

# ROGERS–SZEGŐ POLYNOMIALS AND HALL–LITTLEWOOD SYMMETRIC FUNCTIONS

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ABSTRACT. We use Rogers–Szegő polynomials to unify some well-known identities for Hall–Littlewood symmetric functions due to Macdonald and Kawakana.

## 1. INTRODUCTION AND SUMMARY OF RESULTS

Three classical identities for Schur functions are [18, 16, 17]

$$(1.1a) \quad \sum_{\lambda} s_{\lambda}(x) = \prod_{i \geq 1} \frac{1}{1 - x_i} \prod_{i < j} \frac{1}{1 - x_i x_j}$$

and

$$(1.1b) \quad \sum_{\substack{\lambda \\ \lambda \text{ even}}} s_{\lambda}(x) = \prod_{i \geq 1} \frac{1}{1 - x_i^2} \prod_{i < j} \frac{1}{1 - x_i x_j}$$

and

$$(1.1c) \quad \sum_{\substack{\lambda \\ \lambda' \text{ even}}} s_{\lambda}(x) = \prod_{i < j} \frac{1}{1 - x_i x_j}.$$

Here  $\lambda$  denotes a partition,  $\lambda'$  its conjugate, and the condition “ $\lambda$  even” (or “ $\lambda'$  even”) implies that all parts of  $\lambda$  (or all parts of  $\lambda'$ ) must be even. Furthermore,  $s_{\lambda}(x) = s_{\lambda}(x_1, x_2, \dots)$  is a Schur function of a finite or infinite number of variables.

When  $x = (x_1, \dots, x_n)$  the identities (1.1a)–(1.1c) may be viewed as reciprocals of Weyl denominator formulas; the latter expressing the products

$$\prod_{i=1}^n (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j), \quad \prod_{i=1}^n (1 - x_i^2) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)$$

and

$$\prod_{1 \leq i < j \leq n} (1 - x_i x_j)$$

as sums over the  $B_n$ ,  $C_n$  and  $D_n$  Weyl groups [6].

Probably the most important application of (1.1) was given by Macdonald, who used the bounded form

$$(1.2) \quad \sum_{\substack{\lambda \\ \lambda_1 \leq k}} s_{\lambda}(x_1, \dots, x_n) = \frac{\det(x_i^{j-1} - x_i^{2n+k-j})}{\prod_{i=1}^n (1 - x_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 - x_i x_j)}$$

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of (1.1a) to prove the famous MacMahon conjecture in the theory of plane partitions [3, 17].

Formulae that incorporate all three Schur function identities (1.1) were recently found by Bressoud [4], Ishikawa and Wakayama [8], and by Jouhet and Zeng [9]. If  $m_i(\lambda)$  denotes the multiplicity of the part  $i$  in  $\lambda$ , i.e.,  $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ , then the Bressoud–Ishikawa–Wakayama identity states that

$$(1.3) \quad \sum_{\lambda} f_{\lambda}(a, b) s_{\lambda}(x) = \prod_{i \geq 1} \frac{1}{(1 - ax_i)(1 - bx_i)} \prod_{i < j} \frac{1}{1 - x_i x_j},$$

where

$$f_{\lambda}(a, b) = \prod_{j \text{ odd}} \frac{a^{m_j(\lambda')+1} - b^{m_j(\lambda')+1}}{a - b} \prod_{j \text{ even}} \frac{1 - (ab)^{m_j(\lambda')+1}}{1 - ab}.$$

Similarly, the Jouhet–Zeng formula asserts that

$$(1.4) \quad \sum_{\lambda} f_{\lambda'}(a, b) s_{\lambda}(x) = \prod_{i \geq 1} \frac{(1 + ax_i)(1 + bx_i)}{(1 - x_i)(1 + x_i)} \prod_{i < j} \frac{1}{1 - x_i x_j}.$$

For  $b = 0$  (1.3) and (1.4) reduce to identities of Littlewood [16] combining (1.1a) and (1.1c), or (1.1a) and (1.1b), respectively. Even more general formulae than (1.3) and (1.4), which will not play a role in the present paper, may be found in [9]. A  $\lambda$ -ring approach to the above results may be found in [14].

An important generalization of the Schur functions is given by the Hall–Littlewood symmetric functions  $P_{\lambda}(x; t)$ . Here  $t$  is an additional scalar variable such that  $P_{\lambda}(x; 0) = s_{\lambda}(x)$ . Employing the Hall–Littlewood functions, Macdonald [17] gave the following four generalizations of the identities (1.1a)–(1.1c):

$$(1.5a) \quad \sum_{\lambda} P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1}{1 - x_i} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}$$

and

$$(1.5b) \quad \sum_{\substack{\lambda \\ \lambda \text{ even}}} P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1}{1 - x_i^2} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}$$

and

$$(1.5c) \quad \sum_{\substack{\lambda \\ \lambda' \text{ even}}} c_{\lambda}(t) P_{\lambda}(x; t) = \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}$$

and

$$(1.5d) \quad \sum_{\lambda} d_{\lambda}(t) P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1 - tx_i}{1 - x_i} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j},$$

where for  $\lambda'$  even (so that  $m_i(\lambda)$  is even)

$$c_{\lambda}(t) = \prod_{i \geq 1} (1 - t)(1 - t^3) \cdots (1 - t^{m_i(\lambda)-1}),$$

and for general  $\lambda$

$$d_{\lambda}(t) = \prod_{i \geq 1} (1 - t)(1 - t^3) \cdots (1 - t^{2\lceil m_i(\lambda)/2 \rceil - 1})$$

with  $\lfloor \cdot \rfloor$  and  $\lceil \cdot \rceil$  the usual floor (or integer part) and ceiling functions.

Recently, Kawanaka [12] added two further identities to the list as follows. For  $\lambda$  a partition, let  $\lambda_e$  and  $\lambda_o$  be the partitions containing the even parts and the odd parts of  $\lambda$  respectively. For example, if  $\lambda = (4, 3, 3, 2, 1, 1, 1)$  then  $\lambda_e = (4, 2)$  and  $\lambda_o = (3, 3, 1, 1, 1)$ . As usual  $l(\lambda)$  denotes the length of the partition  $\lambda$  (that is, the number of nonzero parts). Then the Kawanaka identities correspond to the sums

$$(1.5e) \quad \sum_{\lambda} e_{\lambda}(t) P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1 + t^{1/2} x_i}{1 - x_i} \prod_{i < j} \frac{1 - t x_i x_j}{1 - x_i x_j}$$

and

$$(1.5f) \quad \sum_{(\lambda_o)' \text{ even}} f_{\lambda}(t) P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1 - t x_i^2}{1 - x_i^2} \prod_{i < j} \frac{1 - t x_i x_j}{1 - x_i x_j},$$

where

$$e_{\lambda}(t) = \prod_{i \geq 1} (1 + t^{1/2})(1 + t) \cdots (1 + t^{m_i(\lambda)/2}),$$

and, for  $(\lambda_o)'$  even (so that the odd parts of  $\lambda$  have even multiplicity),

$$f_{\lambda}(t) = t^{l(\lambda_o)/2} d_{\lambda}(t).$$

Like their Schur function counterparts the above Hall–Littlewood identities have interesting applications. For example, Kawanaka’s identities have an interpretation in terms of the representation theory of the general linear group over finite fields [11, 12]. In particular, (1.5e) encodes the fact that the symmetric space  $\text{GL}_n(\mathbb{F}_{p^2})/\text{GL}_n(\mathbb{F}_p)$  (where  $\text{GL}_n(\mathbb{F}_p)$  is the general linear group over a finite field of  $p$  elements) is multiplicity free. Similarly, (1.5f), asserts that the symmetric space  $\text{GL}_{2n}(\mathbb{F}_p)/\text{Sp}_{2n}(\mathbb{F}_p)$  (with  $\text{Sp}_{2n}$  the symplectic group) is multiplicity free.

