# THE $\mathrm{A}_{2n}^{(2)}$ ROGERS–RAMANUJAN IDENTITIES

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Abstract. The famous Rogers–Ramanujan and Andrews–Gordon identities are embedded in a doubly-infinite family of Rogers–Ramanujan-type identities labelled by positive integers $m$ and $n$. For fixed $m$ and $n$ the product side corresponds to a specialised character of the affine Kac–Moody algebra $\mathrm{A}_{2n}^{(2)}$ at level $m$, and is expressed as a product of $n^2$ theta functions of modulus $2m+2n+1$, or by level-rank duality, as a product of $m^2$ theta functions. Rogers–Ramanujan-type identities for even moduli, corresponding to the affine Lie algebras $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$, and arbitrary moduli, corresponding to $\mathrm{A}_{n-1}^{(1)}$, are also proven.

## 1. Introduction

The celebrated Rogers–Ramanujan (RR) identities [43]

$$1+\sum_{r=0}^{\infty}\frac{q^{r(r+\sigma)}}{(1-q)\cdots(1-q^r)}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+\sigma+1})(1-q^{5j-\sigma+4})} \tag{1.1}$$

for $\sigma=0,1$ are two of the most important combinatorial identities in all of mathematics, with a remarkably wide range of applications. First recognised by MacMahon and Schur as identities for integer partitions [39, 45], they have since been linked to algebraic geometry [14], K-theory [15], conformal field theory [10, 27], group theory [18], Kac–Moody and double affine Hecke algebras [16, 31–36], knot theory [6, 23, 24], modular forms [13, 41], orthogonal polynomials [7, 12, 20], statistical mechanics [4, 9], probability [19] and transcendental number theory [42].

In 1974 Andrews [1] extended (1.1) to an infinite family of Rogers–Ramanujan-type identities by proving that

$$\sum_{r_1\geq\cdots\geq r_m\geq 0}\frac{q^{r_1^2+\cdots+r_m^2+r_i+\cdots+r_m}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}}=\frac{(q^{2m+3};q^{2m+3})_\infty}{(q)_\infty}\,\theta(q^i;q^{2m+3}), \tag{1.2}$$

where $1\leq i\leq m+1$, $(a)_k=(a;q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1})$ (for $k\in\{0,1,\dots\}\cup\{\infty\}$) a $q$-shifted factorial and $\theta(a;q)=(a;q)_\infty(q/a;q)_\infty$ a modified theta function. The identities (1.2), which can be viewed as the analytic counterpart of Gordon's partition theorem [21], are now commonly referred to as the Andrews–Gordon (AG) identities.

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The various Lie-algebraic interpretations of the Rogers–Ramanujan and Andrews–Gordon identities attach algebras of low rank to (1.1) and (1.2). For example, from the above-cited works of Milne, Lepowsky and Wilson it follows that they arise as principally specialised characters of integrable highest-weight modules of the affine Kac–Moody algebra $\mathrm{A}_1^{(1)}$. This raises the question as to whether (1.1) and (1.2) can be embedded in a larger family of Rogers–Ramanujan-type identities by considering specialised characters of an appropriately chosen affine Lie algebra $\mathrm{X}_N^{(r)}$ for arbitrary $N$. In [5] (see also [17, 49]) some partial results concerning this question were obtained, resulting in Rogers–Ramanujan-type identities for $\mathrm{A}_2^{(1)}$. Unfortunately, the approach of [5] does not in any obvious manner extend to $\mathrm{A}_n^{(1)}$ for all $n$.

In this paper we give a more satisfactory answer to the above question by proving Rogers–Ramanujan and Andrews–Gordon identities for $\mathrm{A}_{2n}^{(2)}$ for arbitrary $n$. In their most compact form, the sum-sides are expressed in terms of Hall–Littlewood polynomials $P_\lambda(x;q)$ evaluated at infinite geometric progressions.

Let $\theta(a_1,\dots,a_k;q)=\theta(a_1;q)\cdots\theta(a_k;q)$ and for $\lambda=(\lambda_1,\lambda_2,\dots)$ an integer partition, let $|\lambda|:=\lambda_1+\lambda_2+\cdots$, $2\lambda:=(2\lambda_1,2\lambda_2,\dots)$ and $\lambda'$ the conjugate of $\lambda$. For example, if $\lambda=(5,3,3,1)$ then $|\lambda|=12$, $2\lambda=(10,6,6,2)$ and $\lambda'=(4,3,3,1,1)$.

**Theorem 1.1** ($\mathrm{A}_{2n}^{(2)}$ RR and AG identities)**.** *For $m$ and $n$ positive integers let $\kappa=2m+2n+1$. Then*

$$
\begin{aligned}
\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big)
&=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n}\prod_{i=1}^n\theta\big(q^{i+m};q^\kappa\big)\prod_{1\leq i<j\leq n}\theta\big(q^{j-i},q^{i+j-1};q^\kappa\big)\\
&=\frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m}\prod_{i=1}^m\theta\big(q^{i+1};q^\kappa\big)\prod_{1\leq i<j\leq m}\theta\big(q^{j-i},q^{i+j+1};q^\kappa\big)
\end{aligned}
\tag{1.3a}
$$

*and*

$$
\begin{aligned}
\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{2|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big)
&=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n}\prod_{i=1}^n\theta\big(q^{i};q^\kappa\big)\prod_{1\leq i<j\leq n}\theta\big(q^{j-i},q^{i+j};q^\kappa\big)\\
&=\frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m}\prod_{i=1}^m\theta\big(q^{i};q^\kappa\big)\prod_{1\leq i<j\leq m}\theta\big(q^{j-i},q^{i+j};q^\kappa\big).
\end{aligned}
\tag{1.3b}
$$

We note the beautiful level-rank duality exhibited by the products on the right, especially those of (1.3b). We also note that for $n=1$ we recover the Rogers–Ramanujan identities and the $i=1$ and $m+1$ instances of the Andrews–Gordon identities in a representation due to Stembridge [47] (see also [18]). The equivalence with (1.1) and (1.2) follows from the specialisation formula [38, p. 213]

$$q^{(\sigma+1)|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q)=\prod_{i\geq 1}\frac{q^{r_i(r_i+\sigma)}}{(q)_{r_i-r_{i+1}}},\qquad r_i:=\lambda_i',$$

and the fact that $\lambda_1\leq m$ implies that $\lambda_i'=r_i=0$ for $i>m$. As shown in the next section, the more general $P_\lambda(1,q,q^2,\dots;q^n)$ is also expressible in terms of $q$-shifted factorials, allowing for a formulation of Theorem 1.1 free of Hall–Littlewood polynomials.

We have also found an even modulus analogue of Theorem 1.1. Surprisingly, the $\sigma=0$ and $\sigma=1$ cases correspond to dual affine Lie algebras.

**Theorem 1.2** ($\mathrm{C}_n^{(1)}$ RR and AG identities). *For $m$ and $n$ positive integers let $\kappa=2m+2n+2$. Then*

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n}\big)
&=\frac{(q^2;q^2)_\infty(q^{\kappa/2};q^{\kappa/2})_\infty(q^\kappa;q^\kappa)_\infty^{n-1}}{(q)_\infty^{n+1}}\\
&\qquad\times\prod_{i=1}^{n}\theta\big(q^i;q^{\kappa/2}\big)\prod_{1\leq i<j\leq n}\theta\big(q^{j-i},q^{i+j};q^\kappa\big)\\
&=\frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m}\prod_{i=1}^{m}\theta\big(q^{i+1};q^\kappa\big)\prod_{1\leq i<j\leq m}\theta\big(q^{j-i},q^{i+j+1};q^\kappa\big).\end{aligned}\tag{1.4}$$

**Theorem 1.3** ($\mathrm{D}_{n+1}^{(2)}$ RR and AG identities). *For $m$ and $n$ positive integers such that $n\geq 2$ let $\kappa=2m+2n$. Then*

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{2|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-2}\big)
&=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q^2;q^2)_\infty(q)_\infty^{n-1}}\prod_{1\leq i<j\leq n}\theta\big(q^{j-i},q^{i+j-1};q^\kappa\big)\\
&=\frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m}\prod_{i=1}^{m}\theta\big(q^{i};q^\kappa\big)\prod_{1\leq i<j\leq m}\theta\big(q^{j-i},q^{i+j};q^\kappa\big).\end{aligned}\tag{1.5}$$

The $(m,n)=(1,2)$ case of (1.5) is equivalent to Milne's modulus 6 Rogers–Ramanujan identity [40, Theorem 3.26].

By combining (1.3)–(1.5) we obtain an identity of mixed type.

**Corollary 1.4.** *For $m$ and $n$ positive integers let $\kappa = 2m+n+2$. Then*

$$\begin{aligned}(1.6)\qquad \sum_{\substack{\lambda \\ \lambda_1\leq m}} & q^{(\sigma+1)|\lambda|} P_{2\lambda}\big(1,q,q^2,\dots;q^n\big) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^m \theta\big(q^{i-\sigma+1};q^\kappa\big) \prod_{1\leq i<j\leq m} \theta\big(q^{j-i},q^{i+j-\sigma+1};q^\kappa\big),\end{aligned}$$

*where $\sigma = 0,1$.*

Rogers–Ramanujan-type identities for $\mathrm{A}_{n-1}^{(1)}$ also exist, although their formulation is perhaps slightly less satisfactory, involving a limit.

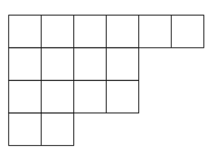
**Theorem 1.5** ($\mathrm{A}_{n-1}^{(1)}$ RR and AG identities)**.** *For $m$ and $n$ positive integers let $\kappa = m+n$. Then*

$$\begin{aligned}\lim_{r\to\infty} q^{-m\binom{r}{2}} P_{(m^r)}(1,q,q^2,\dots;q^n) &= \frac{(q^\kappa;q^\kappa)_\infty^{n-1}}{(q)_\infty^n} \prod_{1\leq i<j\leq n} \theta(q^{j-i};q^\kappa) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^{m-1}}{(q)_\infty^m} \prod_{1\leq i<j\leq m} \theta(q^{j-i};q^\kappa).\end{aligned}$$

The remainder of this paper is organised as follows. In the next section we recall some basic definitions and facts from the theory of Hall–Littlewood polynomials and use this to give an alternative, combinatorial representation for the left-hand side of (1.6). Then, in Sections 3 and 4, we prove Theorems 1.1–1.3 and Theorem 1.5, respectively, and interpret each of the theorems from the point of view of representation theory.

