Elliptic hypergeometric functions associated with root systems

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1.1 Introduction

Let $f = \sum_{n\geq 0} c_n$. The series f is called hypergeometric if the ratio c_{n+1}/c_n, viewed as a function of n, is rational. A simple example is the Taylor series $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. Similarly, if the ratio of consecutive terms of f is a rational function of q^n for some fixed q — known as the base — then f is called a basic hypergeometric series. An early example of a basic hypergeometric series is Euler's q-exponential function $e_q(z) = \sum_{n\geq 0} z^n/((1-q)\cdots(1-q^n))$. If we express the base as $q = \exp(2\pi\mathrm{i}/\omega)$ then c_{n+1}/c_n becomes a trigonometric function in n, with period ω. This motivates the more general definition of an elliptic hypergeometric series as a series f for which c_{n+1}/c_n is a doubly-periodic meromorphic function of n.

Elliptic hypergeometric series first appeared in 1988 in the work of Date et al. on exactly solvable lattice models in statistical mechanics [12]. They were formally defined and identified as mathematical objects of interest in their own right by Frenkel and Turaev in 1997 [19]. Subsequently, Spiridonov introduced the elliptic beta integral, initiating a parallel theory of elliptic hypergeometric integrals [61]. Together with Zhedanov [62, 72] he also showed that Rahman's [43] and Wilson's [74] theory of biorthogonal rational functions — itself a generalization of the Askey scheme [33] of classical orthogonal polynomials — can be lifted to the elliptic level.

All three aspects of the theory of elliptic hypergeometric functions (series, integrals and biorthogonal functions) have been generalized to higher dimensions, connecting them to root systems and Macdonald–Koornwinder theory. In [73] Warnaar introduced elliptic hypergeometric series associated to root systems, including a conjectural series evaluation of type C_n. This was recognized by van Diejen and Spiridonov [14, 15] as a discrete analogue of a multiple elliptic beta integral (or elliptic Selberg integral). They formulated the corresponding integral evaluation, again as a conjecture. This in turn led Rains [44, 46] to develop an elliptic analogue of Macdonald–Koornwinder theory, resulting in continuous as well as discrete biorthogonal elliptic functions attached to the non-reduced root system BC_n. In this theory, the elliptic multiple beta integral and its discrete analogue give the total mass of the biorthogonality measure.

Although a relatively young field, the theory of elliptic hypergeometric functions has already seen some remarkable applications. Many of these involve the multivariable theory.

In 2009, Dolan and Osborn showed that supersymmetric indices of four-dimensional supersymmetric quantum field theories are expressible in terms of elliptic hypergeometric integrals [18]. Conjecturally, such field theories admit electric–magnetic dualities known as Seiberg dualities, such that dual theories have the same index. This leads to non-trivial identities between elliptic hypergeometric integrals (or, for so called confining theories, to integral evaluations). In some cases these are known identities, which thus gives a partial confirmation of the underlying Seiberg duality. However, in many cases it leads to new identities that are yet to be rigorously proved, see e.g. [20, 21, 67, 68, 69] and the recent survey [50]. Another application of elliptic hypergeometric functions is to exactly solvable lattice models in statistical mechanics. We already mentioned the occurrence of elliptic hypergeometric series in the work of Date et al., but more recently it was shown that elliptic hypergeometric integrals are related to solvable lattice models with continuous spin parameters [3, 65]. In the one-variable case, this leads to a generalization of many well-known discrete models such as the two-dimensional Ising model and the chiral Potts model. This relation to solvable lattice models has been extended to multivariable elliptic hypergeometric integrals in [2, 4, 65]. Further applications of multivariable elliptic hypergeometric functions pertain to elliptic Calogero–Sutherland-type systems [51, 64] and the representation theory of elliptic quantum groups [55].

In the current chapter we give a survey of elliptic hypergeometric functions associated with root systems, comprised of three main parts. The first two form in essence an annotated table of the main evaluation and transformation formulas for elliptic hypergeometric integrals and series on root systems. The third and final part gives an introduction to Rains' elliptic Macdonald–Koornwinder theory (in part also developed by Coskun and Gustafson [10]). Due to space limitations, applications will not be covered here and we refer the interested reader to the above-mentioned papers and references therein.

Rather than throughout the text, references for the main results are given in the form of separate notes at the end of each section. These notes also contain some brief historical comments and further pointers to the literature.

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1.1.1 Preliminaries

Elliptic functions are doubly-periodic meromorphic functions on $\mathbb{C}$. That is, a meromorphic function $g:\mathbb{C}\to\mathbb{C}$ is elliptic if there exist ω_1,ω_2 with $\mathrm{Im}(\omega_1/\omega_2)>0$ such that $g(z+\omega_1)=g(z+\omega_2)=g(z)$ for all $z\in\mathbb{C}$. If we define the elliptic nome p by $p=\mathrm{e}^{2\pi\mathrm{i}\omega_1/\omega_2}$ (so that $|p|<1$) then $z\mapsto \mathrm{e}^{2\pi\mathrm{i}z/\omega_2}$ maps the period parallelogram spanned by ω_1,ω_2 to an annulus with radii $|p|$ and 1. Given an elliptic function g with periods ω_1 and ω_2, the function $f:\mathbb{C}^*\to\mathbb{C}$ defined by

$$g(z)=f(\mathrm{e}^{2\pi\mathrm{i}z/\omega_2})$$

is thus periodic in an annulus:

$$f(pz) = f(z).$$

By mild abuse of terminology we will also refer to such f as an elliptic function. A more precise description would be elliptic function in multiplicative form.

The basic building blocks for elliptic hypergeometric functions are

$$\begin{aligned}
\theta(z) = \theta(z;p) &= \prod_{i=0}^{\infty}(1-zp^i)(1-p^{i+1}/z),\\
(z)_k = (z;q,p)_k &= \prod_{i=0}^{k-1}\theta(zq^i;p),\\
\Gamma(z) = \Gamma(z;p,q) &= \prod_{i,j=0}^{\infty}\frac{1-p^{i+1}q^{j+1}/z}{1-zp^iq^j},
\end{aligned}$$

known as the modified theta function, elliptic shifted factorial and elliptic gamma function, respectively. Note that the dependence on the elliptic nome p and base q will mostly be suppressed from our notation. One exception is the q-shifted factorial $(z;q)_\infty = \prod_{i\geq 0}(1-zq^i)$ which, to avoid possible confusion, will never be shortened to $(z)_\infty$.

For simple relations satisfied by the above three functions we refer the reader to [22]. Here we only note that the elliptic gamma function is symmetric in p and q and satisfies

$$\Gamma(pq/z)\Gamma(z) = 1 \quad \text{and} \quad \Gamma(qz) = \theta(z)\Gamma(z).$$

For each of the functions $\theta(z)$, $(z)_k$ and $\Gamma(z)$, we employ condensed notation as exemplified by

$$\begin{aligned}
\theta(z_1,\dots,z_m) &= \theta(z_1)\cdots\theta(z_m),\\
(az^{\pm})_k &= (az)_k(a/z)_k,\\
\Gamma(tz^{\pm}w^{\pm}) &= \Gamma(tzw)\Gamma(tz/w)\Gamma(tw/z)\Gamma(t/zw).
\end{aligned}$$

In the trigonometric case $p = 0$ we have $\theta(z) = 1 - z$, so that $(z)_k$ becomes a standard q-shifted factorial and $\Gamma(z)$ a rescaled version of the q-gamma function.

We also need elliptic shifted factorials indexed by partitions. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of non-negative integers such that only finitely many λ_i are non-zero. The number of positive λ_i is called the length of λ and denoted by $l(\lambda)$. The sum of the λ_i will be denoted by $|\lambda|$. The diagram of λ consists of the points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq i \leq l(\lambda)$ and $1 \leq j \leq \lambda_i$. If these inequalities hold for $(i, j) \in \mathbb{Z}^2$ we write $(i, j) \in \lambda$. Reflecting the diagram in the main diagonal yields the conjugate partition λ'. In other words, the rows of λ are the columns of λ' and vice versa. A standard statistic on partitions is

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda_i'}{2}.$$

For a pair of partitions λ, μ we write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all $i \geq 1$. In particular, when $l(\lambda) \leq n$

and $\lambda_i \le N$ for all $1 \le i \le N$ we write $\lambda \subset (N^n)$. Similarly, we write $\mu \prec \lambda$ if the interlacing conditions $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots$ hold.

With t an additional fixed parameter, we will need the following three types of elliptic shifted factorials index by partitions:

$$\begin{aligned}
(z)_\lambda = (z;q,t;p)_\lambda &= \prod_{(i,j)\in\lambda} \theta(zq^{j-1}t^{1-i}) = \prod_{i\ge 1} (zt^{1-i})_{\lambda_i},\\
C^-_\lambda(z) = C^-_\lambda(z;q,t;p) &= \prod_{(i,j)\in\lambda} \theta(zq^{\lambda_i-j}t^{\lambda'_j-i}),\\
C^+_\lambda(z) = C^+_\lambda(z;q,t;p) &= \prod_{(i,j)\in\lambda} \theta(zq^{\lambda_i+j-1}t^{2-\lambda'_j-i}).
\end{aligned}$$

By $\theta(pz) = -z^{-1}\theta(z)$ it follows that $(a)_\lambda$ is quasi-periodic:

$$(p^k z)_\lambda = \Big[(-z)^{-|\lambda|}q^{-n(\lambda')}t^{n(\lambda)}\Big]^k p^{-\binom{k}{2}|\lambda|}\,(z)_\lambda, \qquad k\in\mathbb{Z}. \tag{1.1.1}$$

Again we use condensed notation so that, for example, $(a_1,\dots,a_k)_\lambda = (a_1)_\lambda\cdots(a_k)_\lambda$.

1.1.2 Elliptic Weyl denominators

Suppressing their p-dependence we define

$$\begin{aligned}
\Delta^{\mathrm{A}}(x_1,\dots,x_{n+1}) &= \prod_{1\le i<j\le n+1} x_j\,\theta(x_i/x_j),\\
\Delta^{\mathrm{C}}(x_1,\dots,x_n) &= \prod_{j=1}^n \theta(x_j^2) \prod_{1\le i<j\le n} x_j\,\theta(x_i x_j^{\pm}),
\end{aligned}$$

which are essentially the Weyl denominators of the affine root systems $\mathrm{A}_n^{(1)}$ and $\mathrm{C}_n^{(1)}$ [29, 37]. Although we have no need for the theory of affine root systems here, it may be instructive to explain the connection to the root system $\mathrm{C}_n^{(1)}$ (the case of $\mathrm{A}_n^{(1)}$ is similar). The Weyl denominator of an affine root system R is the formal product $\prod_{\alpha\in R_+}(1-\mathrm{e}^{-\alpha})^{m(\alpha)}$, where R_+ denotes the set of positive roots and m is a multiplicity function. For $\mathrm{C}_n^{(1)}$, the positive roots are

$$\begin{aligned}
&m\,\delta, && m\ge 1,\\
&m\,\delta + 2\varepsilon_i, && m\ge 0,\ 1\le i\le n,\\
&m\,\delta - 2\varepsilon_i, && m\ge 1,\ 1\le i\le n,\\
&m\,\delta + \varepsilon_i \pm \varepsilon_j, && m\ge 0,\ 1\le i<j\le n,\\
&m\,\delta - \varepsilon_i \pm \varepsilon_j, && m\ge 1,\ 1\le i<j\le n,
\end{aligned}$$

where $\varepsilon_1,\dots,\varepsilon_n$ are the coordinate functions on $\mathbb{R}^n$ and δ is the constant function 1. The roots $m\,\delta$ have multiplicity n, while all other roots have multiplicity 1. Thus, the Weyl denominator for $\mathrm{C}_n^{(1)}$ is

$$\prod_{m=0}^{\infty}\Big((1-\mathrm{e}^{-(m+1)\delta})^n\prod_{i=1}^{n}(1-\mathrm{e}^{-m\delta-2\varepsilon_i})(1-\mathrm{e}^{-(m+1)\delta+2\varepsilon_i})$$
$$\times\prod_{1\le i<j\le n}(1-\mathrm{e}^{-m\delta-\varepsilon_i-\varepsilon_j})(1-\mathrm{e}^{-(m+1)\delta+\varepsilon_i+\varepsilon_j})(1-\mathrm{e}^{-m\delta-\varepsilon_i+\varepsilon_j})(1-\mathrm{e}^{-(m+1)\delta+\varepsilon_i-\varepsilon_j})\Big).$$

It is easy to check that this equals $(p;p)_\infty^n\, x_1^0 x_2^{-1}\cdots x_n^{1-n}\,\Delta^{\mathrm{C}}(x_1,\dots,x_n)$, where $p=\mathrm{e}^{-\delta}$ and $x_i=\mathrm{e}^{-\varepsilon_i}$.

