THE ${}_1\psi_1$ SUMMATION

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1. Introduction

Ramanujan's ${}_1\psi_1$ summation is one of the most important summation formulas in the theory of basic hypergeometric series. It first appeared, without proof, as Entry 17 of Chapter 16 in Ramanujan's second notebook [1, 32], and was popularised by Hardy, who mentioned the result in his twelfth and final lecture on Ramanujans life and work [23, §12.12], calling it "a remarkable formula with many parameters".

The ${}_1\psi_1$ summation is an identity for what nowadays is referred to as a *bilateral basic hypergeometric series* [20]. Neither Ramanujan nor Hardy stated it as such, instead writing it in the symmetric, unilateral form

$$(1.1)\quad 1+\sum_{n=1}^{\infty}\frac{(\alpha^{-1};q^2)_n}{(\beta q^2;q^2)_n}(-\alpha qz)^n+\sum_{n=1}^{\infty}\frac{(\beta^{-1};q^2)_n}{(\alpha q^2;q^2)_n}(-\beta qz^{-1})^n \\ =\frac{(-qz;q^2)_\infty(-qz^{-1};q^2)_\infty(\alpha\beta q^2;q^2)_\infty(q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta z^{-1};q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty},$$

for $\alpha,\beta,z,q\in\mathbb{C}$ such that $|q|<1$, $|\beta q|<|z|<|\alpha q|^{-1}$ and $\alpha,\beta\notin\{q^{-2},q^{-4},\dots\}$. In (1.1), $(a;q)_\infty:=\prod_{k=0}^{\infty}(1-aq^k)$ and $(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k)$ (for a positive integer n) are q-shifted factorials.

Since

$$(1.2)\qquad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

the right-hand side of (1.2) can be used as the definition of $(a;q)_n$ for all integers n. In particular one has $(a;q)_0=1$ and the reflection formula $(a;q)_{-n}(q/a;q)_n=(-q/a)^n q^{\binom{n}{2}}$. Replacing $(\alpha,\beta q^2,-\alpha qz)$ by $(1/a,b,z)$ followed by the substitution $q^2\mapsto q$, (1.2) then allows for (1.1) to be written in the modern, more compact, bilateral form

$$(1.3)\qquad \sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(az;q)_\infty(q/az;q)_\infty(b/a;q)_\infty(q;q)_\infty}{(z;q)_\infty(b/az;q)_\infty(q/a;q)_\infty(b;q)_\infty},$$

for $|q|<1$, $|b/a|<|z|<1$ and $a/q,1/b\notin\{1,q,q^2,\dots\}$. The symmetry $(\alpha,\beta,z)\leftrightarrow(\beta,\alpha,1/z)$ of (1.1) is now given by $(a,b,z)\leftrightarrow(q/b,q/a,b/az)$ and is realised on the left of (1.3) by replacing $n\mapsto -n$ and using the reflection formula.

Ramanujan's ${}_1\psi_1$ summation contains a number of well-known identities as special case. Since for a negative integer n,

$$\frac{1}{(b;q)_n}=(bq^n;q)_{-n}=(1-bq^n)\cdots(1-bq^{-1}),$$

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it follows that the summand of (1.3) vanishes for $n < 0$ if $b = q$. The resulting unilateral summation is the classical q-binomial theorem [20, Equation (II.3)]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{1.4}$$

where $|q| < 1$ and $|z| < 1$. Replacing $z \mapsto z/a$ in (1.3), then letting a tend to infinity and setting $b = 0$, gives Jacobi's triple product identity [26]

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{\binom{n}{2}} = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty,$$

for arbitrary complex z and q such that $z \neq 0$. Finally, for complex z and q such that $|q| < 1$ and $z \neq 0$, let $\theta(z;q) := (z;q)_\infty (q/z;q)_\infty$ be a modified theta function. Substituting $(b,z) \mapsto (aq,b)$ in (1.3), and using the geometric series to write the summand in symmetric form, yields Kronecker's theta function identity [29]

$$\frac{1}{1-a} + \frac{1}{1-b} - 1 + \sum_{k,n=1}^{\infty} \bigl(a^k b^n - a^{-k} b^{-n}\bigr) q^{kn} = \frac{\theta(ab;q)\,(q;q)_\infty^2}{\theta(a;q)\theta(b;q)}, \tag{1.5}$$

for $|q| < |a| < 1$ and $|q| < |b| < 1$.

In the next section a number of different proofs of Ramanujan's ${}_1\psi_1$ summation are reviewed, and in the third and final section several generalisations are discussed. For reasons of space, generalisations of the ${}_1\psi_1$ summation to root systems have been omitted. The interested reader may find these surveyed in [40], which also includes a number of simple applications of the ${}_1\psi_1$ summation to number theory and the theory of special functions. Further surveys of Ramanujan's ${}_1\psi_1$ summation may be found in Ramanujan's edited notebooks [7, pp. 31–34], in [15] (which focuses on proofs) and [27] (which is mostly historical).

