LOGARITHMIC AND COMPLEX
CONSTANT TERM IDENTITIES

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To Jon

ABSTRACT. In recent work on the representation theory of vertex algebras related to the Virasoro minimal models $M(2,p)$, Adamović and Milas discovered logarithmic analogues of (special cases of) the famous Dyson and Morris constant term identities. In this paper we show how the identities of Adamović and Milas arise naturally by differentiating as-yet-conjectural complex analogues of the constant term identities of Dyson and Morris. We also discuss the existence of complex and logarithmic constant term identities for arbitrary root systems, and in particular prove such identities for the root system G_2.
Keywords: Constant term identities; perfect matchings; Pfaffians; root systems; Jon's birthday.

0. JONATHAN BORWEIN

Jon Borwein is known for his love of mathematical *constants*. We hope this paper will spark his interest in *constant terms*.

1. CONSTANT TERM IDENTITIES

The study of constant term identities originated in Dyson's famous 1962 paper *Statistical theory of the energy levels of complex systems* [9]. In this paper Dyson conjectured that for $a_1, \ldots, a_n$ nonnegative integers

$$(1.1) \qquad \mathrm{CT} \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!},$$

where $\mathrm{CT}\, f(X)$ stands for the constant term of the Laurent polynomial (or possibly Laurent series) $f(X) = f(x_1, \ldots, x_n)$. Dyson's conjecture was almost instantly proved by Gunson and Wilson [14, 36]. In a very elegant proof, published several years later [13], Good showed that (1.1) is a direct consequence of Lagrange interpolation applied to $f(X) = 1$.

In 1982 Macdonald generalised the equal-parameter case of Dyson's ex-conjecture, i.e.,

$$(1.2) \qquad \mathrm{CT} \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{k} = \frac{(kn)!}{(k!)^n},$$

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to all irreducible, reduced root systems; here (1.2) corresponds to the root system A_{n-1}. Adopting standard notation and terminology—see [17] or the next section—Macdonald conjectured that [25]

$$(1.3) \qquad \mathrm{CT} \prod_{\alpha \in \Phi} (1 - \mathrm{e}^{\alpha})^k = \prod_{i=1}^{r} \binom{kd_i}{k},$$

where Φ is one of the root systems $A_{n-1}, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ of rank r and $d_1, \ldots, d_r$ are the degrees of its fundamental invariants. For $k = 1$ the Macdonald conjectures are an easy consequence of Weyl's denominator formula

$$\sum_{w \in W} \mathrm{sgn}(w) \, \mathrm{e}^{w(\rho)-\rho} = \prod_{\alpha > 0} \left(1 - \mathrm{e}^{-\alpha}\right)$$

(where W is the Weyl group of Φ and ρ the Weyl vector), and for B_n, C_n, D_n but k general they follow from the Selberg integral. The first uniform proof of (1.3)—based on hypergeometric shift operators—was given by Opdam in 1989 [24].

In his PhD thesis [27] Morris used the Selberg integral to prove a generalisation of (1.2), now commonly referred to as the Morris or Macdonald–Morris constant term identity:

$$(1.4) \quad \mathrm{CT} \left[\prod_{i=1}^{n} (1 - x_i)^a \left(1 - \frac{1}{x_i}\right)^b \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^k \right]$$
$$= \prod_{i=0}^{n-1} \frac{(a + b + ik)!((i+1)k)!}{(a + ik)!(b + ik)!k!},$$

where a and b are arbitrary nonnegative integers.

In their recent representation-theoretic work on W-algebra extensions of the $M(2, p)$ minimal models of conformal field theory [1, 2], Adamović and Milas discovered a novel type of constant term identities, which they termed *logarithmic constant term identities*. Before stating the results of Adamović and Milas, some more notation is needed.

Let $(a)_m = a(a+1) \cdots (a+m-1)$ denote the usual Pochhammer symbol or rising factorial, and let u be either a formal or complex variable. Then the (generalised) binomial coefficient $\binom{u}{m}$ is defined as

$$\binom{u}{m} = (-1)^m \frac{(-u)_m}{m!}.$$

We now interpret $(1 - x)^u$ and $\log(1 - x)$ as the (formal) power series

$$(1.5) \qquad (1 - x)^u = \sum_{m=0}^{\infty} (-x)^m \binom{u}{m}$$

and

$$\log(1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m} = \frac{\mathrm{d}}{\mathrm{d}u}(1 - x)^u \Big|_{u=0}.$$

Finally, for $X = (x_1, \ldots, x_n)$ we define the Vandermonde product

$$\Delta(X) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

One of the discoveries of Adamović and Milas is the following beautiful logarithmic analogue of the equal-parameter case (1.2) of Dyson's identity.

Conjecture 1.1 ([1, Conjecture A.12]). *For n an odd positive integer and k a nonnegative integer define $m := (n-1)/2$ and $K := 2k+1$. Then*

$$(1.6) \quad \mathrm{CT}\left[\Delta(X) \prod_{i=1}^{n} x_i^{-m} \prod_{i=1}^{m} \log\left(1 - \frac{x_{2i}}{x_{2i-1}}\right) \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^k \right] = \frac{(nK)!!}{n!!(K!!)^n}.$$

We remark that the kernel on the left is a Laurent series in X of (total) degree 0. Moreover, in the absence of the term $\prod_{i=1}^{m} \log(1 - x_{2i}/x_{2i-1})$ the kernel is a skew-symmetric Laurent polynomial which therefore has a vanishing constant term. Using identities for harmonic numbers, Adamović and Milas proved (1.6) for $n = 3$, see [1, Corollary 11.11].

Another result of Adamović and Milas, first conjectured in [1, Conjecture 10.3] (and proved for $n = 3$ in (the second) Theorem 1.1 of that paper, see page 3925) and subsequently proved in [2, Theorem 1.4], is the following Morris-type logarithmic constant term identity.

Theorem 1.2. *With the same notation as above,*

(1.7)
$$\mathrm{CT}\left[\Delta(X) \prod_{i=1}^{n} x_i^{2-(k+1)(n+1)}(1-x_i)^a \prod_{i=1}^{m} \log\left(1 - \frac{x_{2i}}{x_{2i-1}}\right) \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^k \right]$$
$$= c_{nk} \prod_{i=0}^{n-1} \binom{a + Ki/2}{(m+1)K - 1},$$

where a is an indeterminate, c_{nk} a nonzero constant, and

$$(1.8) \qquad c_{3,k} = \frac{(3K)!(k!)^3}{6(3k+1)!(K!)^3} \binom{3K-1}{2K-1}^{-1} \binom{5K/2-1}{2K-1}^{-1}.$$

As we shall see later, the above can be generalised to include an additional free parameter resulting in a logarithmic constant term identity more closely resembling Morris' identity, see (1.9) below.

The work of Adamović and Milas raises the following obvious questions:

(1) Can any of the methods of proof of the classical constant term identities, see e.g., [7, 8, 11–15, 19–21, 24, 29–31, 36–40], be utilised to prove the logarithmic counterparts?
(2) Do more of Macdonald's identities (1.3) admit logarithmic analogues?
(3) All of the classical constant term identities have q-analogues [16, 18, 25, 27]. Do such q-analogues also exist in the logarithmic setting?

As to the first and third questions, we can be disappointingly short; we have not been able to successfully apply any of the known methods of proof of constant term identities to also prove Conjecture 1.1, and attempts to find q-analogues have been equally unsuccessful. (In fact, we now believe q-analogues do not exist.)

As to the second question, we have found a very appealing explanation—itself based on further conjectures!—of the logarithmic constant term identities of Adamović and Milas. They arise by differentiating a complex version of Morris' constant term identity. Although such complex constant term identities are conjectured to

exist for other root systems as well—this is actually proved in the case G_2—it seems that only for A_{2n} and G_2 these complex identities imply elegant logarithmic identities.

The remainder of this paper is organised as follows. In the next section we introduce some standard notation related to root systems. Then, in Section 3, we study certain sign functions and prove a related Pfaffian identity needed subsequently. In Section 4, we conjecture a complex analogue of the Morris constant term identity 1.4 for n odd, and prove this for $n = 3$ using Zeilberger's method of creative telescoping [4, 28]. In Section 5 we show that the complex Morris identity implies the following logarithmic analogue of (1.4).

Theorem 1.3 (**Logarithmic Morris constant term identity**)**.** *With the same notation as in Conjecture 1.1 and conditional on the complex Morris constant term identity* (4.5) *to hold, we have*

(1.9)
$$\mathrm{CT}\left[\Delta(X)\prod_{i=1}^{n}x_i^{-m}(1-x_i)^a\left(1-\frac{1}{x_i}\right)^b\prod_{i=1}^{m}\log\left(1-\frac{x_{2i}}{x_{2i-1}}\right)\prod_{1\leq i\neq j\leq n}\left(1-\frac{x_i}{x_j}\right)^k\right]$$
$$=\frac{1}{n!!}\prod_{i=0}^{n-1}\frac{(2a+2b+iK)!!((i+1)K)!!}{(2a+iK)!!(2b+iK)!!K!!},$$

where a, b are nonnegative integers.

In Section 6 we prove complex as well as logarithmic analogues of (1.3) for the root system G_2, and finally, in Section 7 we briefly discuss the classical roots systems B_n, C_n and D_n.

2. Preliminaries on root systems and constant terms

In the final two sections of this paper we consider root systems of types other than A, and below we briefly recall some standard notation concerning root systems and constant term identities. For more details we refer the reader to [17, 25].
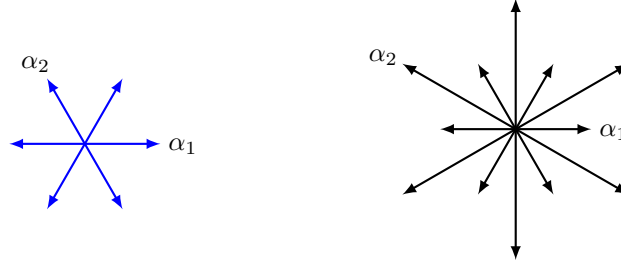

FIGURE 1. The root systems A_2 (left) and G_2 (right) with $\Delta = \{\alpha_1, \alpha_2\}$.

