} \frac{\alpha_i}{(q)_{L-i}(aq)_{L+i}}.$$

Such a pair is called a Bailey pair relative to a. Recalling the definition (1.6) of a conjugate Bailey pair, it follows by a simple interchange of sums that

$$\sum_{L=0}^{\infty} \alpha_L \gamma_L = \sum_{L=0}^{\infty} \beta_L \delta_L. \tag{4.3}$$

Many known q-series identities follow from (4.3) after substitution of suitable Bailey and conjugate Bailey pairs.

Now let L_1, L_2 tend to infinity in (2.1) and replace $i \to i - L$, $\ell \to \ell + 2L$ and $M \to M - L$. This yields

$$\begin{aligned}
&\sum_{i=L}^{M} \frac{q^{i(i+\ell)/N}}{(q)_{i-L}(q)_{i+L+\ell}(q)_{M-i}} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\ \frac{2i+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}} \begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix}\\
&\qquad = \frac{q^{L(L+\ell)/N}}{(q)_{M-L}(q)_{M+L+\ell}} \sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{2L+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}} q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}} \begin{bmatrix}\boldsymbol{\mu}+\boldsymbol{\eta}\\ \boldsymbol{\eta}\end{bmatrix},
\end{aligned}$$

with $(\boldsymbol{m},\boldsymbol{n})$-system (2.2) and $(\boldsymbol{\mu},\boldsymbol{\eta})$-system

$$\boldsymbol{\mu}+\boldsymbol{\eta}=\frac{1}{2}(\mathcal{I}\boldsymbol{\mu}+(M+L+\ell)\boldsymbol{e}_1+(M-L)\boldsymbol{e}_{N-1}). \tag{4.4}$$

Comparing with (1.6) one reads off the following conjugate Bailey pair (which is the special case $\lambda=0$ of [24, Corollary 2.1])

$$\gamma_L=\frac{a^{L/N}q^{L^2/N}}{(q)_{M-L}(aq)_{M+L}}\sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{2L+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}} q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}}\begin{bmatrix}\boldsymbol{\mu}+\boldsymbol{\eta}\\ \boldsymbol{\eta}\end{bmatrix},$$

$$\delta_L=\frac{a^{L/N}q^{L^2/N}}{(q)_{M-L}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\ \frac{2L+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix},$$

with $a=q^\ell$ and where (4.4) and $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(\mathcal{I}\boldsymbol{m}+(2L+\ell)\boldsymbol{e}_1)$ hold.

4.3. **String functions.** Taking the limit $L_1,L_2,M\to\infty$ in Theorem 2.1 we obtain

$$\sum_{i=0}^{\infty}\frac{q^{i(i+\ell)/N}}{(q)_i(q)_{i+\ell}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1}\\ \frac{2i+\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}\boldsymbol{n}}\begin{bmatrix}\boldsymbol{m}+\boldsymbol{n}\\ \boldsymbol{n}\end{bmatrix} =\frac{1}{(q)_\infty}\sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1}\\ \frac{\ell+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}}\frac{q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}}}{(q)_{\boldsymbol{\eta}}}. \tag{4.5}$$

It was shown in Refs. [4, 6, 20, 21, 25] that the left-hand side is proportional to a level-N, $\mathrm{A}_1^{(1)}$ string function $C^N_{m,\ell}$ defined as follows. Let

$$\Theta_{n,m}(z,q)=\sum_{j\in\mathbb{Z}+n/2m} q^{mj^2}z^{-mj}$$

be the classical theta function of degree m and characteristic n. The $\mathrm{A}_1^{(1)}$ character of the highest weight module of highest weight $(N-\ell)\Lambda_0+\ell\Lambda_1$ (where Λ_0 and Λ_1 are the fundamental weights of $\mathrm{A}_1^{(1)}$ and $0\le\ell\le N$) is given by

$$\chi_\ell(z,q)=\frac{\sum_{\sigma=\pm1}\sigma\Theta_{\sigma(\ell+1),N+2}(z,q)}{\sum_{\sigma=\pm1}\sigma\Theta_{\sigma,2}(z,q)}.$$

The level-N $\mathrm{A}_1^{(1)}$ string functions are defined by the expansion

$$\chi_\ell(z,q)=\sum_{m\in2\mathbb{Z}+\ell}C^N_{m,\ell}(q)q^{\frac{m^2}{4N}}z^{-\frac{1}{2}m}.$$

According to the above-cited references

$$
\begin{aligned}
C^N_{m,\ell}(q) &= q^{\frac{(\ell+1)^2}{4(N+2)}-\frac{m^2}{4N}-\frac{1}{8}} \sum_{i=0}^{\infty} \frac{X^{N+2}_{\ell+1}(2i+m)}{(q)_i(q)_{i+m}} \\
&= q^{\frac{(\ell+1)^2}{4(N+2)}-\frac{\ell^2}{4N}-\frac{1}{8}} \sum_{i=0}^{\infty} \frac{q^{i(i+m)/N}}{(q)_i(q)_{i+m}} \\
&\qquad \times \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{N-1} \\ \frac{2i+m+\ell}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}}} q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_\ell)} \begin{bmatrix} \boldsymbol{m}+\boldsymbol{n} \\ \boldsymbol{n} \end{bmatrix},
\end{aligned}
$$

with $\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(\mathcal{I}\boldsymbol{m}+(2i+m)\boldsymbol{e}_1+\boldsymbol{e}_\ell)$ and $X^p_s(L)$ a one-dimensional configuration sum of the $(p-1)$-state Andrews–Baxter–Forrester model in regime I,

$$
X^p_s(L) = \sum_{j=-\infty}^{\infty} q^{j(pj+s)} \left\{ \begin{bmatrix} L \\ \frac{1}{2}(L-s+1)-pj \end{bmatrix} - \begin{bmatrix} L \\ \frac{1}{2}(L-s-1)-pj \end{bmatrix} \right\}.
$$

