Lattice Ising model in a field: E_8 scattering theory

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Zamolodchikov found an integrable field theory related to the Lie algebra E_8, which describes the scaling limit of the Ising model in a magnetic field. He conjectured that there also exist solvable lattice models based on E_8 in the universality class of the Ising model in a field. The dilute A_3 model is a solvable lattice model with a critical point in the Ising universality class. The parameter by which the model can be taken away from the critical point acts like a magnetic field by breaking the Z_2 symmetry between the states. The expected direct relation of the model with E_8 has not been found hitherto. In this letter we study the thermodynamics of the dilute A_3 model and show that in the scaling limit it exhibits an appropriate E_8 structure, which naturally leads to the E_8 scattering theory for massive excitations over the ground state.

1. Introduction

Since the work [1] by A.B. Zamolodchikov it is known that certain perturbations of conformal field theories (CFT's) lead to completely integrable models of massive quantum field theory (QFT). The existence of non-trivial higher integrals of motion and other dynamical symmetries [2–6] in such a QFT allows to compute the spectrum of the particles and their S-matrix explicitly. At the same time, these QFT models can be obtained as the scaling limit of appropriate non-critical solvable lattice models in statistical mechanics (see [7] for an introduction and references on solvable lattice models). In the latter approach the spectrum and the S-matrices can be calculated from the Bethe Ansatz equations for the corresponding lattice model [8–10]. The natural problem arising in this connection is to find lattice counterparts for all known integrable perturbed CFT's and vice versa. A description of known results of such correspondence lies outside the scope of this letter and we refer the interested reader to [1–10] and references therein. Here we consider one particularly important example of this correspondence associated with the Ising model at its critical temperature in a magnetic field, hereafter referred to as the magnetic Ising model.

A.B. Zamolodchikov has shown [11] that the $c = 1/2$ CFT (corresponding to the critical Ising model) perturbed with the spin operator $\phi_{1,2} = \phi_{2,2}$ of dimension $(1/16, 1/16)$ describes an exactly integrable QFT containing eight massive particles with a reflectionless factorised S-matrix. Up to normalisation the masses of

these particles coincide with the components S_i of the Perron–Frobenius vector of the Cartan matrix of the Lie algebra E_8:

$$\frac{m_i}{m_j} = \frac{S_i}{S_j}. \tag{1.1}$$

The element of the S-matrix describing the scattering of the lightest particles, with mass m_1, reads [11]

$$S_{1,1}(\beta) = \frac{\tanh(\beta/2 + i\pi/6)\tanh(\beta/2 + i\pi/5)\tanh(\beta/2 + i\pi/30)}{\tanh(\beta/2 - i\pi/6)\tanh(\beta/2 - i\pi/5)\tanh(\beta/2 - i\pi/30)}, \tag{1.2}$$

with β the rapidity. The other elements are uniquely determined by the bootstrap programme [11].

The aim of this letter is to show that the above QFT describes the scaling limit of the dilute A_3 model of Warnaar, Nienhuis and Seaton [12,13] in the appropriate regime. It should be noted that there were some earlier, rather strong indications supporting the above correspondence. All these parts remarkably fit together with our results, completing a sequence of arguments which can be summarised as follows:

(i) The dilute A_3 model is an interaction-round-a-face model on the square lattice with spins taking three values (detailed definitions are given in equations (2.1)–(2.3)). Admissible values of the adjacent spins are determined by the incidence matrix (2.1), which has largest eigenvalue equal to $1 + \sqrt{2}$.

