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A CRITICAL ISING MODEL IN A MAGNETIC FIELD

S. OLE WARNAAR, BERNARD NIENHUIS and KATHERINE A. SEATON*
Instituut voor Theoretische Fysica, Universiteit van Amsterdam,
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Generalising ideas developed by Owczarek and Baxter and by Pasquier, we establish an equivalence between restricted solid-on-solid models and loop models. Application to the O(n) model leads to a new family of solvable A-D-E models. For the models related to the classical A_L algebra we present an off-critical extension and show that the A_3 model thus obtained can be viewed as a critical spin-1 Ising model in a magnetic field. For this Ising model we calculate the critical exponent $\delta = 15$ without the use of scaling relations.

1. Introduction

A few years ago, Zamolodchikov found an integrable field theory describing the scaling limit of the critical Ising model in a magnetic field [1]. In his paper, Zamolodchikov concluded:

> By the way this E_8 structure strongly suggests that particular integrable "interaction round face" lattice model (of the "restricted" type) associated with the integral weights of E_8 can be constructed whose scaling limit would describe the universality class of the critical Ising model in magnetic field.

In this paper we indeed present a solvable restricted solid-on-solid (RSOS) model describing the scaling limit of the Ising model in a field, though the states of this model are not based on E_8, but on A_3.

The setup of this paper is as follows. First we describe a new method to construct RSOS models defined by graphs starting from loop models. This loop-RSOS equivalence, which is a generalisation of ideas developed by Pasquier [2] and Owczarek and Baxter[3], holds irrespective of the solvability of the models. We then apply our method to the O(n) loop model. This leads to a new family of solvable critical RSOS models labelled by the Dynkin diagrams of the A-D-E Lie algebras. Following Roche [4], who independently obtained the same results, we name these new models *dilute A-D-E* models. For the dilute A model we then give an off-critical extension and show that some of the off-critical models break the Z_2 symmetry

*Present address: *Mathematics Department, University of Melbourne, Parkville Victoria 3052, Australia*

of the underlying Dynkin diagram. Finally, we argue that the A_3 model, in the appropriate regime, can be viewed as a critical spin-1 Ising model in a magnetic field.

2. Loop Models

We consider the partition function defined as the sum over all graphs G consisting of closed non-intersecting polygons (or loops) on a square lattice. An example of such a graph is shown in Fig. 1. Each loop has fugacity n and the nine possible vertices, shown in Fig. 2a, have Boltzmann weights $\rho_1, \ldots, \rho_9$. The partition function of the loop model reads

$$Z = \sum_G \rho_1^{n_1} \ldots \rho_9^{n_9} n^p \,, \tag{2.1}$$

where p is the total number of loops of G and n_i the number of vertices of type i.

If the Boltzmann weights $\rho_1, \ldots, \rho_7$ are all zero, only configurations that densely cover the entire lattice give a non-vanishing contribution to the partition function. Such loop models we call *dense*. This is opposed to the so-called *dilute* loop models, which allow for edges of the lattice to be unoccupied.

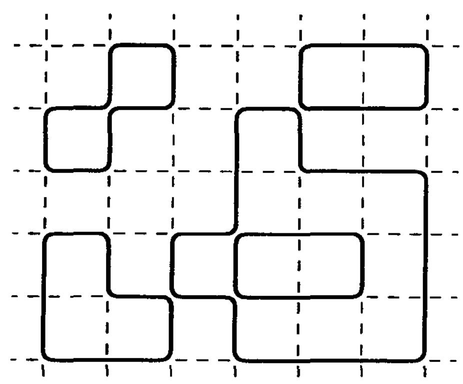


Fig. 1. A configuration of the loop model.

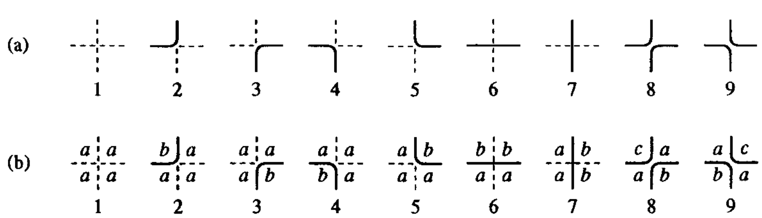


Fig. 2. (a) The vertices of the loop model. (b) The nine possible diagrams in the expansion of (3.3).