Another nice application follows by again considering the bounded versions of the identities of (1.5), see e.g., [7, 10, 17, 19]. For example, (1.5a) has the following bounded form generalizing (1.2). Let  $x = (x_1, \dots, x_n)$  and

$$\Phi(x; t) = \prod_{i=1}^n \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1 - t x_i x_j}{1 - x_i x_j}.$$

Then for  $k$  a positive integer

$$\sum_{\substack{\lambda \\ \lambda_1 \leq k}} P_{\lambda}(x; t) = \sum_{\epsilon \in \{\pm 1\}^n} \Phi(x^{\epsilon}; t) \prod_{i=1}^n x_i^{k(1-\epsilon_i)/2},$$

where  $x^{\epsilon} = (x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$  [17, pp. 232–234]. Making the principal specialization  $x = (z, zt, \dots, zt^{n-1})$  (and then replacing  $t$  by  $q$ ) leads to interesting  $q$ -series identities. The most important ones being the famous Rogers–Ramanujan identities — arising from the bounded form of (1.5b) due to Stembridge [19].

Given the identities (1.5a)–(1.5f) and their striking similarity, an obvious question is whether one can understand all six as special cases of a master identity for Hall–Littlewood functions. We will answer this question in the affirmative in the form of Theorem 1.1 below, generalizing the Jouhet–Zeng identity (1.4) to the level of Hall–Littlewood functions.

For  $m$  a nonnegative integer let  $H_m(z; t)$  be the Rogers–Szegő polynomial [1, Ch. 3, Examples 3–9]

$$(1.6) \quad H_m(z; t) = \sum_{i=0}^m z^i \begin{bmatrix} m \\ i \end{bmatrix}_t.$$

Here  $\begin{bmatrix} m \\ n \end{bmatrix}_t$  is the usual  $t$ -binomial coefficient:

$$\begin{bmatrix} n \\ m \end{bmatrix}_t = \begin{cases} \frac{(t^{n-m+1}; t)_m}{(t; t)_m} & \text{for } m \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $(t; t)_0 = 1$  and  $(t; t)_n = \prod_{i=1}^n (1 - t^i)$  are  $t$ -shifted factorials. We extend the definition of the Rogers–Szegő polynomials to partitions  $\lambda$  by

$$(1.7) \quad h_\lambda(z; t) = \prod_{i \geq 1} H_{m_i(\lambda)}(z; t).$$

For example  $h_{(3,2,2,1)} = H_1^2 H_2$ .

**Theorem 1.1.** *The following formal identity holds:*

$$(1.8) \quad \sum_{\lambda} a^{l(\lambda_e)} h_{\lambda_e}(ab; t) h_{\lambda_o}(b/a; t) P_{\lambda}(x; t) \\ = \prod_{i \geq 1} \frac{(1 + ax_i)(1 + bx_i)}{(1 - x_i)(1 + x_i)} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

It is important to note that the left-hand side satisfies the necessary symmetry under interchange of  $a$  and  $b$ . From the definition of the Rogers–Szegő polynomials it readily follows that  $H_m(z^{-1}; t) = z^{-m} H_m(z; t)$ . Hence, since  $\sum_i m_i(\lambda) = l(\lambda)$ ,

$$h_{\lambda}(z^{-1}; t) = z^{-l(\lambda)} h_{\lambda}(z; t).$$

Applying this to  $h_{\lambda_o}(b/a; t)$  in (1.8) shows that the left is invariant under the interchange of  $a$  and  $b$ .

When  $t = 1$  the Hall–Littlewood functions reduce to the monomial symmetric functions, i.e.,  $P_{\lambda}(x; 1) = m_{\lambda}(x)$ . Since  $h_{\lambda}(z; 1) = (1 + z)^{l(\lambda)}$  this implies the elegant summation

$$\sum_{\lambda} (1 + ab)^{l(\lambda_e)} (a + b)^{l(\lambda_o)} m_{\lambda}(x) = \prod_{i \geq 1} \frac{(1 + ax_i)(1 + bx_i)}{(1 - x_i)(1 + x_i)}.$$

In the following we will show how all six identities stated in (1.5) follow from (1.8). If we take  $a = 1$ , use  $h_{\lambda_e}(b; t) h_{\lambda_o}(b; t) = h_{\lambda}(b; t)$  and finally replace  $b \rightarrow a$  we obtain our first corollary.

**Corollary 1.1.** *There holds*

$$(1.9) \quad \sum_{\lambda} h_{\lambda}(a; t) P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1 + ax_i}{1 - x_i} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

The following explicit evaluations for the Rogers–Szegő polynomials are known, see e.g., [1, 2]:

$$(1.10a) \quad H_m(0; t) = 1$$

$$(1.10b) \quad H_m(-1; t) = \begin{cases} (t; t^2)_{m/2} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

$$(1.10c) \quad H_m(-t; t) = (t; t^2)_{\lceil m/2 \rceil} = (t; t)_m / (t^2; t^2)_{\lfloor m/2 \rfloor}$$

$$(1.10d) \quad H_m(t^{1/2}; t) = (-t^{1/2}; t^{1/2})_m.$$

This immediately yields (in exactly that order) (1.5a), (1.5c)–(1.5e). We note in particular that by taking  $a = -1$  in (1.9) the summand vanishes unless all  $m_i(\lambda)$  are even. That is, all parts of  $\lambda$  must have even multiplicity, or equivalently,  $\lambda'$  must be even.

Next we consider the case  $b = -a$  of (1.8). Using (1.10c) and making the replacement  $a^2 \rightarrow a$  this gives our second corollary.

**Corollary 1.2.** *There holds*

$$(1.11) \quad \sum_{(\lambda_o)^\lambda \text{ even}} a^{l(\lambda_o)/2} h_{\lambda_o}(-a; t) \left( \prod_{i \geq 1} (t; t^2)_{m_i(\lambda_o)/2} \right) P_\lambda(x; t) \\ = \prod_{i \geq 1} \frac{1 - ax_i^2}{1 - x_i^2} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

As remarked before,  $(\lambda_o)'$  being even is equivalent to the odd parts of  $\lambda$  having even multiplicity. We also note that the product on the left-hand side may be replaced by the equivalent

$$\prod_{i \geq 1} (t; t^2)_{m_{2i-1}(\lambda)/2}.$$

Using three of the four specializations of (1.10) gives (1.5b), (1.5c) and (1.5f). This shows that a seventh identity, corresponding to (1.11) with  $a = -t^{1/2}$  has actually been missing from the literature:

$$\sum_{(\lambda_o)^\lambda \text{ even}} k_\lambda(t) P_\lambda(x; t) = \prod_{i \geq 1} \frac{1 + t^{1/2} x_i^2}{1 - x_i^2} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j},$$

with

$$k_\lambda(t) = (-t^{1/2})^{l(\lambda_o)/2} \prod_{i \geq 1} (-t^{1/2}; t^{1/2})_{m_i(\lambda_o)/2}.$$

A further interesting special case of the theorem arises after taking  $b = 0$ .