We will consider elliptic hypergeometric series containing the factor $\Delta^{\mathrm{A}}(xq^k)$ or $\Delta^{\mathrm{C}}(xq^k)$, where $xq^k=(x_1q^{k_1},x_2q^{k_2},\dots,x_rq^{k_r})$, the $k_i\in\mathbb{Z}$ being summation indices, and $r=n+1$ in the case of A_n and $r=n$ in the case of C_n. We refer to these as A_n and C_n series, respectively. In the case of A_n, the summation variables typically satisfy a restriction of the form $k_1+\cdots+k_{n+1}=N$. Eliminating k_{n+1} gives series containing the $\mathrm{A}_{n-1}^{(1)}$ Weyl denominator times $\prod_{i=1}^n\theta(ax_iq^{k_i+|k|})$, where $a=q^{-N}/x_{n+1}$; these will also be viewed as A_n series.

Similarly, A_n integrals contain the factor

$$\frac{1}{\prod_{1\le i<j\le n+1}\Gamma(z_i/z_j,z_j/z_i)},\tag{1.1.2a}$$

where $z_1\cdots z_{n+1}=1$, while C_n integrals contain

$$\frac{1}{\prod_{i=1}^n\Gamma(z_i^{\pm2})\prod_{1\le i<j\le n}\Gamma(z_i^{\pm}z_j^{\pm})}.\tag{1.1.2b}$$

If we denote the expression (1.1.2b) by $g(z)$ then it is easy to verify that, for $k\in\mathbb{Z}^n$,

$$\frac{g(zq^k)}{g(z)}=\left(\prod_{i=1}^n q^{-nk_i-(n+1)k_i^2}z_i^{-2(n+1)k_i}\right)\frac{\Delta^{\mathrm{C}}(zq^k)}{\Delta^{\mathrm{C}}(z)}.$$

A similar relation holds for the A-type factors. This shows that the series can be considered as discrete analogues of the integrals. In fact in many instances the series can be obtained from the integrals via residue calculus.

It is customary to attach a “type” to hypergeometric integrals associated with root systems, although different authors have used slightly different definitions of type. As the terminology will be used here, in type I integrals the only factors containing more than one integration variable are (1.1.2), while type II integrals contain twice the number of such factors. For example, $\mathrm{C}_n^{(\mathrm{II})}$ integrals contain the factor $\prod_{i<j}\Gamma(tz_i^{\pm}z_j^{\pm})/\Gamma(z_i^{\pm}z_j^{\pm})$. It may be noted that, under appropriate assumptions on the parameters,

$$\lim_{q\to1}\lim_{p\to0}\prod_{i<j}\frac{\Gamma(q^{t\pm z_i\pm z_j})}{\Gamma(q^{\pm z_i\pm z_j})}=\lim_{q\to1}\prod_{i<j}\frac{(q^{\pm z_i\pm z_j};q)_\infty}{(q^{t\pm z_i\pm z_j};q)_\infty}$$
$$=\prod_{i<j}(1-z_i^{\pm}z_j^{\pm})^t=\prod_{i<j}\Big((z_i+z_i^{-1})-(z_j+z_j^{-1})\Big)^{2t}.$$

For this reason C_n beta integrals of type II are sometimes referred to as elliptic Selberg integrals. There are also integrals containing an intermediate number of factors. We will refer to these as integrals of mixed type.

1.2 Integrals

Throughout this section we assume that $|q| < 1$. Whenever possible, we have restricted the parameters in such a way that the integrals may be taken over the n-dimensional complex torus $\mathbb{T}^n$. However, all results can be extended to more general parameter domains by appropriately deforming $\mathbb{T}^n$.

When $n = 1$ all the stated A_n and C_n beta integral evaluations reduce to Spiridonov's elliptic beta integral.

1.2.1 A_n *beta integrals*

We will present four A_n beta integrals. In each of these the integrand contains a variable z_{n+1} which is determined from the integration variables $z_1, \dots, z_n$ by the relation $z_1 \cdots z_{n+1} = 1$. To shorten the expressions we define the constant κ_n^{A} by

$$\kappa_n^{\mathrm{A}} = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(n+1)!(2\pi \mathrm{i})^n}.$$

For $1 \le i \le n+2$, let $|s_i| < 1$ and $|t_i| < 1$, such that $ST = pq$, where $S = s_1 \cdots s_{n+2}$ and $T = t_1 \cdots t_{n+2}$. Then we have the type I integral

$$\kappa_n^{\mathrm{A}} \int_{\mathbb{T}^n} \frac{\prod_{i=1}^{n+2} \prod_{j=1}^{n+1} \Gamma(s_i z_j, t_i/z_j)}{\prod_{1 \le i < j \le n+1} \Gamma(z_i/z_j, z_j/z_i)} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} = \prod_{i=1}^{n+2} \Gamma(S/s_i, T/t_i) \prod_{i,j=1}^{n+2} \Gamma(s_i t_j). \tag{1.2.1}$$

Next, let $|s| < 1$, $|t| < 1$, $|s_i| < 1$ and $|t_i| < 1$ for $1 \le i \le 3$, such that $s^{n-1} t^{n-1} s_1 s_2 s_3 t_1 t_2 t_3 = pq$. Then we have the type II integral

$$\begin{aligned}
&\kappa_n^{\mathrm{A}} \int_{\mathbb{T}^n} \prod_{1 \le i < j \le n+1} \frac{\Gamma(s z_i z_j, t/z_i z_j)}{\Gamma(z_i/z_j, z_j/z_i)} \prod_{i=1}^{3} \prod_{j=1}^{n+1} \Gamma(s_i z_j, t_i/z_j) \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} \\
&= \begin{cases}
\displaystyle\prod_{m=1}^{N} \Big(\Gamma(s^m t^m) \prod_{1 \le i < j \le 3} \Gamma(s^{m-1} t^m s_i s_j, s^m t^{m-1} t_i t_j) \Big) \prod_{i,j=1}^{3} \Gamma(s^{m-1} t^{m-1} s_i t_j) \Big) \\
\qquad \times \Gamma(s^{N-1} s_1 s_2 s_3, t^{N-1} t_1 t_2 t_3) \displaystyle\prod_{i=1}^{3} \Gamma(s^N s_i, t^N t_i), & n = 2N, \\
\displaystyle\prod_{m=1}^{N} \Big(\Gamma(s^m t^m) \prod_{1 \le i < j \le 3} \Gamma(s^{m-1} t^m s_i s_j, s^m t^{m-1} t_i t_j) \Big) \prod_{m=1}^{N+1} \prod_{i,j=1}^{3} \Gamma(s^{m-1} t^{m-1} s_i t_j) \\
\qquad \times \Gamma(s^{N+1}, t^{N+1}) \displaystyle\prod_{1 \le i < j \le 3} \Gamma(s^N s_i s_j, t^N t_i t_j), & n = 2N+1.
\end{cases}
\end{aligned} \tag{1.2.2}$$

Let $|t| < 1$, $|t_i| < 1$ for $1 \le i \le n+3$ and $|t| < |s_i| < |t|^{-1}$ for $1 \le i \le n$, where $t^2 t_1 \cdots t_{n+3} = pq$. Then,

$$\kappa_n^{\mathrm{A}} \int_{\mathbb{T}^n} \prod_{1\le i<j\le n+1} \frac{1}{\Gamma(z_i/z_j, z_j/z_i, t^2 z_i z_j)} \prod_{j=1}^{n+1} \Big(\prod_{i=1}^{n} \Gamma(ts_i^{\pm} z_j) \prod_{i=1}^{n+3} \Gamma(t_i/z_j) \Big) \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n}$$
$$= \prod_{i=1}^{n} \prod_{j=1}^{n+3} \Gamma(ts_i^{\pm} t_j) \prod_{1\le i<j\le n+3} \frac{1}{\Gamma(t^2 t_i t_j)}, \quad (1.2.3)$$

which is an integral of mixed type.

Finally, let $|t| < 1$, $|s_i| < 1$ for $1 \le i \le 4$ and $|t_i| < 1$ for $1 \le i \le n+1$ such that $t^{n-1} s_1 \cdots s_4 T = pq$, where $T = t_1 \cdots t_{n+1}$. Then we have a second mixed-type integral:

$$\kappa_n^{\mathrm{A}} \int_{\mathbb{T}^n} \prod_{1\le i<j\le n+1} \frac{\Gamma(tz_i z_j)}{\Gamma(z_i/z_j, z_j/z_i)} \prod_{j=1}^{n+1} \Big(\prod_{i=1}^{4} \Gamma(s_i z_j) \prod_{i=1}^{n+1} \Gamma(t_i/z_j) \Big) \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n}$$
$$= \begin{cases} \Gamma(T) \displaystyle\prod_{i=1}^{4} \frac{\Gamma(t^N s_i)}{\Gamma(t^N T s_i)} \prod_{1\le i<j\le n+1} \Gamma(tt_i t_j) \prod_{i=1}^{4} \prod_{j=1}^{n+1} \Gamma(s_i t_j), & n = 2N, \\ \dfrac{\Gamma(t^{N+1}, T)}{\Gamma(t^{N+1} T)} \displaystyle\prod_{1\le i<j\le 4} \Gamma(t^N s_i s_j) \prod_{1\le i<j\le n+1} \Gamma(tt_i t_j) \prod_{i=1}^{4} \prod_{j=1}^{n+1} \Gamma(s_i t_j), & n = 2N+1. \end{cases} \quad (1.2.4)$$

1.2.2 C_n *beta integrals*

We will give three C_n beta integrals. They all involve the constant

$$\kappa_n^{\mathrm{C}} = \frac{(p;p)_\infty^n (q;q)_\infty^n}{n! 2^n (2\pi \mathrm{i})^n}.$$

Let $|t_i| < 1$ for $1 \le i \le 2n+4$ such that $t_1 \cdots t_{2n+4} = pq$. We then have the following C_n beta integral of type I

$$\kappa_n^{\mathrm{C}} \int_{\mathbb{T}^n} \prod_{1\le i<j\le n} \frac{1}{\Gamma(z_i^{\pm} z_j^{\pm})} \prod_{j=1}^{n} \frac{\prod_{i=1}^{2n+4} \Gamma(t_i z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} = \prod_{1\le i<j\le 2n+4} \Gamma(t_i t_j). \quad (1.2.5)$$

Next, let $|t| < 1$ and $|t_i| < 1$ for $1 \le i \le 6$ such that $t^{2n-2} t_1 \cdots t_6 = pq$. We then have the type II C_n beta integral

$$\kappa_n^{\mathrm{C}} \int_{\mathbb{T}^n} \prod_{1\le i<j\le n} \frac{\Gamma(tz_i^{\pm} z_j^{\pm})}{\Gamma(z_i^{\pm} z_j^{\pm})} \prod_{j=1}^{n} \frac{\prod_{i=1}^{6} \Gamma(t_i z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} = \prod_{m=1}^{n} \Big(\frac{\Gamma(t^m)}{\Gamma(t)} \prod_{1\le i<j\le 6} \Gamma(t^{m-1} t_i t_j) \Big). \quad (1.2.6)$$

This is the elliptic Selberg integral mentioned in the introduction.