Throughout the remainder of this article it is assumed that $|q| < 1$.

2. Proofs of the ${}_1\psi_1$ summation

In the following we adopt the shorthand notation

$$(a_1, \dots, a_k; q)_n := (a_1;q)_n \cdots (a_k;q)_n$$

for $n \in \mathbb{Z} \cup \{\infty\}$, and use the unilateral and bilateral basic hypergeometric series [20]

$${}_r\phi_{r-1}\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; q, z\right] := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r;q)_n}{(q, b_1, \dots, b_{r-1};q)_n}\, z^n, \qquad |z| < 1, \tag{2.1a}$$

$${}_r\psi_r\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; q, z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, \dots, a_r;q)_n}{(b_1, \dots, b_r;q)_n}\, z^n, \qquad \left|\frac{b_1 \cdots b_r}{a_1 \cdots a_r}\right| < |z| < 1. \tag{2.1b}$$

In these definitions it is assumed that the parameters are generic; if k is an arbitrary nonnegative integer then none of the b_i should be of the form q^{-k} and, in the case of (2.1b), none of the a_i should be of the form q^{k+1}. The customary one-line notation for basic hypergeometric and bilateral basic hypergeometric series will also be adopted, so that the left-hand side of (1.3) can be written as ${}_1\psi_1(a;b;q,z)$.

As mentioned in the introduction, Ramanujan recorded the ${}_1\psi_1$ summation in his second notebook without proof. Hardy also did not provide a proof. After stating the result in his lecture [23, Equation (12.12.2)], Hardy remarked "This formula seems to be new. It is however deduced from one which is familiar and probably goes back to Euler". He then stated the q-binomial theorem (1.4). Although Hardy was correct that the $_1\psi_1$ summation was new and could be obtained from the q-binomial theorem, the latter result was almost certainly not known to Euler, with the earliest recording of (1.4) appearing in a 1811 textbook written by Rothe [33], published more than 30 years after Euler's death.

The first published proof of the $_1\psi_1$ summation can be found in Hahn's paper [22] from 1949. Hahn mentioned neither Ramanujan nor Hardy and appears to have been unaware of these earlier occurrences of the result. Hahn was led to the $_1\psi_1$ summation through a study of the first-order homogeneous q-difference equation[1]

$$(b-aqz)f(qz)-q(1-z)f(z)=0, \tag{2.2}$$

for which he sought solutions of the form $f(z)=f(a,b,z;q)=\sum_{n\in\mathbb{Z}}A_n(a,b;q)z^n$ normalised such that $A_0=1$. Equating coefficients of z^n in (2.2) leads to the recurrence $A_{n+1}=A_n(1-aq^n)/(1-bq^n)$, so that $f(a,b,z;q)={}_1\psi_1(a;b;q,z)$. To obtain the product form, Hahn made repeated use of the q-binomial theorem in line with Hardy's remark. As a first step he used a limiting case (which *was* known to Euler) to expand $1/(b;q)_n$ in the summand of f. Thus

$$f(a,b,z;q)=\frac{1}{(b;q)_\infty}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{(a;q)_n z^n q^{\binom{k}{2}}(-bq^n)^k}{(q;q)_k}=\frac{1}{(b;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}(-b)^k}{(q;q)_k}f(a,0,zq^k;q).$$

Since $f(a,0,zq^k;q)=(-az)^{-k}q^{-\binom{k}{2}}(z;q)_k\,f(a,0,z;q)$ by the q-difference equation (2.2),

$$f(a,b,z;q)=\frac{f(a,0,z;q)}{(b;q)_\infty}\,{}_1\phi_0(z;-;q,b/az)=\frac{(b/a;q)_\infty}{(b,b/az;q)_\infty}\,f(a,0,z;q), \tag{2.3}$$

where the second equality follows from the q-binomial theorem (1.4). Again by the q-binomial theorem, $f(a,q,z;q)=(az;q)_\infty/(q;q)_\infty$, which determines $f(a,0,z;q)$ and thus $f(a,b,z;q)$ in full.

Hahn's proof is certainly not the simplest demonstration of the $_1\psi_1$ summation, and a large number of alternative proofs have since been found, many of which are more elementary than Hahn's original proof. It follows from the work of Adiga et al. [1, pp. 26–28] on Ramanujan's second notebook or the monograph on elliptic functions by Venkatachaliengar [38, pp. 20–25] that the second part of Hahn's proof can be modified, eliminating the need for the q-binomial theorem. Since, as a function of z, $(az,q/az;q)_\infty/(z,b/az;q)_\infty$ is analytic in the annulus $|b/a|<|z|<1$ and (for $|b/aq|<|z|<1$) satisfies the q-difference equation (2.2), it follows that

$${}_1\psi_1(a;b;q,z)=C\,\frac{(az,q/az;q)_\infty}{(z,b/az;q)_\infty}, \tag{2.4}$$