**Corollary 1.3.** *There holds*

$$(1.12) \quad \sum_{\lambda} a^{l(\lambda_o)} P_\lambda(x; t) = \prod_{i \geq 1} \frac{1 + ax_i}{1 - x_i^2} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

In the Schur case this reduces to the Littlewood formula mentioned after (1.4), combining (1.1a) and (1.1b).

Observing that

$$H_m(z; 0) = 1 + z + \cdots + z^m = \frac{1 - z^{m+1}}{1 - z}$$

it readily follows that (1.8) simplifies to (1.4) when  $t = 0$ . The reader may wonder whether there perhaps is a companion to Theorem 1.1 extending (1.3) in much the same way. It is certainly possible (see (4.5)) to obtain a formula of the form

$$(1.13) \quad \sum_{\lambda} C_{\lambda}(a, b; t) P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{(1 - atx_i)(1 - btx_i)}{(1 - ax_i)(1 - bx_i)} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

However, for general  $a$  and  $b$  the rational function  $C_{\lambda}(a, b; t)$  does not possess nice characteristics (like factorisation), and we dismiss (1.13) for being insufficiently interesting. Only for  $b = 0$  we have a result elegant enough (although not very deep) to be stated explicitly:

$$(1.14) \quad \sum_{\lambda} a^{l((\lambda')_{\circ})} d_{\lambda}(t) P_{\lambda}(x; t) = \prod_{i \geq 1} \frac{1 - atx_i}{1 - ax_i} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

Here  $(\lambda')_{\circ}$  is the odd part of the conjugate of  $\lambda$  (so that  $l((\lambda')_{\circ})$  is the number of odd columns of the diagram of  $\lambda$ ) and  $d_{\lambda}(t) = h_{\lambda}(-t; t)$  as before. In the Schur case (1.14) reduces to the Littlewood formula mentioned after (1.4), combining (1.1a) and (1.1c).

It may perhaps seem surprising that at the level of Schur functions a pair of equally elegant formulae exists but that only one of these admits an appealing generalization to Hall–Littlewood functions. The explanation for this is however easily given. Let  $\Lambda$  be the ring of symmetric functions and  $\omega : \Lambda \rightarrow \Lambda$  the involution defined by

$$(1.15) \quad \omega(s_{\lambda}) = s_{\lambda'}.$$

**Lemma 1.1.** *Applying  $\omega$  to (1.3) yields (1.4).*

Hence (1.3) and (1.4) may really be viewed as one and the same identity. Since no “good”  $t$ -analogue of  $\omega$  exists there is no guarantee for Hall–Littlewood identities to come in pairs also.

Finally we mention some further results related to (1.8). The first concerns the bounded form of Theorem 1.1, or, to be precise, our failure to find this in full generality. At present we have only been able to find the bounded analogue of (1.9) as follows.

For  $x = (x_1, \dots, x_n)$  define

$$\Phi(x; a, t) = \prod_{i=1}^n \frac{1 + ax_i}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

For  $k$  a positive integer also define a bounded version of  $h_{\lambda}(z; t)$  by

$$h_{\lambda; k}(z; t) = \prod_{i=1}^{k-1} H_{m_i(\lambda)}(z; t).$$

For example  $h_{(3,2,2,1);1} = 1$ ,  $h_{(3,2,2,1);2} = H_1$ ,  $h_{(3,2,2,1);3} = H_1 H_2$  and  $h_{(3,2,2,1);k} = H_1^2 H_2$  for  $k \geq 4$ .

**Proposition 1.1.** *For  $k$  a positive integer and  $x = (x_1, \dots, x_n)$  there holds*

$$(1.16) \quad \sum_{\substack{\lambda \\ \lambda_1 \leq k}} h_{\lambda; k}(z; t) P_\lambda(x; t) = \sum_{\epsilon \in \{\pm 1\}^n} \Phi(x^\epsilon; a, t) \prod_{i=1}^n x_i^{k(1-\epsilon_i)/2}.$$

For  $k = 1$  this can be simplified since

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 1}} P_\lambda(x; t) = \sum_{r=0}^{\infty} P_{(1^r)}(x; t) = \sum_{r=0}^{\infty} e_r(x) = \prod_{i \geq 1} (1 + x_i),$$

with  $e_r$  the  $r$ th elementary symmetric function. Hence

$$\sum_{\epsilon \in \{\pm 1\}^n} \Phi(x^\epsilon; a, t) \prod_{i=1}^n x_i^{(1-\epsilon_i)/2} = \prod_{i=1}^n (1 + x_i).$$

Assuming the specialization  $x = (z, zt, \dots, zt^{n-1})$ , replacing  $t$  by  $q$ , and letting  $n$  tend to infinity yields the  $b = 0$  case of the next proposition.

**Proposition 1.2.** *There holds*

$$\begin{aligned} & \sum_{\substack{\lambda \\ \lambda_1 \leq k}} z^{|\lambda|} (b; q^{-1})_{\lambda_1} h_{\lambda; k}(a; q) P_\lambda(1, q, q^2, \dots; q) \\ &= \frac{(bz^2, -z, -az; q)_\infty}{(z^2, -bz, -abz; q)_\infty} \\ & \quad \times \sum_{r=0}^{\infty} (-1)^r a^r z^{kr} q^{(k+1)\binom{r}{2}} \frac{1 - z^2 q^{2r-1}}{1 - z^2 q^{-1}} \frac{(b; q^{-1})_r (z^2/q, -z/a; q)_r}{(q, -az, bz^2; q)_r}. \end{aligned}$$

Because it lies somewhat outside the scope of the present paper we will not prove this  $q$ -series identity here. (For  $b = 0$  it of course follows from (1.16).) As one application let us take  $b = 0$  and assume that  $z = q^{1/2}$  or  $z = q$  but  $-az \neq q$ . Some simple manipulations then give

$$\begin{aligned} & \sum_{\substack{\lambda \\ \lambda_1 \leq k}} z^{|\lambda|} h_{\lambda; k}(a; q) P_\lambda(1, q, q^2, \dots; q) \\ &= \frac{(-z, -az; q)_\infty}{(q; q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r a^r z^{kr} q^{(k+1)\binom{r}{2}} \frac{(-z/a; q)_r}{(-az; q)_r}. \end{aligned}$$

When  $z = q$  and  $a = 1$  the right may be written as a product using Jacobi's triple product identity, so that we find the Rogers–Ramanujan-type identity

$$\sum_{\substack{\lambda \\ \lambda_1 \leq k}} q^{|\lambda|} h_{\lambda; k}(1; q) P_\lambda(1, q, q^2, \dots; q) = \frac{(-q; q)_\infty^2 (q, q^k, q^{k+1}; q^{k+1})_\infty}{(q; q)_\infty}.$$

Finally we mention that for general  $a$  and  $b$  the generalization of Theorem 1.1 to Macdonald polynomials lacks the necessary elegance, and only (1.12) and (1.14) admit simple  $q$ -generalizations.