At this point it is convenient to introduce notation for more general C_n integrals of type II. For m a non-negative integer, let $|t| < 1$ and $|t_i| < 1$ for $1 \le i \le 2m+6$ such that

$$t^{2n-2} t_1 \cdots t_{2m+6} = (pq)^{m+1}. \quad (1.2.7)$$

We then define

$$J_{\mathrm{C}_n}^{(m)}(t_1, \dots, t_{2m+6}; t) = \kappa_n^{\mathrm{C}} \int_{\mathbb{T}^n} \prod_{1\le i<j\le n} \frac{\Gamma(tz_i^{\pm} z_j^{\pm})}{\Gamma(z_i^{\pm} z_j^{\pm})} \prod_{j=1}^{n} \frac{\prod_{i=1}^{2m+6} \Gamma(t_i z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n}. \quad (1.2.8)$$

Note that (1.2.6) gives a closed-form evaluation for the integral $J^{(0)}_{\mathrm{C}_n}$. As outlined in [46, Appendix], $J^{(m)}_{\mathrm{C}_n}$ can be continued to a single-valued meromorphic function in the parameters t_i and t subject to the constraint (1.2.7). For generic values of the parameters this continuation is obtained by replacing the integration domain with an appropriate deformation of $\mathbb{T}^n$. We can now state the second C_n beta integral of type II as

$$J^{(n-1)}_{\mathrm{C}_n}(t_1,\dots,t_4,s_1,\dots,s_n,pq/ts_1,\dots,pq/ts_n;t) = \Gamma(t)^n\prod_{l=1}^{n}\prod_{1\le i<j\le 4}\Gamma(t^{l-1}t_it_j)\prod_{i=1}^{n}\prod_{j=1}^{4}\frac{\Gamma(s_it_j)}{\Gamma(ts_i/t_j)}, \tag{1.2.9}$$

where $t^{n-2}t_1t_2t_3t_4=1$. In this identity it is necessary to work with an analytic continuation of (1.2.8) since the inequalities $|t_i|$, $|t|<1$ are incompatible with $t^{n-2}t_1t_2t_3t_4=1$ for $n\ge 2$.

1.2.3 Integral transformations

We now turn to integral transformations, starting with integrals of type I. For m a non-negative integer we introduce the notation

$$I^{(m)}_{\mathrm{A}_n}(s_1,\dots,s_{m+n+2};t_1,\dots,t_{m+n+2}) = \kappa^{\mathrm{A}}_n\int_{\mathbb{T}^n}\frac{\prod_{i=1}^{m+n+2}\prod_{j=1}^{n+1}\Gamma(s_iz_j,t_i/z_j)}{\prod_{1\le i<j\le n+1}\Gamma(z_i/z_j,z_j/z_i)}\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n},$$

where $|s_i|<1$ and $|t_i|<1$ for all i, $\prod_{i=1}^{m+n+2}s_it_i=(pq)^{m+1}$ and $z_1\cdots z_{n+1}=1$. We also define

$$I^{(m)}_{\mathrm{C}_n}(t_1,\dots,t_{2m+2n+4}) = \kappa^{\mathrm{C}}_n\int_{\mathbb{T}^n}\prod_{1\le i<j\le n}\frac{1}{\Gamma(z_i^{\pm}z_j^{\pm})}\prod_{j=1}^{n}\frac{\prod_{i=1}^{2m+2n+4}\Gamma(t_iz_j^{\pm})}{\Gamma(z_j^{\pm 2})}\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n},$$

where $|t_i|<1$ for all i and $t_1\cdots t_{2m+2n+4}=(pq)^{m+1}$. The A_n integral satisfies

$$I^{(m)}_{\mathrm{A}_n}(s_1,\dots,s_{m+n+2};t_1,\dots,t_{m+n+2}) = I^{(m)}_{\mathrm{A}_n}(s_1\zeta,\dots,s_{m+n+2}\zeta;t_1/\zeta,\dots,t_{m+n+2}/\zeta)$$

for ζ an $(n+1)$-th root of unity, whereas the C_n integral is invariant under simultaneous negation of all of the t_i. We further note that (1.2.1) and (1.2.5) provide closed-form evaluations of $I^{(0)}_{\mathrm{A}_n}$ and $I^{(0)}_{\mathrm{C}_n}$, respectively.

For the integral $I^{(m)}_{\mathrm{A}_n}$, the following transformation reverses the roles of m and n:

$$I^{(m)}_{\mathrm{A}_n}(s_1,\dots,s_{m+n+2};t_1,\dots,t_{m+n+2}) = \prod_{i,j=1}^{m+n+2}\Gamma(s_it_j)\cdot I^{(n)}_{\mathrm{A}_m}\Big(\frac{\lambda}{s_1},\dots,\frac{\lambda}{s_{m+n+2}};\frac{pq}{\lambda t_1},\dots,\frac{pq}{\lambda t_{m+n+2}}\Big), \tag{1.2.10}$$

where $\lambda^{m+1}=s_1\cdots s_{m+n+2}$, $(pq/\lambda)^{m+1}=t_1\cdots t_{m+n+2}$. Moreover, for $t_1\cdots t_{2m+2n+4}=(pq)^{m+1}$, there is an analogous transformation of type C:

$$I^{(m)}_{\mathrm{C}_n}(t_1,\dots,t_{2m+2n+4}) = \prod_{1\le i<j\le 2m+2n+4}\Gamma(t_it_j)\cdot I^{(n)}_{\mathrm{C}_m}\Big(\frac{\sqrt{pq}}{t_1},\dots,\frac{\sqrt{pq}}{t_{2m+2n+4}}\Big). \tag{1.2.11}$$

It is easy to check that $I^{(m)}_{\mathrm{A}_1}(t_1,\dots,t_{m+3};t_{m+4},\dots,t_{2m+6}) = I^{(m)}_{\mathrm{C}_1}(t_1,\dots,t_{2m+6})$. Thus, combining (1.2.10) and (1.2.11) leads to

$$I^{(1)}_{\mathrm{A}_n}(s_1,\dots,s_{n+3};t_1,\dots,t_{n+3}) = \prod_{1\le i<j\le n+3} \Gamma(S/s_is_j, T/t_it_j)\cdot I^{(1)}_{\mathrm{C}_n}(s_1/v,\dots,s_{n+3}/v,t_1v,\dots,t_{n+3}v), \quad (1.2.12)$$

where $S = s_1\cdots s_{n+3}$, $T = t_1\cdots t_{n+3}$ and $v^2 = S/pq = pq/T$. Since $I^{(1)}_{\mathrm{C}_n}$ is symmetric, (1.2.12) implies non-trivial symmetries of $I^{(1)}_{\mathrm{A}_n}$, such as

$$I^{(1)}_{\mathrm{A}_n}(s_1,\dots,s_{n+3};t_1,\dots,t_{n+3}) = \prod_{i=1}^{n+2}\Gamma(s_it_{n+3}, t_is_{n+3}, S/s_is_{n+3}, T/t_it_{n+3}) \times I^{(1)}_{\mathrm{A}_n}(s_1/v,\dots,s_{n+2}/v, s_{n+3}v^n; t_1v,\dots,t_{n+2}v, t_{n+3}/v^n), \quad (1.2.13)$$

where, with the same definitions of S and T as above, $ST = (pq)^2$ and $v^{n+1} = St_{n+3}/pqs_{n+3}$.

$$J^{(1)}_{\mathrm{C}_n}(t_1,\dots,t_8;t) = \prod_{m=1}^{n}\Big(\prod_{1\le i<j\le 4}\Gamma(t^{m-1}t_it_j)\prod_{5\le i<j\le 8}\Gamma(t^{m-1}t_it_j)\Big) \times J^{(1)}_{\mathrm{C}_n}(t_1v,\dots,t_4v,t_5/v,\dots,t_8/v;t), \quad (1.2.14)$$

where $t^{2n-2}t_1\cdots t_8 = (pq)^2$ and $v^2 = pqt^{1-n}/t_1t_2t_3t_4 = t^{n-1}t_5t_6t_7t_8/pq$. Iterating this transformation yields a symmetry of $J^{(1)}_{\mathrm{C}_n}$ under the Weyl group of type E_7 [46].

We conclude with a transformation between C_n and C_m integrals of type II:

$$\begin{aligned} &J^{(m+n-1)}_{\mathrm{C}_n}(t_1,\dots,t_4,s_1,\dots,s_{m+n},pq/ts_1,\dots,pq/ts_{m+n};t)\\ &\quad= \Gamma(t)^{n-m}\prod_{1\le i<j\le 4}\frac{\prod_{l=1}^{n}\Gamma(t^{l-1}t_it_j)}{\prod_{l=1}^{m}\Gamma(t^{l+n-m-1}t_it_j)}\prod_{i=1}^{m+n}\prod_{j=1}^{4}\frac{\Gamma(s_it_j)}{\Gamma(ts_i/t_j)}\\ &\qquad\times J^{(m+n-1)}_{\mathrm{C}_m}(t/t_1,\dots,t/t_4,s_1,\dots,s_{m+n},pq/ts_1,\dots,pq/ts_{m+n};t), \end{aligned} \quad (1.2.15)$$

where $t_1t_2t_3t_4 = t^{m-n+2}$.

1.2.4 Notes

For $p = 0$ the integrals (1.2.1), (1.2.2), (1.2.5) and (1.2.6) are due to Gustafson [23, 24], the integral (1.2.4) to Gustafson and Rakha [25] and the transformation (1.2.13) to Denis and Gustafson [13]. None of the $p = 0$ instances of (1.2.3), (1.2.9)–(1.2.12), (1.2.14) and (1.2.15) were known prior to the elliptic case.

For general p, van Diejen and Spiridonov conjectured the type I C_n beta integral (1.2.5) and showed that it implies the elliptic Selberg integral (1.2.6) [14, 15]. A rigorous derivation of the classical Selberg integral as a special limit of (1.2.6) is due to Rains [45]. Spiridonov [62] conjectured the type I A_n beta integral (1.2.1) and showed that, combined with (1.2.5), it implies the type II A_n beta integral (1.2.2), as well as the integral (1.2.4) of mixed type. He also showed that (1.2.1) implies (1.2.13). The first proofs of the fundamental type I integrals (1.2.1) and (1.2.5) were obtained by Rains [46]. For subsequent proofs of (1.2.1), (1.2.5) and (1.2.6), see [63], [49, 63] and [27], respectively. In [46] Rains also proved the integral transformations (1.2.10), (1.2.11) and (1.2.14), and gave further transformations analogous to (1.2.14). The integral (1.2.3) of mixed type is due to Spiridonov and Warnaar [70]. The transformation (1.2.15), which includes (1.2.9) as its $m = 0$ case, was conjectured by Rains [47] and also appears in [67]. It was first proved by van der Bult in [8] and subsequently proved and generalized to an identity for the "interpolation kernel" (an analytic continuation of the elliptic interpolation functions R^*_λ of Section 1.4) in [48].

Several of the integral identities surveyed here have analogues for $|q| = 1$. In the case of (1.2.1), (1.2.3) and (1.2.5) these were found in [17], and the unit-circle analogue of (1.2.6) is given in [17].

In [62] Spiridonov gives one more C_n beta integral, which lacks the $p \leftrightarrow q$ symmetry present in all the integrals considered here, and is more elementary in that it follows as a determinant of one-variable beta integrals.

In [47] Rains conjectured several quadratic integral transformations involving the interpolation functions R^*_λ. These conjectures were proved in [9, 48]. In special cases, they simplify to transformations for the function $J^{(2)}_{C_n}$.

Motivated by quantum field theories on lens spaces, Spiridonov [66] evaluated certain finite sums of C_n integrals, both for type I and type II. In closely related work, Kels and Yamazaki [32] obtained transformation formulas for finite sums of A_n and C_n integrals of type I.

As mentioned in the introduction, the recent identification of elliptic hypergeometric integrals as indices in supersymmetric quantum field theory by Dolan and Osborn [18] has led to a large number of conjectured integral evaluations and transformations [20, 21, 67, 68, 69]. It is too early to give a survey of the emerging picture, but it is clear that the identities stated in this section are a small sample from a much larger collection of identities.