## 2. The Hall–Littlewood polynomials

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition [3], i.e., $\lambda_1 \geq \lambda_2 \geq \cdots$ such that only finitely-many $\lambda_i > 0$. The positive $\lambda_i$ are called the parts of $\lambda$ and the number of parts, denoted $l(\lambda)$, is the length of $\lambda$. The size $|\lambda|$ of $\lambda$ is the sum of its parts. The diagram of $\lambda$ consists of $l(\lambda)$ left-aligned rows of squares such that the $i$th row contains $\lambda_i$ squares. For example, the diagram of $\nu = (6,4,4,2)$ of length 4 and size 16 is



The conjugate partition $\lambda'$ follows by transposing the diagram of $\lambda$. For example, $\nu' = (4,4,3,3,1,1)$. The nonnegative integers $m_i = m_i(\lambda)$, $i \geq 1$ give the multiplicities of parts of size $i$, so that $|\lambda| = \sum_i i m_i$. It is easy to see that $m_i = \lambda'_i - \lambda'_{i+1}$. We say that a partition is even if all its parts are even. Note that $\lambda'$ is even if all multiplicities $m_i(\lambda)$ are even. The partition $\nu$ in our example is an even partition. Given two partitions $\lambda, \mu$ we write

$\mu \subseteq \lambda$ if the diagram of $\mu$ is contained in the diagram of $\lambda$, or, equivalently, if $\mu_i \le \lambda_i$ for all $i$. To conclude our discussion of partitions we define the generalised $q$-shifted factorial $b_\lambda(q)$ as

$$b_\lambda(q) = \prod_{i\ge 1} (q)_{m_i} = \prod_{i\ge 1} (q)_{\lambda'_i - \lambda'_{i+1}}. \tag{2.1}$$

Hence $b_\nu(q) = (q)_1^2 (q)_2$.

For a fixed positive integer $n$, let $x = (x_1, \dots, x_n)$. Given a partition $\lambda$ such that $l(\lambda) \le n$, write $x^\lambda$ for the monomial $x_1^{\lambda_1} \dots x_n^{\lambda_n}$ and define $v_\lambda(q) = \prod_{i=0}^n (q)_{m_i}/(1-q)^{m_i}$, where $m_0 := n - l(\lambda)$. The Hall–Littlewood polynomial $P_\lambda(x;q)$ is defined as the symmetric function [38]

$$P_\lambda(x;q) = \frac{1}{v_\lambda(q)} \sum_{w \in \mathfrak{S}_n} w\biggl( x^\lambda \prod_{i<j} \frac{x_i - q x_j}{x_i - x_j} \biggr),$$

where the symmetric group $\mathfrak{S}_n$ acts on $x$ by permuting the $x_i$. It follows from the definition that $P_\lambda(x;q)$ is a homogeneous polynomial of degree $|\lambda|$, a fact used repeatedly in the rest of this paper. $P_\lambda(x;q)$ is defined to be identically 0 if $l(\lambda) > n$. The Hall–Littlewood polynomials may be extended in the usual way to symmetric functions in countably-many variables, see [38]. Below we only need this for $x$ a geometric progression.

For $x = (x_1, x_2, \dots)$ not necessarily finite, let $p_r$ be the $r$-th power sum symmetric function

$$p_r(x) = x_1^r + x_2^r + \cdots,$$

and $p_\lambda = \prod_{i\ge 1} p_{\lambda_i}$. The power sums $\{p_\lambda(x_1, \dots, x_n)\}_{l(\lambda)\le n}$ form a $\mathbb{Q}$-basis of the ring of symmetric functions in $n$ variables. If $\phi_q$ denotes the ring homomorphism $\phi_q(p_r) = p_r/(1-q^r)$, then the modified Hall–Littlewood polynomials $P'_\lambda(x;q)$ are defined as the image of the $P_\lambda(x;q)$ under $\phi_q$:

$$P'_\lambda = \phi_q\bigl(P_\lambda\bigr).$$

We also require the Hall–Littlewood polynomials $Q_\lambda$ and $Q'_\lambda$ defined by

$$Q_\lambda(x;q) = b_\lambda(q) P_\lambda(x;q) \quad \text{and} \quad Q'_\lambda(x;q) = b_\lambda(q) P'_\lambda(x;q). \tag{2.2}$$

Clearly, $Q'_\lambda = \phi_q\bigl(Q_\lambda\bigr)$.

Up to the point where the $x$-variables are specialised, our proof of Theorems 1.1–1.3 features the modified rather than the ordinary Hall–Littlewood polynomials. Through specialisation we arrive at $P_\lambda$ evaluated at a geometric progression thanks to

$$P'_\lambda(1, q, \dots, q^{n-1}; q^n) = P_\lambda(1, q, q^2, \dots; q^n), \tag{2.3}$$

which readily follows from

$$\phi_{q^n}\bigl(p_r(1, q, \dots, q^{n-1})\bigr) = \frac{1-q^{nr}}{1-q^r} \cdot \frac{1}{1-q^{nr}} = p_r(1, q, q^2, \dots).$$

From [28, 50] we may infer the following combinatorial formula for the modified Hall–Littlewood polynomials:

$$Q'_\lambda(x;q)=\sum\prod_{i=1}^{\lambda_1}\prod_{a=1}^{n} x_a^{\mu_i^{(a-1)}-\mu_i^{(a)}} q^{\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}} \begin{bmatrix}\mu_i^{(a-1)}-\mu_{i+1}^{(a)}\\ \mu_i^{(a-1)}-\mu_i^{(a)}\end{bmatrix}_q,$$

where the sum is over partitions $0=\mu^{(n)}\subseteq\cdots\subseteq\mu^{(1)}\subseteq\mu^{(0)}=\lambda'$ and

$$\begin{bmatrix}n\\ m\end{bmatrix}_q=\begin{cases}\dfrac{(q)_n}{(q)_m(q)_{n-m}} & \text{if } m\in\{0,1,\dots,n\}\\ 0 & \text{otherwise}\end{cases}$$

is a $q$-binomial coefficient. Therefore, by (2.1)–(2.3),

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{(\sigma+1)|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q^n) \tag{2.4}$$
$$=\sum\prod_{i=1}^{2m}\Bigg\{\frac{q^{\frac12(\sigma+1)\mu_i^{(0)}}}{(q^n;q^n)_{\mu_i^{(0)}-\mu_{i+1}^{(0)}}}\prod_{a=1}^{n} q^{\mu_i^{(a)}+n\binom{\mu_i^{(a-1)}-\mu_i^{(a)}}{2}}\begin{bmatrix}\mu_i^{(a-1)}-\mu_{i+1}^{(a)}\\ \mu_i^{(a-1)}-\mu_i^{(a)}\end{bmatrix}_{q^n}\Bigg\},$$

where the sum on the right is over partitions $0=\mu^{(n)}\subseteq\cdots\subseteq\mu^{(1)}\subseteq\mu^{(0)}$ such that $(\mu^{(0)})'$ is even and $l(\mu^{(0)})\le 2m$. This may be used to express the sum sides of (1.3)–(1.6) combinatorially. To see that (2.4) indeed generalises the sums in (1.1) and (1.2), we note that the above simplifies for $n=1$ to

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{(\sigma+1)|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q)=\sum\prod_{i=1}^{2m}\frac{q^{\frac12\mu_i(\mu_i+\sigma)}}{(q)_{\mu_i-\mu_{i+1}}}$$

summed on the right over partitions $\mu$ of length at most $2m$ whose conjugates are even. Such partitions are characterised by the restriction $\mu_{2i}=\mu_{2i-1}=:r_i$ so that we get

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{(\sigma+1)|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q)=\sum_{r_1\ge\cdots\ge r_m\ge 0}\prod_{i=1}^{m}\frac{q^{r_i(r_i+\sigma)}}{(q)_{r_i-r_{i+1}}}$$

in accordance with (1.2). If instead we consider $m=1$ and replace $\mu^{(j)}$ by $(r_j,s_j)$ for $j\ge 0$, we find

$$\begin{aligned}\sum_{r=0}^{\infty} & q^{(\sigma+1)r}P_{(2^r)}(1,q,q^2,\dots;q^n)\\ &=\sum\frac{q^{(\sigma+1)r_0}}{(q^n;q^n)_{r_0}}\prod_{j=1}^{n} q^{r_j+s_j+n\binom{r_{j-1}-r_j}{2}+n\binom{s_{j-1}-s_j}{2}}\begin{bmatrix}r_{j-1}-s_j\\ r_{j-1}-r_j\end{bmatrix}_{q^n}\begin{bmatrix}s_{j-1}\\ s_j\end{bmatrix}_{q^n}\\ &=\frac{(q^{n+4};q^{n+4})_\infty}{(q)_\infty}\,\theta\big(q^{2-\sigma};q^{n+4}\big),\end{aligned}$$

where the second sum is over $r_0, s_0, \dots, r_{n-1}, s_{n-1}$ such that $r_0 = s_0$, and $r_n = s_n := 0$.

We conclude this section with a remark about Theorem 1.5. Due to the occurrence of the limit, the left-hand side does not take the form of the usual sum-side of a Rogers–Ramanujan-type identity. For special cases it is, however, possible to eliminate the limit. For example, for partitions of the form $(2^r)$ we found that

$$\begin{aligned}(2.5)\quad & P_{(2^r)}(1,q,q^2,\dots;q^{2n+\delta}) \\ &= \sum_{r\geq r_1\geq\cdots\geq r_n\geq 0} \frac{q^{r^2-r+r_1^2+\cdots+r_n^2+r_1+\cdots+r_n}}{(q)_{r-r_1}(q)_{r_1-r_2}\cdots(q)_{r_{n-1}-r_n}(q^{2-\delta};q^{2-\delta})_{r_n}}\end{aligned}$$

for $\delta = 0, 1$. This turns the $m = 2$ case of Theorem 1.5 into

$$\begin{aligned}\sum_{r_1\geq\cdots\geq r_n\geq 0} & \frac{q^{r_1^2+\cdots+r_n^2+r_1+\cdots+r_n}}{(q)_{r_1-r_2}\cdots(q)_{r_{n-1}-r_n}(q^{2-\delta};q^{2-\delta})_{r_n}} \\ &= \frac{(q^{2n+2+\delta};q^{2n+2+\delta})_\infty}{(q)_\infty}\,\theta(q;q^{2n+2+\delta}).\end{aligned}$$

For $\delta = 1$ this is the $i = 1$ case of the Andrews–Gordon identity (1.2) (with $m$ replaced by $n$). For $\delta = 0$ it corresponds to an identity due to Bressoud [11]. We do not know how to generalise (2.5) to arbitrary rectangular shapes.