Let Φ be an irreducible, reduced root system in a real Euclidean space E with bilinear symmetric form $(\cdot, \cdot)$. Fix a base Δ of Φ and denote by Φ^+ the set of positive roots. Write $\alpha > 0$ if $\alpha \in \Phi^+$. The Weyl vector ρ is defined as half the sum of the positive roots: $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. The height $\mathrm{ht}(\beta)$ of the root β is given by $\mathrm{ht}(\beta) = (\beta, \rho)$. Let r be the rank of Φ (that is, the dimension of E). Then the

degrees $1 < d_1 \le d_2 \le \cdots \le d_r$ of the fundamental invariants of Φ are uniquely determined by

$$\prod_{i \ge 1} \frac{1 - t^{d_i}}{1 - t} = \prod_{\alpha > 0} \frac{1 - t^{\mathrm{ht}(\alpha) + 1}}{1 - t^{\mathrm{ht}(\alpha)}}.$$

For example, in the standard representation of the root system A_{n-1},

$$(2.1) \qquad E = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\},$$

$$\Phi = \{\epsilon_i - \epsilon_j : 1 \le i \ne j \le n\}$$

and

$$\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\} = \{\epsilon_i - \epsilon_{i+1} : 1 \le i \le n - 1\},$$

where ϵ_i denotes the ith standard unit vector in $\mathbb{R}^n$. Since $\mathrm{ht}(\epsilon_i - \epsilon_j) = j - i$,

$$\prod_{\alpha > 0} \frac{1 - t^{\mathrm{ht}(\alpha) + 1}}{1 - t^{\mathrm{ht}(\alpha)}} = \prod_{1 \le i < j \le n} \frac{1 - t^{j - i + 1}}{1 - t^{j - i}} = \prod_{i=1}^{n} \frac{1 - t^i}{1 - t}.$$

The degrees of A_{n-1} are thus $\{2, 3, \ldots, n\}$, and the A_{n-1} case of (1.3) is readily seen to be (1.2).

As a second example we consider the root system G_2 which is made up of two copies of A_2—one scaled. E is (2.1) for $n = 3$, and the canonical choice of simple roots is given by

$$\alpha_1 = \epsilon_1 - \epsilon_2 \quad \text{and} \quad \alpha_2 = 2\epsilon_2 - \epsilon_1 - \epsilon_3.$$

The following additional four roots complete the set of positive root Φ^+:

$$\alpha_1 + \alpha_2 = \epsilon_2 - \epsilon_3,$$
$$2\alpha_1 + \alpha_2 = \epsilon_1 - \epsilon_3,$$
$$3\alpha_1 + \alpha_2 = 2\epsilon_1 - \epsilon_2 - \epsilon_3,$$
$$3\alpha_1 + 2\alpha_2 = \epsilon_1 + \epsilon_2 - 2\epsilon_3.$$

The degrees of G_2 are now easily found to be $\{2, 6\}$ and, after the identification $(\mathrm{e}^{\epsilon_1}, \mathrm{e}^{\epsilon_2}, \mathrm{e}^{\epsilon_2}) = (x, y, z)$, the constant term identity (1.3) becomes

$$(2.2) \qquad \mathrm{CT}\left[\left(1 - \frac{x^2}{yz}\right)^k \left(1 - \frac{y^2}{xz}\right)^k \left(1 - \frac{z^2}{xy}\right)^k \left(1 - \frac{yz}{x^2}\right)^k \left(1 - \frac{xz}{y^2}\right)^k \left(1 - \frac{xy}{z^2}\right)^k \right.$$

$$\left. \times \left(1 - \frac{x}{y}\right)^k \left(1 - \frac{x}{z}\right)^k \left(1 - \frac{y}{x}\right)^k \left(1 - \frac{y}{z}\right)^k \left(1 - \frac{z}{x}\right)^k \left(1 - \frac{z}{y}\right)^k \right]$$

$$= \binom{2k}{k}\binom{6k}{k}.$$

This was first proved, in independent work, by Habsieger and Zeilberger [15, 38], who both utilised the A_2 case of Morris' constant term identity (1.4). They in fact proved a (q-analogue of a) slightly more general result related to another conjecture of Macdonald we discuss next.

Macdonald's (ex-)conjecture (1.3) may be generalised by replacing the exponent k on the left by k_α, where k_α depends only on the length of the root α, i.e., $k_\alpha = k_\beta$ if $\|\alpha\| = \|\beta\|$, where $\|\cdot\| := (\cdot, \cdot)^{1/2}$. If $\alpha^\vee = 2\alpha/\|\alpha\|^2$ is the coroot corresponding to α and $\rho_k = \frac{1}{2}\sum_{\alpha > 0} k_\alpha \alpha$, then Macdonald's generalisation of (1.3) is

$$(2.3) \qquad \mathrm{CT} \prod_{\alpha \in \Phi} (1 - \mathrm{e}^\alpha)^{k_\alpha} = \prod_{\alpha > 0} \frac{|(\rho_k, \alpha^\vee) + k_\alpha|!}{|(\rho_k, \alpha^\vee)|!}.$$

If k_α is independent of α, i.e., $k_\alpha = k$, then $\rho_k = k\rho$ and the above right-hand side may be simplified to that of (1.3).

As an example of (2.3) we consider the full Habsieger–Zeilberger theorem for G_2 [15, 38].

Theorem 2.1. *Let Φ_s and Φ_l denote the set of short and long roots of G_2 respectively. Then*

$$(2.4) \quad \mathrm{CT} \prod_{\alpha \in \Phi_l} (1 - \mathrm{e}^\alpha)^k \prod_{\alpha \in \Phi_s} (1 - \mathrm{e}^\alpha)^m = \frac{(3k + 3m)!(3k)!(2k)!(2m)!}{(3k + 2m)!(2k + m)!(k + m)!k!k!m!}.$$

Note that for $k = 0$ or $m = 0$ this yields (1.2) for $n = 3$. As we shall see in Section 6, it is the above identity, not it equal-parameter case (2.2), that admits a logarithmic analogue.

3. The signatures τ_{ij}

In our discussion of complex and logarithmic constant term identities in Sections 4–7, an important role is played by certain signatures τ_{ij}. For the convenience of the reader, in this section we have collected all relevant facts about the τ_{ij}.

For a fixed odd positive integer n and $m := (n-1)/2$ define τ_{ij} for $1 \le i < j \le n$ by

$$(3.1) \qquad\qquad\qquad \tau_{ij} = \begin{cases} 1 & \text{if } j \le m + i, \\ -1 & \text{if } j > m + i, \end{cases}$$

and extend this to all $1 \le i, j \le n$ by setting $\tau_{ij} = -\tau_{ji}$. Assuming that $1 \le i < n$ we have

$$\tau_{in} = \chi(n \le m + i) - \chi(n > m + i),$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Since $n - m = m + 1$, this is the same as

$$\tau_{in} = \chi(i > m) - \chi(i \le m) = -\tau_{1,i+1} = \tau_{i+1,1}.$$

For $1 \le i, j < n$ we clearly also have $\tau_{ij} = \tau_{i+1,j+1}$. Hence the matrix

$$(3.2) \qquad\qquad\qquad T := (\tau_{ij})_{1 \le i, j \le n}$$

is a *skew-symmetric circulant matrix*. For example, for $n = 5$,

$$T = \begin{pmatrix} 0 & 1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

We note that all of the row-sums (and column-sums) of the above matrix are zero. Because T is a circulant matrix, to verify this property holds for all (odd) n we only needs to verify this for the first row:

$$\sum_{j=1}^{n} \tau_{1j} = \sum_{j=2}^{m+1} 1 - \sum_{j=m+2}^{n} 1 = m - (n - m - 1) = m - m = 0.$$

By the skew symmetry, the vanishing of the row sums may also be stated as follows.

Lemma 3.1. *For* $1 \leq i \leq n$,

$$\sum_{j=1}^{i-1} \tau_{ji} = \sum_{j=i+1}^{n} \tau_{ij}.$$

A property of the signatures τ_{ij}, which will be important in our later discussions, can be neatly clarified by having recourse to Pfaffians.

By a *perfect matching* (or 1-factor) on $[n+1] := \{1, 2, \ldots, n+1\}$ we mean a graph on the vertex set $[n+1]$ such that each vertex has degree one, see e.g., [6,35]. If in a perfect matching π the vertices $i < j$ are connected by an edge we say that $(i,j) \in \pi$. Two edges (i,j) and (k,l) of π are said to be crossing if $i < k < j < l$ or $k < i < l < j$. The crossing number $c(i,j)$ of the edge $(i,j) \in \pi$ is the number of edges crossed by (i,j), and the crossing number $c(\pi)$ is the total number of pairs of crossing edges: $c(\pi) = \frac{1}{2} \sum_{(i,j) \in \pi} c(i,j)$. We can embed perfect matching in the xy-plane, such that (i) the vertex labelled i occurs at the point $(i,0)$, (ii) the edges (i,j) and (k,l) intersect exactly once if they are crossing and do not intersect if they are non-crossing. For example, the perfect matching $\{(1,3), (2,7), (4,5), (6,8)\}$ corresponds to
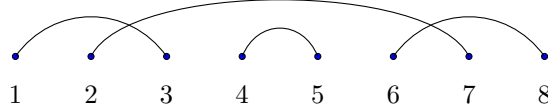
and has crossing number 2 $(c(4,5) = 0$, $c(1,3) = c(6,8) = 1$ and $c(2,7) = 2)$.

The *Pfaffian* of a $(2N) \times (2N)$ skew-symmetric matrix A is defined as [6,22,23,35]:

(3.3)
$$\mathrm{Pf}(A) := \sum_{\pi} (-1)^{c(\pi)} \prod_{(i,j) \in \pi} A_{ij}.$$

After these preliminaries on perfect matching and Pfaffians we now form a second skew-symmetric matrix, closely related to T. First we extend the τ_{ij} to $1 \leq i, j \leq n+1$ by setting $\tau_{i,n+1} = b_i$. We then define the $(n+1) \times (n+1)$ skew-symmetric matrix $Q(a,b) = (Q_{ij}(a,b))_{1 \leq i,j \leq n+1}$, where $a = (a_1, \ldots, a_{n+1})$ and $b = (b_1, \ldots, b_n)$, as follows:

(3.4)
$$Q_{ij}(a,b) = \tau_{ij} a_i a_j \qquad \text{for } 1 \leq i < j \leq n+1.$$

For example, for $n = 5$,

$$Q(a,b) = \begin{pmatrix} 0 & a_1 a_2 & a_1 a_3 & -a_1 a_4 & -a_1 a_5 & a_1 a_6 b_1 \\ -a_2 a_1 & 0 & a_2 a_3 & a_2 a_4 & -a_2 a_5 & a_2 a_6 b_2 \\ -a_3 a_1 & -a_3 a_2 & 0 & a_3 a_4 & a_3 a_5 & a_3 a_6 b_3 \\ a_4 a_1 & -a_4 a_2 & -a_4 a_3 & 0 & a_4 a_5 & a_4 a_6 b_4 \\ a_5 a_1 & a_5 a_2 & -a_5 a_3 & -a_5 a_4 & 0 & a_5 a_6 b_5 \\ -a_6 a_1 b_1 & -a_6 a_2 b_2 & -a_6 a_3 b_3 & -a_6 a_4 b_4 & -a_6 a_5 b_5 & 0 \end{pmatrix}.$$

Note that T is the submatrix of $Q((1^{n+1}), b)$ obtained by deleting the last row and column.