Comparing with (4.5) we obtain the following expression of the string function

$$
C^N_{m,\sigma N}(q) = \frac{q^{\frac{1}{4(N+2)}-\frac{1}{8}}}{(q)_\infty} \sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}^{N-1} \\ \frac{m+\sigma N}{2N}+(C^{-1}\boldsymbol{\eta})_1\in\mathbb{Z}}} \frac{q^{\boldsymbol{\eta}C^{-1}\boldsymbol{\eta}}}{(q)_{\boldsymbol{\eta}}},
$$

which was first derived by Lepowsky and Primc [19].

Acknowledgements. We thank Omar Foda and Trevor Welsh for discussions on the Burge transform. The first author was supported by the "Stichting Fundamenteel Onderzoek der Materie". The second author was supported by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

References

1. G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, (Addison-Wesley, Reading, Massachusetts, 1976).
2. G. E. Andrews, *Schur's theorem, Capparelli's conjecture and q-trinomial coefficients*, Contemp. Math. **166** (1994) 141–154.
3. G. E. Andrews and R. J. Baxter, *Lattice gas generalization of the hard hexagon model. III. q-Trinomial coefficients*, J. Stat. Phys. **47** (1987) 297–330.
4. T. Arakawa, T. Nakanishi, K. Oshima and A. Tsuchiya, *Spectral decomposition of path space in solvable lattice model*, Comm. Math. Phys. **181** (1996) 157–182.
5. W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949) 1–10.

6. P. Bouwknegt, A. W. W. Ludwig and K. Schoutens, *Spinon basis for higher level $SU(2)$ WZW models*, Phys. Lett. B **359** (1995) 304–312.
7. W. H. Burge, *Restricted partition pairs*, J. Combin. Theory Ser. A **63** (1993) 210–222.
8. L. M. Butler, *Subgroup lattices and symmetric functions*, Memoirs of the Amer. Math. Soc., no. 539, vol. **112** (1994).
9. L. Carlitz, *Remark on a combinatorial identity*, J. Combin. Theory Ser. A **17** (1974) 256–257.
10. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Exactly solvable SOS models: local height probabilities and theta function identities*, Nucl. Phys. B **290** (1987) 231–273.
11. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Exactly solvable SOS models. II. Proof of the star-triangle relation and combinatorial identities*, Adv. Stud. Pure Math. **16**, 17–122.
12. O. Foda, K. S. M. Lee and T. A. Welsh, *A Burge tree of Virasoro-type polynomial identities*, Int. J. Mod. Phys. A **13** (1998) 4967–5012.
13. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35, (Cambridge University Press, Cambridge, 1990).
14. I. M. Gessel and C. Krattenthaler, *Cylindric partitions*, Trans. Amer. Math. Soc. **349** (1997) 429–479.
15. H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225** (1967) 154–190.
16. B. Gordon, *Some continued fractions of the Rogers–Ramanujan type*, Duke Math. J. **31** (1965) 741–748.
17. H. W. Gould, *A new symmetrical combinatorial identity*, J. Combin. Theory Ser. A **13** (1972) 278–286.
18. A. N. Kirillov, *Dilogarithm identities*, Prog. Theor. Phys. Suppl. **118** (1995) 61–142.
19. J. Lepowsky and M. Primc, *Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$*, Contemp. Math. Vol. 46 (AMS, Providence, 1985).
20. A. Nakayashiki and Y. Yamada, *Crystallizing the spinon basis*, Comm. Math. Phys. **178** (1996) 179–200.
21. A. Nakayashiki and Y. Yamada, *Crystalline spinon basis for RSOS models*, Int. J. Mod. Phys. A **11** (1996) 395–408.
22. A. Schilling, *Multinomials and polynomial bosonic forms for the branching functions of the $\widehat{su}(2)_M \times \widehat{su}(2)_N/\widehat{su}(2)_{M+N}$ conformal coset models*, Nucl. Phys. B **467** (1996) 247–271.
23. A. Schilling and S. O. Warnaar, *A higher-level Bailey lemma*, Int. J. Mod. Phys. B **11** (1997) 189–195.
24. A. Schilling and S. O. Warnaar, *A higher level Bailey lemma: Proof and application*, The Ramanujan Journal **2** (1998) 327–349.
25. A. Schilling and S. O. Warnaar, *Conjugate Bailey pairs. From configuration sums and fractional-level string functions to Bailey's lemma*, Contemp. Math. **297** (2002) 227–255.
26. L. J. Slater, *Further identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952) 147–167.

27. S. O. Warnaar, *The Andrews–Gordon identities and q-multinomial coefficients*, Comm. Math. Phys. **184** (1997) 203–232.

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands, (Present address: Department of Mathematics, M.I.T., Cambridge MA 02139, USA)

E-mail address: `anne@math.mit.edu`

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

E-mail address: `warnaar@wins.uva.nl`