## 3. Proof of Theorems 1.1–1.3

3.1. **The Watson–Andrews approach.** In 1929 Watson proved the Rogers–Ramanujan identities (1.1) by first proving a new basic hypergeometric series transformation between a terminating balanced ${}_4\phi_3$ series and a terminating very-well-poised ${}_8\phi_7$ series [48]

$$\begin{aligned}(3.1)\quad & \frac{(aq,aq/bc)_N}{(aq/b,aq/c)_N}\sum_{r=0}^{N}\frac{(b,c,aq/de,q^{-N})_r}{(q,aq/d,aq/e,bcq^{-N}/a)_r}\,q^r \\ &= \sum_{r=0}^{N}\frac{1-aq^{2r}}{1-a}\,\frac{(a,b,c,d,e,q^{-N})_r}{(q,aq/b,aq/c,aq/d,aq/e)_r}\left(\frac{a^2q^{N+2}}{bcde}\right)^r.\end{aligned}$$

Here $a, b, c, d, e$ are indeterminates, $N$ is a nonnegative integer and

$$(a_1,\dots,a_m)_k = (a_1,\dots,q_m;q) = (a_1;q)_k\cdots(a_m;q)_k.$$

By letting $b,c,d,e$ tend to infinity and taking the nonterminating limit $N\to\infty$, Watson arrived at what is known as the Rogers–Selberg identity [44,46][1]

$$\sum_{r=0}^{\infty}\frac{a^rq^{r^2}}{(q)_r}=\frac{1}{(aq)_\infty}\sum_{r=0}^{\infty}\frac{1-aq^{2r}}{1-a}\,\frac{(a)_r}{(q)_r}\,(-1)^ra^{2r}q^{5\binom{r}{2}+2r}. \tag{3.2}$$

For $a=1$ or $a=q$ the sum on the right can be expressed in product-form by the Jacobi triple-product identity

$$\sum_{r=-\infty}^{\infty}(-1)^rx^rq^{\binom{r}{2}}=(q)_\infty\,\theta(x;q),$$

resulting in (1.1).

Almost 50 years after Watson's work, Andrews showed that the Andrews–Gordon identities (1.2) for $i=1$ and $i=m+1$ follow in much the same manner from a multiple series generalisation of (3.1) in which the ${}_8\phi_7$ series on the right is replaced by a terminating very-well-poised ${}_{2m+6}\phi_{2m+5}$ series depending on $2m+2$ parameters instead of $b,c,d,e$ [2]. Again the key steps are to let all these parameters tend to infinity, to take the nonterminating limit and express the $a=1$ or $a=q$ instances of the resulting sum as a product by the Jacobi triple-product identity.

Recently, in joint work with Bartlett, we obtained an analogue of Andrews' multiple series transformation for the $\mathrm{C}_n$ root system [8, Theorem 4.2]. Apart from the variables $(x_1,\dots,x_n)$—which play the role of $a$ in (3.1), and are related to the underlying root system—the $\mathrm{C}_n$ Andrews transformation again contains $2m+2$ parameters. Unfortunately, simply following the Andrews–Watson procedure is no longer sufficient. In [40] Milne already obtained the $\mathrm{C}_n$ analogue of the Rogers–Selberg identity (3.2) (the $m=1$ case of (3.3) below) and considered specialisations along the lines of Andrews and Watson. Only for $\mathrm{C}_2$ did this result in a Rogers–Ramanujan-type identity: the modulus 6 case of (1.5) mentioned previously.

The initial two steps towards proof of (1.3)–(1.6), however, are the same as those of Watson and Andrews: we let all $2m+2$ parameters in the $\mathrm{C}_n$ Andrews transformation tend to infinity and take the nonterminating limit. Then, as shown in [8], the right-hand side can be expressed in terms of modified Hall–Littlewood polynomials, resulting in the level-$m$ $\mathrm{C}_n$ Rogers–Selberg identity

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|}P'_{2\lambda}(x;q)=L^{(0)}(x;q) \tag{3.3}$$

[1]Here and elsewhere in the paper we ignore questions of convergence. From an analytic point of view the transition from (3.1) to (3.2) requires the use of the dominated convergence theorem, imposing the restriction $|q|<1$ on the Rogers–Selberg identity. We however choose to view this identity as an identity between formal power series in $q$, in line with the combinatorial and representation-theoretic interpretations of Rogers–Ramanujan-type identities.

for

$$L_m^{(0)}(x;q) := \sum_{r\in\mathbb{Z}_+^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n x_i^{2(m+1)r_i} q^{(m+1)r_i^2+n\binom{r_i}{2}} \times \prod_{i,j=1}^n \Big(-\frac{x_i}{x_j}\Big)^{r_i} \frac{(x_ix_j)_{r_i}}{(qx_i/x_j)_{r_i}}.$$

Here

$$\Delta_{\mathrm{C}}(x) := \prod_{i=1}^n (1-x_i^2) \prod_{1\le i<j\le n} (x_i-x_j)(x_ix_j-1)$$

is the $\mathrm{C}_n$ Vandermonde product and $f(xq^r)$ is shorthand for $f(x_1q^{r_1},\dots,x_nq^{r_n})$. As mentioned previously, (3.3) for $m=1$ is Milne's $\mathrm{C}_n$ Rogers–Selberg formula [40, Corollary 2.21].

Comparing the left-hand side of (3.3) with that of (1.3)–(1.5) it follows that we should make the simultaneous substitutions

$$q \mapsto q^n, \qquad x_i \mapsto q^{(n+\sigma+1)/2-i} \quad (1\le i\le n). \tag{3.4}$$

Then, by the homogeneity and symmetry of the (modified) Hall–Littlewood polynomials and (2.3),

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}(x;q) \longmapsto \sum_{\substack{\lambda\\ \lambda_1\le m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^n).$$

The problem we face is that making the substitution (3.4) on the right-hand side of (3.3) and then writing the resulting $q$-series in product form is very difficult. To get around this problem, we take a rather different route and (up to a small constant) first double the rank of the underlying $\mathrm{C}_n$ root system and then take a limit in which products of pairs of $x$-variables tend to one. To do so we require another result from [8].

First we need to extend our earlier definition of the $q$-shifted factorial to $(a)_k = (a)_\infty/(aq^k)_\infty$. Importantly, $1/(q)_k = 0$ for $k$ a negative integer. Then, for $x=(x_1,\dots,x_n)$, $p$ an integer such that $0\le p\le n$ and $r\in\mathbb{Z}^n$,

$$L_m^{(p)}(x;q) := \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n x_i^{2(m+p+1)r_i} q^{(m+1)r_i^2+(n+p)\binom{r_i}{2}} \times \prod_{i=1}^n \prod_{j=p+1}^n \Big(-\frac{x_i}{x_j}\Big)^{r_i} \frac{(x_ix_j)_{r_i}}{(qx_i/x_j)_{r_i}}. \tag{3.5}$$

Note that the summand of $L_m^{(p)}(x;q)$ vanishes if one of $r_{p+1},\dots,r_n<0$.

**Lemma 3.1** ([8, Lemma A.1])**.** *For $1\le p\le n-1$,*

$$\lim_{x_{p+1}\to x_p^{-1}} L_m^{(p-1)}(x;q) = L_m^{(p)}(x_1,\dots,x_{p-1},x_{p+1},\dots,x_n;q). \tag{3.6}$$

This will be the key to the proof of all four generalised Rogers–Ramanujan identities, although the level of difficulty varies considerably from case to case. We begin with the simplest proof, that of (1.4).

3.2. **Proof of the** (1.4)**.** By iterating (3.6) we obtain

$$\lim_{y_1\to x_1^{-1}}\cdots\lim_{y_n\to x_n^{-1}} L_m^{(0)}(x_1,y_1,\dots,x_n,y_n)=L_m^{(n)}(x_1,\dots,x_n).$$

Hence, after replacing $x\mapsto(x_1,y_1,\dots,x_n,y_n)$ in (3.3) (which corresponds to the doubling of the rank mentioned previously) and taking the $y_i\to x_i^{-1}$ limit for $1\le i\le n$, we find

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|}P'_{2\lambda}(x^{\pm};q)=\frac{1}{(q)_\infty^n\prod_{i=1}^n\theta(x_i^2;q)\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q)} \times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(xq^r)\prod_{i=1}^n x_i^{\kappa r_i-i+1}q^{\frac12\kappa r_i^2-nr_i}, \tag{3.7}$$

where $\kappa=2m+2n+2$ and $f(x^{\pm})=f(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$. Next we make the simultaneous substitutions

$$q\mapsto q^{2n},\qquad x_i\mapsto q^{n-i+1/2}=:\hat{x}_i\ \ (1\le i\le n), \tag{3.8}$$

which corresponds to (3.4) with $(n,\sigma)\mapsto(2n,0)$. By

$$(q^{2n};q^{2n})_\infty^n\prod_{i=1}^n\theta(q^{2n-2i+1};q^{2n})\prod_{1\le i<j\le n}\theta(q^{j-i},q^{2n-i-j+1};q^{2n})=\frac{(q)_\infty^{n+1}}{(q^2;q^2)_\infty},$$

and

$$\begin{aligned} &q^{2n|\lambda|}P'_{2\lambda}(q^{n-1/2},q^{1/2-n},\dots,q^{1/2},q^{-1/2};q^{2n})\\ &\quad=q^{2n|\lambda|}P'_{2\lambda}(q^{1/2-n},q^{3/2-n},\dots,q^{n-1/2};q^{2n}) &&\text{by symmetry}\\ &\quad=q^{|\lambda|}P'_{2\lambda}(1,q,\dots,q^{2n-1};q^{2n}) &&\text{by homogeneity}\\ &\quad=q^{|\lambda|}P_{2\lambda}(1,q,q^2,\dots;q^{2n}) &&\text{by (2.3)}, \end{aligned}$$

this yields

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n}\big)=\frac{(q^2;q^2)_\infty}{(q)_\infty^{n+1}}\,\mathscr{M}, \tag{3.9}$$

where

$$\mathscr{M}:=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(\hat{x}q^{2nr})\prod_{i=1}^n\hat{x}_i^{\kappa r_i-i+1}q^{nr_i^2-2n^2r_i}.$$