1.3 Series

In this section we give the most important summation and transformation formulas for elliptic hypergeometric series associated to A_n and C_n. In the $n = 1$ case all summations except for (1.3.7) simplify to the elliptic Jackson summation of Frenkel and Turaev. Similarly, most transformations may be viewed as generalizations of the elliptic Bailey transformation.

1.3.1 A_n summations

The following A_n elliptic Jackson summation is a discrete analogue of the multiple beta integral (1.2.1):

$$\sum_{\substack{k_1,\dots,k_{n+1}\geq 0\\ k_1+\cdots+k_{n+1}=N}} \frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)} \prod_{i=1}^{n+1} \frac{\prod_{j=1}^{n+2}(x_i a_j)_{k_i}}{(bx_i)_{k_i}\prod_{j=1}^{n+1}(qx_i/x_j)_{k_i}} = \frac{(b/a_1,\dots,b/a_{n+2})_N}{(q,bx_1,\dots,bx_{n+1})_N}, \tag{1.3.1a}$$

where $b = a_1\cdots a_{n+2}x_1\cdots x_{n+1}$. Using the constraint on the summation indices to eliminate k_{n+1}, this identity can be written less symmetrically as

$$\sum_{\substack{k_1,\dots,k_n\geq 0\\ |k|\leq N}} \frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)} \prod_{i=1}^{n}\biggl(\frac{\theta(ax_iq^{k_i+|k|})}{\theta(ax_i)} \frac{(ax_i)_{|k|}\prod_{j=1}^{n+2}(x_ib_j)_{k_i}}{(aq^{N+1}x_i,aqx_i/c)_{k_i}\prod_{j=1}^{n}(qx_i/x_j)_{k_i}}\biggr)\frac{(q^{-N},c)_{|k|}}{\prod_{i=1}^{n+2}(aq/b_i)_{|k|}}\,q^{|k|}$$
$$= c^N\prod_{i=1}^{n}\frac{(aqx_i)_N}{(aqx_i/c)_N}\prod_{i=1}^{n+2}\frac{(aq/cb_i)_N}{(aq/b_i)_N}, \tag{1.3.1b}$$

where $b_1\cdots b_{n+2}cx_1\cdots x_n = a^2q^{N+1}$. By analytic continuation one can then deduce the companion identity

$$\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}\biggl(\frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)}\prod_{i=1}^{n}\biggl(\frac{\theta(ax_iq^{k_i+|k|})}{\theta(ax_i)}\frac{(ax_i)_{|k|}(dx_i,ex_i)_{k_i}}{(aq^{N_i+1}x_i)_{|k|}(aqx_i/b,aqx_i/c)_{k_i}}\biggr)$$
$$\times\frac{(b,c)_{|k|}\,q^{|k|}}{(aq/d,aq/e)_{|k|}}\prod_{i,j=1}^{n}\frac{(q^{-N_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}\biggr) = \frac{(aq/cd,aq/bd)_{|N|}}{(aq/d,aq/bcd)_{|N|}}\prod_{i=1}^{n}\frac{(aqx_i,aqx_i/bc)_{N_i}}{(aqx_i/b,aqx_i/c)_{N_i}}, \tag{1.3.1c}$$

where $bcde = a^2q^{|N|+1}$.

For the discrete analogue of the type II integral (1.2.2) we refer the reader to the Notes at the end of this section.

Our next result corresponds to a discretization of (1.2.3):

$$\sum_{\substack{k_1,\dots,k_{n+1}\geq 0\\ k_1+\cdots+k_{n+1}=N}} \frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)}\prod_{1\leq i<j\leq n+1}\frac{1}{(x_ix_j)_{k_i+k_j}}\prod_{i=1}^{n+1}\frac{q^{\binom{k_i}{2}}x_i^{k_i}\prod_{j=1}^{n}(x_ia_j^{\pm})_{k_i}}{(bx_i,q^{1-N}x_i/b)_{k_i}\prod_{j=1}^{n+1}(qx_i/x_j)_{k_i}}$$
$$= (-bq^{N-1})^N\frac{\prod_{i=1}^{n}(ba_i^{\pm})_N}{(q)_N\prod_{i=1}^{n+1}(bx_i^{\pm})_N}. \tag{1.3.2}$$

Mimicking the steps that led from (1.3.1a) to (1.3.1c), the identity (1.3.2) can be rewritten as a sum over an n-dimensional rectangle, see [53]. Some authors have associated (1.3.2) and related results with the root system D_n rather than A_n.

Finally, the following summation is a discrete analogue of (1.2.4):

$$\sum_{\substack{k_1,\dots,k_{n+1}\geq 0,\\ k_1+\cdots+k_{n+1}=N}} \frac{\Delta^A(xq^k)}{\Delta^A(x)} \prod_{1\leq i<j\leq n+1} q^{k_ik_j}(x_ix_j)_{k_i+k_j} \prod_{i=1}^{n+1} \frac{\prod_{j=1}^4 (x_ib_j)_{k_i}}{x_i^{k_i}\prod_{j=1}^{n+1}(qx_i/x_j)_{k_i}}$$
$$= \begin{cases} \dfrac{(Xb_1,Xb_2,Xb_3,Xb_4)_N}{X^N(q)_N}, & n \text{ odd},\\[2ex] \dfrac{(X,Xb_1b_2,Xb_1b_3,Xb_1b_4)_N}{(Xb_1)^N(q)_N}, & n \text{ even}, \end{cases} \tag{1.3.3}$$

where $X = x_1\cdots x_{n+1}$ and $q^{N-1}b_1\cdots b_4X^2 = 1$.

1.3.2 C_n *summations*

The following C_n elliptic Jackson summation is a discrete analogue of (1.2.5):

$$\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \frac{\Delta^{\mathrm{C}}(xq^k)}{\Delta^{\mathrm{C}}(x)} \prod_{i=1}^n \frac{(bx_i,cx_i,dx_i,ex_i)_{k_i}\,q^{k_i}}{(qx_i/b,qx_i/c,qx_i/d,qx_i/e)_{k_i}} \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j,x_ix_j)_{k_i}}{(qx_i/x_j,q^{N_j+1}x_ix_j)_{k_i}}$$
$$= \frac{\prod_{i,j=1}^n (qx_ix_j)_{N_i}}{\prod_{1\leq i<j\leq n}(qx_ix_j)_{N_i+N_j}} \frac{(q/bc,q/bd,q/cd)_{|N|}}{\prod_{i=1}^n (qx_i/b,qx_i/c,qx_i/d,q^{-N_i}e/x_i)_{N_i}}, \tag{1.3.4}$$

where $bcde = q^{|N|+1}$.

The discrete analogues of the type II integrals (1.2.6) and (1.2.9) are most conveniently expressed in terms of the series

$${}_{r+1}V_r^{(n)}(a;b_1,\dots,b_{r-4}) = \sum_\lambda \Bigg(\prod_{i=1}^n \frac{\theta(at^{2-2i}q^{2\lambda_i})}{\theta(at^{2-2i})} \frac{(at^{1-n},b_1,\dots,b_{r-4})_\lambda}{(qt^{n-1},aq/b_1,\dots,aq/b_{r-4})_\lambda}$$
$$\times \prod_{1\leq i<j\leq n} \left(\frac{\theta(t^{j-i}q^{\lambda_i-\lambda_j},at^{2-i-j}q^{\lambda_i+\lambda_j})}{\theta(t^{j-i},at^{2-i-j})} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}(at^{3-i-j})_{\lambda_i+\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}(aqt^{1-i-j})_{\lambda_i+\lambda_j}}\right) q^{|\lambda|}t^{2n(\lambda)}\Bigg), \tag{1.3.5}$$

where the summation is over partitions $\lambda = (\lambda_1,\dots,\lambda_n)$ of length at most n. Note that this implicitly depends on t as well as p and q. When $b_{r-4} = q^{-N}$ with $N \in \mathbb{Z}_{\geq 0}$, this becomes a terminating series, with sum ranging over partitions $\lambda \subset (N^n)$. The series (1.3.5) is associated with C_n since

$$\prod_{i=1}^n \frac{\theta(at^{2-2i}q^{2\lambda_i})}{\theta(at^{2-2i})} \prod_{1\leq i<j\leq n} \frac{\theta(t^{j-i}q^{\lambda_i-\lambda_j},at^{2-i-j}q^{\lambda_i+\lambda_j})}{\theta(t^{j-i},at^{2-i-j})} = q^{-n(\lambda)}\frac{\Delta^{\mathrm{C}}(xq^\lambda)}{\Delta^{\mathrm{C}}(x)},$$

with $x_i = \sqrt{a}t^{1-i}$.

Using the above notation, the discrete analogue of (1.2.6) is

$${}_{10}V_9^{(n)}(a;b,c,d,e,q^{-N}) = \frac{(aq,aq/bc,aq/bd,aq/cd)_{(N^n)}}{(aq/b,aq/c,aq/d,aq/bcd)_{(N^n)}}, \tag{1.3.6}$$

where $bcdet^{n-1} = a^2q^{N+1}$. This is the C_n summation mentioned in the introduction.

Next, we give a discrete analogue of (1.2.9):

$$
{}_{2r+8}V^{(n)}_{2r+7}\Big(a;t^{1-n}b,\frac{a}{b},c_1q^{k_1},\dots,c_rq^{k_r},\frac{aq}{c_1},\dots,\frac{aq}{c_r},q^{-N}\Big)
= \frac{(aq,qt^{n-1})_{(N^n)}}{(bq,aqt^{n-1}/b)_{(N^n)}}\prod_{i=1}^{r}\frac{(c_ib/a,c_it^{n-1}/b)_{(k_i^n)}}{(c_i,c_it^{n-1}/a)_{(k_i^n)}}, \qquad (1.3.7)
$$

where the k_i are non-negative integers such that $k_1+\cdots+k_r=N$. As the summand contains the factors $(c_iq^{k_i})_\lambda/(c_i)_\lambda$, this is a so-called Karlsson–Minton-type summation.

Finally, we have the C_n summation

$$
\sum_{k_1,\dots,k_n=0}^{N}\frac{\Delta^{\mathrm{C}}(xq^k)}{\Delta^{\mathrm{C}}(x)}\prod_{i=1}^{n}\frac{(x_i^2,bx_i,cx_i,dx_i,ex_i,q^{-N})_{k_i}}{(q,qx_i/b,qx_i/c,qx_i/d,qx_i/e,q^{N+1}x_i^2)_{k_i}}\,q^{k_i}
= \prod_{1\le i<j\le n}\frac{\theta(q^Nx_ix_j)}{\theta(x_ix_j)}\prod_{i=1}^{n}\frac{(qx_i^2,q^{2-i}/bc,q^{2-i}/bd,q^{2-i}/cd)_N}{(qx_i/b,qx_i/c,qx_i/d,q^{-N}e/x_i)_N}, \qquad (1.3.8)
$$

where $bcde=q^{N-n+2}$. There is a corresponding integral evaluation [62], which was mentioned in §1.2.4.

1.3.3 Series transformations

Several of the transformations stated below have companion identities (similar to the different versions of (1.3.1)) which will not be stated explicitly.

The following A_n Bailey transformation is a discrete analogue of (1.2.13):

$$
\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}\Bigg(\frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)}\prod_{i=1}^{n}\Big(\frac{\theta(ax_iq^{k_i+|k|})}{\theta(ax_i)}\,\frac{(ax_i)_{|k|}(ex_i,fx_i,gx_i)_{k_i}}{(aq^{N_i+1}x_i)_{|k|}(aqx_i/b,aqx_i/c,aqx_i/d)_{k_i}}\Big)\\
&\qquad\times\frac{(b,c,d)_{|k|}\,q^{|k|}}{(aq/e,aq/f,aq/g)_{|k|}}\prod_{i,j=1}^{n}\frac{(q^{-N_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}\Bigg)\\
&=\Big(\frac{a}{\lambda}\Big)^{|N|}\frac{(\lambda q/f,\lambda q/g)_{|N|}}{(aq/f,aq/g)_{|N|}}\prod_{i=1}^{n}\frac{(aqx_i,\lambda qx_i/d)_{N_i}}{(\lambda qx_i,aqx_i/d)_{N_i}}\\
&\quad\times\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n}\Bigg(\frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)}\prod_{i=1}^{n}\Big(\frac{\theta(\lambda x_iq^{k_i+|k|})}{\theta(\lambda x_i)}\,\frac{(\lambda x_i)_{|k|}(\lambda ex_i/a,fx_i,gx_i)_{k_i}}{(\lambda q^{N_i+1}x_i)_{|k|}(aqx_i/b,aqx_i/c,\lambda qx_i/d)_{k_i}}\Big)\\
&\qquad\times\frac{(\lambda b/a,\lambda c/a,d)_{|k|}\,q^{|k|}}{(aq/e,\lambda q/f,\lambda q/g)_{|k|}}\prod_{i,j=1}^{n}\frac{(q^{-N_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}}\Bigg),
\end{aligned}
\qquad (1.3.9)
$$

where $bcdefg=a^3q^{|N|+2}$ and $\lambda=a^2q/bce$. For $be=aq$ the sum on the right trivializes and the transformation simplifies to (1.3.1c).