with $C=C(a,b,q)$ independent of z. The product on the right has a simple pole at $z=1$. Moreover, $(a;q)_n z^n/(b;q)_n$ summed over the negative integers is analytic for $|z|>|b/a|$. Multiplying both sides of (2.4) by $(1-z)$ and taking the $z\to 1$ limit, it therefore follows from Abel's theorem that

$$\lim_{z\to 1}(1-z)\,{}_1\psi_1(a;b;q,z)=\lim_{z\to 1}(1-z)\sum_{n=0}^{\infty}\frac{(a;q)_n}{(b;q)_n}\,z^n=\lim_{n\to\infty}\frac{(a;q)_n}{(b;q)_n}=\frac{(a;q)_\infty}{(b;q)_\infty}$$

is equal to

$$C\lim_{z\to 1}(1-z)\,\frac{(az,q/az;q)_\infty}{(z,b/az;q)_\infty}=C\,\frac{(a,q/a;q)_\infty}{(q,b/a;q)_\infty}.$$

[1]Hahn used a slightly different choice of parameters, writing $(a-bz)f(qz)-(1-z)f(z)=0$.

This shows that $C=(b/a,q;q)_\infty/(q/a,b;q)_\infty$.

It is also possible to obtain the ${}_1\psi_1$ summation from the q-binomial theorem in what is essentially a one-line proof. The idea, due to Ismail [24] and commonly referred to as *Ismail's argument*, is to view (1.3) as an identity of analytic functions in the variable b. Since $|b/a|<|z|<1$, both sides are analytic for $|b|<\min\{1,|z/a|\}$. Moreover, for a nonnegative integer k, both sides coincide for $b=q^{k+1}$ by the q-binomial theorem (1.4) with $a\mapsto aq^{-k}$. Since this sequence of b-values has an accumulation point at 0, the result follows by the identity theorem for holomorphic functions. The restriction $|b|<1$ can be lifted by standard analytic continuation arguments.

There are numerous further proofs of Ramanujan's ${}_1\psi_1$ summation that use identities for basic hypergeometric series, and the interested reader is referred to [2,3,6,9,11,12,14,15,19, 25,30,31,34,35]. One particularly appealing example of such a proof is due to Schlosser [35], who applied *Cauchy's method* [8] for turning terminating, unilateral basic hypergeometric series into non-terminating, bilateral series. The starting point is the q-Pfaff–Saalschütz summation [20, Equation (III.12)]

$$ {}_3\phi_2\left[\begin{matrix} a,b,q^{-n} \\ c,abq^{1-n}/c\end{matrix};q,q\right]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{2.5}$$

By the simultaneous substitutions $(n,a,b,c)\mapsto(m+n,aq^{-m},b/az,bq^{-m})$, this can be put in the form

$$\sum_{k=-m}^{n}\frac{(a;q)_k}{(b;q)_k}\,\frac{(z;q)_{n-k}}{(q;q)_{n-k}}\,\frac{(b/az;q)_{m+k}}{(q;q)_{m+k}}\,z^k=\frac{(az;q)_n}{(b;q)_n}\,\frac{(q/az;q)_m}{(q/a;q)_m}\,\frac{(b/a;q)_{m+n}}{(q;q)_{m+n}}, \tag{2.6}$$

for integers m,n such that $m+n\geqslant 0$. The identity (2.6), which is invariant under the substitution $(a,b,z,m,n)\mapsto(q/b,q/a,b/az,n,m)$, is a terminating analogue of the ${}_1\psi_1$ summation. The latter arises by assuming $|b/a|<|z|<1$ and letting m,n tend to infinity. The required interchange of limits and sum is justified by Tannery's theorem. The large-n limit of (2.6), which is implied by the q-Gauss sum (itself the large-n limit of (2.5)), was also independently found in [11,14,30].

Much more elaborate than the q-hypergeometric and/or analytic proofs of the ${}_1\psi_1$ summation are a number of combinatorial proofs, see [10,16,17,41]. With the exception of [10], these establish the identity in the form

$$\frac{(-aq,-bq;q)_\infty}{(abq,q;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1/a;q)_k}{(-bq;q)_k}\,(azq)^k=\frac{(-zq,-1/z;q)_\infty}{(azq,b/z;q)_\infty},\quad |b|<|z|<1/|aq|. \tag{2.7}$$

It would take too much space to fully describe the actual bijections leading to (2.7). Instead it will be explained how the coefficients of z^k on each side of the identity can be interpreted combinatorially. Since the substitution $(a,b,z)\mapsto(b,a,1/qz)$ leaves (2.7) invariant, it suffices to restrict considerations to nonnegative values of k. For a full proof of the above form of the ${}_1\psi_1$ summation the reader is referred to the work of Corteel and Lovejoy [17] and of Yee [41].

