Bounded Littlewood identities

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Abstract

We describe a method, based on the theory of Macdonald–Koornwinder polynomials, for proving bounded Littlewood identities. Our approach provides an alternative to Macdonald's partial fraction technique and results in the first examples of bounded Littlewood identities for Macdonald polynomials. These identities, which take the form of decomposition formulas for Macdonald polynomials of type (R, S) in terms of ordinary Macdonald polynomials, are q,t-analogues of known branching formulas for characters of the symplectic, orthogonal and special orthogonal groups. In the classical limit, our method implies that MacMahon's famous ex-conjecture for the generating function of symmetric plane partitions in a box follows from the identification of $\big(\mathrm{GL}(n,\mathbb{R}), \mathrm{O}(n)\big)$ as a Gelfand pair. As further applications, we obtain combinatorial formulas for characters of affine Lie algebras; Rogers–Ramanujan identities for affine Lie algebras, complementing recent results of Griffin et al.; and quadratic transformation formulas for Kaneko–Macdonald-type basic hypergeometric series.

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CHAPTER 1

Introduction

1.1. Littlewood identities

In his 1950 text on group characters [**85**], D. E. Littlewood presented three identities for Schur functions which can be viewed as reciprocals of the Weyl denominator formulas for the classical groups $\mathrm{B}_n, \mathrm{C}_n$ and D_n. The B_n case—which earlier appeared in an exercise by Schur [**121**]—is given by [**85**, Eq. (11.9; 6)]

$$\sum_{\lambda} s_\lambda(x) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}, \tag{1.1.1}$$

where $s_\lambda(x) = s_\lambda(x_1, \dots, x_n)$ is a Schur function indexed by the partition λ. Almost 30 years later, Macdonald [**90**] proved the following bounded analogue of (1.1.1):

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} s_\lambda(x) = \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{m+2n-j} - x_i^{j-1}\big)}{\prod_{i=1}^{n}(x_i-1)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_i x_j - 1)}, \tag{1.1.2}$$

for m a nonnegative integer, and observed that it implied MacMahon's famous conjecture [**95**] for the generating function of symmetric plane partitions in a box. By reading off the 'sequence of diagonal slices'—an idea more recent than [**90**], see e.g., [**103**]—it immediately follows that the generating function

$$\sum_{\substack{\pi \subset \mathrm{B}(n,n,m) \\ \pi \text{ symmetric}}} q^{|\pi|} \tag{1.1.3}$$

for symmetric plane partitions π contained in a box $\mathrm{B}(n,n,m)$ of size $n \times n \times m$ is given by

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} s_\lambda(q, q^3, \dots, q^{2n-1}). \tag{1.1.4}$$

Hence MacMahon's formula [**95**]

$$\sum_{\substack{\pi \subset \mathrm{B}(n,n,m) \\ \pi \text{ symmetric}}} q^{|\pi|} = \prod_{i=1}^{n} \frac{1-q^{m+2i-1}}{1-q^{2i-1}} \prod_{1\leqslant i<j\leqslant n} \frac{1-q^{2(m+i+j-1)}}{1-q^{2(i+j-1)}} \tag{1.1.5}$$

should follow from the evaluation of the determinant on the right of (1.1.2) in which the variables x_i are specialised as $x_i = q^{2i-1}$ for $1 \leqslant i \leqslant n$. Since the unspecialised determinant is essentially a character of the irreducible $\mathrm{SO}(2n+1,\mathbb{C})$-module of highest weight $m\omega_n$, the required determinant evaluation corresponds to the q-dimension of this module, and follows from the Weyl character formula [**50**]. To prove (1.1.2)—a $\mathrm{SO}(2n+1,\mathbb{C})$ to $\mathrm{GL}(n,\mathbb{C})$ branching formula—Macdonald developed a partial fraction method, resulting in a more general t-analogue for Hall–Littlewood polynomials.

Since the work of Littlewood and Macdonald, many additional Littlewood identities have been discovered and applied to problems in combinatorics, q-series and representation theory. Examples include the enumeration of plane partitions and related combinatorial objects such as tableaux, tilings, longest increasing subsequences and alternating-sign matrices [**7**, **10**, **12**, **13**, **16**, **17**, **31**, **32**, **46**, **66**, **104**, **105**, **107**, **127**, **145**], the computation of characters and branching rules for classical groups and affine Lie algebras [**9**, **53**, **54**, **61**, **62**, **67**, **101**], and applications to Schubert calculus [**70**] Rogers–Ramanujan identities [**47**, **51**, **56**, **127**, **139**] and multiple elliptic hypergeometric series [**71**, **111**]. Surprisingly, despite the interest in Littlewood identities, q,t-analogues of (1.1.2) and other bounded Littlewood identities for Schur and Hall–Littlewood polynomials have remained elusive. In this paper we present an approach to Littlewood identities based on the theory of Macdonald–Koornwinder polynomials. As a result we obtain the missing q,t-analogues, including the following generalisation of Macdonald's determinantal formula (1.1.2).[1]

THEOREM 1.1. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} P_\lambda(x;q,t) \prod_{\substack{s\in\lambda\\ l'(s)\text{ even}}} \frac{1-q^{m-a'(s)}t^{l'(s)}}{1-q^{m-a'(s)-1}t^{l'(s)+1}} \prod_{\substack{s\in\lambda\\ l(s)\text{ even}}} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} = (x_1\cdots x_n)^{\frac{m}{2}}\, P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;q,t,t). \tag{1.1.6}$$

On the left, $P_\lambda(x;q,t)$ is a Macdonald polynomial and $a(s), l(s), a'(s), l'(s)$ are the arm-length, leg-length, arm-colength and leg-colength of the square $s\in\lambda$. The (Laurent) polynomial $P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;q,t,t)$ on the right is a Macdonald polynomial attached to the pair of root systems $(\mathrm{B}_n,\mathrm{B}_n)$, indexed by the rectangular partition or 'half-partition' $(\frac{m}{2},\dots,\frac{m}{2})$ of length n.

Our method also leads to alternative proofs, as well as further examples, of Littlewood identities for the characters of irreducible highest-weight modules of affine Lie algebras, first discovered in [**9**]. In particular we find that such characters arise by taking suitable limits of Hall–Littlewood polynomials of type R. For example, for the twisted affine Lie algebra $\mathrm{A}_{2n}^{(2)}$ we obtain the following formula for the character of the highest-weight module $V(m\varpi_0)$ in terms of modified Hall–Littlewood polynomials $P'_\lambda(x;t)$ and the large-r limit of the Hall–Littlewood polynomial $P^{(\mathrm{B}_r)}_{(\frac{m}{2})^n}(x;t,t_2)$.