(ii) The model has two physically distinct regimes of relevance to our discussion, here denoted as *i)* and *ii)*, depending on the region of the spectral parameter or, equivalently, of a sign of the Hamiltonian of the associated one-dimensional chain. (These are the regimes 2^+ and 3^+ of ref. [13], respectively). The central charges and the conformal dimensions of the leading perturbation computed from exact expressions for the free energy and the local state probabilities of the dilute A_3 model for these two regimes read [14,13]

$$i) \quad c = 1/2, \quad \Delta = 1/16; \qquad ii) \quad c = 6/5, \quad \Delta = 15/16. \tag{1.3}$$

(iii) In refs. [9,15] Bazhanov and Reshetikhin proposed thermodynamic Bethe Ansatz equations (TBAE) related to the A-D-E Lie algebras, corresponding to non-critical models in statistical mechanics. Using standard thermodynamics calculations and the high level Bethe Ansatz (see [8] and references therein) they computed: the central charges of the corresponding scaling field theories, dimensions of the leading perturbations, the spectra and scattering amplitudes of the massive excitations, expressing them through fused Boltzmann weights. In particular, in the case relevant to our discussion ($\mathcal{G} = E_8$, $g = 30$, $p = \ell = 1$, in the notation of [9]) the exponents they found[#1] precisely match (1.3) in both regimes. Furthermore, the TBAE allowing the calculation of the largest eigenvalue of the incidence matrix of the underlying lattice model, gave in this case precisely the value $1 + \sqrt{2}$ [16].

(iv) Finally, the spectrum and S-matrix of the scaling field theory in regime *i)* found in [9] from the high level Bethe Ansatz for E_8 coincide with those of Zamolodchikov's magnetic Ising model.

All the above arguments strongly suggest that the TBAE based on the Lie algebra E_8 as proposed in [9,15], are those of the the dilute A_3 model.

In this paper we present the Bethe Ansatz equations (BAE) for the non-critical, dilute A_L model. As these equations, at criticality, are very similar to those of the Izergin–Korepin model [17,18], it is not surprising that, when specialised to $L = 3$, they do not display any explicit structure related to the root system of E_8. It turns out however that this structure reveals itself in a quite complicated string structure of the solutions to the BAE. Motivated by an extensive numerical investigation of the BAE we formulate an exact conjecture for the thermodynamically significant strings. This leads to TBAE, which, rewritten in a new string basis precisely yield the E_8 based TBAE of ref. [9] discussed under (iii). As a result of (iv) this finalises the correspondence between the dilute A_3 model and the magnetic Ising model.

[#1] Note that the equations (5.1) and (5.4) in [9] have been misprinted. Correcting (5.1) to $c = c^{\mathcal{G}}(l) + c^{\mathcal{G}}(r-l-g) - c^{\mathcal{G}}(r-g) + \text{rank}\,\mathcal{G}$ yields the following result for the central charge in (5.4): $c = 2\,\text{rank}\,\mathcal{G}/(g+2)$. Also the phrases "minimal unitary", just before, and "by the operator $\phi_{(1,3)}$" just after (5.4) should be deleted.

2. The dilute A models

The dilute A_L model, belonging to the more general class of dilute A-D-E models, is an exactly solvable, restricted solid-on-solid model defined on the square lattice. Each site of the lattice can take one of L possible (height) values, subject to the restriction that neighbouring sites of the lattice either have the same height, or differ by ± 1. This adjacency condition can be conveniently expressed by a so-called incidence matrix M:

$$M_{a,b} = \delta_{a,b-1} + \delta_{a,b} + \delta_{a,b+1}, \quad a,b \in \{1,\ldots,L\}, \tag{2.1}$$

where we note that M relates to the Cartan matrix C^{A_L} of the Lie algebra A_L by $M = 3I - C^{A_L}$, with I the identity matrix. The eigenvalues of the incidence matrix are found to be

$$\Lambda_j = 1 + 2\cos\left(\frac{\pi j}{L+1}\right), \quad j = 1,\ldots,L. \tag{2.2}$$

For the case of interest here, $L = 3$, we thus find the largest eigenvalue to be $1 + \sqrt{2}$, in accordance with the prediction for the E_8 TBAE as mentioned in (iii) of the introduction.