An *overpartition* is an ordinary integer partition in which the final occurrence of any part of given size may either be overlined or not [18]. In the following, overpartitions will be considered in which 0 is also an allowed part-size, so that $(3,1,\bar{1},0)$, $(3,1,\bar{1},\bar{0})$ and $(3,1,\bar{1})$ are all viewed as distinct overpartitions of 5. Denote the set of all such overpartitions by $\mathscr{O}$ and, for $\lambda\in\mathscr{O}$, write $|\lambda|$ for the sum of the parts, $l(\lambda)$ for the number of parts, and $p(\lambda)$

for the number of non-overlined parts. For $\lambda = (3,1,\bar{1},0)$, $|\lambda| = 5$, $l(\lambda) = 4$ and $p(\lambda) = 3$, whereas for $\lambda = (3,1,\bar{1})$, $|\lambda| = 5$, $l(\lambda) = 3$ and $p(\lambda) = 2$.

Let $\mathscr{P}$ denote the set of partitions and $\mathscr{D}$ the set of distinct (or strict) partitions, where in both cases partitions are not allowed to have 0 as a part. By adding 1 to each part of an overpartition $\lambda \in \mathscr{O}$, and by then separating the non-overlined and overlined parts to form a pair of partitions, it follows that $\mathscr{O}$ is in bijection with $\mathscr{P} \times \mathscr{D}$. For example

$$(3,1,\bar{1},0) \leftrightarrow (4,2,\bar{2},1) \leftrightarrow \big((4,2,1),(\bar{2})\big) \leftrightarrow \big((4,2,1),(2)\big).$$

Clearly, if $\lambda \in \mathscr{O}$ corresponds to the pair $(\mu,\nu) \in \mathscr{P} \times \mathscr{D}$, then $|\lambda| = |\mu| + |\nu| - l(\mu) - l(\nu)$, $p(\lambda) = l(\mu)$ and $l(\lambda) = l(\mu) + l(\nu)$. By the classical partition identities [4, Equations (5.11) & (5.9)]

$$\sum_{\lambda \in \mathscr{P}} a^{l(\lambda)} q^{|\lambda|} = \frac{1}{(aq;q)_\infty} \qquad \text{and} \qquad \sum_{\lambda \in \mathscr{D}} a^{l(\lambda)} q^{|\lambda|} = (-aq;q)_\infty, \tag{2.8}$$

it thus follows that the three-variable generating function for overpartitions is given by

$$\begin{aligned} O(a,z,q) &:= \sum_{\lambda \in \mathscr{O}} a^{p(\lambda)} z^{l(\lambda)} q^{|\lambda|} \\ &= \bigg(\sum_{\mu \in \mathscr{P}} (az/q)^{l(\mu)} q^{|\mu|}\bigg)\bigg(\sum_{\nu \in \mathscr{D}} (z/q)^{l(\nu)} q^{|\nu|}\bigg) = \frac{(-z;q)_\infty}{(az;q)_\infty}. \end{aligned} \tag{2.9}$$

For nonnegative integers n,r,s and an integer k, let $f(n,k,r,s)$ be the cardinality of the set of pairs $(\lambda,\mu) \in \mathscr{O} \times \mathscr{O}$ subject to the restrictions

$$|\lambda| + |\mu| + l(\lambda) = n, \quad l(\lambda) - l(\mu) = k, \quad p(\lambda) = r, \quad p(\mu) = s.$$

For example, $f(6,2,3,2) = 9$ with the following pairs (λ,μ) of overpartitions (arranged in generalised Frobenius form [17] with the parts of λ in the first row and the parts of μ in the second row) contributing to the count:

$$\begin{pmatrix} \bar{2}\ 0\ 0\ 0 \\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 2\ 0\ 0\ \bar{0} \\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 1\ \bar{1}\ 0\ 0 \\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 1\ 1\ 0\ \bar{0} \\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 0\ 0\ 0\ \bar{0} \\ 2\ 0 \end{pmatrix}, \begin{pmatrix} 0\ 0\ 0\ \bar{0} \\ 1\ 1 \end{pmatrix}, \begin{pmatrix} \bar{1}\ 0\ 0\ 0 \\ 1\ 0 \end{pmatrix}, \begin{pmatrix} 1\ 0\ 0\ \bar{0} \\ 1\ 0 \end{pmatrix}, \begin{pmatrix} \bar{1}\ 0\ 0\ 0\ \bar{0} \\ 0\ 0\ \bar{0} \end{pmatrix}.$$

By (2.9) the coefficient of z^k $(k \in \mathbb{Z})$ of the right-hand side of (2.7) may be expressed in terms of $f(n,k,r,s)$. Specifically,

$$[z^k]\, \frac{(-zq,-1/z;q)_\infty}{(azq,b/z;q)_\infty} = [z^k]\Big(O(a,zq,q)O(b,1/z,q)\Big) = \sum_{n,r,s=0}^{\infty} f(n,k,r,s)\, a^r b^s q^n.$$

For nonnegative integers n,k,r,s, let $g(n,k,r,s)$ be the cardinality of the set of quintuples $(\lambda,\mu,\nu,\omega,\tau) \in \mathscr{P} \times \mathscr{P} \times \mathscr{D} \times \mathscr{D} \times \mathscr{O}$ such that all parts of ω exceed k, all non-overlined parts of τ are equal to 1, all overlined parts of τ are in $\{1,\dots,k\}$, and

$$|\lambda| + |\mu| + |\nu| + |\omega| + |\tau| = n, \quad l(\mu) + l(\nu) + p(\tau) = r, \quad l(\mu) + l(\omega) = s, \quad l(\tau) = k.$$