THEOREM 1.2. *Let $\alpha_0,\dots,\alpha_n$ and $\varpi_0,\dots,\varpi_n$ be the simple roots and fundamental weights of $\mathrm{A}_{2n}^{(2)}$, and $\delta=2\alpha_0+\dots+2\alpha_{n-1}+\alpha_n$ the null root. Set*

$$t=\mathrm{e}^{-\delta} \quad \textit{and} \quad x_i=\mathrm{e}^{-\alpha_i-\dots-\alpha_{n-1}-\alpha_n/2},$$

[1] A second generalisation in terms of $P^{(\mathrm{B}_n,\mathrm{C}_n)}_{(\frac{m}{2})^n}$ is given in Theorem 4.6.

and let $\operatorname{ch} V(\Lambda)$ *denote the character of the integrable highest-weight module* $V(\Lambda)$ *of highest weight* Λ. *Then, for* m *a positive integer,*

$$\begin{aligned}\mathrm{e}^{-m\varpi_0}\operatorname{ch} V(m\varpi_0) &= \lim_{N\to\infty} t^{\frac{1}{2}mnN^2} P^{(\mathrm{B}_{2nN})}_{(\frac{m}{2})^{2nN}}(t^{1/2}X;t,0) \\ &= \sum_{\substack{\lambda \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^\pm,\dots,x_n^\pm;t),\end{aligned} \tag{1.1.7}$$

where $(\dots,ax_i^\pm,\dots) := (\dots,ax_i,ax_i^{-1},\dots)$ *and* $X = X_N(x;t)$ *is the alphabet*

$$X = (x_1^\pm, tx_1^\pm, \dots, t^{N-1}x_1^\pm, \dots\dots, x_n^\pm, tx_n^\pm, \dots, t^{N-1}x_n^\pm) \tag{1.1.8}$$

of cardinality $2nN$.

As shown by Griffin et al. [**47**], character formulas such as (1.1.7) imply Rogers–Ramanujan identities through specialisation. Following their approach, we obtain several new examples of Rogers–Ramanujan identities labelled by affine Lie algebras. For example, from (1.1.7) we obtain the following new identity, where $P_\lambda(x;t)$ is a Hall–Littlewood polynomial, $\theta(x;q)$ a modified theta function and $(q;q)_\infty$ a q-shifted factorial.

Theorem 1.3 ($\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identity). *For* m,n *positive integers, let* $\kappa = m+2n+1$. *Then*

$$\begin{aligned}&\sum_{\substack{\lambda \\ \lambda_1\leqslant m}} q^{|\lambda|/2} P_\lambda(1,q,q^2,\dots;q^{2n}) \\ &\qquad = \frac{(q^\kappa;q^\kappa)_\infty^{n-1}(q^{\kappa/2};q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty} \prod_{i=1}^n \theta(q^i;q^{\kappa/2}) \prod_{1\leqslant i<j\leqslant n} \theta(q^{j-i};q^\kappa)\theta(q^{i+j};q^\kappa).\end{aligned}$$

1.2. Outline

The remainder of this paper is organised as follows. In the next chapter we review some standard material from Macdonald–Koornwinder theory. This includes a discussion of ordinary (or type A) Macdonald polynomials, Koornwinder polynomials and their lifted and virtual analogues, generalised Macdonald polynomials of type (R,S), Hall–Littlewood polynomials and Rogers–Szegő polynomials.

In Chapter 3 we consider two important functionals on the ring of symmetric functions known as virtual Koornwinder integrals. We review a number of earlier results for virtual Koornwinder integrals and prove several new integral evaluations.

In Chapter 4 we present our approach to bounded Littlewood identities. The key idea is to show that each bounded Littlewood identity is equivalent to the closed-form evaluation of a virtual Koornwinder integral. We apply our method to prove several new bounded Littlewood identities, including Theorem 1.1 and a q,t-analogue of the well-known Désarménien–Proctor–Stembridge determinant, see Theorem 4.1.

Chapter 5 contains applications of the main results of Chapter 4. First, in Section 5.1, we show that in the classical limit our approach to bounded Littlewood identities implies that MacMahon's formula (1.1.5) is a consequence of the well-known fact that $\big(\mathrm{GL}(n,\mathbb{R}),\mathrm{O}(n)\big)$ forms a Gelfand pair. Then, in Section 5.2, we show how our bounded Littlewood identities for Hall–Littlewood polynomials give rise to combinatorial formulas for characters of affine Lie algebras, such as

Theorem 1.2. This provides an alternative to the recent approach of Bartlett and the second author based on the C_n Bailey lemma [**9**]. In Section 5.3 we follow recent ideas of Griffin et al. [**47**] and apply the combinatorial character identities to prove new Rogers–Ramanujan identities for the affine Lie algebras $\mathrm{B}_n^{(1)}, \mathrm{A}_{2n-1}^{(2)}, \mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$. The Rogers–Ramanujan identities for $\mathrm{B}_n^{(1)}$, given in Theorem 5.15 and Remark 5.16, generalise Bressoud's well-known Rogers–Ramanujan identities for even moduli [**14, 15**] to arbitrary rank n. As a final application, in Section 5.4 we show that after principal specialisation our bounded q,t-Littlewood identities give rise to new transformation formulas for basic hypergeometric series of Kaneko–Macdonald-type [**58, 94**].

Finally, in Chapter 6, we discuss a number of open problems arising from our work, including several conjectures.

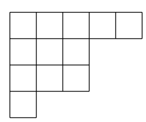
We conclude our paper with two appendices. Appendix A contains some technical lemmas related to the Weyl–Kac character formula, needed in Sections 5.2 and 5.3. In Appendix B we review the known connection between elliptic Selberg integrals and multiple basic hypergeometric series, and use this to prove a number of nonterminating quadratic transformation formulas for such series stated in Section 5.4.