Using standard definitions of $\vartheta_i(u,q)$-functions, suppressing the dependence on the nome $q = e^{-\tau}$, $\tau > 0$, the Boltzmann weights of the allowed height configurations of an elementary face of the lattice are

$$W\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \frac{\vartheta_1(6\lambda - u)\vartheta_1(3\lambda + u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} - \left(\frac{S(a+1)}{S(a)}\frac{\vartheta_4(2a\lambda - 5\lambda)}{\vartheta_4(2a\lambda + \lambda)} + \frac{S(a-1)}{S(a)}\frac{\vartheta_4(2a\lambda + 5\lambda)}{\vartheta_4(2a\lambda - \lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)},$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a \end{pmatrix} = W\begin{pmatrix} a & a \\ a & a\pm 1 \end{pmatrix} = \frac{\vartheta_1(3\lambda - u)\vartheta_4(\pm 2a\lambda + \lambda - u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)},$$

$$W\begin{pmatrix} a & a \\ a\pm 1 & a \end{pmatrix} = W\begin{pmatrix} a & a\pm 1 \\ a & a \end{pmatrix} = \left(\frac{S(a\pm 1)}{S(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_4(\pm 2a\lambda - 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)},$$

$$W\begin{pmatrix} a & a\pm 1 \\ a & a\pm 1 \end{pmatrix} = W\begin{pmatrix} a\pm 1 & a\pm 1 \\ a & a \end{pmatrix} = \left(\frac{\vartheta_4(\pm 2a\lambda + 3\lambda)\vartheta_4(\pm 2a\lambda - \lambda)}{\vartheta_4^2(\pm 2a\lambda + \lambda)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)},$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a\mp 1 \end{pmatrix} = \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)},$$

$$W\begin{pmatrix} a & a\mp 1 \\ a\pm 1 & a \end{pmatrix} = -\left(\frac{S(a-1)S(a+1)}{S^2(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)},$$

$$W\begin{pmatrix} a & a\pm 1 \\ a\pm 1 & a \end{pmatrix} = \frac{\vartheta_1(3\lambda - u)\vartheta_1(\pm 4a\lambda + 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)} + \frac{S(a\pm 1)}{S(a)}\frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)},$$

$$= \frac{\vartheta_1(3\lambda + u)\vartheta_1(\pm 4a\lambda - 4\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)}\left(\frac{S(a\mp 1)}{S(a)}\frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)} - \frac{\vartheta_4(\pm 2a\lambda - 5\lambda)}{\vartheta_4(\pm 2a\lambda + \lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)},$$

$$S(a) = (-)^a\frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)}.$$

The variable λ and the range of the spectral parameter u in the above weights are given by[*2]

$$\lambda = \frac{\pi}{4}\frac{L+2}{L+1} \quad \begin{cases} 0 < u < 3\lambda & \text{regime } i) \\ 3\lambda - \pi < u < 0 & \text{regime } ii). \end{cases} \tag{2.3}$$

[*2] In [13] two more regimes were defined, which are omitted being of no relevance here.

3. Bethe Ansatz

The transfer matrix of the dilute A models is defined in the usual way as

$$T^{\{b\}}_{\{a\}} = \prod_{j=1}^{N} W \begin{pmatrix} b_j & b_{j+1} \\ a_j & a_{j+1} \end{pmatrix}, \tag{3.1}$$

where $\{a\}$ is an admissible path of heights and $a_{N+1} = a_1$, $b_{N+1} = b_1$. The number of admissible paths is given by $\mathrm{Trace}(A^N) = \sum_{j=1}^{L} A_j^N$.

We define two positive integers p and r and a renormalised spectral parameter θ by

$$\lambda = \frac{p}{r}\pi \quad u = \frac{\theta}{2r}\pi, \tag{3.2}$$

where p and r are coprime. Furthermore we introduce the modified ϑ-function

$$h(\theta) = \tfrac{1}{2} q^{-1/4} \vartheta_1\left(\frac{\pi\theta}{2r}, q\right). \tag{3.3}$$