Proposition 3.2. *We have*

$$\mathrm{Pf}(Q(a,b)) = (-1)^{\binom{m}{2}} a_1 a_2 \cdots a_{n+1} (b_1 + b_2 + \cdots + b_n).$$

Proof. The main point of our proof below is to exploit a cyclic symmetry of the terms contributing to $\mathrm{Pf}\big(Q(a,b)\big)$. This reduces the computation of the Pfaffian to that of a sub-Pfaffian of lower order.

Let $S(\pi; a, b)$ denote the summand of $\mathrm{Pf}\big(Q(a,b)\big)$, that is,

$$\mathrm{Pf}\big(Q(a,b)\big) = \sum_{\pi} S(\pi; a, b) \qquad \text{with} \qquad S(\pi; a, b) = (-1)^{c(\pi)} \prod_{(i,j)\in\pi} Q_{ij}(a,b).$$

From the definition (3.4) of $Q_{ij}(a,b)$ and the fact that π is a perfect matching on $[n+1]$,

$$(3.5) \qquad S(\pi; a, b) = (-1)^{c(\pi)} \prod_{(i,j)\in\pi} a_i a_j \tau_{ij} = (-1)^{c(\pi)} a_1 \cdots a_{n+1} \prod_{(i,j)\in\pi} \tau_{ij}.$$

We now observe that $S(\pi; a, b)$ is, up to a cyclic permutation of b, invariant under the permutation w given by $(1, 2, 3, \ldots, n, n+1) \mapsto (n, 1, 2, \ldots, n-1, n+1)$. To see this, denote by π' the image of π under w. For example, the image of the perfect matching given on the previous page is
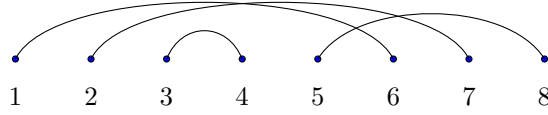
Under the permutation w, all edges not containing the vertices 1 or $n+1$ are shifted one unit to the left: $(i, j) \mapsto (i - 1, j - 1)$. For the edge $(1, j)$ containing vertex 1 we have:

    (i) If $j \leq n$ then $(1, j) \mapsto (j - 1, n)$. This also implies that the edge $(j', n+1)$ $(j' \geq 2)$ containing vertex $n+1$ maps to $(j' - 1, n+1)$.

    (ii) If $j = n+1$ then $(1, j) = (1, n+1) \mapsto (n, n+1) = (j-1, n+1)$.

First we consider (i). If we remove the edge $(1, j)$ from π and carry out w, then the number of crossings of its image is exactly that of π. Hence we only need to focus on the edge $(1, j)$ and its image under w. In π the edge $(1, j)$ has crossing number $c(1, j) \equiv j \pmod 2$, while the edge $(j - 1, n)$ in π' has crossing number $c(j - 1, n) \equiv n - j \equiv j + 1 \pmod 2$. Hence $(-1)^{c(\pi)} = -(-1)^{c(\pi')}$. Since $\tau_{ij} = \tau_{i-1,j-1}$ (for $2 \leq i < j \leq n$) and $\tau_{1,j} = -\tau_{j-1,n}$ it thus follows that π and π' have the same sign. Finally we note that under w, $b_i = \tau_{i,n+1} \mapsto \tau_{i-1,n+1} = b_{i-1}$ (since $i \neq 1$). We thus conclude that

$$(3.6) \qquad S\big(\pi; a, (b_1, \ldots, b_n)\big) \mapsto S\big(\pi'; a, (b_2, \ldots, b_n, b_1)\big),$$

where we note that both sides depend on a single $b_i (\neq b_1)$ only. For example, the perfect matching in the above two figures correspond to

$$S\big((1,3), (2,7), (4,5), (6,8); a, (b_1, \ldots, b_7)\big)$$
$$= (-1)^2 \cdot a_1 a_3 \cdot (-a_2 a_7) \cdot a_4 a_5 \cdot a_6 a_8 b_6 = -a_1 \cdots a_8 b_6$$

and

$$S\big((1,6), (2,7), (3,4), (5,8); a, (b_1, \ldots, b_7)\big)$$
$$= (-1)^3 \cdot (-a_1 a_6) \cdot (-a_2 a_7) \cdot a_3 a_4 \times a_5 a_8 b_5 = -a_1 \cdots a_8 b_5.$$

The case (ii) is even simpler; the edge $(1, n+1)$ in π and its image $(n, n+1)$ in π' both have crossing number 0. The crossing numbers of all other edges do not change by a global shift of one unit to the right, so that $c(\pi) = c(\pi')$:
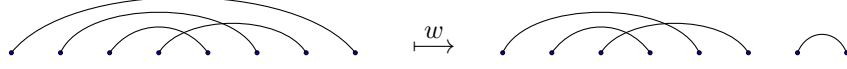
Moreover, $\tau_{ij} = \tau_{i-1,j-1}$ (for $2 \le i < j \le n$) so that π and π' again have the same sign. Finally, from $b_1 = \tau_{1,n+1} \mapsto \tau_{n,n+1} = b_n$ it follows that once again (3.6) holds, where this time both sides depend only on b_1.

From (3.6) it follows that the Pfaffian $\mathrm{Pf}\big(Q(a,b)\big)$ is symmetric under cyclic permutations of the b_i. But since the Pfaffian, viewed as a function of b, has degree 1 it thus follows (see also (3.5)) that

$$\mathrm{Pf}\big(Q(a,b)\big) = C a_1 \cdots a_{n+1}(b_1 + \cdots + b_n)$$

for some yet-unknown constant C. We shall determine C by computing the coefficient of b_n of $\mathrm{Pf}\big(Q((1^{n+1}), b)\big)$, which is equal to the Pfaffian of the $(2m) \times (2m)$ submatrix M of T obtained by deleting its last row and column.

We recall the property $\mathrm{Pf}(M) = \mathrm{Pf}(U^t M U)$ of Pfaffians, where U is a unipotent triangular matrix [35]. Choosing the non-zero entries of the $(2m) \times (2m)$ matrix U to be $U_{ii} = 1$ for $i = 1, \ldots, 2m$, and $U_{i,i+m} = 1$ for $i = 1, \ldots, m$, one transforms M into

$$\left(\begin{array}{c|c} M' & I \\ \hline -I & \varnothing \end{array} \right),$$

where M' is the upper-left $m \times m$ submatrix of M and I is the $m \times m$ identity matrix. The Pfaffian of the above matrix, and hence that of M, is exactly (cf. [35]) $(-1)^{\binom{m}{2}} \det(I) = (-1)^{\binom{m}{2}}$. This, in turn, implies that $C = (-1)^{\binom{m}{2}}$, and the required formula follows. $\qquad\square$

Remark 3.3. By a slight modification of the above proof the following more general Pfaffian results. Let

$$Q_{ij}(X, a, b) := \tau_{ij} a_i a_j (x_i + x_j) \qquad\qquad \text{for } 1 \le i < j \le n$$

and

$$Q_{i,n+1}(X, a, b) := \tau_{i,n+1} a_i a_{n+1} = a_i a_{n+1} b_i \qquad \text{for } 1 \le i \le n,$$

and use this to form the $(n+1) \times (n+1)$ skew-symmetric matrix $Q(X, a, b)$. Then

$$\mathrm{Pf}\big(Q(X, a, b)\big)$$
$$= 2^{m-1}(-1)^{\binom{m}{2}} a_1 a_2 \cdots a_{n+1} \sum_{i=1}^{n} b_i (x_{i+1} \cdots x_{i+m} + x_{i+m+1} \cdots x_{i+n-1}),$$

where $x_{i+n} := x_i$ for $i \ge 1$. For $X = (1/2, \ldots, 1/2)$ this yields Proposition 3.2.

4. The complex Morris constant term identity

Thanks to Lemma 3.1,

(4.1)

$$\prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right) = \prod_{1 \le i < j \le n} \left(-\frac{x_j}{x_i}\right)^{\tau_{ij}} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^2$$

$$= (-1)^{\binom{n}{2}} \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i-1} \tau_{ji} - \sum_{j=i+1}^{n} \tau_{ij}} \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^2$$

$$= (-1)^m \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^2.$$

For odd values of n Morris' constant term identity (1.4) can thus be rewritten in the equivalent form

(4.2) $$\mathrm{CT}\left[\prod_{i=1}^{n}(1 - x_i)^a \left(1 - \frac{1}{x_i}\right)^b \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^{2k}\right]$$

$$= (-1)^{km} \prod_{i=0}^{n-1} \frac{(a + b + ik)!((i+1)k)!}{(a + ik)!(b + ik)!k!}.$$

The crucial point about this rewriting is that in the product

$$\prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^{2k}$$

each of the variables $x_1, x_2, \ldots, x_n$ occurs exactly m times in one of the numerators and m times in one of the denominators. For example,

$$\prod_{1 \le i < j \le 3} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^{2k} = \left(1 - \frac{x_1}{x_2}\right)^{2k} \left(1 - \frac{x_2}{x_3}\right)^{2k} \left(1 - \frac{x_3}{x_1}\right)^{2k}.$$

Obviously, for n even such a rewriting is not possible.

We are now interested in the question as to what happens when $2k$ is replaced by an arbitrary complex variable u. For $n = 3$ we will later prove the following proposition.