The next transformation, which relates an A_n and a C_n series, is a discrete analogue of

(1.2.12):

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \frac{\Delta^{\mathrm{C}}(xq^k)}{\Delta^{\mathrm{C}}(x)} \prod_{i=1}^{n} \frac{(bx_i, cx_i, dx_i, ex_i, fx_i, gx_i)_{k_i}\, q^{k_i}}{(qx_i/b, qx_i/c, qx_i/d, qx_i/e, qx_i/f, qx_i/g)_{k_i}} \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j, x_ix_j)_{k_i}}{(qx_i/x_j, q^{N_j+1}x_ix_j)_{k_i}} \\
&= \frac{\prod_{i,j=1}^{n}(qx_ix_j)_{N_i}}{\prod_{1\le i<j\le n}(qx_ix_j)_{N_i+N_j}} \frac{(\lambda q/e, \lambda q/f, q/ef)_{|N|}}{\prod_{i=1}^{n}(\lambda qx_i, qx_i/e, qx_i/f, q^{-N_i}g/x_i)_{N_i}} \\
&\quad\times \sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \bigg(\frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)} \prod_{i=1}^{n} \Big(\frac{\theta(\lambda x_i q^{k_i+|k|})}{\theta(\lambda x_i)} \frac{(\lambda x_i)_{|k|}(ex_i, fx_i, gx_i)_{k_i}}{(\lambda q^{N_i+1}x_i)_{|k|}(qx_i/b, qx_i/c, qx_i/d)_{k_i}} \Big) \\
&\qquad\qquad \times \frac{(\lambda b, \lambda c, \lambda d)_{|k|}\, q^{|k|}}{(\lambda q/e, \lambda q/f, \lambda q/g)_{|k|}} \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j)_{k_i}}{(qx_i/x_j)_{k_i}} \bigg),
\end{aligned} \tag{1.3.10}$$

where $bcdefg = q^{|N|+2}$ and $\lambda = q/bcd$. For $bc = q$ this reduces to (1.3.4).

The discrete analogue of (1.2.10) provides a duality between A_n and A_m elliptic hypergeometric series:

$$\begin{aligned}
&\sum_{\substack{k_1,\dots,k_{n+1}\ge 0\\ k_1+\dots+k_{n+1}=N}} \frac{\Delta^{\mathrm{A}}(xq^k)}{\Delta^{\mathrm{A}}(x)} \prod_{i=1}^{n+1} \frac{\prod_{j=1}^{m+n+2}(x_ia_j)_{k_i}}{\prod_{j=1}^{m+1}(x_iy_j)_{k_i}\prod_{j=1}^{n+1}(qx_i/x_j)_{k_i}} \\
&\qquad = \sum_{\substack{k_1,\dots,k_{m+1}\ge 0\\ k_1+\dots+k_{m+1}=N}} \frac{\Delta^{\mathrm{A}}(yq^k)}{\Delta^{\mathrm{A}}(y)} \prod_{i=1}^{m+1} \frac{\prod_{j=1}^{m+n+2}(y_i/a_j)_{k_i}}{\prod_{j=1}^{n+1}(y_ix_j)_{k_i}\prod_{j=1}^{m+1}(qy_i/y_j)_{k_i}},
\end{aligned} \tag{1.3.11}$$

where $w_1\cdots w_{m+1} = x_1\cdots x_{n+1}a_1\cdots a_{m+n+2}$. For $m = 0$ this reduces to (1.3.1a).

We next give a discrete analogue of (1.2.11). When M_i and N_i for $i = 1, \dots, n$ are non-negative integers and $bcde = q^{|N|-|M|+1}$, then

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{N_1,\dots,N_n} \bigg(\frac{\Delta^{\mathrm{C}}(xq^k)}{\Delta^{\mathrm{C}}(x)} \prod_{i=1}^{n} \frac{(bx_i, cx_i, dx_i, ex_i)_{k_i}\, q^{k_i}}{(qx_i/b, qx_i/c, qx_i/d, qx_i/e)_{k_i}} \\
&\qquad \times \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(q^{M_j}x_iy_j, qx_i/y_j)_{k_i}}{(x_iy_j, q^{1-M_j}x_i/y_j)_{k_i}} \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j, x_ix_j)_{k_i}}{(qx_i/x_j, q^{N_j+1}x_ix_j)_{k_i}} \bigg) \\
&= q^{-|N||M|} \frac{(q/bc, q/bd, q/cd)_{|N|}}{(q^{-|N|}bc, q^{-|N|}bd, q^{-|N|}cd)_{|M|}} \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{(q^{-N_j}y_i/x_j)_{M_i}}{(y_i/x_j)_{M_i}} \\
&\quad \times \frac{\prod_{i,j=1}^{n}(qx_ix_j)_{N_i}\prod_{1\le i<j\le m}(y_iy_j)_{M_i+M_j}}{\prod_{i,j=1}^{m}(y_iy_j)_{M_i}\prod_{1\le i<j\le n}(qx_ix_j)_{N_i+N_j}} \frac{\prod_{i=1}^{m}(by_i, cy_i, dy_i, q^{1-M_i}/y_ie)_{M_i}}{\prod_{i=1}^{n}(qx_i/b, qx_i/c, qx_i/d, q^{-N_i}e/x_i)_{N_i}} \\
&\quad \times \sum_{k_1,\dots,k_m=0}^{M_1,\dots,M_m} \bigg(\frac{\Delta^{\mathrm{C}}(q^{-1/2}yq^k)}{\Delta^{\mathrm{C}}(q^{-1/2}y)} \prod_{i=1}^{m} \frac{(y_i/b, y_i/c, y_i/d, y_i/e)_{k_i}\, q^{k_i}}{(by_i, cy_i, dy_i, ey_i)_{k_i}} \\
&\qquad\qquad \times \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{(q^{N_j}y_ix_j, y_i/x_j)_{k_i}}{(y_ix_j, q^{-N_j}y_i/x_j)_{k_i}} \prod_{i,j=1}^{m} \frac{(q^{-M_j}y_i/y_j, q^{-1}y_iy_j)_{k_i}}{(qy_i/y_j, q^{M_j}y_iy_j)_{k_i}} \bigg).
\end{aligned} \tag{1.3.12}$$

Recalling the notation (1.3.5), we have the following discrete analogue of (1.2.14):

$$
\begin{aligned}
&{}_{12}V_{11}^{(n)}(a;b,c,d,e,f,g,q^{-N})\\
&\qquad = \frac{(aq,aq/ef,\lambda q/e,\lambda q/f)_{(N^n)}}{(\lambda q,\lambda q/ef,aq/e,aq/f)_{(N^n)}}\,{}_{12}V_{11}^{(n)}\Big(\lambda;\frac{\lambda b}{a},\frac{\lambda c}{a},\frac{\lambda d}{a},e,f,g,q^{-N}\Big), \qquad (1.3.13)
\end{aligned}
$$

where $bcdefgt^{n-1} = a^3q^{N+2}$ and $\lambda = a^2q/bcd$.

Finally, the following Karlsson–Minton-type transformation is an analogue of (1.2.15):

$$
\begin{aligned}
&{}_{2r+8}V_{2r+7}^{(n)}\Big(a;bt^{1-n},\frac{aq^{-M}}{b},c_1q^{k_1},\dots,c_rq^{k_r},\frac{aq}{c_1},\dots,\frac{aq}{c_r},q^{-N}\Big)\\
&\quad = \frac{(aq,t^{n-1}q)_{(N^n)}}{(bq,t^{n-1}aq/b)_{(N^n)}}\,\frac{(bq,t^{n-1}bq/a)_{(M^n)}}{(b^2q/a,t^{n-1}q)_{(M^n)}}\prod_{i=1}^{r}\frac{(bc_i/a,t^{n-1}c_i/b)_{(k_i^n)}}{(c_i,t^{n-1}c_i/a)_{(k_i^n)}}\\
&\qquad \times\,{}_{2r+8}V_{2r+7}^{(n)}\Big(\frac{b^2}{a};bt^{1-n},\frac{bq^{-N}}{a},\frac{bc_1q^{k_1}}{a},\dots,\frac{bc_rq^{k_r}}{a},\frac{bq}{c_1},\dots,\frac{bq}{c_r},q^{-M}\Big), \qquad (1.3.14)
\end{aligned}
$$

where the k_i are non-negative integers such that $k_1+\cdots+k_r = M+N$.

1.3.4 Notes

For $p = 0$ the A_n summations (1.3.1), (1.3.2) and (1.3.3) are due to Milne [39], Schlosser [58] (see also also [5]), and Gustafson and Rakha [25], respectively. The $p = 0$ case of the C_n summation (1.3.4) was found independently by Denis and Gustafson [13], and Milne and Lilly [40]. The $p = 0$ case of the C_n summation (1.3.8) is due to Schlosser [59]. The $p = 0$ case of the transformation (1.3.9) was obtained, again independently, by Denis and Gustafson [13], and Milne and Newcomb [41]. The $p = 0$ cases of (1.3.10) and (1.3.11) are due to Bhatnagar and Schlosser [6] and Kajihara [30], respectively. The $p = 0$ instances of (1.3.6), (1.3.7), (1.3.12), (1.3.13) and (1.3.14) were not known prior to the elliptic cases.

For general p, the A_n summations (1.3.1) and (1.3.2) were first obtained by Rosengren [53] using an elementary inductive argument. A derivation of (1.3.1) from (1.2.1) using residue calculus is given in [62] and a similar derivation of (1.3.2) from (1.2.3) in [70]. The summation (1.3.3) was conjectured by Spiridonov [62] and proved, independently, by Ito and Noumi [28] and by Rosengren [56].

As mentioned in the introduction, Warnaar [73] conjectured the C_n summation (1.3.6). He also proved the more elementary C_n summation (1.3.8). Van Diejen and Spiridonov [14, 16] showed that the C_n summations (1.3.4) and (1.3.6) follow from the (at that time conjectural) integral identities (1.2.5) and (1.2.6). This in particular implied the first proof of (1.3.6) for $p = 0$. For general p, the summations (1.3.4) and (1.3.6) were proved by Rosengren [52, 53], using the case $N = 1$ of Warnaar's identity (1.3.8). Subsequent proofs of (1.3.6) were given in [10, 26, 44, 46]. The proofs in [10, 44] establish the more general sum (1.4.14) for elliptic binomial coefficients. In [46] the identity (1.3.6) arises as a special case of the discrete biorthogonality relation (1.4.27) for the elliptic biorthogonal functions $\tilde{R}_\lambda$.

The transformations (1.3.9) and (1.3.10) were obtained by Rosengren [53], together with two more A_n transformations that are not surveyed here. The transformation (1.3.11) was obtained independently by Kajihara and Noumi [31] and Rosengren [54]. Both these papers contain further transformations that can be obtained by iterating (1.3.11). The transformation (1.3.12) was proved by Rains (personal communication, 2003) by specializing the parameters of [46, Theorem 7.9] to a union of geometric progressions. It appeared explicitly in [34, Theorem 4.2] using a similar approach to Rains. The transformation (1.3.13) was conjectured by Warnaar [73] and established by Rains [44] using the symmetry of the expression (1.4.16) below. The transformation (1.3.14) is stated somewhat implicitly by Rains [47]; it includes (1.3.7) as a special case.