With these definitions the eigenvalues of the transfer matrix are found to be

$$\Lambda(\theta) = \omega \left(\frac{h(4p-\theta)\, h(6p-\theta)}{h(4p)\, h(6p)}\right)^N \prod_{j=1}^{N} \frac{h(\theta + i\theta_j + 2p)}{h(\theta + i\theta_j - 2p)}$$

$$+ \left(\frac{h(\theta)\, h(6p-\theta)}{h(4p)\, h(6p)}\right)^N \prod_{j=1}^{N} \frac{h(\theta + i\theta_j)\, h(\theta + i\theta_j - 6p)}{h(\theta + i\theta_j - 2p)\, h(\theta + i\theta_j - 4p)}$$

$$+ \omega^{-1} \left(-\frac{h(\theta)\, h(2p-\theta)}{h(4p)\, h(6p)}\right)^N \prod_{j=1}^{N} \frac{h(\theta + i\theta_j - 8p)}{h(\theta + i\theta_j - 4p)}, \tag{3.4}$$

where the numbers $\{\theta_j\}$ are given by the following set of BAE:

$$\omega \left(\frac{h(2p + i\theta_j)}{h(2p - i\theta_j)}\right)^N = - \prod_{k=1}^{N} \frac{h(i\theta_j - i\theta_k + 4p)\, h(i\theta_j - i\theta_k - 2p)}{h(i\theta_j - i\theta_k - 4p)\, h(i\theta_j - i\theta_k + 2p)}, \quad j = 1,\ldots,N \tag{3.5}$$

and $\omega = \exp(i\pi\ell/(L+1))$, $\ell = 1,\ldots,L$. Note that if we set the nome q of the function h equal to zero and choose ω to be unity, the above eigenvalue expression and BAE reduce to those of the Izergin–Korepin model [18] in the sector in which the number of roots is equal to the system size N. A proof of equations (3.4) and (3.5) in the critical limit, based on an extension of the mapping of RSOS models onto vertex models found in [19], will be presented elsewhere [20]. In the general elliptic case the equations can be proven using the functional equations for the dilute A models as obtained in ref. [21].

4. Thermodynamic Bethe Ansatz

The remaining part of this letter is devoted to the solution of the BAE in the thermodynamic limit for the dilute A_3 model, corresponding to the case $p = 5$ and $r = 16$.

Let $n^{(t)}$ be a positive integer, $\Delta^{(t)}$ an $n^{(t)}$-dimensional vector with integer coefficients $\Delta_k^{(t)}$ and $\varepsilon^{(t)} = 0, 1$. We then define a string of type t as a set of complex numbers $\{\alpha_{j,k}^{(t)}\}$ with

$$\alpha_{j,k}^{(t)} = \alpha_j^{(t)} + i(\Delta_k^{(t)} + \varepsilon^{(t)} r), \quad k = 1,\ldots,n^{(t)}, \tag{4.1}$$

Table 1
The nine thermodynamically significant string types.

t	$n^{(t)}$	$\Delta^{(t)}/5$	$\varepsilon^{(t)}$
0	1	(0)	0
1	2	$(-1, 1)$	1
2	4	$(-4, -2, 2, 4)$	0
3	6	$(-7, -5, -1, 1, 5, 7)$	1
4	8	$(-10, -8, -4, -2, 2, 4, 8, 10)$	0
5	10	$(-13, -11, -7, -5, -1, 1, 5, 7, 11, 13)$	1
6	7	$(-14, -6, -2, 0, 2, 6, 14)$	1
7	4	$(-3, -1, 1, 3)$	0
8	5	$(-12, -8, 0, 8, 12)$	1

where the real number $\alpha_j^{(t)}$ specifies the centre of the string. Based on an extensive numerical study [20] we find that for $N \to \infty$ each solution $\{\theta_j\}$ of (3.5) consists of a collection of strings, where we have *only nine thermodynamically significant string types* (in the sense that $N^{(t)}/N$ is finite for $N \to \infty$)

$$N = \sum_{t=0}^{8} n^{(t)} N^{(t)} + o(N), \tag{4.2}$$

with $N^{(t)}$ the total number of strings of type t. These nine allowed string types are listed in table 1. We expect the above statement to be exact and claim it as a conjecture.