For example, $g(6,2,3,2) = 9$ with the following quintuples contributing:

$$((2),(1,1),\emptyset,\emptyset,(1,\bar{1})),\ ((1,1),(1,1),\emptyset,\emptyset,(1,\bar{1})),\ ((1),(2,1),\emptyset,\emptyset,(1,\bar{1})),\ ((1),(1,1),\emptyset,\emptyset,(\bar{2},1)),$$
$$(\emptyset,(3,1),\emptyset,\emptyset,(1,\bar{1})),\ (\emptyset,(2,2),\emptyset,\emptyset,(1,\bar{1})),\ (\emptyset,(2,1),\emptyset,\emptyset,(\bar{2},1)),\ (\emptyset,(1,1),(1),\emptyset,(\bar{2},\bar{1})),\ (\emptyset,(1),\emptyset,(3),(1,1)).$$

By (2.8),

$$\sum_{\omega} a^{p(\omega)} q^{|\omega|} = (-aq^{k+1};q)_\infty \quad \text{and} \quad \sum_{\tau} a^{p(\tau)} q^{|\tau|} = (-1/a;q)_k\, (aq)^k,$$

where $\omega \in \mathscr{D}$ and $\tau \in \mathscr{O}$ satisfy the conditions described above, it follows that for $k \geqslant 0$

$$\begin{aligned}[z^k]\, &\frac{(-aq,-bq;q)_\infty}{(abq,q;q)_\infty} \sum_{i=-\infty}^{\infty} \frac{(-1/a;q)_i}{(-bq;q)_i}\,(azq)^i \\ &= \frac{(-aq,-bq^{k+1};q)_\infty}{(q,abq;q)_\infty}\,(-1/a;q)_k\,(aq)^k = \sum_{n,r,s=0}^{\infty} g(n,k,r,s)\,a^r b^s q^n.\end{aligned}$$

The ${}_1\psi_1$ summation is thus equivalent to the combinatorial statement that $f(n,k,r,s) = g(n,k,r,s)$ for all nonnegative integers n,k,r,s.

To conclude our review of proofs of the ${}_1\psi_1$ summation we discuss Kadell's probabilistic approach [28]. For integers n and $n_1,\dots,n_k$, let

$$g_n := \frac{cq^n}{(1+cq^n)(1+cq^{n+1})} \quad \text{and} \quad g_{n_1,\dots,n_k} := g_{n_1}\cdots g_{n_k},$$

where the dependence on c and q has been suppressed. For integers k,M,N such that $k \geqslant 0$, define $G_{0,M,N} := 1$ if $M+N \geqslant 0$, $G_{0,M,N} := 0$ if $M+N<0$, and

$$G_{k,M,N} := \sum_{-N\leqslant n_1\leqslant n_2-2\leqslant\cdots\leqslant n_k-2k+2\leqslant M+k-2} g_{n_1,\dots,n_k}$$

for $k \geqslant 1$. Since

$$G_{k,M,N} = G_{k,M,N-1} + g_{-N}\,G_{k-1,M+1,N-2}$$

for $k \geqslant 1$, it follows that

$$G_{k,M,N} = \frac{q^{\binom{k}{2}}}{(-q^{N-k+1}/c,-cq^M;q)_k} \begin{bmatrix} M+N \\ k \end{bmatrix}. \tag{2.10}$$

This identity will also be used in the large M and/or N limit, and by an abuse of notation the corresponding limits of $G_{k,M,N}$ will be denoted as $G_{k,\infty,N}$, $G_{k,M,\infty}$ and $G_{k,\infty,\infty}$.

Now fix $0<q<1$ and $c>0$, and let X be a random variable, uniformly distributed on $(0,1)$. This may be used to define the discrete random variable $N \in \mathbb{Z}$ by

$$N = \left\lceil \log_q\left(\frac{c^{-1}X}{1-X}\right)\right\rceil,$$

with probability mass function given by[2]

$$P\biggl(q^{N+1} < \frac{c^{-1}X}{1-X} \leqslant q^N\biggr) = P\biggl(\frac{cq^{N+1}}{1+cq^{N+1}} < X \leqslant \frac{cq^N}{1+cq^N}\biggr) = (1-q)\,g_N. \tag{2.11}$$

For $1 \leqslant j \leqslant k$ integers, let $N_1,N_2,\dots,N_k$ be an independent, identically distributed random sample from the distribution (2.11) with order statistics $N_{(1)} < N_{(2)} < \cdots < N_{(k)}$, subject to

$$N_{(i+1)} - N_{(i)} \geqslant 2 \quad \text{for all } 1 \leqslant i \leqslant k-1.$$

Since $G_{k,\infty,\infty} = q^{\binom{k}{2}}/(q;q)_k$, the joint probability mass function of the order statistics is

$$q^{-\binom{k}{2}}(q;q)_k \prod_{i=1}^{k} g_{N_{(i)}} \prod_{i=1}^{k-1} \chi\Bigl(N_{(i+1)} - N_{(i)} \geqslant 2\Bigr).$$

[2]Since $(1-q)\sum_N g_N = (1-q)G_{1,\infty,\infty} = 1$ this is a properly normalised.