CHAPTER 2

Macdonald–Koornwinder theory

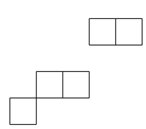
2.1. Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of nonnegative integers such that only finitely many λ_i are positive. The positive λ_i are called the parts of λ, and the number of parts, denoted $l(\lambda)$, is called the length of λ. As is customary, we often ignore the tail of zeros of a partition. If $|\lambda| := \sum_i \lambda_i = n$ we say that λ is a partition of n. The unique partition of 0 is denoted by 0. As usual, we identify a partition with its Young diagram—a collection of left-aligned rows of squares such that the ith row contains λ_i squares. We use the English convention for drawing Young diagrams, with rows labelled from top to bottom and columns from left to right. For example, the partition $(5, 3, 3, 1)$ corresponds to



and its top right-most square has coordinates $(1, 5)$. The conjugate partition λ' is obtained from λ by reflection in the main diagonal, so that the parts of λ' correspond to the columns of λ. For example, the conjugate of $(5, 3, 3, 1)$ is $(4, 3, 3, 1, 1)$. Given a partition λ, the multiplicity $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ counts the number of parts of size i. Clearly, $\sum_{i\geqslant 1} m_i(\lambda) = l(\lambda)$. If λ is a rectangular partition consisting of m rows and n columns we write $\lambda = m^n$. If λ is a partition of length at most n we also write $\lambda + m^n$ for $(\lambda_1 + m, \dots, \lambda_n + m)$. The number of even and odd parts of λ will be denoted by $\mathrm{even}(\lambda)$ and $\mathrm{odd}(\lambda)$ respectively. If $\mathrm{odd}(\lambda) = 0$ we say that λ is even. Given a partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, we write 2λ for the even partition $(2\lambda_1, 2\lambda_2, \dots)$. Similarly, if $\lambda = (\lambda_1, \lambda_2, \dots)$ is even, we write $\lambda/2$ for the partition $(\lambda_1/2, \lambda_2/2, \dots)$.

For two partitions λ, μ we write $\mu \subset \lambda$ if μ is contained in λ, i.e., if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. For $\mu \subset \lambda$, the set-theoretic difference between λ and μ is called a skew shape. To avoid a notational clash with partition complementation to be defined shortly, we write this difference as λ/μ instead of the more common $\lambda - \mu$. For example, the skew shape $(5, 3, 3, 1)/(3, 3, 1)$ is given by



As usual, we identify $\lambda/0$ and λ. A skew shape λ/μ containing at most one square in each column, as in the above example, is referred to as a horizontal strip. Analogously, a vertical strip is a skew diagram with at most one square in each row. If λ/μ is a horizontal strip then the partitions λ and μ are said to be interlacing, which we denote by $\lambda \succ \mu$. Note that $\lambda \succ \mu$ if and only if

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots .$$

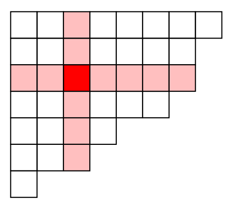
If $\lambda \subset m^n$ we write the complement of λ with respect to m^n as $m^n - \lambda$, that is, $m^n - \lambda = (m - \lambda_n, \dots, m - \lambda_2, m - \lambda_1)$. For example, the complement of $(3,2)$ with respect to 4^3 is $(4,2,1)$.

The dominance order on the set of partitions is defined as follows: $\lambda \geqslant \mu$ if $\lambda_1 + \cdots + \lambda_k \geqslant \mu_1 + \cdots + \mu_k$ for all $k \geqslant 1$. Note that unlike [**91**], we do not assume that $|\lambda| = |\mu|$. If $\lambda \geqslant \mu$ and $\lambda \neq \mu$ we write $\lambda > \mu$.

The arm-length, arm-colength, leg-length and leg-colength of the square $s = (i,j) \in \lambda$ are given by

$$\begin{aligned} a(s) &= a_\lambda(s) := \lambda_i - j, & a'(s) &= a'_\lambda(s) := j - 1, \\ l(s) &= l_\lambda(s) := \lambda'_j - i, & l'(s) &= l'_\lambda(s) := i - 1, \end{aligned}$$

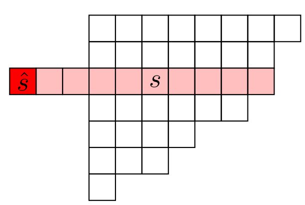
and correspond to the number of squares in the same row or column of s immediately to the east, west, south and north of s respectively. For the square $s = (3,3) \in (8,7,7,6,4,3,1)$ we have $(a(s), l(s), a'(s), l'(s)) = (4,3,2,2)$, as shown in the following diagram:



We also define the closely related 'C$_n$-type' analogues

$$\hat{a}(s) = \hat{a}_\lambda(s) := \lambda_i + j - 1, \qquad \hat{l}(s) = \hat{l}_\lambda(s) := \lambda'_j + i - 1.$$

Diagrammatically, $\hat{a}(s)$ corresponds to the arm-length of the mirror image, say $\hat{s}$, of $s \in \lambda$ upon reflection in the left boundary of λ. In the previous example, $\hat{s} = (3, -2)$ with arm-length $a(\hat{s}) = \hat{a}(s) = 9$.



In much the same way, $\hat{l}(s)$ is the leg-length of the reflection of s in the upper boundary of λ. Note that for all $(A, L) \in \{(a,l), (a',l'), (\hat{a}, \hat{l})\}$, $A_\lambda(i,j) = L_{\lambda'}(j,i)$. The usual hook-length of $s \in \lambda$ is given by $h(s) = a(s) + l(s) + 1$. The statistic $n(\lambda/\mu)$, which will be used repeatedly, is defined as [**73**, page 6]

$$n(\lambda/\mu) := \sum_{s \in \lambda/\mu} l(s) = \sum_{i \geqslant 1} \binom{\lambda'_i - \mu'_i}{2}.$$

For $\mu = 0$ this may also be expressed as the more familiar [**91**, page 3]

$$n(\lambda) = \sum_{s \in \lambda} l'(s) = \sum_{i \geqslant 1} (i-1)\lambda_i.$$

Finally, we say that $\lambda=(\lambda_1,\dots,\lambda_n)$ is a half-partition if $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n>0$ and all λ_i are half-integers. We sometimes write this as $\lambda=\mu+(1/2)^n$ with μ a partition of length at most n. Conversely, if $\lambda=\mu+(1/2)^n$, we also write $\mu=\lambda-(1/2)^n$. The length of a half-partition $\lambda=(\lambda_1,\dots,\lambda_n)$ is by definition n, and $m_i(\lambda)$ for i a positive half-integer is the multiplicity of parts of size i. We use half-partitions to generalise our earlier notion of complementation so that $m^n-\lambda$ makes sense for m an integer or half-integer and $\lambda\subset m^n$ a partition (of length at most n) or half-partition (of length n). For example, $4^3-(5/2,3/2,3/2)=(5/2,5/2,3/2)$ and $(7/2)^3-(5/2,3/2,3/2)=(2,2,1)$. We extend the dominance order to the set of half-partitions in the obvious manner. However, partitions and half-partitions are by definition incomparable.