In the thermodynamic limit the centers of the strings form continuous distributions and the BAE (3.5) lead to integral equations for the densities of strings $\rho_t(\alpha)$ and "holes" $\tilde{\rho}_t(\alpha)$ [22]

$$b_t(\alpha) = -(-)^{\delta_{t,0}} \tilde{\rho}_t(\alpha) + \sum_{s=0}^{8} B_{t,s} * \rho_s(\alpha), \quad t = 0, \ldots, 8, \tag{4.3}$$

where $a * b$ denotes the convolution of the functions a and b

$$a * b\,(\alpha) = \int_{-\tau r/\pi}^{\tau r/\pi} a(\alpha - \beta) b(\beta)\,\mathrm{d}\beta. \tag{4.4}$$

The functions b_t and $B_{t,s}$ in (4.3), which are $2\tau r/\pi$-periodic, read

$$b_t(\alpha) = \sum_{k=1}^{n^{(t)}} \psi_{2p}\left(\alpha + i\Delta_k^{(t)}\right),$$

$$B_{t,s}(\alpha) = -(-)^{\delta_{t,0}} \delta_{t,s} \delta(\alpha) + \sum_{k=1}^{n^{(t)}} \sum_{\ell=1}^{n^{(s)}} \left[\psi_{4p}\left(\alpha + i\Delta_k^{(t)} + i\Delta_l^{(s)}\right) - \psi_{2p}\left(\alpha + i\Delta_k^{(t)} + i\Delta_l^{(s)}\right)\right], \tag{4.5}$$

with ψ_k defined as

$$\psi_k(\alpha) = \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\alpha} \log\left[\frac{h(k + i\alpha)}{h(k - i\alpha)}\right]. \tag{4.6}$$

The function ψ_k has the following Fourier transform (FT)

$$\widehat{\psi}_k(x) = \frac{\sinh(r - k)x}{\sinh rx}, \quad 0 < k < 2r, \tag{4.7}$$

where

$$\widehat{F}(x) = \int\limits_{-\tau r/\pi}^{\tau r/\pi} e^{-i\alpha x} F(\alpha)\, d\alpha, \quad F(\alpha) = \frac{\delta}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\alpha x_n} \widehat{F}(x_n), \tag{4.8}$$

with $\delta = \pi^2/(r\tau)$, $x_n = \delta n$.

As usual, we define the (local) Hamiltonian $\mathcal{H}$ of the associated one-dimensional integrable spin chain as the logarithmic derivative of the transfer matrix at $\theta = 0$. Then, after appropriate normalisation and shift, the spectrum of $\mathcal{H}$, in the limit $N \to \infty$ reads

$$\frac{E}{N} = \epsilon \sum_{t=0}^{8} \int\limits_{-\tau r/\pi}^{\tau r/\pi} b_t(\alpha)\, \rho_t(\alpha)\, d\alpha + O\left(e^{-\mu N}\right) \quad \mu > 0. \tag{4.9}$$

where $\epsilon = -1$ for regime *i)* and $\epsilon = 1$ for regime *ii)* in (2.3).

The densities ρ_t are normalised such that

$$\int\limits_{-\tau r/\pi}^{\tau r/\pi} \rho_t(\alpha)\, d\alpha = N^{(t)}/N. \tag{4.10}$$

Therefore, from equation (4.2), we have

$$\sum_{t=0}^{8} \int\limits_{-\tau r/\pi}^{\tau r/\pi} n^{(t)} \rho_t(\alpha)\, d\alpha = 1. \tag{4.11}$$

This relation together with equation (4.3) for $t = 0$ implies

$$\tilde{\rho}_0(\alpha) = 0. \tag{4.12}$$

Hence we conclude that the strings of type 0 have no holes in any state, and we eliminate $\rho_0(\alpha)$ from (4.3). After a tedious calculation we find that the resulting integral equations can naturally be described in terms of the E_8 root system as follows.