Proposition 4.1. *For a, b nonnegative integers and $\mathrm{Re}(1 + \frac{3}{2}u) > 0$,*

(4.3) $$\mathrm{CT}\left[(1-x)^a(1-y)^a(1-z)^a \left(1 - \frac{1}{x}\right)^b \left(1 - \frac{1}{y}\right)^b \left(1 - \frac{1}{z}\right)^b\right.$$

$$\left. \times \left(1 - \frac{x}{y}\right)^u \left(1 - \frac{y}{z}\right)^u \left(1 - \frac{z}{x}\right)^u\right]$$

$$= \cos\left(\tfrac{1}{2}\pi u\right) \frac{\Gamma(1 + \frac{3}{2}u)}{\Gamma^3(1 + \frac{1}{2}u)} \prod_{i=0}^{2} \frac{(1 + \frac{1}{2}iu)_{a+b}}{(1 + \frac{1}{2}iu)_a (1 + \frac{1}{2}iu)_b}.$$

As follows from its proof, a slightly more general result in fact holds. Using $(z)_{n+m} = (z)_n (z+n)_m$ and $(1-x)^a(1-x^{-1})^b = (-x)^{-b}(1-x)^{a+b}$, then replacing $a \mapsto a - b$, and finally using $(z - b)_b = (-1)^b(1 - z)_b$, the identity (4.3) can also be

stated as

$$(4.4) \quad \left[x^b y^b z^b\right] \left[(1-x)^a (1-y)^a (1-z)^a \left(1-\frac{x}{y}\right)^u \left(1-\frac{y}{z}\right)^u \left(1-\frac{z}{x}\right)^u \right]$$

$$= \cos\left(\tfrac{1}{2}\pi u\right) \frac{\Gamma(1+\tfrac{3}{2}u)}{\Gamma^3(1+\tfrac{1}{2}u)} \prod_{i=0}^{2} \frac{(-a-\tfrac{1}{2}iu)_b}{(1+\tfrac{1}{2}iu)_b},$$

where $\left[X^c\right] f(X)$ (with $X^c = x_1^{c_1} \cdots x_n^{c_n}$) denotes the coefficient of X^c in $f(X)$. This alternative form of (4.3) is true for all $a, u \in \mathbb{C}$ such that $\operatorname{Re}(1 + \tfrac{3}{2}u) > 0$.

In view of Proposition 4.1 it seems reasonable to make the following more general conjecture.

Conjecture 4.2 (**Complex Morris constant term identity**). *Let n be an odd positive integer, a, b nonnegative integers and $u \in \mathbb{C}$ such that $\operatorname{Re}(1 + \tfrac{1}{2}nu) > 0$. Then there exists a polynomial $P_n(x)$, independent of a and b, such that $P_n(0) = 1/(n-2)!!$, $P_n(1) = 1$, and*

$$(4.5) \quad \mathrm{CT} \left[\prod_{i=1}^{n} (1-x_i)^a \left(1-\frac{1}{x_i}\right)^b \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^u \right]$$

$$= x^m P_n(x^2) \frac{\Gamma(1+\tfrac{1}{2}nu)}{\Gamma^n(1+\tfrac{1}{2}u)} \prod_{i=0}^{n-1} \frac{(1+\tfrac{1}{2}iu)_{a+b}}{(1+\tfrac{1}{2}iu)_a (1+\tfrac{1}{2}iu)_b},$$

where $x = x(u) := \cos\left(\tfrac{1}{2}\pi u\right)$ and $m := (n-1)/2$.

Note that for u an odd positive integer the kernel on the left of (4.5) is a skew-symmetric function, so that its constant term trivially vanishes. When u is an even integer, say $2k$ then $x = \cos(\pi k) = (-1)^k$ so that $x^m P_n(x^2) = (-1)^{km} P_n(1) = (-1)^{km}$ in accordance with (4.2). Similar to the case $n = 3$, in the form

$$\left[(x_1 \cdots x_n)^b\right] \left[\mathrm{CT} \left[\prod_{i=1}^{n} (1-x_i)^a \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^u \right] \right]$$

$$= x^m P_n(x^2) \frac{\Gamma(1+\tfrac{1}{2}nu)}{\Gamma^n(1+\tfrac{1}{2}u)} \prod_{i=0}^{n-1} \frac{(-a-\tfrac{1}{2}iu)_b}{(1+\tfrac{1}{2}iu)_b}$$

Conjecture 4.2 should hold for all $a \in \mathbb{C}$.

For $n = 1$ the left-side of (4.5) does not depend on u so that $P_1(x) = 1$. Moreover, from Proposition 4.1 it follows that also $P_3(x) = 1$. Extensive numerical computations leave little doubt that the next two instances of $P_n(x)$ are given by

$$P_5(x) = \frac{1}{3}(1 + 2x)$$

$$P_7(x) = \frac{1}{45}(3 + 26x - 16x^2 + 32x^3).$$

Conjecturally, we also have $\deg(P_n(x)) = \binom{m}{2}$ and

$$P_n'(0) = 2\binom{m}{2} \frac{2n-1}{9(n-2)!!}$$

$$P_n'(1) = \frac{2}{3}\binom{m}{2}, \qquad P_n''(1) = \frac{2}{45}\binom{m}{3}(19m + 23),$$

but beyond this we know very little about $P_n(x)$.

To conclude our discussion of the polynomials $P_n(x)$ we note that if $z_i = z_i(u) := \cos(i\pi u)$, then

$$P_5\big(x^2(u)\big) = \frac{1}{3}(2 + z_1)$$

$$P_7\big(x^2(u)\big) = \frac{1}{45}(20 + 20z_1 + 4z_2 + z_3),$$

suggesting that the coefficients of z_i admit a combinatorial interpretation.

As will be shown in the next section, the complex Morris constant term identity (4.5) implies the logarithmic Morris constant term identity (1.9), and the only properties of $P_n(x)$ that are essential in the proof are $P_n(0) = 1/(n-2)!!$ and $P_n(1) = 1$.

To conclude this section we give a proof of Proposition 4.1. The reader unfamiliar with the basic setup of the method of creative telescoping is advised to consult the text [28].

Proof of Proposition 4.1. Instead of proving (4.3) we establish the slightly more general (4.4).

By a six-fold use of the binomial expansion (1.5), the constant term identity (4.4) can be written as the following combinatorial sum:

$$\sum_{m_0,m_1,m_2=0}^{\infty} \prod_{i=0}^{2} (-1)^{m_i} \binom{u}{m_i}\binom{a}{b+m_i-m_{i+1}}$$

$$= \cos\left(\tfrac{1}{2}\pi u\right) \frac{\Gamma(1+\frac{3}{2}u)}{\Gamma^3(1+\frac{1}{2}u)} \prod_{i=0}^{2} \frac{(-a-\frac{1}{2}iu)_b}{(1+\frac{1}{2}iu)_b},$$

where $m_3 := m_0$ and where $a, u \in \mathbb{C}$ such that $\mathrm{Re}(1+\frac{3}{2}u) > 0$ and b is a nonnegative integer. If we denote the summand of this identity by $f_b(\frac{1}{2}u, -1-a; m)$ where $m := (m_0, m_1, m_2)$, then we need to prove that

$$(4.6) \qquad F_b(u,v) := \sum_{m \in \mathbb{Z}^3} f_b(u,v;m) = \cos\left(\pi u\right) \frac{\Gamma(1+3u)}{\Gamma^3(1+u)} \prod_{i=0}^{2} \frac{(1+v-iu)_b}{(1+iu)_b},$$

for $\mathrm{Re}(1+3u) > 0$.

In our working below we suppress the dependence of the various functions on the variables u and v. In particular we write F_b and $f_b(m)$ for $F_b(u,v)$ and $f_b(u,v;m)$.

The function $f_0(m)$ vanishes unless $m_0 = m_1 = m_2$. Hence

$$F_0 = \sum_{m=0}^{\infty} (-1)^m \binom{2u}{m}^3 = {}_3F_2\left[\begin{matrix} -2u, -2u, -2u \\ 1, 1 \end{matrix}; 1\right],$$

where we adopt standard notation for (generalised) hypergeometric series, see e.g., [3, 5]. The ${}_3F_2$ series is summable by the $2a = b = c = -2u$ case of Dixon's sum [3, Eq. (2.2.11)]

$$(4.7) \quad {}_3F_2\left[\begin{matrix} 2a, b, c \\ 1+2a-b, 1+2a-c \end{matrix}; 1\right]$$

$$= \frac{\Gamma(1+a)\Gamma(1+2a-b)\Gamma(1+2a-c)\Gamma(1+a-b-c)}{\Gamma(1+2a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+2a-b-c)}$$

for $\mathrm{Re}(1 + a - b - c) > 0$. As a result,

$$F_0 = \frac{\Gamma(1-u)\Gamma(1+u)}{\Gamma(1-2u)\Gamma(1+2u)} \cdot \frac{\Gamma(1+3u)}{\Gamma^3(1+u)} = \cos(\pi u)\frac{\Gamma(1+3u)}{\Gamma^3(1+u)},$$

proving the $b = 0$ instance of (4.6).

In the remainder we assume that $b \geq 1$.

Let $\mathcal{C}$ be the generator of the cyclic group C_3 acting on m as $\mathcal{C}(m) = (m_2, m_0, m_1)$. With the help of the multivariable Zeilberger algorithm [4], one discovers the (humanly verifiable) rational function identity

$$(4.8) \quad t_b(m)\prod_{i=0}^{2}(b+iu) + \prod_{i=0}^{2}(b+v-iu)$$

$$= \sum_{i=0}^{2}\Big(r_b\big(e_1 + \mathcal{C}^i(m)\big)\,s_b\big(\mathcal{C}^i(m)\big) + r_b\big(\mathcal{C}^i(m)\big)\Big),$$

where

$$r_b(m) = -\frac{m_0(b+v+m_2-m_0)}{6(b+m_1-m_2)(b+m_2-m_0)}$$
$$\times \big((2b+v)(3b^2+3bv+2uv) + 2(m_1-m_2)(3b^2+3bv+v^2-uv)\big),$$

$$s_b(m) = -\frac{f_{b-1}(e_1+m)}{f_{b-1}(m)} = \frac{(2u-m_0)(b+v+m_0-m_1)(b+m_2-m_0-1)}{(1+m_0)(b+m_0-m_1)(b+v+m_2-m_0-1)},$$

$$t_b(m) = -\frac{f_b(m)}{f_{b-1}(m)} = \prod_{i=0}^{2}\frac{b+v+m_i-m_{i+1}}{b+m_i-m_{i+1}},$$

and $e_1 + m := (1+m_0, m_1, m_2)$. If we multiply (4.8) by $-f_{b-1}(m)$ and use that $f_b(m) = f_b(\mathcal{C}^i(m))$ we find that

$$f_b(m)\prod_{i=0}^{2}(b+iu) - f_{b-1}(m)\prod_{i=0}^{2}(b+v-iu)$$

$$= \sum_{i=0}^{2}\Big[r_b\big(e_1+\mathcal{C}^i(m)\big)\,f_{b-1}\big(e_1+\mathcal{C}^i(m)\big) - r_b\big(\mathcal{C}^i(m)\big)f_{b-1}\big(\mathcal{C}^i(m)\big)\Big],$$

Summing this over $m \in \mathbb{Z}^3$ the right-hand side telescopes to zero, resulting in

$$F_b = F_{b-1}\prod_{i=0}^{2}\frac{(b+v-iu)}{(b+iu)}.$$

By b-fold iteration this yields

$$F_b = F_0\prod_{i=0}^{2}\frac{(1+v-iu)_b}{(1+iu)_b}. \qquad \square$$

5. The logarithmic Morris constant term identity

This section contains three parts. In the first very short part, we present an integral analogue of the logarithmic Morris constant term identity. This integral may be viewed as a logarithmic version of the well-known Morris integral. The second and third, more substantial parts, contain respectively a proof of Theorem 1.3

and, exploiting some further results of Adamović and Milas, a strengthening of this theorem.