Therefore, the $(k-j+1)$th order statistic $N_{(j)}$ has probability mass function

$$P\big(N_{(j)}=n\big)=q^{-\binom{k}{2}}(q;q)_k\,g_n\,G_{j-1,\infty,n-j+1}\,G_{k-j,-n-2,\infty}$$
$$=\frac{(q;q)_k(-c;q)_{k-2j+2}(-1/c;q)_{2j-k-1}}{(q;q)_{j-1}(q;q)_{k-j}(-c;q)_{k-j+2}(-1/c;q)_j}\,\frac{(-cq^{1-j};q)_n}{(-cq^{k-j+2};q)_n}\,q^{(k-j+1)n},$$

where the second line follows from (2.10). When n is summed over the integers this should give 1, thus proving the ${}_1\psi_1$ summation (1.3) for $(a,b,z)=(-cq^{1-j},-cq^{k-j+2},q^{k-j+1})$, where $c>0$, $0<q<1$ and j,k are integers such that $1\leqslant j\leqslant k$. Making the substitution $(c,k)\mapsto(-cq^{j-k-1},k+j-1)$ and appealing to analytic continuation, it follows that (1.3) holds with $(a,b,z)=(cq^{-k},cq^j,q^k)$ for complex c,q such that $|q|<1$, and positive integers j,k. Since ${}_1\psi_1(cq^{-k},b;q,q^k)$ is analytic in b for $|b|<\min\{1,|c|\}$ and the sequence $b=cq^j$ for $j\geqslant 1$ has 0 as accumulation point, this establishes (1.3) with $(a,z)=(cq^{-k},q^k)$, where $|b/c|<1$ and $|q|<1$. Finally, since ${}_1\psi_1(c/z,b;q,z)$ is analytic in z for $|z|<1$ and the sequence $z=q^k$ for $k\geqslant 1$ once again has an accumulation point at 0, this implies (1.3) with $a=c/z$. Replacing $c\mapsto az$ completes the proof.

3. Generalisations

There are very few summation or transformation formulas for bilateral or unilateral basic hypergeometric series that include the ${}_1\psi_1$ summation as a special case. For bilateral series there only appears to be Chu's q-analogue of Dougall's ${}_2H_2$ sum, given by [13, Theorem 2]

$$\tag{3.1} {}_2\psi_2\left[\begin{matrix}a,b\\c,d\end{matrix};q,\frac{cd}{abq}\right]=\frac{(cd/bq,bq^2/cd,c/a,d/a,b,q;q)_\infty}{(cd/abq,bq/c,bq/d,q/a,c,d;q)_\infty}+b\,\frac{(bq/a,q/c,q/d,q;q)_\infty}{(bq/c,bq/d,q/a,q/b;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}bq/c,bq/d\\bq/a\end{matrix};q,q\right]$$

for $a,b,c,d\in\mathbb{C}^*$ such that $|cd/abq|<1$. Chu's proof of (3.1) uses Abel's lemma, Heine's ${}_2\phi_1$ transformation [20, Equation (III.1)] and the ${}_1\psi_1$ summation (1.3). The latter is recovered by taking $d=abzq/c$ (so that $|z|<1$) and then letting b tend to 0 (which requires that $|c/az|<1$). By

$$\lim_{b\to 0}{}_2\psi_2\left[\begin{matrix}a,b\\c,abzq/c\end{matrix};q,z\right]={}_1\psi_1\left[\begin{matrix}a\\c\end{matrix};q,z\right]\qquad\text{and}\qquad\lim_{b\to 0}b\,\frac{(c/abz;q)_\infty}{(q/b;q)_\infty}=0,$$

the ${}_1\psi_1$ summation then follows with b replaced by c. Chu's identity generalises several other well-known identities, such as the q-Gauss sum (obtained for $c=q$ or $d=q$) and the non-terminating q-Chu–Vandermonde sum (obtained for $a=1$). For $d=abq^2/c$ the ${}_2\phi_1$ series on the right of (3.1) may be summed by a special case of the q-Gauss sum, resulting in a 'balanced' ${}_2\psi_2$ summation that appears to be missing from the literature:

$${}_2\psi_2\left[\begin{matrix}a,b\\c,abq^2/c\end{matrix};q,q\right]=\frac{q/c}{(1-aq/c)(1-bq/c)}\left(\frac{(a,b;q)_\infty}{(c,abq^2/c;q)_\infty}-ab\,\frac{(q/c,c/abq;q)_\infty}{(q/a,q/b;q)_\infty}\right).$$