What remains is to express $\mathscr{M}$ in product form. As a first step we use the $\mathrm{C}_n$ Weyl denominator formula [29, Lemma 2]

$$\Delta_{\mathrm{C}}(x)=\det_{1\le i,j\le n}\big(x_i^{j-1}-x_i^{2n-j+1}\big) \tag{3.10}$$

as well as multilinearity, to write $\mathscr{M}$ as

(3.11)
$$\mathscr{M}=\det_{1\le i,j\le n}\biggl(\sum_{r\in\mathbb{Z}}\hat{x}_i^{\kappa r-i+1}q^{n\kappa r^2-2n^2r}\Bigl((\hat{x}_iq^{2nr})^{j-1}-(\hat{x}_iq^{2nr})^{2n-j+1}\Bigr)\biggr).$$

We now replace $(i,j)\mapsto(n-j+1,n-i+1)$ and, viewing the resulting determinant as being of the form $\det\bigl(\sum_r u_{ij;r}-\sum_r v_{ij;r}\bigr)$, we change the summation index $r\mapsto -r-1$ in the sum over $v_{ij;r}$. Then

(3.12)
$$\mathscr{M}=\det_{1\le i,j\le n}\biggl(q^{a_{ij}}\sum_{r\in\mathbb{Z}}y_i^{2nr-i+1}q^{2n\kappa\binom{r}{2}+\frac{1}{2}\kappa r}\Bigl((y_iq^{\kappa r})^{j-1}-(y_iq^{\kappa r})^{2n-j}\Bigr)\biggr),$$

where $y_i=q^{\kappa/2-i}$ and $a_{ij}=j^2-i^2+(i-j)(\kappa+1)/2$. Since the factor $q^{a_{ij}}$ does not contribute to the determinant, we can apply the $\mathrm{B}_n$ Weyl denominator formula [29]

(3.13)
$$\det_{1\le i,j\le n}\bigl(x_i^{j-1}-x_i^{2n-j}\bigr)=\prod_{i=1}^n(1-x_i)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1)=:\Delta_{\mathrm{B}}(x)$$

to obtain
$$\mathscr{M}=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(yq^{\kappa r})\prod_{i=1}^n y_i^{2nr_i-i+1}q^{2n\kappa\binom{r_i}{2}+\frac{1}{2}\kappa r_i}.$$

By the $\mathrm{D}_{n+1}^{(2)}$ Macdonald identity [37]

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(xq^{r})\prod_{i=1}^n x_i^{2nr_i-i+1}q^{2n\binom{r_i}{2}+\frac{1}{2}r_i}$$
$$=(q^{1/2};q^{1/2})_\infty(q)_\infty^{n-1}\prod_{i=1}^n\theta(x_i;q^{1/2})_\infty\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q)$$

with $(q,x)\mapsto(q^\kappa,y)$ this results in

(3.14)
$$\mathscr{M}=(q^{\kappa/2};q^{\kappa/2})_\infty(q^\kappa;q^\kappa)_\infty^{n-1}\prod_{i=1}^n\theta\bigl(q^i;q^{\kappa/2}\bigr)\prod_{1\le i<j\le n}\theta\bigl(q^{j-i},q^{i+j};q^\kappa\bigr),$$

where we have also used the simple symmetry $\theta(q^{a-b};q^a)=\theta(q^b;q^a)$. Substituting (3.14) into (3.9) proves the first equality of (1.4).

To show that the second equality holds is a straightforward exercise in manipulating infinite products, and we omit the details.

There is a somewhat different approach to (1.4) based on the representation theory of the affine Kac–Moody algebra $\mathrm{C}_n^{(1)}$ [26]. Let $I=\{0,1,\dots,n\}$, and $\alpha_i$, $\alpha_i^\vee$ and $\Lambda_i$ for $i\in I$ the simple roots, simple coroots and fundamental weights of $\mathrm{C}_n^{(1)}$. Let $\langle\cdot,\cdot\rangle$ denote the usual pairing between the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, so that $\langle\Lambda_i,\alpha_j^\vee\rangle=\delta_{ij}$. Finally, let $V(\Lambda)$ be the integrable highest-weight module of $\mathrm{C}_n^{(1)}$ of highest weight $\Lambda$ with character $\operatorname{ch} V(\Lambda)$.

The homomorphism

$$F_{\mathbb{1}}:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q]],\qquad F_{\mathbb{1}}(\mathrm{e}^{-\alpha_i})=q\quad\text{for all } i\in I \tag{3.15}$$

is known as principal specialisation. In [25] Kac showed that the principally specialised characters admit a product form. Let $\rho$ be the Weyl vector (that is $\langle\rho,\alpha_i^{\vee}\rangle=1$ for $i\in I$) and $\operatorname{mult}(\alpha)$ the multiplicity of $\alpha$. Then Kac's formula is given by

$$F_{\mathbb{1}}\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)=\prod_{\alpha\in\Delta_+^{\vee}}\bigg(\frac{1-q^{\langle\Lambda+\rho,\alpha\rangle}}{1-q^{\langle\rho,\alpha\rangle}}\bigg)^{\operatorname{mult}(\alpha)}, \tag{3.16}$$

where $\Delta_+^{\vee}$ is the set of positive coroots. This result, which is valid for all types $\mathrm{X}_N^{(r)}$, can be rewritten in terms of theta functions. Assuming $\mathrm{C}_n^{(1)}$ and setting

$$\Lambda=(\lambda_0-\lambda_1)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n,$$

for $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$ a partition, this rewriting takes the form

$$\begin{aligned}F_{\mathbb{1}}\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)&=\frac{(q^2;q^2)_\infty(q^{\kappa/2};q^{\kappa/2})_\infty(q^\kappa;q^\kappa)_\infty^{n-1}}{(q;q)_\infty^{n+1}}\\&\quad\times\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};q^{\kappa/2}\big)\prod_{1\le i<j\le n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n+2-i-j};q^\kappa\big),\end{aligned} \tag{3.17}$$

where $\kappa=2n+2\lambda_0+2$.

The earlier product form now arises by recognising (see e.g., [8, Lemma 2.1]) the right-hand side of (3.7) as

$$\mathrm{e}^{-m\Lambda_0}\operatorname{ch}V(m\Lambda_0)$$

upon the identification

$$q=\mathrm{e}^{-\alpha_0-2\alpha_1-\cdots-2\alpha_{n-1}-\alpha_n}\quad\text{and}\quad x_i=\mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}-\alpha_n/2}\ \ (1\le i\le n).$$

Since (3.8) corresponds exactly to the principal specialisation (3.15) it follows from (3.17) with $\lambda=(m,0^n)$ that

$$\begin{aligned}F_{\mathbb{1}}\big(\mathrm{e}^{-m\Lambda_0}\operatorname{ch}V(m\Lambda_0)\big)&=\frac{(q^2;q^2)_\infty(q^{\kappa/2};q^{\kappa/2})_\infty(q^\kappa;q^\kappa)_\infty^{n-1}}{(q;q)_\infty^{n+1}}\\&\quad\times\prod_{i=1}^n\theta\big(q^{n-i+1};q^{\kappa/2}\big)\prod_{1\le i<j\le n}\theta\big(q^{j-i},q^{i+j};q^\kappa\big).\end{aligned}$$

We should remark that this representation-theoretic approach is not essentially different from our earlier $q$-series proof. Indeed, the principal specialisation formula (3.17) itself is an immediate consequence of the $\mathrm{D}_{n+1}^{(2)}$

Macdonald identity, and if, instead of the right-hand side of (3.7), we consider the more general

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{1}{(q)_\infty^n\prod_{i=1}^n\theta(x_i^2;q)\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q)}$$
$$\times\sum_{r\in\mathbb{Z}^n}\det_{1\le i,j\le n}\Big((x_iq^{r_i})^{j-\lambda_i-1}-(x_iq^{r_i})^{2n-j+\lambda_i+1}\Big)\prod_{i=1}^n x_i^{\kappa r_i+\lambda_i-i+1}q^{\frac{1}{2}\kappa r_i^2-nr_i}$$

for $\kappa=2n+2\lambda_0+2$, then all of the steps carried out between (3.7) and (3.14) carry over to this more general setting. The only notable changes are that (3.11) generalises to

$$\mathscr{M}=\det_{1\le i,j\le n}\bigg(\sum_{r\in\mathbb{Z}}\hat{x}_i^{\kappa r+\lambda_i-i+1}q^{n\kappa r^2-2n^2r}$$
$$\times\Big((\hat{x}_iq^{2nr})^{j-\lambda_i-1}-(\hat{x}_iq^{2nr})^{2n-j+\lambda_i+1}\Big)\bigg).$$

and that in (3.12) we have to redefine $y_i$ and $a_{ij}$ as $q^{\kappa/2-\lambda_{n-i+1}-i}$ and $j^2-i^2+(i-j)(\kappa+1)/2+(j-1/2)\lambda_{n-j+1}-(i-1/2)\lambda_{n-i+1}$.

3.3. **Proof of the** (1.3a)**.** Again we iterate (3.6), but this time the variable $x_n$, remains unpaired:

$$\lim_{y_1\to x_1^{-1}}\cdots\lim_{y_{n-1}\to x_{n-1}^{-1}}L_m^{(0)}(x_1,y_1,\dots,x_{n-1},y_{n-1},x_n)=L_m^{(n-1)}(x_1,\dots,x_n).$$

Therefore, if we replace $x\mapsto(x_1,y_1,\dots,x_{n-1},y_{n-1},x_n)$ in (3.3) (changing the rank from $n$ to $2n-1$) and take the $y_i\to x_i^{-1}$ limit for $1\le i\le n-1$, we obtain

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\le m}}q^{|\lambda|}P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-1}^{\pm},x_n;q\big)&\\ =\frac{1}{(q)_\infty^{n-1}(qx_n^2)_\infty\prod_{i=1}^{n-1}(qx_i^{\pm}x_n,qx_i^{\pm2})_\infty\prod_{1\le i<j\le n-1}(qx_i^{\pm}x_j^{\pm})_\infty}&\\ \times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\bigg(-\frac{x_i^\kappa}{x_n}\bigg)^{r_i}q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i}\frac{(x_ix_n)_{r_i}}{(qx_i/x_n)_{r_i}},&\end{aligned}\tag{3.18}$$

where $\kappa=2m+2n+1$, $(ax_i^{\pm})_\infty:=(ax_i)_\infty(ax_i^{-1})_\infty$ and

$$(ax_i^{\pm}x_j^{\pm})_\infty:=(ax_ix_j)_\infty(ax_i^{-1}x_j)_\infty(ax_ix_j^{-1})_\infty(ax_i^{-1}x_j^{-1})_\infty.$$

Recalling the comment immediately after (3.5), the summand of (3.18) vanishes unless $r_n\ge 0$.