5.1. **A logarithmic Morris integral.** By a repeated use of Cauchy's integral formula, constant term identities such as (1.4) or (1.9) may be recast in the form of multiple integral evaluations. In the case of (1.4) this leads to the well-known Morris integral [10, 27]

$$\int_{[-\frac{1}{2}\pi,\frac{1}{2}\pi]^n} \prod_{i=1}^{n} e^{i(a-b)\theta_i} \sin^{a+b}(\theta_i) \prod_{1 \leq i < j \leq n} \sin^{2k}(\theta_i - \theta_j) \, d\theta_1 \cdots d\theta_n$$

$$= \big(B_{k,n}(a,b)\big)^n \prod_{i=0}^{n-1} \frac{(a+b+ik)!((i+1)k)!}{(a+ik)!(b+ik)!k!},$$

where $B_{k,n}(a,b) = \pi\, i^{a-b}\, 2^{-k(n-1)-a-b}$. The Morris integral may be shown to be a simple consequence of the Selberg integral [10, 32]. Thanks to (1.9) we now have a logarithmic analogue of the Morris integral as follows:

$$\int_{[-\frac{1}{2}\pi,\frac{1}{2}\pi]^n} \prod_{i=1}^{n} e^{i(a-b)\theta_i} \sin^{a+b}(\theta_i) \prod_{i=1}^{m} \log\big(1 - e^{2\,i(\theta_{2i} - \theta_{2i-1})}\big)$$

$$\times \prod_{1 \leq i < j \leq n} \sin^{K}(\theta_i - \theta_j) \, d\theta_1 \cdots d\theta_n$$

$$= \big(C_{k,n}(a,b)\big)^n \frac{1}{n!!} \prod_{i=0}^{n-1} \frac{(2a+2b+iK)!!((i+1)K)!!}{(2a+iK)!!(2b+iK)!!K!!},$$

where $C_{k,n}(a,b) = \pi\, i^{a-b-m}\, 2^{-Km-a-b}$. Unfortunately, this cannot be rewritten further in a form that one could truly call a logarithmic Selberg integral.

5.2. **Proof of Theorem 1.3.** In this subsection we prove that the logarithmic Morris constant term identity (1.9) is nothing but the mth derivative of the complex Morris constant term identity (4.5) evaluated at $u = K := 2k + 1$.

To set things up we first prepare a technical lemma. For $\mathfrak{S}_n$ the symmetric group on n letters and $w \in \mathfrak{S}_n$, we denote by $\text{sgn}(w)$ the signature of the permutation w, see e.g., [26]. The identity permutation in $\mathfrak{S}_n$ will be written as $\mathbb{1}$.

Lemma 5.1. *For n an odd integer, set $m := (n-1)/2$. Let t_{ij} for $1 \leq i < j \leq n+1$ be a collection of signatures (i.e., each t_{ij} is either $+1$ or -1) such that $t_{i,n+1} = 1$, and $\tilde{Q}$ a skew-symmetric matrix with entries $\tilde{Q}_{ij} = t_{ij}$ for $1 \leq i < j \leq n+1$.*

If $f(X)$ is a skew-symmetric polynomial in $X = (x_1, \ldots, x_n)$, $g(z)$ a Laurent polynomial or Laurent series in the scalar variable z, and $g_{ij}(X) := g((x_i/x_j)^{t_{ij}})$, then the following statements hold.

(1) *For $w \in \mathfrak{S}_n$, denote $g(w; X) := \prod_{k=1}^{m} g(x_{w_{2k-1}}/x_{w_{2k}})$. Then*

$$\text{CT}\big[f(X)g(w; X)\big] = \text{sgn}(w)\,\text{CT}\big[f(X)g(\mathbb{1}; X)\big].$$

(2) *For π a perfect matching on $[n + 1]$,*

$$(5.1) \qquad \sum_{\pi} \text{CT}\bigg[f(X) \prod_{\substack{(i,j)\in\pi \\ j \neq n+1}} g_{ij}(X)\bigg] = \text{Pf}(\tilde{Q})\,\text{CT}\big[f(X)g(\mathbb{1}; X)\big].$$

We will be needing a special case of this lemma corresponding to $t_{ij} = \tau_{ij}$ for $1 \leq i, j \leq n$, with the τ_{ij} defined in (3.1). Then the matrix $\tilde{Q}$ is coincides with $Q\big((1^{n+1}), (1^n)\big)$ of (3.4), so that by Lemma 3.2, $\mathrm{Pf}(\tilde{Q}) = (-1)^{\binom{m}{2}} n$. We summarise this in the following corollary.

Corollary 5.2. *If in Lemma 5.1 we specialise $t_{ij} = \tau_{ij}$ for $1 \leq i < j \leq n$, then*

$$(5.2) \qquad \sum_{\pi} \mathrm{CT}\left[f(X) \prod_{\substack{(i,j)\in\pi \\ j\neq n+1}} g_{ij}(X) \right] = (-1)^{\binom{m}{2}} n\, \mathrm{CT}\left[f(X) g(\mathbb{1}; X) \right].$$

Proof of Lemma 5.1. (1) According to the "Stanton–Stembridge trick" [33, 34, 39],

$$\mathrm{CT}\big[h(X) \big] = \mathrm{CT}\big[w\big(h(X)\big) \big] \quad \text{for } w \in \mathfrak{S}_n,$$

where $w(h(X))$ is shorthand for $h(x_{w_1}, \ldots, x_{w_n})$.

For our particular choice of h, the skew-symmetric factor $f(X)$ produces the claimed sign.

(2) A permutation $w \in \mathfrak{S}_n$ may be interpreted as a signed perfect matching $(-1)^{d(w)}(w_1, w_2) \cdots (w_{n-2}, w_{n-1})(w_n, w_{n+1})$, where $d(w)$ counts the number $|\{k \leq m : w_{2k-1} > w_{2k}\}|$. By claim (1), the left hand-side of (5.1) is a multiple of $\mathrm{CT}\big[f(X) g(\mathbb{1}; X) \big]$; the factor is exactly the sum $\sum_{\pi} (-1)^{c(\pi)} \prod t_{ij}$, in which one recognises the Pfaffian of $\tilde{Q}$. $\qquad \square$

Conditional proof of (1.9). Suppressing the a and b dependence, denote the left and right-hand sides of (4.5) by $L_n(u)$ and $R_n(u)$ respectively. We then wish to show that (1.9) is identical to

$$L_n^{(m)}(K) = R_n^{(m)}(K).$$

Let us first consider the right hand side, which we write as $R_n(u) = p_n(u) r_n(u)$, where

$$p_n(u) = x^m P_n(x^2), \qquad x = x(u) = \cos\left(\tfrac{1}{2}\pi u\right)$$

and

$$(5.3) \qquad r_n(u) = \frac{\Gamma(1 + \tfrac{1}{2}nu)}{\Gamma^n(1 + \tfrac{1}{2}u)} \prod_{i=0}^{n-1} \frac{(1 + \tfrac{1}{2}iu)_{a+b}}{(1 + \tfrac{1}{2}iu)_a (1 + \tfrac{1}{2}iu)_b}.$$

Since $x(K) = 0$, it follows that for $0 \leq j \leq m$,

$$(5.4) \qquad p_n^{(j)}(K) = (-1)^{km+m} \left(\frac{\pi}{2}\right)^m \frac{m!}{(n-2)!!}\, \delta_{jm}.$$

Therefore, since $r_n(u)$ is m times differentiable at $u = K$,

$$(5.5) \qquad R_n^{(m)}(K) = p_n^{(m)}(K) r_n(K).$$

By the functional equation for the gamma function

$$(5.6) \qquad \Gamma(1 + \tfrac{1}{2}N) = \begin{cases} N!!\, 2^{-N/2} \sqrt{\pi/2} & \text{if } N > 0 \text{ is odd,} \\ N!!\, 2^{-N/2} & \text{if } N \geq 0 \text{ is even,} \end{cases}$$

and, consequently,

$$(5.7) \qquad (1 + \tfrac{1}{2}N)_a = \frac{(N + 2a)!!}{2^a N!!}$$

for any nonnegative integer N. Applying these formulae to (5.3) with $u = K$, we find that

$$r_n(K) = \left(\frac{2}{\pi}\right)^m \prod_{i=0}^{n-1} \frac{(2a + 2b + iK)!!((i+1)K)!!}{(2a + iK)!!(2b + iK)!!K!!}.$$

Combined with (5.4) and (5.5) this implies

$$(5.8) \qquad R_n^{(m)}(K) = (-1)^{(k+1)m} \frac{m!}{(n-2)!!} \prod_{i=0}^{n-1} \frac{(2a + 2b + iK)!!((i+1)K)!!}{(2a + iK)!!(2b + iK)!!K!!}.$$