Let $C_{t,s}^{E_8}$ $t,s = 1,\ldots,8$ be the elements of the Cartan matrix for E_8, where we use the following enumeration of the nodes of the corresponding Dynkin diagram:
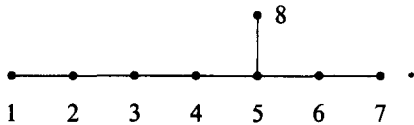
Furthermore, define the functions $K_{t,s}^{E_8}$, $A_{t,s}^{E_8}$, $a_{t,s}^{E_8}$ and s by their FT

$$\widehat{K}_{t,s}^{E_8}(x) = \delta_{t,s} + \widehat{s}(x)\left(C_{t,s}^{E_8} - 2\delta_{t,s}\right), \quad \widehat{A}_{t,s}^{E_8}(x) = \left[\widehat{K}^{E_8}(x)\right]_{t,s}^{-1},$$

$$\widehat{a}_{t,s}^{E_8}(x) = \widehat{s}(x)\widehat{A}_{t,s}^{E_8}(x), \quad \widehat{s}(x) = \frac{1}{2\cosh x}. \tag{4.13}$$

With these definitions, and after eliminating ρ_0, the integral equations (4.3) and the energy expression (4.9) take the form

$$a_{1,t}^{E_8}(\alpha) = \tilde{\rho}_t(\alpha) + \sum_{s=1}^{8} A_{t,s}^{E_8} * \rho_s(\alpha), \quad t = 1, \ldots, 8,$$

$$\frac{E}{N} = -\epsilon \sum_{t=1}^{8} \int_{-\tau r/\pi}^{\tau r/\pi} a_{1,t}^{E_8}(\alpha)\, \rho_t(\alpha)\, \mathrm{d}\alpha + \mathrm{const}. \tag{4.14}$$

We can now use (4.14) to study the scaling limit of the model. In fact, all relevant calculations have already been carried out in ref. [9] and we only need to refer to the appropriate results therein. To make the correspondence with ref. [9] somewhat more transparent, let us give the expression for the equilibrium free energy $F(T)$ of the one-dimensional spin chain at finite temperature T, as it follows from (4.14) via standard TBA calculations [22],

$$\frac{F(T)}{N} = -\sum_{t=1}^{8} \int_{-\tau r/\pi}^{\tau r/\pi} a_{1,t}^{E_8}(\alpha)\, T \log\left(1 + \mathrm{e}^{-\beta \epsilon_t(\alpha)}\right) \mathrm{d}\alpha + \mathrm{const}, \tag{4.15}$$

where $\beta = 1/T$ is the inverse temperature. The functions $\epsilon_t = T \log(\tilde{\rho}_t/\rho_t)$ are the solutions of the integral equation

$$\epsilon \delta_{1,t} s(\alpha) = T \log\left(1 + \mathrm{e}^{-\beta \epsilon_t(\alpha)}\right) - \sum_{s=1}^{8} K_{t,s}^{E_8} * T \log\left(1 + \mathrm{e}^{\beta \epsilon_s(\alpha)}\right)(\alpha). \tag{4.16}$$

The above two equations are equivalent to (3.20) and (3.21) of ref. [9], respectively, with their $\mathcal{G} = E_8$, $r = 32$, $g = 30$, $p = \ell = 1$, their nome q replaced by $q^{1/2}$ and with their ϵ_j^a negated. This last difference reflects the fact that our TBA equations are dual to those of ref. [9] in the sence that the densities of strings and holes are interchanged. From (4.15) and (4.16) it follows that for $T = 0$

$$\begin{aligned} &i)\epsilon = -1, \quad \epsilon_t(\alpha) = a_{1,t}^{E_8}, \\ &ii) \quad \epsilon = +1 \quad \epsilon_t(\alpha) = -\delta_{t,1} s(\alpha). \end{aligned} \tag{4.17}$$

The functions $|\epsilon_t(\alpha)|$ are the energies of the excitations over the ground state.

For $\epsilon = -1$ the ground state is formed by type 0 strings. As was remarked after equation (4.12), these strings have no holes for any state. Therefore the Dirac sea is "frozen", and the excitations correspond to the remaining eight string types. The phenomenon of "freezing" of the Dirac sea which can be interpreted as the confinement of "holes" has been first observed in the TBAE calculations of ref. [23] for the RSOS models of Andrews, Baxter and Forrester [24].