A discrete analogue of the type II A_n beta integral (1.2.2) has been conjectured by Spiridonov and Warnaar in [71]. Surprisingly, this conjecture contains the C_n identity (1.3.6) as a special case.

The summation formula (1.3.8) can be obtained as a determinant of one-dimensional summations. Further summations and transformations of determinantal type are given in [57]. The special case $t = q$ of (1.3.6) and (1.3.13) is also closely related to determinants, see [60].

Transformations related to the sum (1.3.3) are discussed in [56]. In their work on elliptic Bailey lemmas on root systems, Bhatnagar and Schlosser [7] discovered two further elliptic Jackson summations for A_n, as well as corresponding transformation formulas. For none of these an integral analogue is known. Langer, Schlosser and Warnaar [36] proved a curious A_n transformation formula, which is new even in the one-variable case.

1.4 Elliptic Macdonald–Koornwinder theory

A function f on $(\mathbb{C}^*)^n$ is said to be BC_n-symmetric if it is invariant under the action of the hyperoctahedral group $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$. Here the symmetric group S_n acts by permuting the variables and $\mathbb{Z}/2\mathbb{Z}$ by replacing a variable with its reciprocal. The interpolation functions

$$R^*_\lambda(x_1, \ldots, x_n; a, b; q, t; p), \tag{1.4.1}$$

introduced independently by Rains [44, 46] and by Coskun and Gustafson [10], are BC_n-symmetric elliptic functions that generalize Okounkov's BC_n interpolation Macdonald polynomials [42] as well as the Macdonald polynomials of type A [38]. They form the building blocks of Rains' more general BC_n-symmetric functions [44, 46]

$$\tilde{R}_\lambda(x_1, \ldots, x_n; a : b, c, d; u, v; q, t; p).$$

The $\tilde{R}_\lambda$ are an elliptic generalization of the Koornwinder polynomials [35], themselves a generalization to BC_n of the Askey–Wilson polynomials [1]. The price one pays for ellipticity is that the functions R^*_λ and $\tilde{R}_\lambda$ are neither polynomial nor orthogonal. The latter do however form a biorthogonal family, and for $n = 1$ they reduce to the continuous biorthogonal functions of Spiridonov (elliptic case) [62] and Rahman (the $p = 0$ case) [43] and, appropriately specialized, to the discrete biorthogonal functions of Spiridonov and Zhedanov (elliptic case) [72] and Wilson (the $p = 0$ case) [74].

There are a number of ways to define the elliptic interpolation functions. Here we will describe them via a branching rule. The branching coefficient $c_{\lambda\mu}$ is a complex function on $(\mathbb{C}^*)^7$, indexed by a pair of partitions λ, μ. It is defined to be zero unless $\lambda \succ \mu$, in which case

$$\begin{aligned} c_{\lambda\mu}(z; a, b; q, t, T; p) &= \frac{(aTz^{\pm}, pqa/bt)_\lambda}{(aTz^{\pm}, pqa/bt)_\mu} \frac{(pqz^{\pm}/bt, T)_\mu}{(pqz^{\pm}/b, tT)_\lambda} \\ &\quad\times \prod_{\substack{(i,j)\in\lambda \\ \lambda'_j=\mu'_j}} \frac{\theta(q^{\lambda_i+j-1}t^{2-i-\lambda'_j}aT/b)}{\theta(pq^{\mu_i-j+1}t^{\mu'_j-i})} \prod_{\substack{(i,j)\in\lambda \\ \lambda'_j\neq\mu'_j}} \frac{\theta(q^{\lambda_i-j}t^{\lambda'_j-i+1})}{\theta(pq^{\mu_i+j}t^{-i-\mu'_j}aT/b)} \\ &\quad\times \prod_{\substack{(i,j)\in\mu \\ \lambda'_j=\mu'_j}} \frac{\theta(pq^{\lambda_i-j+1}t^{\lambda'_j-i})}{\theta(q^{\mu_i+j-1}t^{1-i-\mu'_j}aT/b)} \prod_{\substack{(i,j)\in\mu \\ \lambda'_j\neq\mu'_j}} \frac{\theta(pq^{\lambda_i+j}t^{1-i-\lambda'_j}aT/b)}{\theta(q^{\mu_i-j}t^{\mu'_j-i+1})}. \end{aligned} \tag{1.4.2}$$

From (1.1.1) and the invariance under the substitution $z \mapsto z^{-1}$ it follows that $c_{\lambda\mu}$ is a BC_1-symmetric elliptic function of z. The elliptic interpolation functions are uniquely determined by the branching rule

$$R^*_\lambda(x_1, \dots, x_{n+1}; a, b; q, t; p) = \sum_\mu c_{\lambda\mu}(x_{n+1}; a, b; q, t, t^n; p) R^*_\mu(x_1, \dots, x_n; a, b; q, t; p), \tag{1.4.3}$$

subject to the initial condition $R^*_\lambda(-; a, b; q, t; p) = \delta_{\lambda,0}$. It immediately follows that the interpolation function (1.4.1) vanishes if $l(\lambda) > n$. From the symmetry and ellipticity of the branching coefficient it also follows that the interpolation functions are BC_1-symmetric and elliptic in each of the x_i. S_n-symmetry (and thus BC_n-symmetry), however, is not manifest and is a consequence of the non-trivial fact that

$$\sum_\mu c_{\lambda\mu}(z; a, b; q, t, T; p) c_{\mu\nu}(w; a, b; q, t, T/t; p) \tag{1.4.4}$$

is a symmetric function in z and w; see also the discussion around (1.4.17) below.

In the remainder of this section $x = (x_1, \dots, x_n)$. Comparison of their respective branching rules shows that Okounkov's BC_n interpolation Macdonald polynomials $P^*_\lambda(x; q, t, s)$ and the ordinary Macdonald polynomials $P_\lambda(x; q, t)$ arise in the limit as

$$P^*_\lambda(x; q, t, s) = \lim_{p\to 0} \big(-s^2 t^{2n-2}\big)^{-|\lambda|} q^{-n(\lambda')} t^{2n(\lambda)} \frac{(t^n)_\lambda}{C^-_\lambda(t)} R^*_\lambda(st^\delta x; s, p^{1/2}b; q, t; p),$$

and

$$P_\lambda(x; q, t) = \lim_{z\to\infty} z^{-|\lambda|} \lim_{p\to 0} \big(-at^{n-1}\big)^{-|\lambda|} q^{-n(\lambda')} t^{2n(\lambda)} \frac{(t^n)_\lambda}{C^-_\lambda(t)} R^*_\lambda(zx; a, p^{1/2}b; q, t; p),$$

where $\delta = (n-1, \dots, 1, 0)$ is the staircase partition of length $n-1$, $st^\delta x = (st^{n-1}x_1, \dots, st^0 x_n)$ and $zx = (zx_1, \dots, zx_n)$.

Many standard properties of $P_\lambda(x; q, t)$ and $P^*_\lambda(x; q, t, s)$ have counterparts for the elliptic interpolation functions. Here we have space for only a small selection. Up to normalization, Okounkov's BC_n interpolation Macdonald polynomials are uniquely determined by symmetry and vanishing properties. The latter carry over to the elliptic case as follows:

$$R^*_\mu(aq^\lambda t^\delta; a, b; q, t; p) = 0 \tag{1.4.5}$$

if $\mu \not\subset \lambda$. For $q, t, a, b, c, d \in \mathbb{C}^*$ the elliptic difference operator $D^{(n)}(a, b, c, d; q, t; p)$, acting on BC_n-symmetric functions, is given by

$$(D^{(n)}(a, b, c, d; q, t; p)f)(x) = \sum_{\sigma\in\{\pm1\}^n} f(q^{\sigma/2}x) \prod_{i=1}^{n} \frac{\theta(ax_i^{\sigma_i}, bx_i^{\sigma_i}, cx_i^{\sigma_i}, dx_i^{\sigma_i})}{\theta(x_i^{2\sigma_i})} \prod_{1\le i<j\le n} \frac{\theta(tx_i^{\sigma_i}x_j^{\sigma_j})}{\theta(x_i^{\sigma_i}x_j^{\sigma_j})},$$

where $q^{\sigma/2}x = (q^{\sigma_1/2}x_1, \dots, q^{\sigma_n/2}x_n)$. Then

$$\begin{aligned} &D^{(n)}(a, b, c, d; q, t; p)R^*_\lambda(x; aq^{1/2}, bq^{1/2}; q, t; p) \\ &\qquad = \prod_{i=1}^{n} \theta(abt^{n-i}, acq^{\lambda_i}t^{n-i}, bcq^{-\lambda_i}t^{i-1}) \cdot R^*_\lambda(x; a, b; q, t; p) \end{aligned} \tag{1.4.6}$$

provided that $t^{n-1}abcd = p$. Like the Macdonald polynomials, there is no simple closed-form expression for the elliptic interpolation functions. When indexed by rectangular partitions of length n, however, they do admit a simple form, viz.

$$R^*_{(N^n)}(x; a, b; q, t; p) = \prod_{i=1}^{n} \frac{(ax_i^\pm)_N}{(pqx_i^\pm/b)_N}. \tag{1.4.7}$$

The principal specialization formula for the elliptic interpolation functions is

$$R^*_\lambda(zt^\delta; a, b; q, t; p) = \frac{(t^{n-1}az, a/z)_\lambda}{(pqt^{n-1}z/b, pq/bz)_\lambda}. \tag{1.4.8}$$

The R^*_λ satisfy numerous symmetries, all direct consequence of symmetries of the branching coefficients $c_{\lambda\mu}$. Two of the most notable ones are

$$R^*_\lambda(x; a, b; q, t; p) = R^*_\lambda(-x; -a, -b; q, t; p) \tag{1.4.9a}$$

$$= \Big(\frac{qt^{n-1}a}{b}\Big)^{2|\lambda|} q^{4n(\lambda')}t^{-4n(\lambda)}R^*_\lambda(x; 1/a, 1/b; 1/q, 1/t; p). \tag{1.4.9b}$$

Specializations of R^*_μ give rise to elliptic binomial coefficients. Before defining these we introduce the function

$$\Delta_\lambda(a|b_1, \dots, b_k) = \frac{(pqa)_{2\lambda^2}}{C^-_\lambda(t, pq)C^+_\lambda(a, pqa/t)} \frac{(b_1, \dots, b_k)_\lambda}{(pqa/b_1, \dots, pqa/b_k)_\lambda},$$

where the dependence on q, t and p has been suppressed and where $2\lambda^2$ is shorthand for the partition $(2\lambda_1, 2\lambda_1, 2\lambda_2, 2\lambda_2, \dots)$. Explicitly, for λ such that $l(\lambda) \le n$,

$$\Delta_\lambda(a|b_1, \dots, b_k) = \Big(\frac{(-1)^k a^{k-3}q^{k-3}t}{b_1\cdots b_k}\Big)^{|\lambda|} q^{(k-4)n(\lambda')}t^{-(k-6)n(\lambda)} \frac{(at^{1-n}, aqt^{-n}, b_1, \dots, b_k)_\lambda}{(qt^{n-1}, t^n, aq/b_1, \dots, aq/b_k)_\lambda}$$

$$\times \prod_{i=1}^{n} \frac{\theta(at^{2-2i}q^{2\lambda_i})}{\theta(at^{2-2i})} \prod_{1\le i<j\le n} \biggl(\frac{\theta(t^{j-i}q^{\lambda_i-\lambda_j}, at^{2-i-j}q^{\lambda_i+\lambda_j})}{\theta(t^{j-i}, at^{2-i-j})} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}(at^{3-i-j})_{\lambda_i+\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}(aqt^{1-i-j})_{\lambda_i+\lambda_j}}\biggr),$$

so that

$$_{r+1}V_r^{(n)}(a;b_1,\dots,b_{r-4}) = \sum_{\lambda} \frac{(b_3,\dots,b_{r-4},qt^{n-1}b_1b_2)_\lambda}{(aq/b_3,\dots,aq/b_{r-4},at^{1-n}/b_1b_2)_\lambda} \Delta_\lambda\Bigl(a\Big|t^n,b_1,b_2,\frac{at^{1-n}}{b_1b_2}\Bigr). \tag{1.4.10}$$