Next we focus on the calculation of $L_n^{(m)}(K)$. To keep all equations in check we define

$$f_{ab}(X) := \prod_{i=1}^{n} (1 - x_i)^a \left(1 - \frac{1}{x_i}\right)^b.$$

and

$$(5.9) \qquad F_{ab}(X) := \Delta(X) \prod_{i=1}^{n} x_i^{-m} (1 - x_i)^a \left(1 - \frac{1}{x_i}\right)^b \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^k.$$

Let $i := (i_1, \ldots, i_m)$ and $j := (j_1, \ldots, j_m)$. Then, by a straightforward application of the product rule,

$$L_n^{(m)}(u) = \sum_{1 \le i_1 < j_1 \le n} \cdots \sum_{1 \le i_m < j_m \le n} L_{n;i,j}(u),$$

where

$$L_{n;i,j}(u) = \mathrm{CT}\left[f_{ab}(X) \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^u \prod_{\ell=1}^{m} \log\left(1 - \left(\frac{x_{i_\ell}}{x_{j_\ell}}\right)^{\tau_{i_\ell j_\ell}}\right)\right].$$

For $u = K$ the kernel without the product over logarithms is a skew-symmetric function in X, so that $L_{n;i,j}(K) = 0$ if there exists a pair of variables, say x_r and x_s, that does not occur in the product of logarithms. In other words, $L_{n;i,j}(K)$ vanishes unless all of the $2m = n - 1$ entries of i and j are distinct:

$$L_n^{(m)}(K) = \sum \mathrm{CT}\left[f_{ab}(X) \prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^K \prod_{\ell=1}^{m} \log\left(1 - \left(\frac{x_{i_\ell}}{x_{j_\ell}}\right)^{\tau_{i_\ell j_\ell}}\right)\right],$$

where the sum is over $1 \le i_\ell < j_\ell \le n$ for $1 \le \ell \le m$ such that all of $i_1, \ldots, i_m, j_1, \ldots, j_m$ are distinct. By (4.1) and

$$\prod_{1 \le i < j \le n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right) = (-1)^{\binom{m}{2}} \Delta(X) \prod_{i=1}^{n} x_i^{-m}$$

this can be simplified to

$$L_n^{(m)}(K) = (-1)^{km + \binom{m}{2}} \sum \mathrm{CT}\left[F_{ab}(X) \prod_{\ell=1}^{m} \log\left(1 - \left(\frac{x_{i_\ell}}{x_{j_\ell}}\right)^{\tau_{i_\ell j_\ell}}\right)\right].$$

Using the $\mathfrak{S}_m$ symmetry of the product over the logarithmic terms, this reduces further to

$$L_n^{(m)}(K) = (-1)^{km + \binom{m}{2}} m! \sum \mathrm{CT}\left[F_{ab}(X) \prod_{\ell=1}^{m} \log\left(1 - \left(\frac{x_{i_\ell}}{x_{j_\ell}}\right)^{\tau_{i_\ell j_\ell}}\right)\right],$$

where $1 \leq i_\ell < j_\ell \leq n$ for $1 \leq \ell \leq m$ such that $i_1 < i_2 < \cdots < i_m$, and all of $i_1, \ldots, i_m, j_1, \ldots, j_m$ are pairwise distinct.

For the term in the summand corresponding to i, j there is exactly one integer ℓ in $[n]$ such that $\ell \notin i, j$. Pair this integer with $n+1$ to form the edge $(\ell, n+1)$ in a perfect matching on $[n+1]$. The other edges of this perfect matching are given by the m distinct pairs $(i_1, j_1), \ldots, (i_m, j_m)$. Hence

$$L_n^{(m)}(K) = (-1)^{km+\binom{m}{2}} m! \sum_\pi \mathrm{CT}\left[F_{ab}(X) \prod_{\substack{(i,j) \in \pi \\ j \neq n+1}} \log\left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)\right].$$

Since $F_{ab}(X)$ is a skew-symmetric function (it is the product of a symmetric function times the skew-symmetric Vandermonde product $\Delta(X)$) we are in a position to apply Corollary 5.2. Thus

$$L_n^{(m)}(K) = (-1)^{km} n\, m!\, \mathrm{CT}\left[F_{ab}(X) \prod_{i=1}^m \log\left(1 - \frac{x_{2i-1}}{x_{2i}}\right)\right].$$

Finally we replace $X \mapsto X^{-1}$ using $F_{ab}(X^{-1}) = (-1)^m F_{ba}(X)$, and use the symmetry in a and b to find

$$L_n^{(m)}(K) = (-1)^{(k+1)m} n\, m!\, \mathrm{CT}\left[F_{ab}(X) \prod_{i=1}^m \log\left(1 - \frac{x_{2i}}{x_{2i-1}}\right)\right].$$

Equating this with (5.8) completes the proof of (1.9). $\qquad\qquad\square$

5.3. **A strengthening of Theorem 1.3.** As will be described in more detail below, using some further results of Adamović and Milas, it follows that the logarithmic Morris constant term identity (1.9) holds provided it holds for $a = b = 0$, i.e., provided the logarithmic analogue (1.2) of Dyson's identity holds. The proof of Theorem 1.3 given in the previous subsection implies that the latter follows from what could be termed the complex analogue of Dyson's identity, i.e., the $a = b = 0$ case of (4.5):

$$(5.10) \qquad \mathrm{CT}\left[\prod_{i=1}^n \prod_{1 \leq i < j \leq n} \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^u\right] = x^m P_n(x^2) \frac{\Gamma(1 + \frac{1}{2}nu)}{\Gamma^n(1 + \frac{1}{2}u)}.$$

As a consequence of all this, Theorem 1.3 can be strengthened as follows.

Theorem 5.3 (**Logarithmic Morris constant term identity, strong version**)**.** *The complex Dyson constant term identity* (5.10) *implies the logarithmic Morris constant term identity.*

To justify this claim, let $e_r(X)$ for $r = 0, 1, \ldots, n$ denote the rth elementary symmetric function. The $e_r(X)$ have generating function [26]

$$(5.11) \qquad \sum_{r=0}^n z^r e_r(X) = \prod_{i=1}^n (1 + zx_i).$$

Recalling definition (5.9) of F_{ab}, we now define $f_r(a) = f_r(a, b, k, n)$ by

$$f_r(a) = \mathrm{CT}\left[(-1)^r e_r(X) G_{ab}(X)\right],$$

where

$$G_{ab}(X) = F_{ab}(X) \prod_{i=1}^m \log\left(1 - \frac{x_{2i}}{x_{2i-1}}\right).$$

In the following b may be viewed as a formal or complex variable, but a must be taken to be an integer.

From (5.11) with $z = -1$ it follows that

$$(5.12) \qquad \sum_{r=0}^{n} f_r(a) = \mathrm{CT}\left[G_{a+1,b}(X)\right] = f_0(a+1).$$

According to [2, Theorem 7.1] (translated into the notation of this paper) we also have

$$(5.13) \qquad (n-r)(2b+rK)f_r(a) = (r+1)(2a+2+(n-r-1)K)f_{r+1}(a),$$

where we recall that $K := 2k+1$. Iterating this recursion yields

$$f_r(a) = f_0(a)\binom{n}{r}\prod_{i=0}^{r-1}\frac{2b+iK}{2a+2+(n-i-1)K}.$$

Summing both sides over r and using (5.12) leads to

$$f_0(a+1) = f_0(a)\, {}_2F_1\left[\begin{matrix}-n,\, 2b/K \\ 1-n-(2a+2)/K\end{matrix};1\right].$$

The ${}_2F_1$ series sums to $((2a+2b+2)/K)_n/((2a+2)K)_n$ by the Chu–Vandermonde sum [3, Corollary 2.2.3]. Therefore,

$$f_0(a+1) = f_0(a)\prod_{i=0}^{n-1}\frac{2a+2b+2+iK}{2a+2+iK}.$$

This functional equation can be solved to finally yield

$$f_0(a) = f_0(0)\prod_{i=0}^{n-1}\frac{(2a+2b+iK)!!(iK)!!}{(2b+iK)!!(2a+iK)!!}.$$

To summarise the above computations, we have established that

$$\mathrm{CT}\left[G_{ab}(X)\right] = \mathrm{CT}\left[G_{0,b}(X)\right]\prod_{i=0}^{n-1}\frac{(2a+2b+iK)!!(iK)!!}{(2b+iK)!!(2a+iK)!!}.$$

But since $G_{0,0}(X)$ is homogeneous of degree 0 it follows that

$$\mathrm{CT}\left[G_{0,b}(X)\right] = \mathrm{CT}\left[G_{0,0}(X)\right],$$

so that indeed the logarithmic Morris constant term identity is implied by its $a = b = 0$ case.

We finally remark that it seems highly plausible that the recurrence (5.13) has the following analogue for the complex Morris identity (enhanced by the term $(-1)^r e_r(X)$ in the kernel):

$$(n-r)(2b+ru)f_r(a) = (r+1)(2a+2+(n-r-1)u)f_{r+1}(a).$$

However, the fact that for general complex u the kernel is not a skew-symmetric function seems to prevent the proof of [2, Theorem 7.1] carrying over to the complex case in a straightforward manner.

6. The root system G_2

In this section we prove complex and logarithmic analogues of the Habsieger–Zeilberger identity (2.4).

Theorem 6.1 (Complex G_2 constant term identity). *For $u, v \in \mathbb{C}$ such that $\mathrm{Re}(1 + \frac{3}{2}u) > 0$ and $\mathrm{Re}(1 + \frac{3}{2}(u + v)) > 0$,*

$$
(6.1) \quad \mathrm{CT} \left[\left(1 - \frac{yz}{x^2}\right)^u \left(1 - \frac{xz}{y^2}\right)^u \left(1 - \frac{xy}{z^2}\right)^u \left(1 - \frac{x}{y}\right)^v \left(1 - \frac{y}{z}\right)^v \left(1 - \frac{z}{x}\right)^v \right]
$$
$$
= \frac{\cos\left(\frac{1}{2}\pi u\right) \cos\left(\frac{1}{2}\pi v\right) \Gamma(1 + \frac{3}{2}(u + v)) \Gamma(1 + \frac{3}{2}u) \Gamma(1 + u) \Gamma(1 + v)}{\Gamma(1 + \frac{3}{2}u + v) \Gamma(1 + u + \frac{1}{2}v) \Gamma(1 + \frac{1}{2}(u + v)) \Gamma^2(1 + \frac{1}{2}u) \Gamma(1 + \frac{1}{2}v)}.
$$

Proof. We adopt the method of proof employed by Habsieger and Zeilberger [15,38] in their proof of Theorem 2.1.