2.2. Generalised q-shifted factorials

Let

$$(z;q)_\infty:=\prod_{i\geqslant 0}(1-zq^i)\quad\text{and}\quad(z;q)_n:=\frac{(z;q)_\infty}{(zq^n;q)_\infty}$$

be the standard q-shifted factorials [**43**]. In this paper we mostly view q-series as formal power series, but occasionally we require q to be a complex variable such that $|q|<1$. The modified theta function is defined as

$$\theta(z;q):=(z;q)_\infty(q/z;q)_\infty=\frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}(-z)^kq^{\binom{k}{2}}\quad\text{for } z\neq 0,\tag{2.2.1}$$

where the equality between the product and the sum is known as the Jacobi triple product identity [**43**, Equation (II.28)]. We also need more general q-shifted factorials indexed by partitions:

$$(z;q,t)_\lambda:=\prod_{s\in\lambda}\bigl(1-zq^{a'(s)}t^{-l'(s)}\bigr)=\prod_{i=1}^n(zt^{1-i};q)_{\lambda_i},\tag{2.2.2a}$$

$$\begin{aligned}C^-_\lambda(z;q,t)&:=\prod_{s\in\lambda}\bigl(1-zq^{a(s)}t^{l(s)}\bigr)\\&=\prod_{i=1}^n(zt^{n-i};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{(zt^{j-i-1};q)_{\lambda_i-\lambda_j}}{(zt^{j-i};q)_{\lambda_i-\lambda_j}},\end{aligned}\tag{2.2.2b}$$

$$\begin{aligned}C^+_\lambda(z;q,t)&:=\prod_{s\in\lambda}\bigl(1-zq^{\hat{a}(s)}t^{1-\hat{l}(s)}\bigr)\\&=\prod_{i=1}^n\frac{(zt^{2-2i};q)_{2\lambda_i}}{(zt^{2-i-n};q)_{\lambda_i}}\prod_{1\leqslant i<j\leqslant n}\frac{(zt^{2-i-j};q)_{\lambda_i+\lambda_j}}{(zt^{3-i-j};q)_{\lambda_i+\lambda_j}}.\end{aligned}\tag{2.2.2c}$$

In all three cases, the choice of n on the right is irrelevant as long as $n\geqslant l(\lambda)$. We note that $(a;q,t)_\lambda$ is sometimes denoted as $C^0_\lambda(a;q,t)$, see e.g., [**113**], and that $C^-_\lambda(t;q,t)=c_\lambda(q,t)$ and $C^-_\lambda(q;q,t)=c'_\lambda(q,t)$, with c_λ and c'_λ the hook-length polynomials of Macdonald [**91**, page 352]. In particular, $C^-_\lambda(q;q,q)=c_\lambda(q,q)=c'_\lambda(q,q)=H_\lambda(q)$ with

$$H_\lambda(q):=\prod_{s\in\lambda}\bigl(1-q^{h(s)}\bigr)\tag{2.2.3}$$

the classical hook-length polynomial. For $s \in \lambda$, let

$$b_\lambda(s;q,t) := \frac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}}. \tag{2.2.4}$$

Then

$$b_\lambda(q,t) := \frac{c_\lambda(q,t)}{c'_\lambda(q,t)} = \prod_{s\in\lambda} b_\lambda(s;q,t). \tag{2.2.5}$$

For both q-shifted factorials and theta functions, we use condensed notation such as

$$(z_1,\dots,z_k;q,t)_\lambda = (z_1;q,t)_\lambda \cdots (z_k;q,t)_\lambda.$$

It is an elementary exercise to verify the following identities, which will be used repeatedly throughout this paper:

$$(a;q,t)_{\lambda'} = (-a)^{|\lambda|}q^{n(\lambda)}t^{-n(\lambda')}(a^{-1};t,q)_\lambda, \tag{2.2.6a}$$

$$C^-_{\lambda'}(a;q,t) = C^-_\lambda(a;t,q), \tag{2.2.6b}$$

$$C^+_{\lambda'}(a;q,t) = (-aq)^{|\lambda|}q^{3n(\lambda)}t^{-3n(\lambda')}C^+_\lambda((aqt)^{-1};t,q), \tag{2.2.6c}$$

$$(a;q,t)_{2\lambda} = (a,aq;q^2,t)_\lambda, \tag{2.2.7a}$$

$$C^-_{2\lambda}(a;q,t) = C^-_\lambda(a,aq;q^2,t), \tag{2.2.7b}$$

and

$$(a;q,t)_{m^n-\lambda} = (-q^{1-m}t^{n-1}/a)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\frac{(a;q,t)_{m^n}}{(q^{1-m}t^{n-1}/a;q,t)_\lambda}, \tag{2.2.8a}$$

$$C^-_{m^n-\lambda}(a;q,t) = (-q^{1-m}/a)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\frac{(at^{n-1};q,t)_{m^n}C^-_\lambda(a;q,t)}{(at^{n-1},q^{1-m}/a;q,t)_\lambda}. \tag{2.2.8b}$$

2.3. Rogers–Szegő polynomials

For integers k,n such that $0 \leqslant k \leqslant n$, let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

be a q-binomial coefficient. Then the Rogers–Szegő polynomials $H_m(z;q)$ are defined as [**132**]

$$H_m(z;q) := \sum_{k=0}^m z^k \begin{bmatrix} m \\ k \end{bmatrix}_q, \tag{2.3.1}$$

for m a nonnegative integer. They have generating function [**2**, page 49]

$$\sum_{m\geqslant 0} \frac{H_m(z;q)t^m}{(q;q)_m} = \frac{1}{(t,tz;q)_\infty}, \tag{2.3.2}$$

and satisfy the three-term recurrence relation

$$H_{m+1}(z;q) = (1+z)H_m(z;q) - (1-q^m)zH_{m-1}(z;q)$$

subject to the initial conditions $H_{-1}=0$, $H_0=1$. For $0<q<1$, the Rogers–Szegő polynomials satisfy the orthogonality relation

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} H_m(zq^{-1/2};q)H_n(\bar{z}q^{-1/2};q)\,\big|(zq^{1/2};q)_\infty\big|^2\,\frac{\mathrm{d}z}{z}=q^{-m}(q;q)_m\,\delta_{m,n},$$

where $\mathbb{T}$ is the positively-oriented unit circle.

The Rogers–Szegő polynomials are closely related to symmetric functions and may, for example, be expressed in terms of Schur functions as

$$H_m(z;q)=(q)_m\sum_{\lambda\vdash m}\frac{q^{n(\lambda)}}{H_\lambda(q)}\,s_\lambda(1,z), \tag{2.3.3}$$

with $H_\lambda(q)$ the hook-length polynomial (2.2.3). Indeed, by (2.3.2) and the $n=2$ case of [**91**, page 66]

$$\sum_\lambda\frac{q^{n(\lambda)}}{H_\lambda(q)}\,s_\lambda(x_1,\dots,x_n)=\prod_{i=1}^n\frac{1}{(x_i;q)_\infty},$$

the formula (2.3.3) immediately follows.