Let $\hat{x}:=(-x_1,\dots,-x_{n-1},-1)$ and

$$\phi_r=\begin{cases}1 & \text{if } r=0\\ 2 & \text{if } r=1,2,\dots.\end{cases}\tag{3.19}$$

Letting $x_n$ tend to 1 in (3.18) using

$$\lim_{x_n\to 1}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\frac{(x_ix_n)_{r_i}}{(qx_i/x_n)_{r_i}}=\phi_{r_n}\frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})},$$

we find

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leq m}} & q^{|\lambda|}P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-1}^{\pm},1;q\big)\\ &=\frac{1}{(q)_\infty^n\prod_{i=1}^{n-1}(qx_i^{\pm},qx_i^{\pm 2})_\infty\prod_{1\leq i<j\leq n-1}(qx_i^{\pm}x_j^{\pm})_\infty}\\ &\qquad\times\sum_{r_1,\dots,r_{n-1}=-\infty}^{\infty}\sum_{r_n=0}^{\infty}\phi_{r_n}\frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})}\prod_{i=1}^n\hat{x}_i^{\kappa r_i}q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i}.\end{aligned}$$

It is easily checked that the summand on the right (without the factor $\phi_{r_n}$) is invariant under the variable change $r_n\mapsto -r_n$. Using the elementary relations

(3.20)
$$\theta(-1;q)=2(-q)_\infty^2,\quad (-q)_\infty(q;q^2)_\infty=1,\quad \theta(z,-z;q)\theta(qz^2;q^2)=\theta(z^2),$$

we can thus simplify the above to

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leq m}} & q^{|\lambda|}P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-1}^{\pm},1;q\big)\\ &=\frac{1}{(q)_\infty^n\prod_{i=1}^{n}\theta(\hat{x}_i;q)\theta(q\hat{x}_i^2;q^2)\prod_{1\leq i<j\leq n}\theta(\hat{x}_i/\hat{x}_j,\hat{x}_i\hat{x}_j;q)}\\ &\qquad\times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(\hat{x}q^r)\prod_{i=1}^n\hat{x}_i^{\kappa r_i-i+1}q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i}.\end{aligned}\tag{3.21}$$

The remainder of the proof is similar to that of (1.4). We make the simultaneous substitutions

$$q\mapsto q^{2n-1},\qquad x_i\mapsto q^{n-i}\ \ (1\leq i\leq n),\tag{3.22}$$

so that from here on $\hat{x}_i:=-q^{n-i}$. By

$$\begin{aligned}(q^{2n-1};q^{2n-1})_\infty^n&\prod_{i=1}^n\theta(-q^{n-i};q^{2n-1})\theta(q^{2n-2i+1};q^{4n-2})\\ &\times\prod_{1\leq i<j\leq n}\theta(q^{j-i},q^{2n-i-j};q^{2n-1})=2(q)_\infty^n\end{aligned}$$

and (2.3), this results in

$$\sum_{\substack{\lambda\\ \lambda_1\leq m}}q^{|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big)=\frac{\mathscr{M}}{2(q)_\infty^n},$$

for

$$\mathscr{M} := \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}\big(\hat{x}q^{(2n-1)r}\big) \prod_{i=1}^{n} \hat{x}_i^{\kappa r_i-i+1} q^{\frac{1}{2}(2n-1)\kappa r_i^2-\frac{1}{2}(2n-1)^2 r_i}.$$

By (3.13) and multilinearity $\mathscr{M}$ can be rewritten in the form

$$\mathscr{M} = \det_{1\le i,j\le n}\bigg(\sum_{r\in\mathbb{Z}} \hat{x}_i^{\kappa r-i+1} q^{\frac{1}{2}(2n-1)\kappa r^2-\frac{1}{2}(2n-1)^2 r} \times \Big(\big(\hat{x}_i q^{(2n-1)r}\big)^{j-1} - \big(\hat{x}_i q^{(2n-1)r}\big)^{2n-j}\Big)\bigg).$$

Following the same steps that led from (3.11) to (3.12) yields

$$\mathscr{M} = \det_{1\le i,j\le n}\bigg((-1)^{i-j} q^{b_{ij}} \sum_{r\in\mathbb{Z}} (-1)^r y_i^{(2n-1)r-i+1} q^{(2n-1)\kappa\binom{r}{2}} \times \Big((y_i q^{\kappa r})^{j-1} - (y_i q^{\kappa r})^{2n-j}\Big)\bigg), \tag{3.23}$$

where

$$y_i = q^{\frac{1}{2}(\kappa+1)-i} \quad\text{and}\quad b_{ij} := j^2 - i^2 + \frac{1}{2}(i-j)(\kappa+3). \tag{3.24}$$

Again the factor $(-1)^{i-j}q^{b_{ij}}$ does not contribute and application of (3.13) gives

$$\mathscr{M} = \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}(y_i q^{\kappa r}) \prod_{i=1}^{n} (-1)^{r_i} y_i^{(2n-1)r_i-i+1} q^{(2n-1)\kappa\binom{r_i}{2}}.$$

To complete the proof we apply the following variant of the $\mathrm{B}_n^{(1)}$ Macdonald identity[2]

$$\sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{B}}(xq^r) \prod_{i=1}^{n} (-1)^{r_i} x_i^{(2n-1)r_i-i+1} q^{(2n-1)\binom{r_i}{2}} = 2(q)_\infty^n \prod_{i=1}^{n} \theta(x_i;q) \prod_{1\le i<j\le n} \theta(x_i/x_j, x_i x_j; q), \tag{3.25}$$

with $(q,x) \mapsto (q^\kappa, y)$.

Again (1.3a) can be understood representation-theoretically, but this time the relevant Kac–Moody algebra is $\mathrm{A}_{2n}^{(2)}$. According to [8, Lemma 2.3] the right-hand side of (3.21) with $\hat{x}$ interpreted not as $\hat{x} = (-x_1,\dots,-x_{n-1},-1)$ but as

$$\hat{x}_i = \mathrm{e}^{-\alpha_0-\dots-\alpha_{n-i}} \quad (1\le i\le n)$$

[2]The actual $\mathrm{B}_n^{(1)}$ Macdonald identity has the restriction $|r| \equiv 0 \pmod 2$ in the sum over $r\in\mathbb{Z}^n$, which eliminates the factor 2 on the right. To prove the form used here it suffices to take the $a_1,\dots,a_{2n-1}\to 0$ and $a_{2n}\to -1$ limit in Gustafson's multiple ${}_6\psi_6$ summation for the affine root system $\mathrm{A}_{2n-1}^{(2)}$, see [22].

and $q$ as

$$q = \mathrm{e}^{-2\alpha_0-\cdots-2\alpha_{n-1}-\alpha_n} \tag{3.26}$$

is the $\mathrm{A}_{2n}^{(2)}$ character

$$\mathrm{e}^{-m\Lambda_n}\operatorname{ch} V(m\Lambda_n).$$

The substitution (3.22) corresponds to

$$\mathrm{e}^{\alpha_0} \mapsto -1 \quad \text{and} \quad \mathrm{e}^{\alpha_i} \mapsto q \;\; (1 \le i \le n). \tag{3.27}$$

Denoting this by $F$, we have the general specialisation formula

$$\begin{aligned} F\big(\mathrm{e}^{-\Lambda}\operatorname{ch} V(\Lambda)\big) &= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q)_\infty^n}\prod_{i=1}^n \theta\big(q^{\lambda_0-\lambda_i+i};q^\kappa\big) \\ &\quad\times \prod_{1\le i<j\le n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j-i-j+2n+1};q^\kappa\big), \end{aligned} \tag{3.28}$$

where $\kappa = 2n+2\lambda_0+1$ and

$$\Lambda = 2\lambda_n\Lambda_0 + (\lambda_{n-1}-\lambda_n)\Lambda_1 + \cdots + (\lambda_1-\lambda_2)\Lambda_{n-1} + (\lambda_0-\lambda_1)\Lambda_n$$

for $\lambda = (\lambda_0,\lambda_1,\dots,\lambda_n)$ a partition. For $\lambda = (m,0^n)$ (so that $\Lambda = m\Lambda_n$) this is in accordance with (1.3a).