If $A(x, y, z; a, u)$ denotes the kernel on the left of the complex Morris identity (4.4) for $n = 3$, and if and $G(x, y, z; u, v)$ denotes the kernel on the left of (6.1), then

$$
G(x, y, z; u, v) = A(x/y, y/z, z/x, v, u).
$$

Therefore,

$$
\begin{aligned}
\mathrm{CT}\, G(x, y, z; u, v) &= \mathrm{CT}\, A(x/y, y/z, z/x; v, u) \\
&= \mathrm{CT}\, A(x, y, z; v, u)\big|_{xyz=1} \\
&= \sum_{b=0}^{\infty} \left[x^b y^b z^b\right] A(x, y, z; v, u) \\
&= \cos\left(\tfrac{1}{2}\pi u\right) \frac{\Gamma(1 + \frac{3}{2}u)}{\Gamma^3(1 + \frac{1}{2}u)} \,_3F_2 \left[\begin{matrix} -v, -\frac{1}{2}u - v, -u - v \\ 1 + \frac{1}{2}u, 1 + u \end{matrix}; 1\right],
\end{aligned}
$$

where the last equality follows from (4.4). Summing the $_3F_2$ series by Dixon's sum (4.7) with $(2a, b, c) \mapsto (-v, -\frac{1}{2}u - v, -u - v)$ completes the proof. $\qquad \square$

Just as for the root system A_{n-1}, we can use the complex G_2 constant term identity to proof a logarithmic analogue of (2.4).

Theorem 6.2. *Assume the representation of the G_2 root system as given in Section 2, and let Φ_s^+ and Φ_l^+ denote the set of positive short and positive long roots respectively. Define*

$$
G(K, M) = \frac{1}{3} \frac{(3K + 3M)!!(3K)!!(2K)!!(2M)!!}{(3K + 2M)!!(2K + M)!!(K + M)!!K!!K!!M!!}.
$$

Then for k, m nonnegative integers,

$$
\mathrm{CT}\left[\mathrm{e}^{-3\alpha_1 - 2\alpha_2} \log(1 - \mathrm{e}^{\alpha_2}) \prod_{\alpha \in \Phi_l^+} (1 - \mathrm{e}^{\alpha}) \prod_{\alpha \in \Phi_l} (1 - \mathrm{e}^{\alpha})^k \prod_{\alpha \in \Phi_s} (1 - \mathrm{e}^{\alpha})^m \right] = G(K, M),
$$

where $(K, M) := (2k + 1, 2m)$, and

$$
\mathrm{CT}\left[\mathrm{e}^{-2\alpha_1 - \alpha_2} \log(1 - \mathrm{e}^{\alpha_1}) \prod_{\alpha \in \Phi_s^+} (1 - \mathrm{e}^{\alpha}) \prod_{\alpha \in \Phi_l} (1 - \mathrm{e}^{\alpha})^k \prod_{\alpha \in \Phi_s} (1 - \mathrm{e}^{\alpha})^m \right] = G(K, M),
$$

where $(K, M) := (2k, 2m + 1)$.

We can more explicitly write the kernels of the two logarithmic G_2 constant term identities as

$$\frac{z^2}{xy}\left(1-\frac{x^2}{yz}\right)\left(1-\frac{y^2}{xz}\right)\left(1-\frac{xy}{z^2}\right)\log\left(1-\frac{y^2}{xz}\right)$$

$$\times\left(\left(1-\frac{x^2}{yz}\right)\left(1-\frac{y^2}{xz}\right)\left(1-\frac{z^2}{xy}\right)\left(1-\frac{yz}{x^2}\right)\left(1-\frac{xz}{y^2}\right)\left(1-\frac{xy}{z^2}\right)\right)^k$$

$$\times\left(\left(1-\frac{x}{y}\right)\left(1-\frac{x}{z}\right)\left(1-\frac{y}{x}\right)\left(1-\frac{y}{z}\right)\left(1-\frac{z}{x}\right)\left(1-\frac{z}{y}\right)\right)^m$$

and

$$\frac{z}{x}\left(1-\frac{x}{y}\right)\left(1-\frac{y}{z}\right)\left(1-\frac{x}{z}\right)\log\left(1-\frac{x}{y}\right)$$

$$\times\left(\left(1-\frac{x^2}{yz}\right)\left(1-\frac{y^2}{xz}\right)\left(1-\frac{z^2}{xy}\right)\left(1-\frac{yz}{x^2}\right)\left(1-\frac{xz}{y^2}\right)\left(1-\frac{xy}{z^2}\right)\right)^k$$

$$\times\left(\left(1-\frac{x}{y}\right)\left(1-\frac{x}{z}\right)\left(1-\frac{y}{x}\right)\left(1-\frac{y}{z}\right)\left(1-\frac{z}{x}\right)\left(1-\frac{z}{y}\right)\right)^m$$

respectively.

Proof of Theorem 6.2. If we differentiate (6.1) with respect to u, use the cyclic symmetry in (x,y,z) of the kernel on the left, and finally set $u=2k+1=K$, we get

$$3\,\mathrm{CT}\left[\log\left(1-\frac{xz}{y^2}\right)\left(1-\frac{yz}{x^2}\right)^K\left(1-\frac{xz}{y^2}\right)^K\left(1-\frac{xy}{z^2}\right)^K\left(1-\frac{x}{y}\right)^v\left(1-\frac{y}{z}\right)^v\left(1-\frac{z}{x}\right)^v\right]$$

$$=(-1)^{k+1}\frac{\pi}{2}\frac{\cos\left(\frac{1}{2}\pi v\right)\Gamma(1+\frac{3}{2}(K+v))\Gamma(1+\frac{3}{2}K)\Gamma(1+K)\Gamma(1+v)}{\Gamma(1+\frac{3}{2}K+v)\Gamma(1+K+\frac{1}{2}v)\Gamma(1+\frac{1}{2}(K+v))\Gamma^2(1+\frac{1}{2}K)\Gamma(1+\frac{1}{2}v)}.$$

Setting $v=2m=M$ and carrying out some simplifications using (5.6) and (5.7) completes the proof of the first claim.

In much the same way, if we differentiate (6.1) with respect to v, use the cyclic symmetry in (x,y,z) and set $v=2m+1=M$, we get

$$3\,\mathrm{CT}\left[\log\left(1-\frac{x}{y}\right)\left(1-\frac{yz}{x^2}\right)^u\left(1-\frac{xz}{y^2}\right)^u\left(1-\frac{xy}{z^2}\right)^u\left(1-\frac{x}{y}\right)^M\left(1-\frac{y}{z}\right)^M\left(1-\frac{z}{x}\right)^M\right]$$

$$=(-1)^{m+1}\frac{\pi}{2}\frac{\cos\left(\frac{1}{2}\pi u\right)\Gamma(1+\frac{3}{2}(u+M))\Gamma(1+\frac{3}{2}u)\Gamma(1+u)\Gamma(1+M)}{\Gamma(1+\frac{3}{2}u+M)\Gamma(1+u+\frac{1}{2}M)\Gamma(1+\frac{1}{2}(u+M))\Gamma^2(1+\frac{1}{2}u)\Gamma(1+\frac{1}{2}M)}.$$

Setting $u=2k=K$ yields the second claim. $\square$

7. Other root systems

Although further root systems admit complex analogues of the Macdonald constant term identities (1.3) or (2.3), it seems the existence of elegant logarithmic identities is restricted to A_{2n} and G_2. To see why this is so, we will discuss the root systems B_n, C_n and D_n. In order to treat all three simultaneously, it will be convenient to consider the more general *non-reduced* root system BC_n. With ϵ_i again denoting the ith standard unit vector in $\mathbb{R}^n$, this root system is given by

$$\Phi=\{\pm\epsilon_i:\ 1\leq i\leq n\}\cup\{\pm2\epsilon_i:\ 1\leq i\leq n\}\cup\{\pm\epsilon_i\pm\epsilon_j:\ 1\leq i<j\leq n\}.$$

Using the Selberg integral, Macdonald proved that [25]

$$(7.1) \quad \mathrm{CT}\left[\prod_{i=1}^{n}(1-x_i^{\pm})^a(1-x_i^{\pm 2})^b\prod_{1\leq i<j\leq n}(1-x_i^{\pm}x_j^{\pm})^k\right]$$

$$=\prod_{i=0}^{n-1}\frac{(k+ik)!(2a+2b+2ik)!(2b+2ik)!}{k!(a+b+ik)!(b+ik)!(a+2b+(n+i-1)k)!},$$

where a, b, k are nonnegative integers and where we have adopted the standard shorthand notation $(1-x^{\pm}):=(1-x)(1-1/x)$, $(1-x^{\pm 2}):=(1-x^2)(1-1/x^2)$, $(1-x^{\pm}y^{\pm}):=(1-xy)(1-x/y)(1-y/x)(1-1/xy)$. For $b=0$ the above identity is the B_n case of (2.3), for $a=0$ it is the C_n case of (2.3) and for $a=b=0$ it is the D_n case of (2.3) (and also (1.3)).

A first task in finding a complex analogue of (7.1) is to fix signatures τ_{ij} and σ_{ij} for $1\leq i,j\leq n$ such that

$$(7.2) \quad \prod_{1\leq i<j\leq n}(1-x_i^{\pm}x_j^{\pm})=\prod_{1\leq i<j\leq n}\left(1-(x_ix_j)^{\sigma_{ij}}\right)^2\left(1-\left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^2.$$

This would allow the rewriting of (7.1) as

$$(7.3) \quad \mathrm{CT}\left[\prod_{i=1}^{n}(1-x_i^{\pm})^a(1-x_i^{\pm 2})^b\prod_{1\leq i<j\leq n}\left(1-(x_ix_j)^{\sigma_{ij}}\right)^{2k}\left(1-\left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^{2k}\right]$$

$$=\prod_{i=0}^{n-1}\frac{(k+ik)!(2a+2b+2ik)!(2b+2ik)!}{k!(a+b+ik)!(b+ik)!(a+2b+(n+i-1)k)!},$$

after which $2k$ can be replaced by the complex variable u.