Let  $P_\lambda(x; q, t)$  be Macdonald's symmetric function and let  $b_\lambda^{\text{oa}}(q, t)$  and  $b_\lambda^{\text{el}}(q, t)$  be the rational functions defined in (2.2) of the next section.

**Proposition 1.3.** *The following formal identities hold:*

$$(1.17) \quad \sum_{\lambda} a^{l(\lambda_{\circ})} b_{\lambda}^{\text{oa}}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(1 + ax_i)(qtx_i^2; q^2)_{\infty}}{(x_i^2; q^2)_{\infty}} \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}}$$

and

$$(1.18) \quad \sum_{\lambda} a^{l((\lambda')_{\circ})} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(atx_i; q)_{\infty}}{(ax_i; q)_{\infty}} \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}}.$$

In the next section we give a brief introduction to Hall–Littlewood functions, Section 3 contains a proof of the claims of the first section, and, finally, in Section 4 we restate some of our results in the language of  $\lambda$ -rings.

## 2. HALL–LITTLEWOOD FUNCTIONS

Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition, i.e.,  $\lambda_1 \geq \lambda_2 \geq \dots$  with finitely many  $\lambda_i$  unequal to zero. The length and weight of  $\lambda$ , denoted by  $l(\lambda)$  and  $|\lambda|$ , are the number and sum of the non-zero  $\lambda_i$  (called parts), respectively. The unique partition of weight zero is denoted by 0, and the multiplicity of the part  $i$  in the partition  $\lambda$  is denoted by  $m_i(\lambda)$ .

We identify a partition with its (Young) diagram or Ferrers graph in the usual way, and the conjugate  $\lambda'$  of  $\lambda$  is the partition obtained by reflecting the diagram of  $\lambda$  in the main diagonal. Hence  $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ .

If  $\lambda$  and  $\mu$  are two partitions then  $\mu \subset \lambda$  iff  $\lambda_i \geq \mu_i$  for all  $i \geq 1$ , i.e., the diagram of  $\lambda$  contains the diagram of  $\mu$ . If  $\mu \subset \lambda$  then the skew-diagram  $\lambda - \mu$  denotes the set-theoretic difference between  $\lambda$  and  $\mu$ , and  $|\lambda - \mu| = |\lambda| - |\mu|$ . A skew diagram  $\theta$  is a horizontal/vertical  $r$ -strip if it contains exactly  $r$  squares, i.e.,  $|\theta| = r$ , and has at most one square in each of its columns/rows. For example, if  $\lambda = (6, 3, 3, 1)$  and  $\mu = (4, 3, 1)$  then  $\lambda - \mu$  is a horizontal 5-strip and  $\lambda' - \mu'$  a vertical 5-strip.

Let  $s = (i, j)$  be a square in the diagram of  $\lambda$ . Then  $a(s)$ ,  $a'(s)$ ,  $l(s)$  and  $l'(s)$  are the arm-length, arm-colength, leg-length and leg-colength of  $s$ , defined by

$$(2.1a) \quad a(s) = \lambda_i - j, \quad a'(s) = j - 1$$

$$(2.1b) \quad l(s) = \lambda'_j - i, \quad l'(s) = i - 1.$$

From this we may define several standard rational functions on partitions:

$$b_{\lambda}(s; q, t) = \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}},$$

and

$$(2.2) \quad b_{\lambda}^{\text{el}}(q, t) = \prod_{\substack{s \in \lambda \\ l(s) \text{ even}}} b_{\lambda}(s; q, t), \quad b_{\lambda}^{\text{oa}}(q, t) = \prod_{\substack{s \in \lambda \\ a(s) \text{ odd}}} b_{\lambda}(s; q, t),$$

$$b_{\lambda}^{\text{ol}}(q, t) = \prod_{\substack{s \in \lambda \\ l(s) \text{ odd}}} b_{\lambda}(s; q, t), \quad b_{\lambda}^{\text{ea}}(q, t) = \prod_{\substack{s \in \lambda \\ a(s) \text{ even}}} b_{\lambda}(s; q, t).$$

Obviously,

$$(2.3) \quad b_{\lambda}(q, t) := b_{\lambda}^{\text{el}}(q, t) b_{\lambda}^{\text{ol}}(q, t) = b_{\lambda}^{\text{ea}}(q, t) b_{\lambda}^{\text{oa}}(q, t)$$



and

$$(2.4) \quad b_{\lambda'}^{\text{el}}(q, t)b_{\lambda}^{\text{ea}}(t, q) = b_{\lambda'}^{\text{ol}}(q, t)b_{\lambda}^{\text{oa}}(t, q) = 1.$$

Let  $S_n$  be the symmetric group,  $\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{S_n}$  be the ring of symmetric polynomials in  $n$  independent variables and  $\Lambda$  the ring of symmetric functions in countably many variables.

For  $x = (x_1, \dots, x_n)$  and  $\lambda$  a partition such that  $l(\lambda) \leq n$  the Hall–Littlewood polynomial  $P_{\lambda}(x; t)$  is defined by

$$(2.5) \quad P_{\lambda}(x; t) = \sum_{w \in S_n / S_n^{\lambda}} w \left( x^{\lambda} \prod_{\lambda_i > \lambda_j} \frac{x_i - tx_j}{x_i - x_j} \right).$$

Here  $S_n^{\lambda}$  is the subgroup of  $S_n$  consisting of those permutations that leave  $\lambda$  invariant, and  $w(f(x)) = f(w(x))$ . When  $l(\lambda) > n$ ,

$$(2.6) \quad P_{\lambda}(x; t) = 0.$$

The Hall–Littlewood polynomials are symmetric polynomials in  $x$ , homogeneous of degree  $|\lambda|$ , with coefficients in  $\mathbb{Z}[t]$ , and form a  $\mathbb{Z}[t]$  basis of  $\Lambda_n[t]$ . Thanks to the stability property  $P_{\lambda}(x_1, \dots, x_n, 0; t) = P_{\lambda}(x_1, \dots, x_n; t)$  the Hall–Littlewood polynomials may be extended to the Hall–Littlewood functions in an infinite number of variables  $x_1, x_2, \dots$  in the usual way, to form a  $\mathbb{Z}[t]$  basis of  $\Lambda[t]$ . The parameter  $t$  in the Hall–Littlewood symmetric functions serves to interpolate between the Schur functions and monomial symmetric functions;  $P_{\lambda}(x; 0) = s_{\lambda}(x)$  and  $P_{\lambda}(x; 1) = m_{\lambda}(x)$ . We also introduce a second Hall–Littlewood function  $Q_{\lambda}$  by

$$(2.7) \quad Q_{\lambda}(x; t) = b_{\lambda}(t)P_{\lambda}(x; t),$$

where  $b_{\lambda}(t) = b_{\lambda}(0, t) = \prod_{i \geq 1} (t; t)_{m_i(\lambda)}$ . Then the Cauchy identity for Hall–Littlewood functions takes the form

$$(2.8) \quad \sum_{\lambda} P_{\lambda}(x; t)Q_{\lambda}(y; t) = \prod_{i, j \geq 1} \frac{1 - tx_i y_j}{1 - x_i y_j}.$$

When  $\lambda = (1^r)$  and  $\lambda = (r)$  the Hall–Littlewood polynomials reduce to the  $r$ th elementary and  $r$ th complete symmetric functions

$$(2.9) \quad P_{(1^r)} = e_r \quad \text{and} \quad P_{(r)} = h_r.$$

These functions may be defined by their generating functions as

$$(2.10) \quad \sum_{r=0}^{\infty} z^r e_r(x) = \prod_{i \geq 1} (1 + tx_i)$$

and

$$(2.11) \quad \sum_{r=0}^{\infty} z^r h_r(x) = \prod_{i \geq 1} \frac{1}{1 - tx_i}.$$

Since  $e_r = s_{(1^r)}$  and  $h_r = s_{(r)}$  we have

$$(2.12) \quad \omega(e_r) = h_r,$$

with  $\omega$  the involution (1.15).