3.4. **Proof of** (1.3b)**.** In (3.18) we set $x_n = q^{1/2}$ so that

$$\begin{aligned} &\sum_{\substack{\lambda \\ \lambda_1\le m}} q^{|\lambda|}P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-1}^{\pm},q^{1/2};q\big) \\ &\quad= \frac{1}{(q)_\infty^{n-1}(q^2)_\infty\prod_{i=1}^{n-1}(q^{3/2}x_i^{\pm},qx_i^{\pm 2})_\infty\prod_{1\le i<j\le n-1}(qx_i^{\pm}x_j^{\pm})_\infty} \\ &\qquad\times \sum_{r_1,\dots,r_{n-1}=-\infty}^{\infty}\sum_{r_n=0}^{\infty}\frac{\Delta_{\mathrm{C}}(\hat{x}q^r)}{\Delta_{\mathrm{C}}(\hat{x})}\prod_{i=1}^n(-1)^{r_i}\hat{x}_i^{\kappa r_i}q^{\frac{1}{2}\kappa r_i^2-nr_i}, \end{aligned}$$

where $\kappa = 2m+2n+1$ and $\hat{x} = (x_1,\dots,x_{n-1},q^{1/2})$. The $r_n$-dependent part of the summand is

$$(-1)^{r_n}q^{\kappa\binom{r_n+1}{2}-nr_n}\frac{1-q^{2r_n+1}}{1-q}\prod_{i=1}^{n-1}\frac{x_iq^{r_i}-q^{r_n+1/2}}{x_i-q^{1/2}}\cdot\frac{x_iq^{r_n+r_i+1/2}-1}{x_iq^{1/2}-1},$$

which is readily checked to be invariant under the substitution $r_n \mapsto -r_n-1$. Hence

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\le m}} & q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-1}^{\pm},q^{1/2};q\big)\\ &= \frac{1}{2(q)_\infty^n\prod_{i=1}^{n-1}(-1)\theta(q^{1/2}x_i,x_i^2;q)\prod_{1\le i<j\le n-1}\theta(x_i/x_j,x_ix_j;q)}\\ &\qquad\times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(\hat{x}q^r)\prod_{i=1}^n(-1)^{r_i}\hat{x}_i^{\kappa r_i-i}q^{\frac12\kappa r_i^2-nr_i+\frac12}.\end{aligned}$$

Our next step is to replace $x_i\mapsto x_{n-i+1}$ and $r_i\mapsto r_{n-i+1}$. By $\theta(x;q)=-x\theta(x^{-1};q)$ and (3.20), this leads to

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\le m}} & q^{|\lambda|} P'_{2\lambda}\big(q^{1/2},x_2^{\pm},\dots,x_n^{\pm};q\big)\\ &= \frac{1}{(q)_\infty^n\prod_{i=1}^{n}\theta(-q^{1/2}\hat{x}_i;q)\theta(\hat{x}_i^2;q^2)\prod_{1\le i<j\le n}\theta(\hat{x}_i/\hat{x}_j,\hat{x}_i\hat{x}_j;q)}\\ &\qquad\times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(\hat{x}q^r)\prod_{i=1}^n(-1)^{r_i}\hat{x}_i^{\kappa r_i-i+1}q^{\frac12\kappa r_i^2-nr_i},\end{aligned}\tag{3.29}$$

where now $\hat{x}=(q^{1/2},x_2,\dots,x_n)$. Again we are at the point where we can specialise, letting

$$q\mapsto q^{2n-1},\qquad x_i\mapsto q^{n-i+1/2}=:\hat{x}_i\ \ (1\le i\le n).\tag{3.30}$$

This is consistent, since $x_1=q^{1/2}\mapsto q^{n-1/2}$. By

$$\begin{aligned}(q^{2n-1};q^{2n-1})_\infty^n&\prod_{i=1}^n\theta(-q^{2n-i};q^{2n-1})\theta(q^{2n-2i+1};q^{4n-2})\\ &\times\prod_{1\le i<j\le n}\theta(q^{j-i},q^{2n-i-j+1};q^{2n-1})=2(q)_\infty^n\end{aligned}$$

this gives rise to

$$\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{2|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-1}\big)=\frac{\mathscr{M}}{2(q)_\infty^n},$$

where

$$\mathscr{M}:=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{C}}(\hat{x}q^{(2n-1)r})\prod_{i=1}^n(-1)^{r_i}\hat{x}_i^{\kappa r_i-i+1}q^{\frac12(2n-1)\kappa r_i^2-(2n-1)nr_i}.$$

Expressing $\mathscr{M}$ in determinantal form using (3.10) yields

$$\mathscr{M} = \det_{1\le i,j\le n}\biggl(\sum_{r\in\mathbb{Z}}(-1)^r\hat{x}_i^{\kappa r-i+1}q^{\frac{1}{2}(2n-1)\kappa r^2-(2n-1)nr} \\ \times\Bigl(\bigl(\hat{x}_i q^{(2n-1)r}\bigr)^{j-1}-\bigl(\hat{x}_i q^{(2n-1)r}\bigr)^{2n-j+1}\Bigr)\biggr).$$

We now replace $(i,j)\mapsto(j,i)$ and, viewing the resulting determinant as of the form $\det\bigl(\sum_r u_{ij;r}-\sum_r v_{ij;r}\bigr)$, we change the summation index $r\mapsto -r$ in the sum over $u_{ij;r}$. The expression for $\mathscr{M}$ we obtain is exactly (3.23) except that $(-1)^{i-j}q^{b_{ij}}$ is replaced by $q^{c_{ij}}$ and $y_i$ is given by $q^{n-i+1}$ instead of $q^{(\kappa+1)/2-i}$. Following the previous proof results in (1.3b).

To interpret (1.3b) in terms of $\mathrm{A}_{2n}^{(2)}$, we note that by [8, Lemma 2.2] the right-hand side of (3.29) in which $\hat{x}$ is interpreted as

$$\hat{x}_i=-q^{1/2}\,\mathrm{e}^{\alpha_0+\cdots+\alpha_{i-1}}\quad(1\le i\le n)$$

(and $q$ again as (3.26)) corresponds to the $\mathrm{A}_{2n}^{(2)}$ character

$$\mathrm{e}^{-2m\Lambda_0}\,\mathrm{ch}\,V(2m\Lambda_0).$$

The specialisation (3.30) is then again consistent with (3.27). From (3.28) with $\lambda=(m^{n+1})$ the first product-form on the right of (1.3b) immediately follows.

By level-rank duality we can also identify (1.3b) as a specialisation of the $\mathrm{A}_{2m}^{(2)}$ character $\mathrm{e}^{-2n\Lambda_0}\,\mathrm{ch}\,V(2n\Lambda_0)$.

3.5. **Proof of (1.5).** Our final proof is the most complicated of the four. Once again we iterate (3.6), but now both $x_{n-1}$ and $x_n$ remain unpaired:

$$\lim_{y_1\to x_1^{-1}}\cdots\lim_{y_{n-2}\to x_{n-2}^{-1}} L_m^{(0)}(x_1,y_1,\dots,x_{n-2},y_{n-2},x_{n-1},x_n) \\ = L_m^{(n-2)}(x_1,\dots,x_n).$$

Accordingly, if we replace $x\mapsto(x_1,y_1,\dots,x_{n-2},y_{n-2},x_{n-1},x_n)$ in (3.3) (thereby changing the rank from $n$ to $2n-2$) and take the $y_i\to x_i^{-1}$ limit for $1 \le i \le n-2$, we obtain

$$\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-2}^{\pm},x_{n-1},x_n;q\big)\\
&\quad=\frac{1}{(q)_\infty^{n-2}(qx_{n-1}^2,qx_{n-1}x_n,qx_n^2)_\infty}\\
&\qquad\times\frac{1}{\prod_{i=1}^{n-2}(qx_i^{\pm 2},qx_i^{\pm}x_{n-1},qx_i^{\pm}x_n)_\infty\prod_{1\le i<j\le n-2}(qx_i^{\pm}x_j^{\pm})_\infty}\\
&\qquad\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\bigg(\frac{x_i^{\kappa}}{x_{n-1}x_n}\bigg)^{r_i} q^{\frac{1}{2}\kappa r_i^2-(n-1)r_i}\frac{(x_ix_{n-1},x_ix_n)_{r_i}}{(qx_i/x_{n-1},qx_i/x_n)_{r_i}},
\end{aligned}$$

where $\kappa=2m+2n$. It is important to note that the summand vanishes unless $r_{n-1}$ and $r_n$ are both nonnegative. Next we let $(x_{n-1},x_n)$ tend to $(q^{1/2},1)$ using

$$\lim_{(x_{n-1},x_n)\to(q^{1/2},1)}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\frac{(x_ix_{n-1},x_ix_n)_{r_i}}{(qx_i/x_{n-1},qx_i/x_n)_{r_i}}=\phi_{r_n}\frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})},$$

with $\phi_r$ as in (3.19) and $\hat{x}:=(-x_1,\dots,-x_{n-2},-q^{1/2},-1)$. Hence

$$\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-2}^{\pm},q^{1/2},1;q\big)\\
&\quad=\frac{1}{(q)_\infty^{n-1}(q^{3/2};q^{1/2})_\infty\prod_{i=1}^{n-2}(qx_i^{\pm};q^{1/2})_\infty(qx_i^{\pm 2})_\infty\prod_{1\le i<j\le n-2}(qx_i^{\pm}x_j^{\pm})_\infty}\\
&\qquad\times\sum_{r_1,\dots,r_{n-2}=-\infty}^{\infty}\ \sum_{r_{n-1},r_n=0}^{\infty}\phi_{r_n}\frac{\Delta_{\mathrm{B}}(\hat{x}q^r)}{\Delta_{\mathrm{B}}(\hat{x})}\prod_{i=1}^n\hat{x}_i^{\kappa r_i}q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i}.
\end{aligned}$$

Since the summand (without the factor $\phi_{r_n}$) is invariant under the variable change $r_n\mapsto -r_n$ as well as the change $r_{n-1}\mapsto -r_{n-1}-1$, we can rewrite this as

$$\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\le m}} q^{|\lambda|} P'_{2\lambda}\big(x_1^{\pm},\dots,x_{n-2}^{\pm},q^{1/2},1;q\big)\\
&\quad=\frac{1}{(q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty\prod_{i=1}^{n}\theta(\hat{x}_i;q^{1/2})\prod_{1\le i<j\le n}\theta(\hat{x}_i/\hat{x}_j,\hat{x}_i\hat{x}_j)}\\
&\qquad\times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(\hat{x}q^r)\prod_{i=1}^n\hat{x}_i^{\kappa r_i-i+1}q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i},
\end{aligned}$$

where, once again, we have used (3.20) to clean up the infinite products. Before we can carry out the usual specialisation we need to relabel $x_1,\dots,x_{n-2}$ as $x_2,\dots,x_{n-1}$ and, accordingly, we redefine $\hat{x}$ as $(-q^{1/2},-x_2,\dots,-x_{n-1},-1)$.