3. Restricted Solid-On-Solid Models

In this section we define a RSOS model on the square lattice and show how it is related to the loop model defined previously.

3.1. *Definition of the model*

Consider an arbitrary connected graph $\mathcal{G}$. Such a graph consists of L nodes, labelled by an integer *height* $a \in \{1, \dots, L\}$, and a number of bonds between the nodes. We do not allow for more than one bond between two nodes and for simplicity we only consider *simple* graphs, that is, graphs that have no bonds connecting a node to itself (a tadpole)[1]. Two nodes are called adjacent ($\sim$) on $\mathcal{G}$ if they are connected via a single bond.

We can represent $\mathcal{G}$ by an adjacency matrix A as follows

$$A_{a,b} = \begin{cases} 1 & a \sim b \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

The largest eigenvalue of A is denoted by Λ and the corresponding Perron-Frobenius vector by S.

With the above definitions, the Boltzmann weight of an elementary face of the RSOS model is given by

$$\begin{aligned} W\begin{pmatrix} d & c \\ a & b \end{pmatrix} &= \rho_1 \delta_{a,b,c,d} + \rho_2 \delta_{a,b,c} A_{a,d} + \rho_3 \delta_{a,c,d} A_{a,b} + \left(\frac{S_a}{S_b}\right)^{1/2} \rho_4 \delta_{b,c,d} A_{a,b} \\ &+ \left(\frac{S_c}{S_a}\right)^{1/2} \rho_5 \delta_{a,b,d} A_{a,c} + \rho_6 \delta_{a,b} \delta_{c,d} A_{a,c} + \rho_7 \delta_{a,d} \delta_{b,c} A_{a,b} \\ &+ \rho_8 \delta_{a,c} A_{a,b} A_{a,d} + \left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} \rho_9 \delta_{b,d} A_{b,c} A_{a,b}\,, \end{aligned} \tag{3.2}$$

where S_a is the a-th entry of S and a, b, c and d can take any of the L heights of the graph $\mathcal{G}$. The generalised Kronecker-δ is defined as $\delta_{i_1,\dots,i_m} \equiv \prod_{j=2}^{m} \delta_{i_1,i_j}$.

In analogy with the loop model, if $\rho_1 = \cdots = \rho_7 = 0$, we call the RSOS model dense, opposed to dilute RSOS models for which $\rho_1, \dots, \rho_7$ are not all zero. Dilute RSOS models allow for height configurations in which neighbouring sites of the lattice have equal height. Since we only consider simple graphs, the Boltzmann weight of such configurations vanishes for dense RSOS models.

3.2. *Loop-RSOS equivalence*

We now show that the partition function of the RSOS model can be mapped onto the loop model if we identify $\Lambda = n$.

[1]For a discussion of non-simple graphs, we refer to [6].

We start by substituting expression (3.2) into the partition function of the RSOS model, defined as

$$Z = \sum_{\text{heights}} \prod_{\text{faces}} W\begin{pmatrix} d & c \\ a & b \end{pmatrix} . \tag{3.3}$$

We then expand this into a sum of 9^N terms, where N is the number of faces of the lattice. A given term in the expansion has one of the nine terms of (3.2) for each elementary face of the lattice. These nine possible terms can be represented by the diagrams shown in Fig. 2b, in which the lines indicate local domain walls separating regions of different height that are adjacent on $\mathcal{G}$.

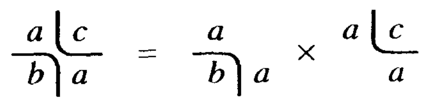
Due to the δ-functions, only configurations in which the local domain walls form global domain walls, separating regions of the lattice of different height, contribute to the partition function. Hence, the partition sum is given as the sum over all configurations G of domain walls and a sum over heights consistent with G

$$Z = \sum_G \rho_1^{n_1} \cdots \rho_9^{n_9} \sum_{\text{heights}} \prod_{a,b=1}^{L} \left(\frac{S_b}{S_a}\right)^{m_{ba}} . \tag{3.4}$$

Here m_{ba} is the total power of S_b/S_a arising from the vertices of type 4, 5 and 9, where we count the powers of S_b/S_a and S_a/S_b separately.

To avoid technical difficulties, we demand that all boundary sites of the lattice carry the same height. All domain walls then form closed polygons or loops.