The Pieri formula for Hall–Littlewood polynomials states that

$$(2.13) \quad P_{\mu}(x; t)e_r(x) = \sum_{\lambda} f_{\mu(1^r)}^{\lambda}(t)P_{\lambda}(x; t),$$

where the coefficient  $f_{\mu(1^r)}^\lambda(t)$  is zero unless  $\mu \subset \lambda$  such that the skew diagram  $\lambda - \mu$  is a horizontal  $r$ -strip. An explicit expression for  $f_{\mu(1^r)}^\lambda(t)$  is given by [17, p. 215]

$$(2.14) \quad f_{\mu(1^r)}^\lambda(t) = \prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t \quad \text{for } |\lambda - \mu| = r$$

and zero otherwise.

The more general structure constants of the Hall–Littlewood functions are defined by

$$(2.15) \quad P_\mu(x; t)P_\nu(x; t) = \sum_{\lambda} f_{\mu\nu}^\lambda(t)P_\lambda(x; t).$$

These may be utilized to define the skew function  $Q_{\lambda/\mu}$  by

$$(2.16) \quad Q_{\lambda/\mu}(x; t) = \sum_{\nu} f_{\mu\nu}^\lambda(t)Q_\nu(x; t).$$

### 3. PROOFS

**3.1. Proof of Theorem 1.1.** We first prove the  $b = 0$  case of the theorem, corresponding to Corollary 1.3, and then use this to obtain the theorem for general  $a$  and  $b$ .

Our point of departure is (1.5b). Replacing the summation index  $\lambda$  by  $\mu$  and multiplying both sides by  $\prod_i(1 + ax_i)$  yields

$$(3.1) \quad \sum_{\substack{\mu \\ \mu \text{ even}}} P_\mu(x; t) \prod_{i \geq 1} (1 + ax_i) = \prod_{i \geq 1} \frac{1 + ax_i}{1 - x_i^2} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

By (2.10) we can expand the left-hand side of (3.1) as

$$\text{LHS}(3.1) = \sum_{r=0}^{\infty} \sum_{\substack{\mu \\ \mu \text{ even}}} a^r P_\mu(x; t) e_r(x).$$

Next we use the Pieri formula (2.13) to rewrite this as

$$\text{LHS}(3.1) = \sum_{r=0}^{\infty} \sum_{\substack{\lambda, \mu \\ \mu \text{ even}}} a^r f_{\mu(1^r)}^\lambda(t) P_\lambda(x; t).$$

Since  $f_{\mu(1^r)}^\lambda(t) = 0$  when  $|\lambda - \mu| \neq r$  this may also be written as

$$\text{LHS}(3.1) = \sum_{\substack{\lambda, \mu \\ \mu \text{ even}}} a^{|\lambda - \mu|} f_{\mu(1^{|\lambda - \mu|})}^\lambda(t) P_\lambda(x; t).$$

Since  $f_{\mu(1^{|\lambda - \mu|})}^\lambda(t)$  is zero unless  $\lambda - \mu$  is a vertical strip, only those partitions  $\mu$  contribute to the sum for which  $0 \leq \lambda_i - \mu_i \leq 1$ . Combined with the fact that  $\mu$  must be even this completely fixes  $\mu$  as  $\mu_i = 2[\lambda_i/2]$  (so that  $|\lambda - \mu| = l(\lambda_o)$ , the number of parts of  $\lambda$  of odd length). For example if  $\lambda = (7, 5, 5, 4, 3, 1)$  then the only contributing  $\mu$  to the sum is  $\mu = (6, 4, 4, 4, 2)$ . In terms of conjugate partitions this implies that if  $\lambda'_i > \lambda'_{i+1}$  then  $\mu'_i = \lambda'_{i+1}$ . For the partitions in our example

$\lambda' = (6, 5, 5, 4, 3, 1, 1)$  and  $\mu' = (5, 5, 4, 4, 1, 1)$  and  $\lambda_1 > \lambda_2$  so that  $\lambda_2 = \mu_2$ ,  $\lambda_3 > \lambda_4$  so that  $\lambda_4 = \mu_4$ , et cetera. From (2.14) we infer that

$$(3.2) \quad f_{\mu(1^{|\lambda-\mu|})}^\lambda(t) = \prod_{i \geq 1} \left[ \begin{matrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{matrix} \right]_t.$$

By the above considerations regarding  $\lambda$  and  $\mu$ , we find that whenever an upper index of a  $t$ -binomial coefficient in the above product is positive the lower index must be zero. Hence we simplify to

$$\text{LHS(3.1)} = \sum_{\lambda} a^{l(\lambda_o)} P_{\lambda}(x; t).$$

Equating this with the right-hand side of (3.1) completes the proof of Corollary 1.3.

Next we use (1.12) to prove the full theorem. To this end we multiply both sides of (1.12) by  $\prod_i (1 + bx_i)$  and replace  $\lambda$  by  $\mu$  to get

$$(3.3) \quad \sum_{\mu} a^{l(\mu_o)} P_{\mu}(x; t) \prod_{i \geq 1} (1 + bx_i) = \prod_{i \geq 1} \frac{(1 + ax_i)(1 + bx_i)}{(1 - x_i)(1 + x_i)} \prod_{i < j} \frac{1 - tx_i x_j}{1 - x_i x_j}.$$

Following exactly the same steps as before, again using (2.10), (2.13) and (3.2), the left-hand side may be rewritten as

$$\text{LHS(3.3)} = \sum_{\lambda, \mu} a^{l(\mu_o)} b^{|\lambda-\mu|} P_{\lambda}(x; t) \prod_{i \geq 1} \left[ \begin{matrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{matrix} \right]_t.$$

Next we replace the sum over the partition  $\mu$  by a sum over a sequence  $k = (k_1, k_2, \dots)$  of nonnegative integers as follows:  $\mu'_i = \lambda'_i - k_i$ . Using  $\lambda'_i - \lambda'_{i+1} = m_i(\lambda)$  and

$$\begin{aligned} l(\mu_o) &= \sum_{i \geq 1} m_{2i-1}(\mu) \\ &= \sum_{i \geq 1} (\mu'_{2i-1} - \mu'_{2i}) \\ &= \sum_{i \geq 1} (\lambda'_{2i-1} - \lambda'_{2i} - k_{2i-1} + k_{2i}) \\ &= l(\lambda_o) - \sum_{i \geq 1} (k_{2i-1} - k_{2i}) \\ &= l(\lambda_o) + \sum_{i \geq 1} (-1)^i k_i, \end{aligned}$$

we then obtain

$$\begin{aligned} \text{LHS(3.3)} &= \sum_{\lambda, k} a^{l(\lambda)} P_{\lambda}(x; t) \prod_{i \geq 1} a^{(-1)^i k_i} b^{k_i} \left[ \begin{matrix} m_i(\lambda) \\ k_i \end{matrix} \right]_t \\ &= \sum_{\lambda} a^{l(\lambda)} P_{\lambda}(x; t) \prod_{i \geq 1} \sum_{k_i=0}^{m_i(\lambda)} a^{(-1)^i k_i} b^{k_i} \left[ \begin{matrix} m_i(\lambda) \\ k_i \end{matrix} \right]_t. \end{aligned}$$