Then

$$\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{|\lambda|}P'_{2\lambda}\big(q^{1/2},x_2^{\pm},\dots,x_{n-1}^{\pm},1;q\big) \tag{3.31}$$

$$= \frac{1}{(q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty\prod_{i=1}^n\theta(\hat{x}_i;q^{1/2})\prod_{1\leq i<j\leq n}\theta(\hat{x}_i/\hat{x}_j,\hat{x}_i\hat{x}_j)}$$

$$\times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(\hat{x}q^r)\prod_{i=1}^n\hat{x}_i^{\kappa r_i-i+1}q^{\frac{1}{2}\kappa r_i^2-\frac{1}{2}(2n-1)r_i},$$

for $n\geq 2$. We are now ready to make the substitutions

$$q\mapsto q^{2n-2},\qquad x_i\mapsto q^{n-i}\ \ (2\leq i\leq n-1),$$

so that $\hat{x}_i:=-q^{n-i}$ for $1\leq i\leq n$. By

$$(q^{2n-2};q^{2n-2})_\infty^{n-1}(q^{n-1};q^{n-1})_\infty\prod_{i=1}^n\theta(-q^{n-i};q^{n-1})$$
$$\times\prod_{1\leq i<j\leq n}\theta(q^{j-i},q^{2n-i-j};q^{2n-2})=4(q^2;q^2)_\infty(q)_\infty^{n-1}$$

and (2.3) this results in

$$\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{2|\lambda|}P_{2\lambda}\big(1,q,q^2,\dots;q^{2n-3}\big)=\frac{\mathscr{M}}{4(q^2;q^2)_\infty(q)_\infty^{n-1}},$$

with $\mathscr{M}$ given by

$$\mathscr{M}:=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(\hat{x}q^{2(n-1)r})\prod_{i=1}^n\hat{x}_i^{\kappa r_i-i+1}q^{(n-1)\kappa r_i^2-(n-1)(2n-1)r_i}.$$

By the $\mathrm{B}_n$ determinant (3.13),

$$\mathscr{M}=\det_{1\leq i,j\leq n}\bigg(\sum_{r\in\mathbb{Z}}\hat{x}_i^{\kappa r-i+1}q^{(n-1)\kappa r^2-(n-1)(2n-1)r}$$
$$\times\Big(\big(\hat{x}_iq^{2(n-1)r}\big)^{j-1}-\big(\hat{x}_iq^{2(n-1)r}\big)^{2n-j}\Big)\bigg).$$

By the same substitutions that transformed (3.11) into (3.12) we obtain

$$\mathscr{M}=\det_{1\leq i,j\leq n}\bigg((-1)^{i-j}q^{b_{ij}}\sum_{r\in\mathbb{Z}}y_i^{2(n-1)r-i+1}q^{2(n-1)\kappa\binom{r}{2}}$$
$$\times\Big(\big(y_iq^{\kappa r}\big)^{j-1}+\big(y_iq^{\kappa r}\big)^{2n-j-1}\Big)\bigg),$$

where $y_i$ and $b_{ij}$ are as in (3.24). Recalling the Weyl denominator formula for $\mathrm{D}_n$ [29]

$$\frac{1}{2}\det_{1\le i,j\le n}\bigl(x_i^{j-1}+x_i^{2n-j-1}\bigr)=\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1)=:\Delta_{\mathrm{D}}(x)$$

we can rewrite $\mathscr{M}$ in the form

$$\mathscr{M}=2\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(xq^r)\prod_{i=1}^n y_i^{2(n-1)r_i-i+1}q^{2(n-1)\kappa\binom{r_i}{2}}.$$

Taking the $a_1,\dots,a_{2n-2}\to 0$, $a_{2n-1}\to 1$ and $a_{2n}\to -1$ limit in Gustafson's multiple ${}_6\psi_6$ summation for the affine root system $\mathrm{A}_{2n-1}^{(2)}$ [22] leads to the following variant of the $\mathrm{D}_n^{(1)}$ Macdonald identity[3]

$$\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(xq^r)\prod_{i=1}^n x_i^{2(n-1)r_i-i+1}q^{2(n-1)\binom{r_i}{2}}=2(q)_\infty^n\prod_{1\le i<j\le n}\theta(x_i/x_j,x_ix_j;q).$$

This implies the claimed product form for $\mathscr{M}$ and completes our proof.

Again (1.5) has a simple representation-theoretic interpretation. According to [8, Lemma 2.4] the right-hand side of (3.31) in which $\hat{x}$ is interpreted not as $\hat{x}=(-q^{1/2},-x_1,\dots,-x_{n-1},-1)$ but as

$$\hat{x}_i=\mathrm{e}^{-\alpha_i-\dots-\alpha_n}\quad(1\le i\le n)$$

and $q$ as

$$q=\mathrm{e}^{-2\alpha_0-\dots-2\alpha_n}$$

yields the $\mathrm{D}_{n+1}^{(2)}$ character

$$\mathrm{e}^{-2m\Lambda_0}\,\mathrm{ch}\,V(2m\Lambda_0).$$

The specialisation (3.5) then corresponds to

$$\mathrm{e}^{\alpha_0},\ \mathrm{e}^{\alpha_n}\mapsto -1\quad\text{and}\quad \mathrm{e}^{\alpha_i}\mapsto q\ \ (2\le i\le n-1).$$

Denoting this by $F$, we have

$$\begin{aligned}F\bigl(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda)\bigr)&=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q^2;q^2)_\infty(q)_\infty^{n-1}}\\&\qquad\times\prod_{1\le i<j\le n}\theta\bigl(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j-i-j+2n+1};q^\kappa\bigr),\end{aligned}$$

where $\kappa=2n+2\lambda_0$ and

$$\Lambda=2(\lambda_0-\lambda_1)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\dots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+2\lambda_n\Lambda_n,$$

for $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_n)$ a partition or half-partition (i.e., all $\lambda_i\in\mathbb{Z}+1/2$). For $\lambda=(m,0^n)$ this agrees with (1.5).

[3]As in the $\mathrm{B}_n^{(1)}$ case, the actual $\mathrm{D}_n^{(1)}$ Macdonald identity contains the restriction $|r|\equiv 0 \pmod 2$ on the sum over $r$.

## 4. Proof of Theorem 1.5

For $k$ and $m$ integers such that $0 \le k \le m$ we write the near-rectangular partition $(\underbrace{m,\dots,m}_{r \text{ times}},k)$ as $(m^r,k)$.

**Theorem 4.1** ($\mathrm{A}_{n-1}^{(1)}$ RR and AG identities)**.** *Let $m$ and $n$ be positive integers and $k$ a nonnegative integer not exceeding $m$. Then*

$$\begin{aligned}(4.1)\quad &\lim_{r\to\infty} q^{-m\binom{r}{2}-kr} Q_{(m^r,k)}(1,q,q^2,\dots;q^n)\\ &\qquad = \frac{(q^n;q^n)_\infty (q^\kappa;q^\kappa)_\infty^{n-1}}{(q)_\infty^n} \prod_{i=1}^{n-1} \theta(q^{i+k};q^\kappa) \prod_{1\le i<j\le n-1} \theta(q^{j-i};q^\kappa),\end{aligned}$$

*where $\kappa = m+n$.*

For $k=0$ (or $k=m$) this yields Theorem 1.5. Before we give a proof of the above theorem we remark that by a similar calculation it also follos that

$$\begin{aligned}&\lim_{r\to\infty} q^{-m\binom{r+1}{2}} Q_{(k,m^r)}(1,q,q^2,\dots;q^n)\\ &\qquad = \begin{bmatrix} k-m+n-1 \\ n-1 \end{bmatrix}_q \frac{(q^n;q^n)_\infty (q^\kappa;q^\kappa)_\infty^{n-1}}{(q)_\infty^n} \prod_{1\le i<j\le n} \theta(q^{j-i};q^\kappa),\end{aligned}$$

for $k \ge m$.

*Proof of Theorem 4.1.* The following identity for the modified Hall–Littlewood polynomials indexed by near-rectangular partitions is a special case of [8, Corollary 3.2]:

$$\begin{aligned}Q'_{(m^r,k)}(x;q) = (q)_r (q)_1 &\sum_{\substack{u\in\mathbb{Z}_+^n \\ |u|=r+1}} \sum_{\substack{v\in\mathbb{Z}_+^n \\ |v|=r}} \prod_{i=1}^n x_i^{ku_i+(m-k)v_i} q^{k\binom{u_i}{2}+(m-k)\binom{v_i}{2}}\\ &\times \prod_{i,j=1}^n \frac{(qx_i/x_j)_{u_i-u_j}}{(qx_i/x_j)_{u_i-v_j}} \cdot \frac{(qx_i/x_j)_{v_i-v_j}}{(qx_i/x_j)_{v_i}}.\end{aligned}$$

It suffices to compute the limit on the left-hand side of (4.1) for $r$ a multiple of $n$. Hence we replace $r$ by $nr$ in the above expression, and then shift $u_i \mapsto u_i + r$ and $v_i \mapsto v_i + r$ for all $1 \le i \le n$, to obtain

$$\begin{aligned}Q'_{(m^{nr},k)}(x;q) &= (x_1\cdots x_n)^{mr} q^{mn\binom{r}{2}+kr} (q)_{nr}(q)_1\\ &\quad\times \sum_{\substack{u\in\mathbb{Z}^n \\ |u|=1}} \sum_{\substack{v\in\mathbb{Z}^n \\ |v|=0}} \prod_{i=1}^n x_i^{ku_i+(m-k)v_i} q^{k\binom{u_i}{2}+(m-k)\binom{v_i}{2}}\\ &\quad\times \prod_{i,j=1}^n \frac{(qx_i/x_j)_{u_i-u_j}}{(qx_i/x_j)_{u_i-v_j}} \cdot \frac{(qx_i/x_j)_{v_i-v_j}}{(qx_i/x_j)_{r+v_i}}.\end{aligned}$$

Since the summand vanishes unless $u_i \geq v_i$ for all $i$ and $|u| = |v| + 1$ it follows that $u = v + \epsilon_\ell$, for some $\ell = 1, \dots, n$, where $(\epsilon_\ell)_i = \delta_{\ell i}$. Hence

$$\begin{aligned} Q'_{(m^{nr},k)}(x;q) &= (x_1 \cdots x_n)^{mr} q^{mn\binom{r}{2}+kr} (q)_{nr} \\ &\quad \times \sum_{\substack{v \in \mathbb{Z}^n \\ |v|=0}} \prod_{i=1}^n x_i^{mv_i} q^{m\binom{v_i}{2}} \prod_{i,j=1}^n \frac{(qx_i/x_j)_{v_i-v_j}}{(qx_i/x_j)_{r+v_i}} \\ &\quad \times \sum_{\ell=1}^n \big(x_\ell q^{v_\ell}\big)^k \prod_{\substack{i=1 \\ i \neq k}}^n \frac{1}{1 - q^{v_i - v_\ell} x_i / x_\ell}. \end{aligned}$$