We require two generalisations of the Rogers–Szegő polynomials to polynomials indexed by partitions. First,

$$\begin{aligned} h_\lambda^{(m)}(a,b;q) &:= \prod_{\substack{i=1\\ i\text{ odd}}}^{m-1}(-a)^{m_i(\lambda)}H_{m_i(\lambda)}(b/a;q)\prod_{\substack{i=1\\ i\text{ even}}}^{m-1}H_{m_i(\lambda)}(ab;q) \\ &= (-a)^{\mathrm{odd}(\lambda)}\prod_{\substack{i=1\\ i\text{ odd}}}^{m-1}H_{m_i(\lambda)}(b/a;q)\prod_{\substack{i=1\\ i\text{ even}}}^{m-1}H_{m_i(\lambda)}(ab;q), \end{aligned} \tag{2.3.4}$$

where m is a nonnegative integer. Compared to earlier definitions in [**9**, **139**] the parameters a and b have been replaced by their negatives. Since H_m is a self-reciprocal polynomial, i.e., $z^mH_m(1/z;q)=H_m(z;q)$, it follows that $h_\lambda^{(m)}(a,b;q)$ is symmetric in a and b:

$$h_\lambda^{(m)}(a,b;q)=h_\lambda^{(m)}(b,a;q).$$

Since $m_i(\lambda)=0$ for $i>\lambda_1$, the upper bound on the products over i in (2.3.4) may be dropped if $m\geqslant\lambda_1+1$. For such m we simply write $h_\lambda(a,b;q)$. That is,

$$h_\lambda(a,b;q):=\prod_{\substack{i\geqslant1\\ i\text{ odd}}}(-a)^{m_i(\lambda)}H_{m_i(\lambda)}(b/a;q)\prod_{\substack{i\geqslant1\\ i\text{ even}}}H_{m_i(\lambda)}(ab;q). \tag{2.3.5}$$

We further define

$$h_\lambda^{(m)}(a;t):=h_\lambda^{(m)}(a,-1;t)=\prod_{i=1}^{m-1}H_{m_i(\lambda)}(-a;t). \tag{2.3.6}$$

It follows from

$$H_m(0;t) = 1, \tag{2.3.7a}$$

$$H_m(-1;t) = \begin{cases} (t;t^2)_{m/2} & m \text{ even,} \\ 0 & m \text{ odd,} \end{cases} \tag{2.3.7b}$$

$$H_m(-t;t) = (t;t^2)_{\lceil m/2 \rceil}, \tag{2.3.7c}$$

$$H_m(t^{1/2};t) = (-t^{1/2};t^{1/2})_m, \tag{2.3.7d}$$

(see [**139**, Equation (1.10)]) that for special values of a and b the polynomials (2.3.4)–(2.3.6) completely factor. This will be important in Section 5.2 in our discussion of character formulas for affine Lie algebras.

2.4. Plethystic notation

Let $\mathfrak{S}_n$ be the symmetric group on n letters, $\Lambda_n = \mathbb{F}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ the ring of symmetric functions in the alphabet $x_1, \dots, x_n$ with coefficients in $\mathbb{F}$, and Λ the corresponding ring of symmetric functions in countably many variables, see e.g., [**91**, **126**]. We will mostly consider $\mathbb{F} = \mathbb{Q}(q,t)$ and $\mathbb{F} = \mathbb{Q}(q,t,t_0,t_1,t_2,t_3)$, or minor variations thereof.

To facilitate computations in Λ we frequently employ plethystic or λ-ring notation [**49**, **72**]. This is most easily described in terms of the Newton power sums

$$p_r(x) := x_1^r + x_2^r + \cdots, \qquad r \geqslant 1,$$

with generating function

$$\Psi_z(x) := \sum_{r=1}^{\infty} z^{r-1} p_r(x) = \sum_{i \geqslant 1} \frac{x_i}{1 - z x_i}.$$

The power sums form an algebraic basis of Λ, that is, $\Lambda = \mathbb{F}[p_1, p_2, \dots]$.

If $x = (x_1, x_2, \dots)$, we additively write $x = x_1 + x_2 + \cdots$, and to indicate the latter notation, we use plethystic brackets:

$$f(x) = f(x_1, x_2, \dots) = f[x_1 + x_2 + \cdots] = f[x], \qquad f \in \Lambda.$$

A power sum whose argument is the sum, difference or Cartesian product of two alphabets x and y is then defined as

$$p_r[x+y] := p_r[x] + p_r[y], \tag{2.4.1a}$$

$$p_r[x-y] := p_r[x] - p_r[y], \tag{2.4.1b}$$

$$p_r[xy] := p_r[x] p_r[y], \tag{2.4.1c}$$

respectively. In particular, if x is the empty alphabet then $p_r[-y] = -p_r[y]$, (which should not be confused with $p(-y) = p(-y_1, -y_2, \dots) = (-1)^r p_r(y)$), so that

$$f[-(-x)] = f[x]. \tag{2.4.2}$$

Occasionally we need to also use an ordinary minus sign in plethystic notation. To distinguish this from a plethystic minus sign, we denote by ε the alphabet consisting of the single letter -1, so that for $f \in \Lambda$

$$f(-x) = f(-x_1, -x_2, \dots) = f[\varepsilon x_1 + \varepsilon x_2 + \cdots] = f[\varepsilon x].$$

Hence

$$p_r[\varepsilon x] = (-1)^r p_r[x], \qquad p_r[-\varepsilon x] = (-1)^{r-1} p_r[x]$$

and

$$f[x+\varepsilon] = f(-1, x_1, x_2, \dots).$$

For indeterminates a, b, t and $f \in \Lambda$, we further define $f[(a-b)/(1-t)]$ by

$$p_r\Big[\frac{a-b}{1-t}\Big] = \frac{a^r - b^r}{1-t^r}, \tag{2.4.3}$$

and note that $(a-b)/(1-t)$ may be viewed as the difference between the alphabets $a(1+t+t^2+\cdots)$ and $b(1+t+t^2+\cdots)$, where $a(1+t+t^2+\cdots)$ is the Cartesian product of the single-letter alphabet a and the infinite alphabet $1+t+t^2+\cdots$. Alternatively, $(a-b)/(1-t)$ may be interpreted as the Cartesian product of $a-b$ and $1+t+t^2+\cdots$. We can of course combine (2.4.1) and (2.4.3), and for example

$$p_r\Big[x + \frac{a-b}{1-t}\Big] = p_r[x] + p_r\Big[\frac{a-b}{1-t}\Big].$$

For $r \geqslant 0$, the complete symmetric functions are defined by

$$h_r(x) := \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r} x_{i_1} x_{i_2} \cdots x_{i_r},$$

and admit the generating function

$$\sigma_z(x) := \sum_{r \geqslant 0} z^r h_r(x) = \prod_{i \geqslant 1} \frac{1}{1 - z x_i}. \tag{2.4.4}$$