Factoring the product over  $i$  into a product over even values of  $i$  and a product over odd values of  $i$  and then using that

$$m_i(\lambda_e) = \begin{cases} m_i(\lambda) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad m_i(\lambda_o) = \begin{cases} m_i(\lambda) & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even,} \end{cases}$$

we obtain the further rewriting

$$\text{LHS(3.3)} = \sum_{\lambda} a^{l(\lambda_o)} P_{\lambda}(x; t) \prod_{i \geq 1} \sum_{k=0}^{m_i(\lambda_e)} (ab)^k \begin{bmatrix} m_i(\lambda_e) \\ k \end{bmatrix}_t \sum_{k=0}^{m_i(\lambda_o)} (b/a)^k \begin{bmatrix} m_i(\lambda_o) \\ k \end{bmatrix}_t.$$

Finally, by (1.6) and (1.7), this becomes

$$\begin{aligned} \text{RHS(3.3)} &= \sum_{\lambda} a^{l(\lambda_o)} P_{\lambda}(x; t) \prod_{i \geq 1} H_{m_i(\lambda_e)}(ab; t) H_{m_i(\lambda_o)}(b/a; t) \\ &= \sum_{\lambda} a^{l(\lambda_o)} h_{\lambda_e}(ab; t) h_{\lambda_o}(b/a; t) P_{\lambda}(x; t), \end{aligned}$$

completing the proof.

**3.2. Proof of Lemma 1.1.** Acting with  $\omega$  on the left-hand side of (1.4) yields

$$\begin{aligned} \omega(\text{LHS(1.4)}) &= \sum_{\lambda} f_{\lambda'}(a, b) \omega(s_{\lambda})(x) \\ &= \sum_{\lambda} f_{\lambda'}(a, b) s_{\lambda'}(x) \\ &= \sum_{\lambda} f_{\lambda}(a, b) s_{\lambda}(x) \\ &= \text{LHS(1.3)}. \end{aligned}$$

where in the second-last step we have changed the summation index from  $\lambda$  to its conjugate and used the fact that summing over  $\lambda$  is equivalent to summing over  $\lambda'$ .

Dealing with the right-hand side requires a few more steps but is equally elementary. By (1.1b) and (2.10) we have

$$\text{RHS(1.4)} = \sum_{u, v} \sum_{\substack{\lambda \\ \lambda \text{ even}}} a^u b^v e_u(x) e_v(x) s_{\lambda}(x).$$

Therefore

$$\begin{aligned} \omega(\text{RHS(1.4)}) &= \sum_{u, v} \sum_{\substack{\lambda \\ \lambda \text{ even}}} a^u b^v \omega(e_u) \omega(e_v) \omega(s_{\lambda})(x) \\ &= \sum_{u, v} \sum_{\substack{\lambda \\ \lambda \text{ even}}} a^u b^v h_u(x) h_v(x) s_{\lambda'}(x) \\ &= \sum_{u, v} \sum_{\substack{\lambda \\ \lambda' \text{ even}}} a^u b^v h_u(x) h_v(x) s_{\lambda}(x) \\ &= \sum_{u, v} a^u b^v h_u(x) h_v(x) \prod_{i < j} \frac{1}{1 - x_i x_j}, \end{aligned}$$

where the last equality follows from (1.1c). Finally using (2.11) we get

$$\omega(\text{RHS(1.4)}) = \text{RHS(1.3)}.$$

**3.3. Proof of Proposition 1.1.** The proof follows [7, 10, 17, 19] mutatis mutandis.

**3.4. Proof of Proposition 1.3.** We will assume the reader is familiar with the theory of Macdonald polynomials. All notations and definitions used in the proof may be found in Chapter VI of [17]. Whenever possible we have indicated the precise page in [17] where a particular result or definition may be found.

Both (1.17) and (1.18) may simply be proved using their  $a = 0$  specializations established in [17]. It is however more instructive to only prove (1.18) in this way, and to obtain (1.17) by acting on the former with the automorphism  $\omega_{q,t}$  of  $\Lambda_F$ . This automorphism acts on the Macdonald polynomials as [17, p. 327]

$$\omega_{q,t}P_\lambda(x; q, t) = Q_{\lambda'}(x; t, q),$$

where  $Q_\lambda(x; q, t) = b_\lambda(q, t)P_\lambda(x; q, t)$ .

*Proof of (1.18).* We may assume the  $a = 0$  case of (1.18) given by [17, p. 349]

$$(3.4) \quad \sum_{\substack{\lambda \\ \lambda' \text{ even}}} b_\lambda^{\text{el}}(q, t)P_\lambda(x; q, t) = \prod_{i < j} \frac{(tx_i x_j; q)_\infty}{(x_i x_j; q)_\infty}.$$

Since [17, p. 311]

$$(3.5) \quad \sum_{r=0}^{\infty} g_r(x; q, t)a^r = \prod_{i \geq 1} \frac{(atx_i; q)_\infty}{(ax_i; q)_\infty}$$

this implies that

$$\text{RHS}(1.18) = \sum_{\substack{\mu, r \\ \mu' \text{ even}}} b_\mu^{\text{el}}(q, t)a^r P_\mu(x; q, t)g_r(x; q, t).$$

By the Pieri formula [17, p. 340]

$$P_\mu(x; q, t)g_r(x; q, t) = \sum_{\substack{\lambda \\ \lambda - \mu \text{ hor. } r\text{-strip}}} \varphi_{\lambda/\mu}(q, t)P_\lambda(x; q, t)$$

this becomes

$$\text{RHS}(1.18) = \sum_{\substack{\lambda, \mu \\ \mu' \text{ even} \\ \lambda - \mu \text{ hor. strip}}} a^{|\lambda - \mu|} b_\mu^{\text{el}}(q, t) \varphi_{\lambda/\mu}(q, t) P_\lambda(x; q, t).$$

Reasoning as before (see the proof of Theorem 1.1) it follows that for given  $\lambda$  the partition  $\mu$  is uniquely fixed as  $\mu'_i = 2\lfloor \lambda'_i/2 \rfloor$ . Assuming such  $\mu$  we thus obtain

$$\text{RHS}(1.18) = \sum_{\lambda} a^{l((\lambda')^\circ)} b_\mu^{\text{el}}(q, t) \varphi_{\lambda/\mu}(q, t) P_\lambda(x; q, t).$$

Since [17, p. 351]

$$b_\mu^{\text{el}}(q, t) \varphi_{\lambda/\mu}(q, t) = b_\lambda^{\text{el}}(q, t)$$

(for  $\mu'_i = 2\lfloor \lambda'_i/2 \rfloor$ ) we arrive at

$$\text{RHS}(1.18) = \sum_{\lambda} a^{l((\lambda')^\circ)} b_\lambda^{\text{el}}(q, t) P_\lambda(x; q, t)$$

completing the proof.  $\square$

A slightly different proof in the context of  $\lambda$ -rings will be presented in the next section.