The elliptic binomial coefficients $\binom{\lambda}{\mu}_{[a,b]} = \binom{\lambda}{\mu}_{[a,b];q,t;p}$ may now be defined as

$$\binom{\lambda}{\mu}_{[a,b]} = \Delta_\mu(a/b|t^n,1/b)\,R^*_\mu(x_1,\dots,x_n;a^{1/2}t^{1-n},ba^{-1/2};q,t;p)\Big|_{x_i=a^{1/2}q^{\lambda_i}t^{1-i}}, \tag{1.4.11}$$

where on the right n can be chosen arbitrarily provided that $n \ge l(\lambda), l(\mu)$. Apart from their n-independence, the elliptic binomial coefficients are also independent of the choice of square root of a. Although $\binom{\lambda}{0}_{[a,b]} = 1$ they are not normalized like ordinary binomial coefficients, and

$$\binom{\lambda}{\lambda}_{[a,b]} = \frac{(1/b,pqa/b)_\lambda}{(b,pqa)_\lambda} \frac{C^+_\lambda(a)}{C^+_\lambda(a/b)}. \tag{1.4.12}$$

The elliptic binomial coefficients vanish unless $\mu \subset \lambda$, are elliptic in both a and b, and invariant under the simultaneous substitution $(a,b,q,t) \mapsto (1/a,1/b,1/q,1/t)$. They are also conjugation symmetric:

$$\binom{\lambda}{\mu}_{[a,b];q,t;p} = \binom{\lambda'}{\mu'}_{[aq/t,b];1/t,1/q;p}. \tag{1.4.13}$$

A key identity is

$$\binom{\lambda}{\nu}_{[a,c]} = \frac{(b,ce,cd,bde)_\lambda}{(cde,bd,be,c)_\lambda} \frac{(1/c,bd,be,cde)_\nu}{(bcde,e,d,b/c)_\nu} \sum_\mu \frac{(c/b,d,e,bcde)_\mu}{(bde,ce,cd,1/b)_\mu} \binom{\lambda}{\mu}_{[a,b]} \binom{\mu}{\nu}_{[a/b,c/b]} \tag{1.4.14}$$

for generic parameters such that $bcde = aq$. The $c \to 1$ limit of $\binom{\lambda}{\nu}_{[a,c]}(c)_\lambda/(1/c)_\nu$ exists and is given by $\delta_{\lambda\nu}$. Multiplying both sides of (1.4.14) by $(c)_\lambda/(1/c)_\nu$ and then letting c tend to 1 thus yields the orthogonality relation

$$\sum_\mu \binom{\lambda}{\mu}_{[a,b]} \binom{\mu}{\nu}_{[a/b,1/b]} = \delta_{\lambda\mu}. \tag{1.4.15}$$

For another important application of (1.4.14) we note that by (1.4.8)

$$\binom{(N^n)}{\mu}_{[a,b]} = \Delta_\mu(a/b|t^n,1/b,aq^Nt^{1-n},q^{-N}).$$

Setting $\lambda = (N^n)$ and $\nu = 0$ in (1.4.14) and recalling (1.4.10) yields the C_n Jackson summation (1.3.6). Also (1.3.8) may be obtained as a special case of (1.4.14) but the details of the derivation are more intricate, see [44]. As a final application of (1.4.14) it can be shown that

$$\frac{(b,b'e)_\lambda}{(b'de,bd)_\lambda}\,\frac{(b'de,bf/c)_\nu}{(b/c,b'de/f)_\nu}\sum_\mu \frac{(c/b,d,e,bb'de,b'g/c,b'f/c)_\mu}{(bb'de/c,b'e,b'd,1/b,f,g)_\mu}\binom{\lambda}{\mu}_{[a,b]}\binom{\mu}{\nu}_{[a/b,c/b]} \tag{1.4.16}$$

is symmetric in b and b', where $bb'de = aq$ and $cde = fg$. Setting $\lambda = (N^n)$ and $\nu = 0$ results in the transformation formula (1.3.13). Now assume that $bcd = b'c'd'$. Twice using the symmetry of (1.4.16) it follows that

$$\sum_\mu \frac{(c,d,aq/c',aq/d')_\lambda}{(c,d,aq/c',aq/d')_\mu}\,\frac{(c'/b,d'/b,aq/bc,aq/bd)_\mu}{(c'/b,d'/b,aq/bc,aq/bd)_\nu} \times \frac{(1/b,aq/b)_\nu}{(1/b,aq/bb')_\mu}\,\frac{(b',aq)_\mu}{(b',aq/b)_\lambda}\binom{\lambda}{\mu}_{[a,b]}\binom{\mu}{\nu}_{[a/b,b']} \tag{1.4.17}$$

is invariant under the simultaneous substitution $(b,c,d)\leftrightarrow(b',c',d')$. The branching coefficient (1.4.2) may be expressed as an elliptic binomial coefficient as

$$c_{\lambda\mu}(z;a,b;q,t,T;p) = \frac{(aTz^\pm,pqa/bt,t)_\lambda}{(aTz^\pm,pqa/bt,1/t)_\mu}\,\frac{(pqz^\pm/bt,T,pqaT/b)_\mu}{(pqz^\pm/b,tT,pqaT/bt)_\lambda}\binom{\lambda}{\mu}_{[aT/b,t]}, \tag{1.4.18}$$

so that, up to a simple change of variables and the use of (1.1.1), the z,w-symmetry of (1.4.4) corresponds to the $b = b'$ case of the symmetry of (1.4.17). To conclude our discussion of the elliptic binomial coefficients we remark that they also arise as connection coefficients between the interpolation functions. Specifically,

$$R^*_\lambda(x;a,b;q,t;p) = \sum_\mu \binom{\lambda}{\mu}_{[t^{n-1}a/b,a/a']}\frac{(a/a',t^{n-1}aa')_\lambda}{(a'/a,t^{n-1}aa')_\mu}\,\frac{(pqt^{n-1}a/b,pq/ab)_\mu}{(pqt^{n-1}a'/b,pq/a'b)_\lambda}\,R^*_\mu(x;a',b;q,t;p). \tag{1.4.19}$$

Let a,b,c,d,u,v,q,t be complex parameters such that $t^{2n-2}abcduv = pq$, and λ a partition of length at most n. Then the BC_n-symmetric biorthogonal functions $\tilde{R}_\lambda$ are defined as

$$\tilde{R}_\lambda(x;a\!:\!b,c,d;u,v;q,t;p) = \sum_{\mu\subset\lambda}\binom{\lambda}{\mu}_{[1/uv,t^{1-n}/av]}\frac{(pq/bu,pq/cu,pq/du,pq/uv)_\mu}{(t^{n-1}ab,t^{n-1}ac,t^{n-1}ad,t^{n-1}av)_\mu}\,R^*_\mu(x;a,u;q,t;p). \tag{1.4.20}$$

By (1.4.15) this relation between the two families of BC_n elliptic functions can be inverted. We also note that from (1.4.19) it follows that

$$\tilde{R}_\lambda(x;a\!:\!b,c,d;u,t^{1-n}/b;q,t;p) = \frac{(pq/au,pqt^{n-1}a/u)_\lambda}{(b/a,t^{n-1}ab)_\lambda}\,R^*_\mu(x;b,u;q,t;p), \tag{1.4.21}$$

so that the interpolation functions are a special case of the biorthogonal functions. Finally, from Okounkov's binomial formula for Koornwinder polynomials [42] it follows that in the $p\to 0$ limit the $\tilde{R}_\lambda$ simplify to the Koornwinder polynomials $K_\lambda(x;a,b,c,d;q,t)$:

$$K_\lambda(x;a,b,c,d;q,t) = \lim_{p\to 0}(at^{n-1})^{-|\lambda|}t^{n(\lambda)}\frac{(t^n,t^{n-1}ab,t^{n-1}ac,t^{n-1}ad)_\lambda}{C^-_\lambda(t)C^+_\lambda(abcdt^{2n-2}/q)} \times \tilde{R}_\lambda(x;a\!:\!b,c,d;up^{1/2},vp^{1/2},q,t;p).$$

Most of the previously-listed properties of the interpolation functions have implications for the biorthogonal functions. For example, using (1.4.8) and (1.4.14) one can prove the principal specialization formula

$$\tilde{R}_\lambda(bt^\delta;a\!:\!b,c,d;u,v;q,t;p) = \frac{(t^{n-1}bc,t^{n-1}bd,t^{1-n}/bv,pqt^{n-1}a/u)_\lambda}{(t^{n-1}ac,t^{n-1}ad,t^{1-n}/av,pqt^{n-1}b/u)_\lambda}. \qquad (1.4.22)$$

Another result that carries over is the elliptic difference equation (1.4.6). Combined with (1.4.20) it yields

$$D^{(n)}(a,u,b,pt^{1-n}/uab;q,t;p)\tilde{R}^*_\lambda(x;aq^{1/2}\!:\!bq^{1/2},cq^{-1/2},dq^{-1/2};uq^{1/2},vq^{-1/2},q,t;p) = \prod_{i=1}^n \theta(abt^{n-i},aut^{n-i},but^{n-i})\cdot \tilde{R}_\lambda(x;a\!:\!b,c,d;u,v,q,t;p). \qquad (1.4.23)$$

The Koornwinder polynomials are symmetric in the parameters a,b,c,d. From (1.4.20) it follows that $\tilde{R}_\lambda$ is symmetric in b,c,d but the choice of normalization breaks the full S_4 symmetry. Instead,

$$\tilde{R}_\lambda(x;a\!:\!b,c,d;u,v;q,t;p) = \tilde{R}_\lambda(x;b\!:\!a,c,d;u,v;q,t;p)\tilde{R}_\lambda(bt^\delta;a\!:\!b,c,d;u,v;q,t;p). \qquad (1.4.24)$$

For partitions λ,μ such that $l(\lambda),l(\mu)\le n$ the biorthogonal functions satisfy evaluation symmetry:

$$\tilde{R}_\lambda(at^\delta q^\mu;a\!:\!b,c,d;u,v;q,t;p) = \tilde{R}_\mu(\hat{a}t^\delta q^\lambda;\hat{a}\!:\!\hat{b},\hat{c},\hat{d};\hat{u},\hat{v};q,t;p), \qquad (1.4.25)$$

where

$$\hat{a} = \sqrt{abcd/pq}, \quad \hat{a}\hat{b} = ab, \quad \hat{a}\hat{c} = ac, \quad \hat{a}\hat{d} = ad, \quad a\hat{u} = \hat{a}u, \quad a\hat{v} = v\hat{a}.$$

Given a pair of partitions λ,μ such that $l(\lambda),l(\mu)\le n$, define

$$\tilde{R}_{\lambda\mu}(x;a\!:\!b,c,d;u,v;t;p,q) = \tilde{R}_\lambda(x;a\!:\!b,c,d;u,v;p,t;q)\tilde{R}_\mu(x;a\!:\!b,c,d;u,v;q,t;p).$$