Since $\Psi_z(x) = \frac{\mathrm{d}}{\mathrm{d}z} \log \sigma_z(x)$, it follows that

$$\sigma_1[x+y] = \sigma_1[x]\sigma_1[y] = \prod_{i \geqslant 1} \frac{1}{(1-x_i)(1-y_i)}, \tag{2.4.5a}$$

$$\sigma_1[x-y] = \frac{\sigma_1[x]}{\sigma_1[y]} = \prod_{i \geqslant 1} \frac{1-y_i}{1-x_i}, \tag{2.4.5b}$$

$$\sigma_1\Big[\frac{a-b}{1-t}\Big] = \prod_{k \geqslant 0} \frac{\sigma_1[at^k]}{\sigma_1[bt^k]} = \frac{(b;t)_\infty}{(a;t)_\infty}. \tag{2.4.5c}$$

These three formulas allow various infinite products to be expressed in terms of symmetric functions. For example, it follows immediately from the generating function (2.3.2) that the Rogers–Szegő polynomials $H_m(z;q)$ may be identified as

$$H_m(z;q) = (q;q)_m \, h_m\Big[\frac{1+z}{1-q}\Big].$$

Finally, for $\mathbb{F} = \mathbb{Q}(q,t)$, $\omega_{q,t}$ is the $\mathbb{F}$-algebra endomorphism of Λ defined by [**91**, page 312]

$$\omega_{q,t}\, p_r = (-1)^{r-1} \frac{1-q^r}{1-t^r}\, p_r.$$

Note that $\omega_{t,q} = \omega_{q,t}^{-1}$ and, plethystically,

$$\omega_{q,t}\, f(x) = f\Big(\Big[-\varepsilon\, \frac{1-q}{1-t}\, x\Big]\Big), \qquad f \in \Lambda. \tag{2.4.6}$$

2.5. Macdonald polynomials

Let $\mathbb{F} = \mathbb{Q}(q,t)$. The power sums $p_\lambda := \prod_{i=1}^{l(\lambda)} p_{\lambda_i}$ may be used to define Macdonald's q,t-analogue of the Hall scalar product on Λ as [**91**]

$$\langle p_\lambda, p_\mu\rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{n} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},$$

where $z_\lambda = \prod_{i\geqslant 1} m_i(\lambda)!\, i^{m_i(\lambda)}$. The Macdonald polynomials $P_\lambda(q,t) = P_\lambda(x;q,t)$ are the unique family of symmetric functions such that [**91**]

$$P_\lambda(q,t) = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}(q,t) m_\mu \tag{2.5.1}$$

and

$$\langle P_\lambda(q,t), P_\mu(q,t)\rangle_{q,t} = 0 \qquad \text{if} \quad \lambda \neq \mu.$$

Here the m_λ are the monomial symmetric functions, defined by

$$m_\lambda(x) := \sum_\alpha x^\alpha,$$

where α is summed over distinct permutations of $\lambda = (\lambda_1, \lambda_2, \dots)$ and $x^\alpha := \prod_{i\geqslant 1} x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. By the triangularity of (2.5.1) and the fact that the m_λ form a basis of Λ, it immediately follows that the Macdonald polynomials form a basis of Λ as well. When $l(\lambda) > n$, $P_\lambda(x_1,\dots,x_n;q,t) = 0$ and the polynomials $P_\lambda(x_1,\dots,x_n;q,t)$ indexed by partitions λ of length at most n form a basis of Λ_n.

The skew Macdonald polynomials $P_{\lambda/\mu}(q,t)$ are defined by

$$\langle P_{\lambda/\mu}(q,t), P_\nu(q,t)\rangle_{q,t} = \langle P_\lambda(q,t), P_\mu(q,t)P_\nu(q,t)\rangle_{q,t},$$

and vanish unless $\mu \subset \lambda$. For $q = t$ the Macdonald polynomials simplify to the Schur functions: $P_{\lambda/\mu}(t,t) = s_{\lambda/\mu}$.

For later comparison with the Koornwinder and (R,S) Macdonald polynomials, we remark that an alternative description of the Macdonald polynomials in n variables is as the unique family of polynomials (2.5.1) such that [**91**]

$$\langle P_\lambda, P_\mu\rangle'_{q,t} = 0 \qquad \text{if } \lambda \neq \mu,$$

where, for $|q|,|t| < 1$, $\langle\cdot,\cdot\rangle'_{q,t}$ is the scalar product on $\mathbb{F}[x] = \mathbb{F}[x_1,\dots,x_n]$ defined by

$$\langle f,g\rangle'_{q,t} := \frac{1}{n!(2\pi\mathrm{i})^n} \int_{\mathbb{T}^n} f(x) g(x^{-1}) \Delta(x;q,t) \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

Here $f(x^{-1}) := f(x_1^{-1},\dots,x_n^{-1})$, $\Delta(x;q,t)$ is the Macdonald density

$$\Delta(x;q,t) := \prod_{1\leqslant i<j\leqslant n} \frac{(x_i/x_j, x_j/x_i;q)_\infty}{(tx_i/x_j, tx_j/x_i;q)_\infty} \tag{2.5.2}$$

and $\mathbb{T}^n$ is the n-dimensional complex torus:

$$\mathbb{T}^n := \{(x_1,\dots,x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}.$$

Below we list a number of standard results from Macdonald polynomial theory needed later. First of all, defining a second family of Macdonald polynomials $Q_{\lambda/\mu}(q,t) = Q_{\lambda/\mu}(x;q,t)$ as

$$Q_{\lambda/\mu}(q,t) := \frac{b_\lambda(q,t)}{b_\mu(q,t)} P_{\lambda/\mu}(q,t), \tag{2.5.3}$$

with $b_\lambda(q,t)$ given by (2.2.5), we have the duality [**91**, page 327]

$$\omega_{q,t}\, P_\lambda(q,t) = Q_{\lambda'}(t,q), \tag{2.5.4}$$

as well as the orthogonality [**91**, page 324]

$$\langle P_\lambda(q,t), Q_\mu(q,t)\rangle_{q,t} = \delta_{\lambda\mu}.$$

This last equation is equivalent to the Cauchy identity [**91**, page 324]

$$\sum_\lambda P_\lambda(x;q,t)Q_\lambda(y;q,t) = \prod_{i,j\geqslant 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty},$$

which we repeatedly require in the dual form [**91**, page 329]

$$\sum_{\lambda\subset m^n} (-1)^{|\lambda|} P_\lambda(x_1,\dots,x_n;q,t)P_{\lambda'}(y_1,\dots,y_m;t,q) = \prod_{i=1}^n\prod_{j=1}^m (1-x_iy_j). \tag{2.5.5}$$