For $\epsilon = 1$ the Dirac sea is formed by the type 1 strings, and the only excitations correspond to holes in the Dirac sea. These excitations are of the kink type.

Now we consider the scaling limit. We introduce a dimensional spacing parameter d for our chain and take the limit $N \to \infty$, $d \to 0$, keeping the (dimensional) length of the chain $L = Nd$ to be macroscopically bigger than the correlation length: $L >> R_c = q^{-\xi}d$, where ξ is the index of the correlation length. In the scaling limit we thus have $d \sim q^\xi$, $N >> q^{-\xi}$, $q \to 0$, and we obtain the massive relativistic spectrum of excitations. To find this, one has to compute the energy dispersion law for the physical excitations in the $q \to 0$ limit keeping the rapidities α of the order of $\alpha_0 = \tau r/\pi$, where the functions $|\epsilon_t(\alpha)|$ have their minima. Taking into account the correspondence in notation discussed after (4.16), one gets from (4.1) and (4.2) of ref. [9]

$$i)\epsilon_t\left(\tfrac{30}{\pi}\beta + \alpha_0\right) = m_t \cosh\beta + o(q^\xi), \quad m_t = \mathrm{const}\, S_t q^\xi, \quad \xi = \tfrac{8}{15},$$

$$ii)\,|\epsilon_1\left(\tfrac{2}{\pi}\beta + \alpha_0\right)| = m \cosh\beta + o(q^\xi), \quad m = \mathrm{const}\, q^\xi, \quad \xi = 8, \tag{4.18}$$

where β here denotes the rapidity variable and S_t was defined just before equation (1.1).

Using the scaling relation $\zeta = (2 - 2\Delta)^{-1}$ it is seen that the values of ζ in (4.18) lead exactly to the dimensions of the leading perturbations as given in (1.3).

The values of the central charges of the corresponding (ultraviolet) conformal field theories listed in (1.3) have also been previously calculated. For regime $i)$ in [9,25,26] and for regime $ii)$ in [9].

Finally, the logarithmic derivative of the S-matrix for regime $i)$, where all the string excitations for $t = 1 \ldots 8$ correspond to distinct particles, can be found from equation (4.14). The remaining phase ambiguity for each matrix element can be fixed by imposing $S_{t,t} = -1$, and exploiting the bootstrap procedure. The result is [9,26]

$$S_{t,s}(\beta) = (-)^{\delta_{t,s}} \exp \left\{ -i \int_{-\infty}^{\infty} \tilde{A}_{t,s}^{E_8}(x) \, \frac{\sin(30\beta x/\pi)}{x} \, dx \right\}, \tag{4.19}$$

which coincides with Zamolodichkov's E_8 S-matrix [11].

For regime $ii)$ the kink–kink S-matrix is of the RSOS type related to the E_7 Lie algebra [9], and will be discussed elsewhere [20].

5. Summary and conclusion

In this paper we have established the final link between Zamolodchikov's E_8 S-matrix of the critical Ising model in a field [11] and its underlying lattice model. By making a conjecture for the possible string solutions of its Bethe Ansatz equations, we have derived a system of thermodynamic BAE for the dilute A_3 lattice model of Warnaar et al. [12]. After a suitable transformation we have recast these TBAE in terms of the root system of the Lie algebra E_8. These E_8 TBAE are found to be precisely those conjectured earlier by Bazhanov and Reshetikhin [9], and using their results, the correspondence between the dilute A_3 model and the E_8 S-matrix is made.

To conclude we mention that two more remarkable integrable $\phi_{1,2}$ perturbations of CFT's are known, notably those related to S-matrices with hidden E_7 $(c = 7/10)$ and E_6 $(c = 6/7)$ structure [27]. Like the E_8 case, the underlying lattice models of these integrable QFT's correspond to models in the dilute A hierarchy. The working for these two extra cases, corresponding to dilute A_4 and A_6, respectively, as well as some additional results for the dilute A_3 model will be the subject of a future publication [20]

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