Next we use

$$\prod_{i,j=1}^n (qx_i/x_j)_{v_i-v_j} = \frac{\Delta(xq^v)}{\Delta(x)} (-1)^{(n-1)|v|} q^{-\binom{|v|}{2}} \prod_{i=1}^n x_i^{nv_i - |v|} q^{n\binom{v_i}{2}+(i-1)v_i},$$

where $\Delta(x) := \prod_{1 \leq i < j \leq n} (1 - x_i/x_j)$, and

$$\sum_{\ell=1}^n x_\ell^k \prod_{\substack{i=1 \\ i \neq k}}^n \frac{1}{1 - x_i/x_\ell} = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} = h_k(x) = s_{(k)}(x),$$

where $h_k$ and $s_\lambda$ are the complete symmetric and Schur function, respectively. Thus

$$\begin{aligned} Q'_{(m^{nr},k)}(x;q) &= (x_1 \cdots x_n)^{mr} q^{mn\binom{r}{2}+kr} (q)_{nr} \\ &\quad \times \sum_{\substack{v \in \mathbb{Z}^n \\ |v|=0}} s_{(k)}(xq^v) \frac{\Delta(xq^v)}{\Delta(x)} \prod_{i=1}^n x_i^{\kappa v_i} q^{\frac{1}{2}\kappa v_i^2 + i v_i} \prod_{i,j=1}^n \frac{1}{(qx_i/x_j)_{r+v_i}}, \end{aligned}$$

where $\kappa := m + n$. Note that the summand vanishes unless $v_i \geq -r$ for all $i$. This implies the limit

$$\begin{aligned} &\lim_{r \to \infty} q^{-mn\binom{r}{2}-kr} \frac{Q'_{(m^{nr},k)}(x;q)}{(x_1 \cdots x_n)^{mr}} \\ &\quad = \frac{1}{(q)_\infty^{n-1} \prod_{1 \leq i < j \leq n} \theta(x_i/x_j;q)} \sum_{\substack{v \in \mathbb{Z}^n \\ |v|=0}} s_{(k)}(xq^v) \Delta(xq^v) \prod_{i=1}^n x_i^{\kappa v_i} q^{\frac{1}{2}\kappa v_i^2 + i v_i}. \end{aligned}$$

The expression on the right is exactly the Weyl–Kac formula for the level-$m$ $A_{n-1}^{(1)}$ character [26]

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda), \qquad \Lambda = (m-k)\Lambda_0 + k\Lambda_1,$$

provided we identify

$$q = \mathrm{e}^{-\alpha_0 - \alpha_1 - \cdots - \alpha_{n-1}} \quad \text{and} \quad x_i/x_{i+1} = \mathrm{e}^{-\alpha_i} \;\; (1 \leq i \leq n-1).$$

Hence

$$\lim_{r\to\infty} q^{-mn\binom{r}{2}-kr}\,\frac{Q'_{(m^{nr},k)}(x;q)}{(x_1\cdots x_n)^{mr}} = \mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda),$$

with $\Lambda$ as above. For $m=1$ and $k=0$ this was obtained in [28] by more elementary means. The simultaneous substitutions $q\mapsto q^n$ and $x_i\mapsto q^{n-i}$ correspond to the principal specialisation (3.15). From (3.16) we can then read off the product form claimed in (4.1). $\square$

## References


1. G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. USA **71** (1974), 4082–4085.
2. G. E. Andrews, *Problems and prospects for basic hypergeometric functions*, in *Theory and Application of Special Functions*, pp. 191–224, Math. Res. Center, Univ. Wisconsin, 35, Academic Press, New York, 1975.
3. G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading, Massachusetts, 1976.
4. G. E. Andrews, R. J. Baxter and P. J. Forrester, *Eight-vertex SOS model and generalized Rogers–Ramanujan-type identities*, J. Statist. Phys. **35** (1984), 193–266.
5. G. E. Andrews, A. Schilling and S. O. Warnaar, *An $A_2$ Bailey lemma and Rogers–Ramanujan-type identities*, J. Amer. Math. Soc. **12** (1999), 677–702.
6. C. Armond and O. T. Dasbach, *Rogers–Ramanujan identities and the head and tail of the colored Jones polynomial*, arXiv:1106.3948.
7. R. Askey and M. E. H. Ismail, *A generalization of ultraspherical polynomials*, in *Studies in pure mathematics*, pp. 55–78, Birkhäuser, Basel, 1983.
8. N. Bartlett and S. O. Warnaar, *Hall–Littlewood polynomials and characters of affine Lie algebras*, arXiv:1304.1602.
9. R. J. Baxter, *Rogers–Ramanujan identities in the hard hexagon model*, J. Statist. Phys. 26 (1981), 427–452.
10. A. Berkovich and B. M. McCoy, *Rogers–Ramanujan identities: a century of progress from mathematics to physics*, Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998), Doc. Math. 1998, Extra Vol. III, 163–172.
11. D. M. Bressoud, *An analytic generalization of the Rogers–Ramanujan identities with interpretation*, Quart. J. Maths. Oxford (2) **31** (1980), 385–399.
12. D. M. Bressoud, *On partitions, orthogonal polynomials and the expansion of certain infinite products*, Proc. London Math. Soc. (3) **42** (1981), 478–500.
13. K. Bringmann, K. Ono and R. Rhoades, *Eulerian series as modular forms*, J. Amer. Math. Soc. **21** (2008), 1085–1104.
14. C. Bruschek, H. Mourtada and J. Schepers, *Arc spaces and the Rogers–Ramanujan identities*, Ramanujan J. **30** (2013), 9–38.
15. J. L. Dupont and C.-H. Sah, *Dilogarithm identities in conformal field theory and group homology*, Comm. Math. Phys. **161** (1994), 265–282.
16. I. Cherednik and B. Feigin, *Rogers–Ramanujan type identities and Nil-DAHA*, Adv. in Math. to appear, arXiv:1209.1978.
17. B. Feigin, O. Foda Ond T. Welsh, *Andrews–Gordon type identities from combinations of Virasoro characters*, Ramanujan J. **17** (2008), 33–52.
18. J. Fulman, *The Rogers–Ramanujan identities, the finite general linear groups, and the Hall–Littlewood polynomials*, Proc. Amer. Math. Soc. **128** (2000), 17–25.
19. J. Fulman, *A probabilistic proof of the Rogers–Ramanujan identities*, Bull. London Math. Soc. **33** (2001), 397–407.
20. K. Garrett, M. E. H. Ismail and D. Stanton, *Variants of the Rogers–Ramanujan identities*, Adv. in Appl. Math. **23** (1999), 274–299.

21. B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.
22. R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very-well-poised on semisimple Lie algebras*, in *Ramanujan International Symposium on Analysis*, pp. 187–224, Macmillan of India, New Delhi, 1989.
23. K. Hikami, *Volume conjecture and asymptotic expansion of q-series*, Experiment. Math. **12** (2003), 319–337.
24. K. Hikami, *q-Series and L-functions related to half-derivatives of the Andrews–Gordon identity*, Ramanujan J. **11** (2006), 175–197.
25. V. G. Kac, *Infinite-dimensional Lie algebras, and the Dedekind $\eta$-function*, Funkcional. Anal. i Priložen **8** (1974), 77–78.
26. V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd Edition, Cambridge University Press, Cambridge, 1990.
27. R. Kedem, B. M. McCoy and E. Melzer, *The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in $(1+1)$-dimensional quantum field theory*, in *Recent Progress in Statistical Mechanics and Quantum Field Theory*, pp. 195–219, World Sci. Publ., River Edge, NJ, 1995.
28. A. N. Kirillov, *New combinatorial formula for modified Hall–Littlewood polynomials*, in *q-Series from a Contemporary Perspective*, pp. 283–333, Contemp. Math., 254, AMS, Providence, RI, 2000.
29. C. Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** (1999), Art. B42q, 41pp.
30. A. Lascoux and M.-P. Schützenberger, *Sur une conjecture de H. O. Foulkes*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), A323–A324.
31. J. Lepowsky and S. Milne, *Lie algebras and classical partition identities*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 578–579.
32. J. Lepowsky and S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. in Math. **29** (1978), 15–59.
33. J. Lepowsky and R. L. Wilson, *The Rogers–Ramanujan identities: Lie theoretic interpretation and proof*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), 699–701.
34. J. Lepowsky and R. L. Wilson, *A new family of algebras underlying the Rogers–Ramanujan identities and generalizations*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), 7254–7258.
35. J. Lepowsky and R. L. Wilson, *A Lie theoretic interpretation and proof of the Rogers–Ramanujan identities*, Adv. in Math. **45** (1982), 21–72.
36. J. Lepowsky and R. L. Wilson, *The structure of standard modules. I. Universal algebras and the Rogers–Ramanujan identities*, Inv. Math. **77** (1984), 199–290.
37. I. G. Macdonald, *Affine root systems and Dedekind's $\eta$-function*, Inv. Math. **15** (1972), 91–143.
38. I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, New York, 1995.
39. P. A. MacMahon, *Combinatory Analysis*, Vol. I and II, Chelsea Publishing Co., New York 1960.
40. S. C. Milne, *The $C_l$ Rogers–Selberg identity*, SIAM J. Math. Anal. **25** (1994), 571–595.
41. K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, in *Current Developments in Mathematics, 2008*, pp. 347–454, Int. Press, Somerville, MA, 2009.
42. B. Richmond and G. Szekeres, *Some formulas related to dilogarithms, the zeta function and the Andrews–Gordon identities*, J. Austral. Math. Soc. Ser. A **31** (1981), 362–373.
43. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

44. L. J. Rogers, *Proof of cerytain identities in combinatory analysis*, Proc. Camb. Phil. Soc. **19** (1919), 214–216.
45. I. Schur, *Ein Beitrag zur Additiven Zahlen theorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys. Math. Klasse (1917), 302–321.
46. A. Selberg, *Über einige arithmetische Identitäten*, Avh. Norske Vid.-Akad. Oslo. I. Mat.-Naturv. Kl. **8** (1936), 1–23.
47. J. R. Stembridge, *Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities*, Trans. Amer. Math. Soc. **319** (1990), 469–498.
48. G. N. Watson, *A new proof of the Rogers–Ramanujan identities*, J. London Math. Soc. **4**, 4–9.
49. S. O. Warnaar, *Hall–Littlewood functions and the $A_2$ Rogers–Ramanujan identities*, Adv. in Math. **200** (2006), 403–434.
50. S. O. Warnaar and W. Zudilin, *Dedekind's $\eta$-function and Rogers–Ramanujan identities*, Bull. Lond. Math. Soc. **44** (2012), 1–11.

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