Note that $\tilde{R}_{\lambda\mu}(x;a\!:\!b,c,d;u,v;t;p,q)$ is invariant under the simultaneous substitutions $\lambda\leftrightarrow\mu$ and $p\leftrightarrow q$. The functions $\tilde{R}_{\lambda\mu}(x;a\!:\!b,c,d;u,v;t;p,q)$ form a biorthogonal family, with continuous biorthogonality relation

$$\kappa^C_n\int_{C_{\lambda\nu,\mu\omega}}\tilde{R}_{\lambda\mu}(z_1,\dots,z_n;t_1\!:\!t_2,t_3,t_4;t_5,t_6;t;p,q)\tilde{R}_{\nu\omega}(z_1,\dots,z_n;t_1\!:\!t_2,t_3,t_4;t_6,t_5;t;p,q) \times \prod_{1\le i<j\le n}\frac{\Gamma(tz_i^\pm z_j^\pm)}{\Gamma(z_i^\pm z_j^\pm)}\prod_{j=1}^n\frac{\prod_{i=1}^6\Gamma(t_iz_j^\pm)}{\Gamma(z_j^{\pm 2})}\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}$$

$$= \delta_{\lambda\nu}\delta_{\mu\omega}\prod_{m=1}^n\left(\frac{\Gamma(t^m)}{\Gamma(t)}\prod_{1\le i<j\le 6}\Gamma(t^{m-1}t_it_j)\right)$$

$$\times \frac{1}{\Delta_\lambda(1/t_5t_6|t^n, t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{1-n}/t_0t_5, t^{1-n}/t_0t_6; p, t; q)}$$
$$\times \frac{1}{\Delta_\mu(1/t_5t_6|t^n, t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{1-n}/t_0t_5, t^{1-n}/t_0t_6; q, t; p)}. \tag{1.4.26}$$

Here, $C_{\lambda\nu,\mu\omega}$ is a deformation of $\mathbb{T}^n$ which separates sequences of poles of the integrand tending to zero from sequences tending to infinity. The location of these poles depends on the choice of partitions, see [46] for details. Provided $|t| < 1$ and $|t_i| < 1$ for $1 \le i \le 6$ we can take $C_{00,00} = \mathbb{T}^n$ so that for $\lambda = \mu = \nu = \omega = 0$ one recovers the type $\mathrm{C}_n^{(\mathrm{II})}$ integral (1.2.6). The summation (1.3.6), which is the discrete analogue of (1.2.6), follows in a similar manner from the discrete biorthogonality relation

$$\sum_{\mu\subset(N^n)} \Delta_\mu(t^{2n-2}a^2|t^n, t^{n-1}ac, t^{n-1}ad, t^{n-1}au, t^{n-1}av, q^{-N})$$
$$\times \tilde{R}_\lambda(aq^\mu t^\delta; a\!:\!b,c,d;u,v;t;p,q)\tilde{R}_\nu(aq^\mu t^\delta; a\!:\!b,c,d;v,u;t;p,q)$$
$$= \frac{\delta_{\lambda\nu}}{\Delta_\lambda(1/uv|t^n, t^{n-1}ab, t^{n-1}ac, t^{n-1}ad, t^{1-n}/au, t^{1-n}/av)}$$
$$\times \frac{(b/a, pq/uc, pq/ud, pq/uv)_{(N^n)}}{(pqt^{n-1}a/u, t^{n-1}bc, t^{n-1}bd, t^{n-1}bv)_{(N^n)}}, \tag{1.4.27}$$

where $t^{2n-2}abcduv = pq$ and $q^Nt^{n-1}ab = 1$. The discrete biorthogonality can also be lifted to the functions $\tilde{R}_{\lambda\mu}$ but since the resulting identity factors into two copies of (1.4.27) — the second copy with q replaced by p and N by a second discrete parameter M — this is no more general than the above.

The final result listed here is a (dual) Cauchy identity which incorporates the Cauchy identities for the Koornwinder polynomials, BC_n interpolation Macdonald polynomials and ordinary Macdonald polynomials:

$$\sum_{\lambda\subset(N^n)} \Delta_\lambda(q^{1-2N}/uv|t^n, q^{-N}, q^{1-N}t^{1-n}/av, a/u)$$
$$\times \tilde{R}_\lambda(x; a\!:\!b,c,d; q^Nu, q^{N-1}v; q,t;p)\tilde{R}_{\hat{\lambda}}(y; a\!:\!b,c,d; t^nu, t^{n-1}v; t,q;p)$$
$$= \frac{(a/u, pq^{1-N}/au, pq^{1-N}/bu, pq^{1-N}/cu, pq^{1-N}/du, pq^{2-2N}/uv)_{(N^n)}}{(t^{n-1}ab, t^{n-1}ac, t^{n-1}ad, q^{N-1}t^{n-1}av)_{(N^n)}}$$
$$\times \prod_{i=1}^n\prod_{j=1}^N \theta(x_i^\pm y_j) \prod_{i=1}^n \frac{1}{(ux_i^\pm)_m} \prod_{j=1}^N \frac{1}{(p/uy_j, y_j/u; 1/t, p)_n}, \tag{1.4.28}$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_N)$, $\hat{\lambda} = (n-\lambda'_m, \dots, n-\lambda'_1)$ and $abcduvq^{2m-2}t^{2n-2} = p$.

1.4.1 Notes

Instead of $R^*_\lambda(x_1, \dots, x_n, a, b; q, t; p)$, Rains denotes the BC_n-symmetric interpolation functions as $R^{*(n)}_\lambda(x_1, \dots, x_n, a, b; q, t; p)$, see [44, 46, 47]. An equivalent family of functions is defined by Coskun and Gustafson in [10] (see also [11]). They refer to these as well-poised

Macdonald functions, denoted as $W_\lambda(x_1,\dots,x_n;q,p,t,a,b)$. The precise relation between the two families is given by

$$W_\lambda(x_1/a,\dots,x_n/a;q,p,t,a^2,a/b) = \Big(\frac{t^{1-n}b^2}{q^2}\Big)^{|\lambda|} q^{-2n(\lambda')}t^{2n(\lambda)}\frac{(t^n)_\lambda}{C^-_\lambda(t)}\,\frac{(qt^{n-2}a/b;q,t^2;p)_{2\lambda}}{(qa/tb)_\lambda\,C^+_\lambda(qt^{n-2}a/b)}\,R^*_\lambda(x_1,\dots,x_n;a,b;q,t;p).$$

Similarly, Rains writes $\tilde{R}^{(n)}_\lambda(x_1,\dots,x_n;a:b,c,d;u,v;q,t;p)$ for the biorthogonal functions instead of $\tilde{R}_\lambda(x_1,\dots,x_n;a:b,c,d;u,v;q,t;p)$, see again [44, 46, 47].

The branching rule (1.4.3) is the $k=1$ instance of [44, Eq. (4.40)] or the $\mu=0$ case of [10, Eq. (2.14)]. The vanishing property (1.4.5) is [46, Corollary 8.12] combined with (1.4.21), or [10, Theorem 2.6]. The elliptic difference equation (1.4.6) is [44, Eq. (3.34)]. The formula (1.4.7) for the interpolation function indexed by a rectangular partition of length n is the $\lambda=0$ case of [44, Eq. (3.42)] or [10, Corollary 2.4]. The principal specialization formula (1.4.8) is [44, Eq. (3.35)]. The symmetry (1.4.9a) is [44, Eq. (3.39)] and the symmetry (1.4.9b) is [44, Eq. (3.38)] or [10, Proposition 2.8]. The definition of the elliptic binomial coefficients (1.4.11) is due to Rains, see [44, Eq. (4.1)]. Coskun and Gustafson define so-called elliptic Jackson coefficients

$$\omega_{\lambda/\mu}(z;r;a,b)=\omega_{\lambda/\mu}(z;r,q,p;a,b),$$

see [10, Eq. (2.38)]. Up to normalization these are the elliptic binomials coefficients:

$$\omega_{\lambda/\mu}(z;r;a,b)=\frac{(1/z,az)_\lambda}{(qbz,qb/az)_\lambda}\,\frac{(qbz/r,qb/azr,bq,r)_\mu}{(1/z,az,qb/r^2,1/r)_\mu}\begin{pmatrix}\lambda\\ \mu\end{pmatrix}_{[b,r]}.$$

The value of the elliptic binomials (1.4.12) is [44, Eq. (4.8)]. It is equivalent to [10, Eq. (2.9)] and also [10, Eq. (4.23)]. The conjugation symmetry (1.4.13) of the elliptic binomial coefficients is [44, Corollary 4.4]. The summation (1.4.14) is [44, Theorem 4.1], and is equivalent to the “cocycle identity” [10, Eq. (3.7)] for the elliptic Jackson coefficients. The orthogonality relation (1.4.15) is [44, Corollary 4.3] or [10, Eq. (4.16)]. The symmetry of (1.4.16) is [44, Theorem 4.9] or [10, Eq. (3.8)], and the symmetry of (1.4.17) is [44, Corollary 4.11]. The expression (1.4.18) for the branching coefficients is a consequence of [44, Corollary 4.5] or [10, Lemma 3.11]. The connection coefficient identity (1.4.19) is [44, Corollary 4.14]. It is equivalent to the “Jackson sum” [10, Eq. (3.6)] for the well-poised Macdonald functions W_λ.

Definition (1.4.20) of the birthogonal functions is [44, Eq. (5.1)], its principal specialization (1.4.22) is [44, Eq. (5.4)] and the difference equation (1.4.23) is [44, Lemma 5.2]. The parameter and evalation symmetries (1.4.24) and (1.4.25) are [44, Theorem 5.1] and [44, Theorem 5.4], respectively. The important biorthogonality relation (1.4.26) is a combination of [46, Theorem 8.4] and [46, Theorem 8.10]. Its discrete analogue (1.4.27) is [44, Theorem 5.8], see also [44, Theorem 8.11]. Finally, the Cauchy identity (1.4.28) is [44, Theorem 5.11].

The BC_n-symmetric interpolation functions satisfy several further important identities not covered in the main text, such as a “bulk” branching rule [44, Theorem 4.16] which extends (1.4.3), and a generalized Pieri rule [44, Theorem 4.17]. In [11] Coskun applies the elliptic binomial coefficients (elliptic Jackson coefficients in his language) to formulate an elliptic

Bailey lemma of type BC_n. The interpolation functions further admit a generalization to skew interpolation functions [47]

$$\mathcal{R}_{\lambda/\mu}([v_1,\dots,v_{2n}];a,b;q,t;p), \qquad \mu\subseteq\lambda.$$

These are elliptic functions, symmetric in the variables $v_1,\dots,v_{2n}$, such that [47, Theorem 2.5]

$$R^*_\lambda(x_1,\dots,x_n;a,b;q,t;p)=\frac{(pqa/tb)_\lambda}{(t^n)_\lambda}\,\mathcal{R}^*_{\lambda/0}([t^{1/2}x_1^\pm,\dots,t^{1/2}x_n^\pm];t^{n-1/2}a,t^{1/2}b;q,t;p).$$

They also generalize the n-variable skew elliptic Jackson coefficients [10, Eq. (2.43)]

$$\omega_{\lambda/\mu}(x_1,\dots,x_n;r,q,p;a,b)$$

of Coskun and Gustafson:

$$\begin{aligned}
&\omega_{\lambda/\mu}(r^{n-1/2}x_1/a,\dots,r^{n-1/2}x_n/a;r;a^2r^{1-2n},ar^{1-n}/b)\\
&\qquad=\Big(-\frac{b^3}{q^3a}\Big)^{|\lambda|-|\mu|}q^{3n(\mu')-3n(\lambda')}t^{3n(\lambda)-3n(\mu)}r^{-n|\mu|}\,\frac{(aq/br)_\lambda}{(aqr^{-n-1}/b)_\mu}\,\frac{(r)_\mu}{(r)_\lambda}\\
&\qquad\qquad\times\mathcal{R}^*_{\lambda/\mu}([r^{1/2}x_1^\pm,\dots r^{1/2}x_n^\pm];a,b;q,t;p).
\end{aligned}$$

A very different generalization of the interpolation functions in given in [48] in the form of an interpolation kernel $\mathcal{K}_c(x_1,\dots,x_n;y_1,\dots,y_n;q,t;p)$. By specialising $y_i=q^{\lambda_i}t^{n-i}a/c$ with $c=\sqrt{t^{n-1}ab}$ for all $1\le i\le n$ one recovers, up to a simple normalising factor, $R^*_\lambda(x_1,\dots,x_n;a,b;q,t;p)$. In the same paper Rains uses this kernel to prove quadratic transformation formulas for elliptic Selberg integrals.

Also for the biorthogonal functions we have omitted a number of further results, such as a “quasi”-Pieri formula [44, Theorem 5.10] and a connection coefficient formula of Askey–Wilson type [44, Theorem 5.6], generalizing (1.4.19).

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