We also need the g- and e-Pieri rules for Macdonald polynomials [**91**, page 340], expressed in generating function form. First, in the g-Pieri case

$$P_\mu(x;q,t)\prod_{i\geqslant 1}\frac{(atx_i;q)_\infty}{(ax_i;q)_\infty} = \sum_{\lambda\succ\mu} a^{|\lambda/\mu|}\varphi_{\lambda/\mu}(q,t)P_\lambda(x;q,t), \tag{2.5.6}$$

where the Pieri coefficient $\varphi_{\lambda/\mu}(q,t)$ is given by [**91**, page 342]

$$\begin{aligned}\varphi_{\lambda/\mu}(q,t) = \prod_{1\leqslant i\leqslant j\leqslant l(\lambda)} &\frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}}{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}\cdot\frac{(qt^{j-i};q)_{\mu_i-\mu_{j+1}}}{(t^{j-i+1};q)_{\mu_i-\mu_{j+1}}} \\ &\times \frac{(t^{j-i+1};q)_{\lambda_i-\mu_j}}{(qt^{j-i};q)_{\lambda_i-\mu_j}}\cdot\frac{(t^{j-i+1};q)_{\mu_i-\lambda_{j+1}}}{(qt^{j-i};q)_{\mu_i-\lambda_{j+1}}}.\end{aligned} \tag{2.5.7}$$

Similarly, the e-Pieri rule is given by

$$P_\mu(x;q,t)\prod_{i\geqslant 1}(1+ax_i) = \sum_{\lambda'\succ\mu'} a^{|\lambda/\mu|}\psi'_{\lambda/\mu}(q,t)P_\lambda(x;q,t), \tag{2.5.8}$$

where [**91**, page 336]

$$\psi'_{\lambda/\mu}(q,t) = \prod \frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}\cdot\frac{1-q^{\lambda_i-\lambda_j}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j}t^{j-i}}. \tag{2.5.9}$$

The product in (2.5.9) is over all $i<j$ such that $\lambda_i=\mu_i$ and $\lambda_j>\mu_j$. An alternative expression for $\psi'_{\lambda/\mu}(q,t)$ is given by [**91**, page 340]

$$\psi'_{\lambda/\mu}(q,t) = \prod \frac{b_\lambda(s;q,t)}{b_\mu(s;q,t)}, \tag{2.5.10}$$

where $b_\lambda(s;q,t)$ is given by (2.2.4) and where the product is over all squares $s=(i,j)\in\mu\subset\lambda$ such that $i<j$, $\mu_i=\lambda_i$ and $\lambda'_j>\mu'_j$.

For λ a partition, define

$$b^{\mathrm{ea}}_\lambda(q,t) := \prod_{\substack{s\in\lambda\\ a(s)\text{ even}}} b_\lambda(s;q,t) = \prod_{\substack{s\in\lambda\\ a(s)\text{ even}}} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}, \tag{2.5.11}$$

where 'ea' stands for 'even arm(-length)'.

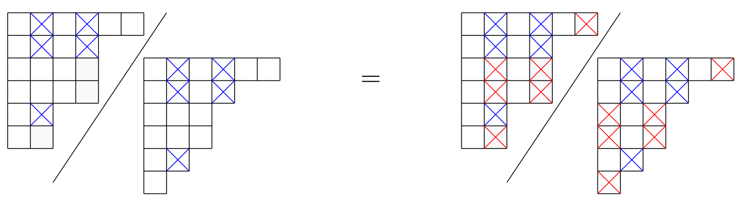
LEMMA 2.1. *For partitions $\lambda \succ \mu$ such that*

$$\lambda = 2\lceil \mu/2 \rceil := (2\lceil \mu_1/2 \rceil, 2\lceil \mu_2/2 \rceil, \dots),$$

we have

$$\psi'_{\lambda/\mu}(q,t) = \frac{C^-_{\lambda/2}(t;q^2,t)}{C^-_{\lambda/2}(q;q^2,t)} \cdot \frac{1}{b^{\mathrm{ea}}_\mu(q,t)}.$$

PROOF. The product in (2.5.10) is over all squares s in μ for which λ and μ have the same row length but different column lengths. If $\lambda = 2\lceil \mu/2 \rceil$ then λ is obtained from μ by adding a square to each row of odd length. Hence the product is over all squares $s = (i,j)$ such that μ_i and j are even and such that there exists a $k > i$ with $\mu_k = j-1$. For example, if $\mu = (6,4,3,3,2,1)$ then $\lambda = (6,4,4,4,2,2)$. In the diagram on the left, shown below, the squares contributing to $\prod b_\lambda(s;q,t)/b_\mu(s;q,t)$ are marked with a blue cross. Because each marked square (i,j) occurs in a rows of even length and has even j-coordinate, each marked square must have even arm-length. We can, however, include all other squares of λ and μ with even arm-length, since their respective contributions to $b_\lambda(s;q,t)$ and $b_\mu(s;q,t)$ trivially cancel, as indicated by the added red squares in the diagram on the right:



Hence

$$\begin{aligned}\psi'_{\lambda/\mu}(q,t) &= \prod_{\substack{s\in\lambda \\ a(s)\text{ even}}} b_\lambda(s;q,t) \prod_{\substack{s\in\mu \\ a(s)\text{ even}}} \frac{1}{b^{\mathrm{ea}}_\mu(s;q,t)} \\ &= \bigg(\prod_{s\in\lambda/2} \frac{1-q^{2a(s)}t^{l(s)+1}}{1-q^{2a(s)+1}t^{l(s)}}\bigg) \cdot \frac{1}{b^{\mathrm{ea}}_\mu(q,t)},\end{aligned}$$

where the second equality uses the fact that λ is an even partition. By (2.2.2) we are done. □

If in (2.5.8) we set $x_i = 0$ for $i > n$ and equate terms of degree $|\mu| + n$, we obtain

$$P_\mu(x_1,\dots,x_n;q,t)\, x_1\cdots x_n = \psi'_{(\mu+1^n)/\mu}(q,t) P_{\mu+1^n}(x_1,\dots,x_n;q,t).$$

By (2.5.9), the Pieri coefficient on the right is 1, so that [**91**, page 325]

$$P_\mu(x_1,\dots,x_n;q,t)\, x_1\cdots x_n = P_{\mu+1^n}(x_1,\dots,x_n;q,t). \tag{2.5.12}$$

Closely related to the Pieri formulas is the branching rule [**91**, page 346]

$$P_\lambda(x_1,\dots,x_n;q,t) = \sum_{\mu\prec\lambda} x_n^{|\lambda/\mu|} \psi_{\lambda/\mu}(q,t) P_\mu(x_1,\dots,x_{n-1};q,t), \tag{2.5.13}$$

where [**91**, page 341]

$$\psi_{\lambda/\mu}(q,t) = \psi'_{\lambda'/\mu'}(t,q). \tag{2.5.14}$$

Together with the initial condition $P_\mu(-;q,t)=\delta_{\mu,0}$, this uniquely determines the Macdonald polynomials.