In the following we abbreviate (7.2) as $L(X)=R_{\sigma\tau}(X)$. In order to satisfy this equation, we note that for an arbitrary choice of the signatures σ_{ij} and τ_{ij},

$$L(X)=\prod_{1\leq i<j\leq n}\left(1-(x_ix_j)^{\pm\sigma_{ij}}\right)\left(1-\left(\frac{x_i}{x_j}\right)^{\pm\tau_{ij}}\right)$$

$$=\prod_{1\leq i<j\leq n}(x_ix_j)^{-\sigma_{ij}}\left(\frac{x_i}{x_j}\right)^{-\tau_{ij}}\left(1-(x_ix_j)^{\sigma_{ij}}\right)^2\left(1-\left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^2$$

$$=R_{\sigma\tau}(X)\prod_{i=1}^{n}x_i^{-\sum_{j>i}(\sigma_{ij}+\tau_{ij})-\sum_{j<i}(\sigma_{ji}-\tau_{ji})}.$$

We must therefore fix the σ_{ij} and τ_{ij} such that

$$(7.4) \quad \sum_{j=i+1}^{n}(\sigma_{ij}+\tau_{ij})+\sum_{j=1}^{i-1}(\sigma_{ji}-\tau_{ji})=0$$

for all $1\leq i\leq n$. If we sum this over all i this gives

$$0=\sum_{1\leq i<j\leq n}\sigma_{ij}\equiv\binom{n}{2}\pmod 2.$$

We thus conclude that a necessary condition for (7.4), and hence (7.2), to hold is that $n\equiv 0,1\pmod 4$. As we shall show next it is also a sufficient condition, as there are many solutions to (7.4) for the above two congruence classes.

Lemma 7.1. *For $n \equiv 1 \pmod 4$ define $m := (n-1)/2$ and $p := m/2$. If we choose τ_{ij} as in (3.1) and σ_{ij}, $1 \le i < j \le n$, as*

$$(7.5) \qquad \sigma_{ij} = \begin{cases} -1 & \text{if } p < j - i \le 3p, \\ 1 & \text{otherwise,} \end{cases}$$

then (7.4), and thus (7.2), is satisfied.

We can extend the definition of σ_{ij} to all $1 \le i, j \le n$ by setting $\sigma_{ij} = -\sigma_{ji}$. Then the matrix $\Sigma = (\sigma_{ij})_{1 \le i,j \le n}$ is a skew-symmetric Toeplitz matrix. For example, for $n = 5$ the above choice for the σ_{ij} generates

$$\Sigma = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix}.$$

Proof of Lemma 7.1. Note that by Lemma 3.1 we only need to prove that

$$\sum_{j=i+1}^{n} \sigma_{ij} + \sum_{j=1}^{i-1} \sigma_{ji} = 0.$$

If for $1 \le j \le i - 1$ we define $\sigma_{i,n+j} := -\sigma_{ij} = \sigma_{ji}$ then this becomes

$$(7.6) \qquad \sum_{j=i+1}^{n+i-1} \sigma_{ij} = 0.$$

We now observe that $\sigma_{i+1,j+1} = \sigma_{i,j}$. For $j < n$ or $j > n$ this follows immediately from (7.5). For $j = n$ it follows from $\sigma_{1,i+1} = \sigma_{i,n}$, which again follows from (7.5) since $p < n - i \le 3p$ is equivalent to $p < i \le 3p$. Thanks to $\sigma_{i+1,j+1} = \sigma_{i,j}$ we only need to check (7.6) for $i = 1$. Then

$$\sum_{j=2}^{n} \sigma_{ij} = \sum_{j=2}^{p+1} 1 - \sum_{j=p+2}^{3p+1} 1 + \sum_{j=3p+2}^{n} 1 = n - 4p - 1 = 0. \qquad \square$$

Lemma 7.2. *For $n \equiv 0 \pmod 4$ define $m := (n-2)/2$. If we choose τ_{ij} as in (3.1) and σ_{ij} as*

$$\sigma_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is even or } i + j = m + 2, \\ -1 & \text{if } i + j \text{ is odd and } i + j \ne m + 2, \end{cases}$$

then (7.4), and thus (7.2), is satisfied.

Proof. By a simple modification of Lemma 3.1 it follows that for n even and $m = (n-2)/2$,

$$\sum_{j=i+1}^{n} \tau_{ij} - \sum_{j=1}^{i-1} \tau_{ji} = \begin{cases} -1 & \text{if } 1 \le i \le m + 1, \\ 1 & \text{if } m + 1 < i < n. \end{cases}$$

Hence we must show that

$$\sum_{j=i+1}^{n} \sigma_{ij} + \sum_{j=1}^{i-1} \sigma_{ji} = \begin{cases} 1 & \text{if } 1 \le i \le m + 1, \\ -1 & \text{if } m + 1 < i < n. \end{cases}$$

But this is obvious. The sum on the left is over $n - 1$ terms, with one more odd $i + j$ then even $i + j$. Hence, without the exceptional condition on $i + j = m + 2$, the sum would always be -1. To have $i + j = m + 2$ as part of one of the two sums we must have $i \leq m + 1$, in which case one -1 is changed to a $+1$ leading to a sum of $+1$ instead of -1. $\qquad \square$

Lemmas 7.1 and 7.2 backed up with numerical data for $n = 4$ and $n = 5$ suggest the following generalisation of (7.3).

Conjecture 7.3 (**Complex BC$_n$ constant term identity**). *Let $n \equiv \zeta \pmod 4$ where $\zeta = 0, 1$, and let $u \in \mathbb{C}$ such that $\min\{\mathrm{Re}(1 + 2b + (n-1)u), \mathrm{Re}(1 + \frac{1}{2}nu)\} > 0$. Assume that τ_{ij} and σ_{ij} for $1 \leq i < j \leq n$ are signatures satisfying (7.4). Then there exists a polynomial $P_n(x)$, independent of a and b, such that $P_n(1) = 1$ and*

$$(7.7) \quad \mathrm{CT}\left[\prod_{i=1}^{n}(1 - x_i^{\pm})^a(1 - x_i^{\pm 2})^b \prod_{1 \leq i < j \leq n} \left(1 - (x_i x_j)^{\sigma_{ij}}\right)^u \left(1 - \left(\frac{x_i}{x_j}\right)^{\tau_{ij}}\right)^u \right]$$

$$= x^{n - \zeta} P_n(x^2) \frac{\Gamma(1 + \frac{1}{2}nu)}{\Gamma(1 + \frac{1}{2}(n-1)u)\Gamma^n(1 + \frac{1}{2}u)} \prod_{i=1}^{n-1} \frac{\Gamma(1 + iu)}{\Gamma(1 + (i - \frac{1}{2})u)}$$

$$\times \prod_{i=0}^{n-1} \frac{(\frac{1}{2} + \frac{1}{2}iu)_{a+b}(\frac{1}{2} + \frac{1}{2}iu)_b}{(1 + \frac{1}{2}(n + i - 1)u)_{a+2b}}$$

where $x = x(u) := \cos\left(\frac{1}{2}\pi u\right)$. Trivially, $P_1(x) = 1$. Conjecturally,

$$P_4(x) = 1 \quad and \quad P_5(x) = \frac{1}{15}(3 + 4x + 8x^2).$$

We note that the D$_4$ case of the conjecture, i.e., $a = b = 0$ and $n = 4$, is equivalent to the following new hypergeometric multisum identity

$$\sum \prod_{1 \leq i < j \leq 4} (-1)^{k_{ij}} \binom{u}{k_{ij}} \binom{u}{m_{ij}} = \cos^4(\tfrac{1}{2}\pi u) \frac{\Gamma(1 + u)\Gamma^2(1 + 2u)\Gamma(1 + 3u)}{\Gamma^5(1 + \frac{1}{2}u)\Gamma^2(1 + \frac{3}{2}u)\Gamma(1 + \frac{5}{2}u)},$$

where the sum is over $\{k_{ij}\}_{1 \leq i < j \leq 4}$ and $\{m_{ij}\}_{1 \leq i < j \leq 4}$ subject to the constraints

$$k_{12} - k_{13} - k_{14} + m_{12} + m_{13} - m_{14} = 0,$$
$$k_{12} - k_{23} + k_{24} - m_{12} + m_{23} - m_{24} = 0,$$
$$k_{13} - k_{23} + k_{34} + m_{13} - m_{23} - m_{34} = 0,$$
$$k_{14} + k_{24} - k_{34} - m_{14} + m_{24} - m_{34} = 0,$$

or, equivalently,

$$\sum_{\substack{1 \leq i < j \leq 4 \\ i = p \text{ or } j = p}} (\tau_{ij} k_{ij} + \sigma_{ij} m_{ij}) = 0 \qquad \text{for } 1 \leq p \leq 4.$$

Unfortunately, from the point of view of logarithmic constant term identities, (7.7) is not good news. On the right-hand side the exponent $n - \zeta$ of x is too high relative to the rank n of the root system. (Compare with $m = (n - 1)/2$ versus $n - 1$ for A$_{n-1}$.) If we write (7.7) as $L_n(u) = R_n(u)$ and define $K := 2k + 1$, then due to the factor $x^{n - \zeta}$, $R_n^{(j)}(K) = 0$ for all $1 \leq j < n - \zeta$. Much like the Morris

case, $R_n^{(n-\zeta)}(K)$ yields a ratio of products of double factorials:

$$R_n^{(n-\zeta)}(K) = (n-\zeta)! P_n(0) \frac{(nK)!!}{((n-1)K)!!(K!!)^n} \prod_{i=1}^{n-1} \frac{(2iK)!!}{((2i-1)K)!!}$$

$$\times \prod_{i=0}^{n-1} \frac{(2b+iK-1)!!(2a+2b+iK-1)!!((n+i-1)K)!!}{(2a+4b+(n+i-1)K)!!(iK-1)!!(iK-1)!!}.$$

However, if we differentiate $L_n(u)$ as many as $n-\zeta$ times, a large number of different types of logarithmic terms give a nonvanishing contribution to $L_n^{(n-\zeta)}(K)$—unlike type A where only terms with the same functional form (corresponding to perfect matchings) survive the specialisation $u = K$. For example, for $n = 4$ terms such as

$$\log^3\left(1 - \frac{x_1}{x_2}\right)\log\left(1 - \frac{1}{x_2 x_3}\right),$$

$$\log^2\left(1 - \frac{x_1}{x_2}\right)\log(1 - x_1 x_2)\log\left(1 - \frac{1}{x_2 x_3}\right),$$

$$\log^2\left(1 - \frac{x_1}{x_2}\right)\log^2(1 - x_3 x_4),$$

and many similar such terms, all give nonvanishing contributions.

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