*Proof of (1.17).* Acting with  $\omega_{q,t}$  on the left of (1.18) yields

$$\begin{aligned}\omega_{q,t}(\text{LHS}(1.18)) &= \sum_{\lambda} a^{l((\lambda')^\circ)} b_{\lambda'}^{\text{el}}(q, t) Q_{\lambda'}(x; t, q) \\ &= \sum_{\lambda} a^{l((\lambda')^\circ)} b_{\lambda'}^{\text{el}}(q, t) b_{\lambda}(t, q) P_{\lambda}(x; t, q) \\ &= \sum_{\lambda} a^{l(\lambda^\circ)} b_{\lambda}^{\text{oa}}(t, q) P_{\lambda}(x; t, q),\end{aligned}$$

where the last equality follows by (2.3) and (2.4).

On the other hand, by (3.5) the right-hand side of (1.18) may be written as

$$\text{RHS}(1.18) = \sum_{r=0}^{\infty} g_r(x; q, t) a^r \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}}.$$

Applying  $\omega_{q,t}$  and using [17, p. 312]

$$\omega_{q,t}(g_r(x; q, t)) = e_r(x)$$

and [17, p. 351]

$$\omega_{q,t} \left( \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}} \right) = \prod_{i \geq 1} \frac{(qtx_i^2; t^2)_{\infty}}{(x_i^2; t^2)_{\infty}} \prod_{i < j} \frac{(qx_i x_j; t)_{\infty}}{(x_i x_j; t)_{\infty}}$$

gives

$$\begin{aligned}\omega_{q,t}(\text{RHS}(1.18)) &= \sum_{r=0}^{\infty} e_r(x) a^r \prod_{i \geq 1} \frac{(qtx_i^2; t^2)_{\infty}}{(x_i^2; t^2)_{\infty}} \prod_{i < j} \frac{(qx_i x_j; t)_{\infty}}{(x_i x_j; t)_{\infty}} \\ &= \prod_{i \geq 1} \frac{(1 + ax_i)(qtx_i^2; t^2)_{\infty}}{(x_i^2; t^2)_{\infty}} \prod_{i < j} \frac{(qx_i x_j; t)_{\infty}}{(x_i x_j; t)_{\infty}}. \quad \square\end{aligned}$$

#### 4. $\lambda$ -RINGS

Lascoux recently revisited the Schur function identities of the introduction from the point of view of  $\lambda$ -rings [14]. In this section we adopt Lascoux's approach, and restate some of our results in  $\lambda$ -ring (or plethystic) notation. For an introduction to symmetric functions and  $\lambda$ -rings we refer to [13, 15].

Given two alphabets  $\mathbb{X}$  and  $\mathbb{Y}$  we denote by  $\mathbb{X} + \mathbb{Y}$  and  $\mathbb{X}\mathbb{Y}$  their disjoint union and Cartesian product. Decomposing an alphabet as the sum of its letters, we follow the convention of writing  $\mathbb{X} = \sum_{x \in \mathbb{X}} x$  instead of  $\mathbb{X} = \sum_{x \in \mathbb{X}} \{x\}$ .

The complete symmetric function  $h_r[\mathbb{X} - \mathbb{Y}]$  is defined by its generating series

$$(4.1) \quad \sigma_z[\mathbb{X} - \mathbb{Y}] := \frac{\prod_{y \in \mathbb{Y}} (1 - zy)}{\prod_{x \in \mathbb{X}} (1 - zx)} = \sum_{r=0}^{\infty} z^r h_r[\mathbb{X} - \mathbb{Y}].$$

Here we use the plethystic brackets to distinguish from our earlier notation of (2.11). In particular,  $h_r(x_1, x_2, \dots) = h_r[\mathbb{X}]$  and  $e_r(x_1, x_2, \dots) = e_r[\mathbb{X}] = (-1)^r h_r[-\mathbb{X}]$  for  $\mathbb{X} = \{x_1, x_2, \dots\}$ . We also define  $h_r[(1 - q)\mathbb{X}/(1 - t)]$  by

$$\prod_{x \in \mathbb{X}} \frac{(tx; q)_{\infty}}{(zx; q)_{\infty}} = \sum_{r \geq 0} z^r h_r[(1 - t)\mathbb{X}/(1 - q)].$$

(We mostly use this with  $t = 0$  and  $q$  replaced by  $t$ .) Then, by (4.1),  $\mathbb{X}/(1-t) = \mathbb{X}\{1, t, t^2, \dots\}$ , so that by Euler's  $q$ -exponential sum [5, Equation (II.1)]

$$(4.2) \quad h_r[a/(1-t)] = \frac{a^r}{(t; t)_r}.$$

For our present purposes it is important to note that the Rogers–Szegő polynomials actually arise as complete symmetric functions [14, Exercise 2.22]:

$$a^r H_r(b/a; t) = \frac{h_r[\mathbb{X}/(1-t)]}{h_r[1/(1-t)]} = (t; t)_r h_r[\mathbb{X}/(1-t)], \quad \mathbb{X} = \{a, b\}.$$

Indeed, since  $\mathbb{X} = \{a, b\} = a + b$ , the factorization of the left-hand side of (4.1) implies the convolution

$$(4.3) \quad \begin{aligned} h_r[\mathbb{X}/(1-t)] &= \sum_{i=0}^r h_{r-i}[a/(1-t)] h_i[b/(1-t)] \\ &= a^r \sum_{i=0}^r \frac{(b/a)}{(t; t)_i (t; t)_{r-i}} \quad (\text{by (4.2)}) \\ &= a^r \frac{H_r(b/a)}{(t; t)_r}. \end{aligned}$$

Next we turn to Theorem 1.1. Let  $Q'_\lambda$  be the modified Hall–Littlewood function

$$Q'_\lambda[\mathbb{X}; t] = Q_\lambda[\mathbb{X}/(1-t); t].$$

From (2.8) it follows that

$$\sum_{\lambda} P_\lambda[\mathbb{X}; t] Q'_\lambda[\mathbb{Y}; t] = \prod_{\substack{x \in \mathbb{X} \\ y \in \mathbb{Y}}} \frac{1}{1-xy} = \sigma_1[\mathbb{X}\mathbb{Y}].$$

Consequently,

$$\begin{aligned} \sigma_1[\mathbb{X}\mathbb{Y}] P_\mu[\mathbb{X}; t] &= \sum_{\nu} P_\nu[\mathbb{X}; t] P_\mu[\mathbb{X}; t] Q'_\nu[\mathbb{Y}; t] \\ &= \sum_{\lambda, \nu} f_{\mu\nu}^\lambda(t) P_\lambda[\mathbb{X}; t] Q'_\nu[\mathbb{Y}; t] \quad (\text{by (2.15)}) \\ &= \sum_{\lambda} P_\lambda[\mathbb{X}; t] Q'_{\lambda/\mu}[\mathbb{Y}; t] \quad (\text{by (2.16)}). \end{aligned}$$