For each configuration G of domain walls, the dependence of the Boltzmann weight on the heights can be factorised into separate contributions associated with each loop. For that purpose we make the following decomposition:



$$\left(\frac{S_b S_c}{S_a S_a}\right)^{1/2} \qquad \left(\frac{S_b}{S_a}\right)^{1/2} \qquad \left(\frac{S_c}{S_a}\right)^{1/2} .$$

If we call the height immediately inside (outside) a loop the *inner* (*outer*) height of the loop, we find that as a result of this factorisation, the total contribution to $m_{ab} - m_{ba}$ of a loop with inner height a and outer height b is always 1.

We can now start to sum over the inner height of loops that do not surround other loops. Let a be the inner and b the outer height of such a loop. We then have

$$\sum_{a\sim b} \frac{S_a}{S_b} = \sum_{a=1}^{L} A_{b,a} \frac{S_a}{S_b} = \Lambda \equiv n . \tag{3.5}$$

The result of this summation is twofold. First of all we see that all polygons not surrounding other polygons contribute a factor n. Secondly, we observe that the dependence on the outer height has disappeared. Hence, we can now sum over the height of regions immediately outside these polygons. Repeating this procedure from the inside out we find that completely summing out the heights yields a factor n^p. This proves the equivalence of the partition functions of the RSOS model (3.3) and the loop model (2.1).

4. Solvable Cases

In general the loop model (2.1) and hence the related RSOS models are not solvable. In the following we consider two cases for which the loop model satisfies the Yang-Baxter equation. This generalisation of the Yang-Baxter equation [7] to loop variables is treated in [8, 6].

4.1. *The Temperley-Lieb model*

The first solvable example is the Temperley-Lieb (TL) loop model [9]. For this dense loop model the loop-RSOS equivalence was first established independently by Pasquier [2] and Owczarek and Baxter [3]. The weights and fugacity of the model are given by

$$\rho_1 = \cdots = \rho_7 = 0 \quad \rho_8 = \frac{\sin(\lambda - u)}{\sin \lambda} \quad \rho_9 = \frac{\sin u}{\sin \lambda} \quad n = 2 \cos \lambda \,. \tag{4.1}$$

We note that for each adjacency graph $\mathcal{G}$ the related RSOS model provides a representation of the Temperley-Lieb algebra [10].

4.2. *The O(n) model*

As a second example we consider the O(n) model [11]. This loop model, related to the Izergin-Korepin model [12], is defined by

$$\begin{aligned}
\rho_1 &= (\sin 2\lambda \sin 3\lambda + \sin u \sin(3\lambda - u))/(\sin 2\lambda \sin 3\lambda) \\
\rho_2 = \rho_3 &= \sin(3\lambda - u)/\sin 3\lambda \\
\rho_4 = \rho_5 &= \epsilon_1 \sin u / \sin 3\lambda \\
\rho_6 = \rho_7 &= \epsilon_2 \sin u \sin(3\lambda - u)/(\sin 2\lambda \sin 3\lambda) \\
\rho_8 &= \sin(2\lambda - u) \sin(3\lambda - u)/(\sin 2\lambda \sin 3\lambda) \\
\rho_9 &= - \sin u \sin(\lambda - u)/(\sin 2\lambda \sin 3\lambda) \\
n &= -2 \cos 4\lambda \,,
\end{aligned} \tag{4.2}$$

where $\epsilon_1^2 = \epsilon_2^2 = 1$. The dilute RSOS models related to this loop model were found independently by Roche [4] and Warnaar *et al.* [13].

4.3. A-D-E *classification*

Both the TL and the O(n) loop model are critical for $n \leq 2$. Restricting ourselves to critical RSOS models, we are interested in connected simple graphs that have adjacency matrices with largest eigenvalue less or equal than two. All such graphs have been classified [14] and are given by the Dynkin diagrams of the classical and affine simply-laced Lie algebras. The Dynkin diagrams of these so-called A-D-E algebras are shown in Fig. 3. The A-D-E models based on the TL loop model are simply known as the A-D-E models. Following Roche [4] we term the A-D-E models based on the O(n) model dilute A-D-E models.