Two examples of identities for unilateral basic hypergeometric series that generalise the ${}_1\psi_1$ summation in the form (1.1) are

$$\tag{3.2} {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]-\frac{(bz,q/bz,c/b,a;q)_\infty}{(z,q/z,a/b,c;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}bq/c,b\\bq/a\end{matrix};q,\frac{cq}{abz}\right]=\frac{(az,q/az,c/a,b;q)_\infty}{(z,cq/abz,b/a,c;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}q/b,c/b\\aq/b\end{matrix};q,\frac{q}{z}\right],$$

for $\max\{|q|, |cq/ab|\} < |z| < 1$, and

$$ {}_3\phi_2\left[\begin{matrix} a, b, q \\ c, abzq/c \end{matrix}; q, z\right] - \frac{(1-c/abz)(1-c/q)}{(1-q/a)(1-c/bq)}\, {}_3\phi_2\left[\begin{matrix} q^2/c, bzq/c, q \\ q^2/a, bq^2/c \end{matrix}; q, \frac{c}{az}\right] = \frac{(az, q/az, c/a, b, bzq/c, q; q)_\infty}{(z, c/az, q/a, c, abzq/c, bq/c; q)_\infty}, \tag{3.3} $$

for $|c/a| < |z| < 1$. The first of these identities simplifies to the ${}_1\psi_1$ summation (with $b \mapsto c$) when $b = q$ and to the q-binomial theorem when $a = 1$ or $b = c$. It follows by applying Heine's transformation [20, Equation (III.3)] to the last term in Watson's three-term transformation formula [20, Equation (III.32)] for ${}_2\phi_1$ series. The second identity, which was first stated by Andrews in [5, Theorem 6], simplifies to the ${}_1\psi_1$ summation (again with $b \mapsto c$) in the $b \to 0$ limit and to the q-Gauss sum when $a = 1$ or $c = q$. It follows by specialising $c = q$ in Gasper and Rahman's three-term transformation formula [20, Equation (III.33)] for ${}_3\phi_2$ series and by noting that the second ${}_3\phi_2$ series on the right simplifies to a ${}_2\phi_1$ which can be evaluated by the q-Gauss sum.

There are several further generalisations of the ${}_1\psi_1$ summation that are not q-hypergeometric in nature. The simplest example is Vildanov's identity [39][3]

$$ \sum_{n=-\infty}^{\infty} \frac{(bq^n, q^{m-n}/c; q^m)_\infty}{(aq^n, q^{m-n}/a; q^m)_\infty}\, z^n = \frac{(az, q/az, q, q; q)_\infty}{(z, q/z, a, q/a; q)_\infty}\, \frac{(z^m, q^m/z^m, b/c; q^m)_\infty}{(q^m, az^m/c, b/az^m; q^m)_\infty}, \tag{3.4} $$

where m is a positive integer and $|b/a| < |z^m| < |c/a|$. Vildanov's identity is invariant under the simultaneous substitution $(a, b, c, z) \mapsto (q^m/a, q^m/c, q^m/b, 1/z)$, and for $m = 1$ is (1.3) with $(a, z) \mapsto (c, az/c)$. The identity can be proved by replacing $n \mapsto nm + k$ for $0 \leqslant k \leqslant m-1$ and, for fixed k, carrying out the sum over n by the ${}_1\psi_1$ summation. The standard theta function addition formula

$$ \sum_{k=0}^{m-1} \frac{\theta(az^m q^k; q^m)}{\theta(aq^k; q^m)}\, z^k = \frac{(q;q)_\infty^2}{(q^m;q^m)_\infty^2}\, \frac{\theta(az;q)\theta(z^m;q^m)}{\theta(a;q)\theta(z;q)} $$

(which may be viewed as a terminating analogue of Kronecker's identity (1.5)) then yields the right-hand side of (3.4).

One level up in complexity is a generalisation due to Guo and Schlosser [21], which is presented below in a somewhat simplified form. For complex a, b, c, z such that $|b/ac| < |z| < 1$,

$$ \sum_{n=-\infty}^{\infty} \frac{1 - az_n q^n}{1 - azq^n}\, \frac{(qz_n, b/az_n; q)_\infty}{(az_n, q/az_n; q)_\infty}\, \frac{(a;q)_n}{(b;q)_n}\, z_n^n = \frac{1}{1-z}\, \frac{(b/a, q; q)_\infty}{(q/a, b; q)_\infty}, \tag{3.5} $$

where $z_n = z_n(a) := z(1 - aczq^n)/(1 - azq^n)$. If $f_n(a, b, c, z; q)$ denotes the ratio of the summand on the left and the evaluation on the right, then, since $z_{n+1}(a) = z_n(aq)$, it follows that $f_{n+1}(a, b, c, z; q) = f_n(aq, bq, c, z; q)$. This implies that $f(a) := \sum_n f_n(a, ab, c, z; q)$ is periodic along annuli, i.e., $f(a) = f(aq)$, so that $f(a)$ is an elliptic function in multiplicative form. The upshot of (3.5) is that this elliptic function is bounded and thus a constant, and that this constant is equal to 1. If $c = 1$ then $z_n = z$, and (3.5) simplifies to Ramanujan's ${}_1\psi_1$ summation. If c is replaced by $-c/a^2z^2$ and then the $z \to 0$ limit is taken (so that $z_n \to cq^n/a$), the Guo–Schlosser summation reduces to a limiting case of Bailey's ${}_6\psi_6$ summation [20, Equation (II.33)]. Guo and Schlosser prove (3.5) by once again appealing