(The above equation also follows by the substitution  $\mathbb{X} \rightarrow \mathbb{X}/(1-t)$  in an identity on page 227 of [17]). Summing  $\mu$  over the even partitions and replacing  $\mathbb{Y}$  by  $-\mathbb{Y}$ , we thus find

$$\sum_{\lambda} P_\lambda[\mathbb{X}; t] \sum_{\substack{\mu \\ \mu \text{ even}}} Q'_{\lambda/\mu}[-\mathbb{Y}; t] = \sigma_1[-\mathbb{X}\mathbb{Y}] \sum_{\substack{\mu \\ \mu \text{ even}}} P_\mu[\mathbb{X}; t].$$

Finally note that the sum on the right may be performed by (1.5b). Hence we arrive at

$$(4.4) \quad \sum_{\lambda} P_\lambda[\mathbb{X}; t] B_\lambda[\mathbb{Y}; t] = \sigma_1[(1-t)e_2[\mathbb{X}] + h_2[\mathbb{X}] - \mathbb{X}\mathbb{Y}],$$

with

$$B_\lambda[\mathbb{Y}; t] = \sum_{\substack{\mu \\ \mu \text{ even}}} Q'_{\lambda/\mu}[-\mathbb{Y}; t].$$

Dispensing with the plethystic notation we may write (4.4) as

$$\sum_{\lambda} P_{\lambda}(x; t) B_{\lambda}(y; t) = \prod_{i, j \geq 1} (1 - x_i y_j) \prod_{i \geq 1} \frac{1}{1 - x_i^2} \prod_{i < j} \frac{1 - t x_i x_j}{1 - x_i x_j},$$

Theorem 1.1 (with  $a \rightarrow -a$  and  $b \rightarrow -b$ ) is thus equivalent to the following closed form expression for  $B_{\lambda}$  in the case of a two-letter alphabet.

**Theorem 4.1.** *Let  $\mathbb{Y} = \{a, b\}$ . Then*

$$B_{\lambda}(a, b; t) = \sum_{\mu \text{ even}}^{\mu} Q'_{\lambda/\mu}[-\mathbb{Y}; t] = (-a)^{l(\lambda_o)} h_{\lambda_e}(ab; t) h_{\lambda_o}(b/a; t).$$

For example, when  $\lambda = (1^r)$  the only non-vanishing contribution to the sum over  $\mu$  comes from  $\mu = 0$  (since  $Q'_{\lambda/\mu}$  vanishes if  $\mu \not\subseteq \lambda$ ). Hence

$$\begin{aligned} B_{(1^r)}(a, b; t) &= Q'_{(1^r)}[-\mathbb{Y}; t] \\ &= Q_{(1^r)}[-\mathbb{Y}/(1-t); t] \\ &= b_{(1^r)}(t) e_r[-\mathbb{Y}/(1-t)] && \text{(by (2.7) and (2.9))} \\ &= (-1)^r (t; t)_r h_r[\mathbb{Y}/(1-t)] \\ &= (-a)^r H_r(b/a) && \text{(by (4.3)),} \end{aligned}$$

which is in accordance with the right hand side of Theorem 4.1 for  $\lambda = (1^r)$ .

In much the same way it follows that

$$(4.5a) \quad \sum_{\lambda} P_{\lambda}(x; t) C_{\lambda}(y; t) = \prod_{i, j \geq 1} \frac{1 - t x_i y_j}{1 - x_i y_j} \prod_{i < j} \frac{1 - t x_i x_j}{1 - x_i x_j},$$

with

$$(4.5b) \quad C_{\lambda}(y; t) = \sum_{\mu' \text{ even}}^{\mu} c_{\mu}(t) Q_{\lambda/\mu}(y; t),$$

but as remarked after (1.13), only for  $y = (a)$  does the sum on the right of (4.5b) simplify. Perhaps the best way to understand this case (corresponding to (1.14)) is however not through (4.5) but by adding  $a$  to the alphabet  $\mathbb{X}$  as explained below in the Macdonald polynomial setting.

Consider the identity (1.18). For  $a = 0$  this is (3.4) which may be expressed in  $\lambda$ -ring notation as

$$\sum_{\substack{\lambda \\ \lambda' \text{ even}}} b_{\lambda}^{\text{el}}(q, t) P_{\lambda}[\mathbb{X}; q, t] = \sigma_1 \left[ \frac{1-t}{1-q} e_2[\mathbb{X}] \right] =: f[\mathbb{X}].$$

Replacing  $\mathbb{X}$  by  $\mathbb{X} + \mathbb{Y}$  and using that  $e_2[\mathbb{X} + \mathbb{Y}] = e_2[\mathbb{X}] + e_2[\mathbb{Y}] + e_1[\mathbb{X}\mathbb{Y}]$ ,  $\sigma_1[\mathbb{X} + \mathbb{Y}] = \sigma_1[\mathbb{X}] \sigma_1[\mathbb{Y}]$  and  $P_{\lambda}[\mathbb{X} + \mathbb{Y}; q, t] = \sum_{\mu} P_{\mu}[\mathbb{X}; q, t] P_{\lambda/\mu}[\mathbb{Y}; q, t]$  (for this last result see [17, p. 345]), this implies

$$\sum_{\mu} P_{\mu}[\mathbb{X}; q, t] \sum_{\substack{\lambda \\ \lambda' \text{ even}}} b_{\lambda}^{\text{el}}(q, t) P_{\lambda/\mu}[\mathbb{Y}; q, t] = f[\mathbb{X}] f[\mathbb{Y}] \prod_{\substack{x \in \mathbb{X} \\ y \in \mathbb{Y}}} \frac{(txy; q)_{\infty}}{(xy; q)_{\infty}}.$$



When  $\mathbb{Y}$  contains a single letter  $a$ , so that we have effectively added  $a$  to  $\mathbb{X}$ , this simplifies to

$$\sum_{\mu} P_{\mu}[\mathbb{X}; q, t] \sum_{\substack{\lambda \\ \lambda' \text{ even} \\ \lambda - \mu \text{ hor. strip}}} a^{|\lambda - \mu|} b_{\lambda}^{\text{el}}(q, t) \psi_{\lambda/\mu}(q, t) = f[\mathbb{X}] \prod_{x \in \mathbb{X}} \frac{(atx; q)_{\infty}}{(ax; q)_{\infty}}.$$

To get the expression on the left we have used [17, p. 346]. The partition  $\lambda$  in the sum on the left is fixed by  $\mu$  as  $\lambda'_i = 2\lceil \mu'_i/2 \rceil$ . Assuming such  $\mu$ , we get

$$\sum_{\mu} a^{l((\lambda')^{\circ})} b_{\lambda}^{\text{el}}(q, t) \psi_{\lambda/\mu}(q, t) P_{\mu}[\mathbb{X}; q, t] = f[\mathbb{X}] \prod_{x \in \mathbb{X}} \frac{(atx; q)_{\infty}}{(ax; q)_{\infty}}.$$

But putting together two combinatorial identities on pages 350 and 351 of [17] yields

$$b_{\lambda}^{\text{el}}(q, t) \psi_{\lambda/\mu}(q, t) = b_{\mu}^{\text{el}}(q, t)$$

so that (1.18) follows.

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