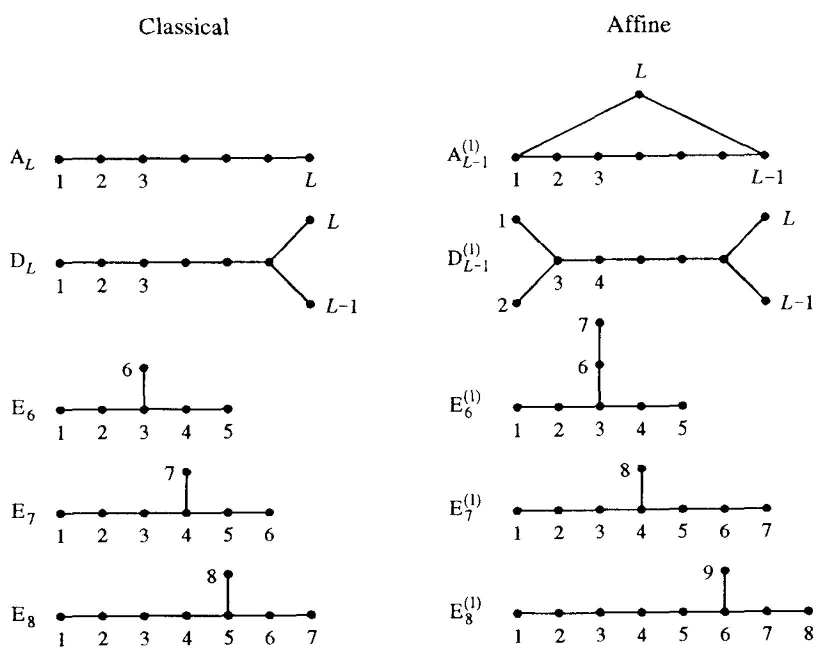


Fig. 3. Dynkin diagrams of the simply-laced Lie algebras.

5. The Dilute A Model

It is well-known that the critical A and D models admit an off-critical extension while remaining solvable [5, 15, 16, 17]. For the dilute A-D-E models, only the models based on the classical A algebras allow an off-critical extension. With the usual definition of the ϑ-functions [18]

$$\vartheta_1(u) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n})$$

$$\vartheta_4(u) = \prod_{n=1}^{\infty} (1 - 2p^{2n-1} \cos 2u + p^{4n-2})(1 - p^{2n}), \tag{5.1}$$

where we have suppressed the dependence on the *nome* p, $|p| < 1$, the weights of the dilute A model are given by[2]

$$W\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \frac{\vartheta_1(6\lambda - u)\vartheta_1(3\lambda + u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} - \left(\frac{S(a+1)}{S(a)}\frac{\vartheta_4(2a\lambda - 5\lambda)}{\vartheta_4(2a\lambda + \lambda)} + \frac{S(a-1)}{S(a)}\frac{\vartheta_4(2a\lambda + 5\lambda)}{\vartheta_4(2a\lambda - \lambda)}\right) \times \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a \end{pmatrix} = W\begin{pmatrix} a & a \\ a & a\pm 1 \end{pmatrix} = \frac{\vartheta_1(3\lambda - u)\vartheta_4(\pm 2a\lambda + \lambda - u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)}$$

$$W\begin{pmatrix} a & a \\ a\pm 1 & a \end{pmatrix} = W\begin{pmatrix} a & a\pm 1 \\ a & a \end{pmatrix} = \left(\frac{S(a\pm 1)}{S(a)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_4(\pm 2a\lambda - 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)}$$

$$W\begin{pmatrix} a & a\pm 1 \\ a & a\pm 1 \end{pmatrix} = W\begin{pmatrix} a\pm 1 & a\pm 1 \\ a & a \end{pmatrix} = \left(\frac{\vartheta_4(\pm 2a\lambda + 3\lambda)\vartheta_4(\pm 2a\lambda - \lambda)}{\vartheta_4^2(\pm 2a\lambda + \lambda)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a\mp 1 \end{pmatrix} = \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

$$W\begin{pmatrix} a & a\mp 1 \\ a\pm 1 & a \end{pmatrix} = \left(\frac{S(a-1)S(a+1)}{S^2(a)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

[2] Following our notation in [19], we have relabelled the states by $a \to L + 1 - a$ and replaced λ by $\frac{1}{2}\pi - \lambda$ in comparison with [13].