[3]When z is an integral power of q the right-hand side of (3.4) should be interpreted in the appropriate limiting sense. For example, for $z = 1$ the right hand side is $m(b/c, q^m; q^m)_\infty/(a/c, b/a; q^m)_\infty$.

to Ismail's argument; both sides are analytic in b for $|b|<1$, and for $b=q^{k+1}$ the identity reduces to the $a\mapsto aq^{-k}$ case of the unilateral summation obtained by setting $b=q$ in (3.5). This extension of the q-binomial theorem (obtained when $c=1$) is then proved using inversion techniques.

A final non-hypergeometric generalisation is Schlosser's noncommutative ${}_1\psi_1$ summation. Let $\mathcal{A}$ be a unital complex Banach algebra with norm $\|\cdot\|$, identity 1 and group of invertible elements $G(\mathcal{A})$. Assume that $a,b,q,z\in\mathcal{A}$ such that b,q are central elements, $a,q,z\in G(\mathcal{A})$, and

$$\max\{\|q\|,\|z\|,\|bz^{-1}a^{-1}\|\}<1. \tag{3.6}$$

Let $\mathscr{C}:=\{b,qa^{-1},qz^{-1}a^{-1}z\}$, and further assume that the binomials $1-cq^n\in G(\mathcal{A})$ for $c\in\mathscr{C}$ and nonnegative integer n. (By (3.6), the binomials $1-q^n$, $1-zq^n$ and $bz^{-1}a^{-1}q^n$ are also invertible, with inverses given by Neumann series.) For, $a_1,\dots,a_r,b_1,\dots,b_r\in\mathcal{A}$ such that $1-b_jq^n\in G(\mathcal{A})$, define

$$\left\lfloor\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_r\end{matrix};q\right\rfloor_\infty:=\gamma_1\gamma_2\gamma_3\cdots\quad\text{and}\quad\left\lceil\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_r\end{matrix};q\right\rceil_\infty:=\cdots\gamma_3\gamma_2\gamma_1,$$

where

$$\gamma_i:=(1-b_1q^{i-1})^{-1}(1-a_1q^{i-1})\cdots(1-b_rq^{i-1})^{-1}(1-a_rq^{i-1}).$$

Then [36, Equation (3.3)][4]

$$\begin{aligned}1+\sum_{n=1}^\infty\prod_{i=1}^n\Big((1-bq^{n-i})^{-1}(1-aq^{n-i})z\Big)+\sum_{n=1}^\infty\prod_{i=1}^n\Big(z^{-1}(a-q^{n-i+1})^{-1}(b-q^{n-i+1})\Big)\\ =\left\lfloor\begin{matrix}q,bz^{-1}a^{-1}z\\ b,bz^{-1}a^{-1}\end{matrix};q\right\rfloor_\infty\left\lceil\begin{matrix}qz^{-1}a^{-1}\\ qz^{-1}a^{-1}z\end{matrix};q\right\rceil_\infty\left\lfloor\begin{matrix}az\\ z\end{matrix};q\right\rfloor_\infty.\end{aligned}$$

When $\mathcal{A}=\mathbb{C}$, this is exactly Ramanujan's ${}_1\psi_1$ summation. Schlosser proves his summation by modifying the Andrews–Askey proof [6] of (1.3) to the setting of Banach algebras. In particular, he shows that the left-hand side satisfies the three functional equations

$$\begin{aligned}F(a,b,z;q)&=F(aq,bq,z;q)(1-b)^{-1}(1-a)z,\\ z\,F(a,b,z;q)&=az\,F(a,b,zq;q)+F(aq,b,z;q)(1-a)z,\\ F(a,bq,z;q)&=b\,F(a,bq,zq;q)+(1-b)\,F(a,b,z;q),\end{aligned}$$

which may be combined to yield the following analogue of (2.3):

$$F(a,b,z;q)=\left\lfloor\begin{matrix}bz^{-1}a^{-1}z,0\\ bz^{-1}a^{-1},b\end{matrix};q\right\rfloor_\infty F(a,0,z;q).$$

By mimicking the steps taken on page 3, the evaluation of $F(a,0,z;q)$ then follows from the noncommutative q-binomial theorem [37, Theorem 7.2]

$$F(a,q,z;q)=\left\lfloor\begin{matrix}az\\ z\end{matrix};q\right\rfloor_\infty.$$

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[4] The conditions that $q\in G(\mathcal{A})$ and $\|q\|<1$ are inadvertently missing in [36].

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