$$W\begin{pmatrix} a & a\pm1 \\ a\pm1 & a \end{pmatrix} = \frac{\vartheta_1(3\lambda-u)\vartheta_1(\pm4a\lambda+2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)} + \frac{S(a\pm1)}{S(a)}\frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)}$$

$$= \frac{\vartheta_1(3\lambda+u)\vartheta_1(\pm4a\lambda-4\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda-4\lambda)} + \left(\frac{S(a\mp1)}{S(a)}\frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)} - \frac{\vartheta_4(\pm2a\lambda-5\lambda)}{\vartheta_4(\pm2a\lambda+\lambda)}\right) \times \frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda-4\lambda)}$$

$$S(a) = (-)^a\frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)}\,. \tag{5.2}$$

When the nome of the elliptic functions is taken to zero, the critical weights (3.2) are recovered.

6. An Ising Model in a Field

There are four different critical branches in terms of the parameters u and λ. For all four, the central charge is known from the equivalence with the O(n) model [20],

$$\lambda = \frac{\pi}{4}\left(1\pm\frac{1}{L+1}\right)\begin{cases} 0<u<3\lambda & c = 1-\dfrac{6}{(L+1)(L+1\mp1)} \\ -\pi+3\lambda<u<0 & c=\dfrac{3}{2}-\dfrac{6}{(L+1)(L+1\pm1)}\,. \end{cases} \tag{6.1}$$

At criticality the dilute A models satisfy the Z_2 symmetry of the underlying Dynkin diagram, but off criticality this symmetry is broken, for L odd,

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} \neq W\begin{pmatrix} L+1-d & L+1-c \\ L+1-a & L+1-b \end{pmatrix} \quad L \text{ odd}\,. \tag{6.2}$$

The simplest model that exhibits this broken symmetry is realised when $L=3$. This is a three-state model with states $\{+,0,-\}$, of which a $+$ and $-$ spin cannot be neighbours. For $\lambda=\frac{5}{16}\pi$ and $0<u<\frac{15}{16}\pi$, this model has central charge $c=\frac{1}{2}$ and belongs to the universality class of the ordinary Ising model. In fact, calculating the free energy and the magnetisation as a function of the nome p, we find [19]

$$\begin{aligned} f_{\text{sing}}(p) &\sim |p|^{16/15} \\ m(p) &\sim \pm|p|^{1/15}\,. \end{aligned} \tag{6.3}$$

If we compare this with the definition of the critical exponent δ for spin systems

$$\begin{aligned} f_{\rm sing}(h,T_c) &\sim |h|^{1+1/\delta} \\ m(h,T_c) &\sim \pm|h|^{1/\delta}, \end{aligned} \tag{6.4}$$

where h is the magnetic field, we conclude that the nome of the elliptic functions in (5.2) serves as a magnetic field and that for the Ising model $\delta = 15$. Although this value is universally accepted, this is the first time that it has been calculated directly, rather than from scaling relations with other exponents.

7. Summary and Discussion

In this paper we have described a new method to construct RSOS models out of loop models. Application of this method to the $O(n)$ model led to a new family of RSOS models labelled by Dynkin diagrams. Independently these models have been found by Roche [4], who suggested the name dilute A-D-E models.

For the models based on the Dynkin diagrams of the classical A_L algebras we have presented an off-critical extension. Away from criticality and for odd values of L, the Boltzmann weights of this dilute A model break the Z_2 symmetry of the underlying Dynkin diagram.

We have shown that the dilute A_3 model, in the appropriate regime, can be viewed as a critical spin-1 Ising model in a magnetic field. For this model we obtained central charge $c = \frac{1}{2}$ and magnetic critical exponent $\delta = 15$.

As mentioned in the introduction, Zamolodchikov obtained an integrable field theory describing the scaling limit of the critical Ising model in a field. Smirnov showed [21] that Zamolodichkov's S-matrix, which has a hidden E_8 structure, relates to a RSOS projection of the Izergin-Korepin R-matrix. Indeed the dilute A_3 model is such a restricted Izergin-Korepin model, but the precise relation with Zamolodchikov's S-matrix remains unclear. By investigating the excitation spectrum of the dilute A model, we hope to establish this connection.

Finally, we would like to mention that Kuniba found RSOS models related to the Izergin-Korepin model that are very similar to the dilute A models [22]. His models, which belong to the more general class of $A_n^{(2)}$ RSOS models, do not, however, break the symmetry of the adjacency diagrams.

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