We conclude our list of results for Macdonald polynomials with the principal specialisation formula [**91**, page 338]

$$P_\lambda(1,t,\dots,t^{n-1};q,t)=P_\lambda\Big(\Big[\frac{1-t^n}{1-t}\Big];q,t\Big)=t^{n(\lambda)}\frac{(t^n;q,t)_\lambda}{C^-_\lambda(t;q,t)}. \tag{2.5.15}$$

2.6. Koornwinder polynomials

2.6.1. Koornwinder polynomials. The Koornwinder polynomials [**65**] are a generalisation of the Macdonald polynomials to the root system BC_n. They depend on six parameters, except for $n=1$ when they correspond to the 5-parameter Askey–Wilson polynomials [**6**].

Throughout this section $x=(x_1,\dots,x_n)$. Then the Koornwinder density is given by

$$\Delta(x;q,t;t_0,t_1,t_2,t_3):=\prod_{i=1}^n\frac{(x_i^{\pm 2};q)_\infty}{\prod_{r=0}^3(t_rx_i^{\pm};q)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{(x_i^{\pm}x_j^{\pm};q)_\infty}{(tx_i^{\pm}x_j^{\pm};q)_\infty}, \tag{2.6.1}$$

where

$$\begin{aligned}(x_i^{\pm};q)_\infty&:=(x_i,x_i^{-1};q)_\infty,\\(x_i^{\pm}x_j^{\pm};q)_\infty&:=(x_ix_j,x_ix_j^{-1},x_i^{-1}x_j,x_i^{-1}x_j^{-1};q)_\infty.\end{aligned}$$

For complex $q,t,t_0,\dots,t_3$ such that $|q|,|t|,|t_0|,\dots,|t_3|<1$, this density may be used to define a scalar product on $\mathbb{C}[x^{\pm 1}]$ as

$$\langle f,g\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3}:=\int_{\mathbb{T}^n}f(x)g(x^{-1})\Delta(x;q,t;t_0,t_1,t_2,t_3)\,\mathrm{d}T(x),$$

where

$$\mathrm{d}T(x):=\frac{1}{2^nn!(2\pi\mathrm{i})^n}\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_n}{x_n}. \tag{2.6.2}$$

Let $W=\mathfrak{S}_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$ be the hyperoctahedral group with natural action on $\mathbb{C}[x^{\pm}]$. For λ a partition of length at most n, let m_λ^W be the W-invariant monomial symmetric function indexed by λ:

$$m_\lambda^W(x):=\sum_\alpha x^\alpha.$$

Here the sum is over all α in the W-orbit of λ. In analogy with Macdonald polynomials, the Koornwinder polynomials $K_\lambda=K_\lambda(x;q,t;t_0,t_1,t_2,t_3)$ are defined as the unique family of polynomials in $\Lambda^{\mathrm{BC}_n}:=\mathbb{C}[x^{\pm}]^W$ such that [**65**]

$$K_\lambda=m_\lambda^W+\sum_{\mu<\lambda}c_{\lambda\mu}m_\mu^W$$

and

$$\langle K_\lambda,K_\mu\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3}=0\qquad\text{if }\lambda\neq\mu. \tag{2.6.3}$$

From the definition it follows that the K_λ are symmetric under permutation of the t_r. The quadratic norm was first evaluated in [**35**] (self-dual case) and [**120**] (general case). For our purposes we only need

$$\langle 1,1\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3}=\prod_{i=1}^{n}\frac{(t,t_0t_1t_2t_3t^{n+i-2};q)_\infty}{(q,t^i;q)_\infty\prod_{0\leqslant r<s\leqslant 3}(t_rt_st^{i-1};q)_\infty},\tag{2.6.4}$$

known as Gustafson's integral [**48**].

The BC_n analogue of the Cauchy identity (2.5.5) is given by [**98**, Theorem 2.1]

$$\begin{aligned}\sum_{\lambda\subset m^n}(-1)^{|\lambda|}&K_{m^n-\lambda}(x;q,t;t_0,t_1,t_2,t_3)K_{\lambda'}(y;t,q;t_0,t_1,t_2,t_3)\\&=\prod_{i=1}^{n}\prod_{j=1}^{m}\big(x_i+x_i^{-1}-y_j-y_j^{-1}\big)=\prod_{i=1}^{n}\prod_{j=1}^{m}x_i^{-1}\big(1-x_iy_j^{\pm}\big),\end{aligned}\tag{2.6.5}$$

where $y=(y_1,\dots,y_m)$ and $(1-ab^{\pm}):=(1-ab)(1-ab^{-1})$.

2.6.2. Lifted and virtual Koornwinder polynomials. The lifted Koornwinder polynomials $\tilde K_\lambda=\tilde K_\lambda(q,t,T;t_0,t_1,t_2,t_3)=\tilde K_\lambda(x;q,t,T;t_0,t_1,t_2,t_3)$ are a 7-parameter family of inhomogeneous symmetric functions [**108**]. They are invariant under permutations of the t_r and form a $\mathbb{Q}(q,t,T,t_0,t_1,t_2,t_3)$-basis of Λ. For example, $\tilde K_0=1$ and

$$\tilde K_1=m_1+\frac{1-T}{(1-t)(1-t_0t_1t_2t_3T^2/t^2)}\sum_{r=0}^{3}\Big(\frac{t_0t_1t_2t_3T}{t_rt}-t_r\Big).$$

As a function of the t_r, the lifted Koornwinder polynomial $\tilde K_\lambda$ has poles at

$$t_0t_1t_2t_3=q^{1-\hat a(s)}t^{\hat l(s)+1}T^{-2}=q^{2-\lambda_i-j}t^{i+\lambda'_j}T^{-2}\tag{2.6.6}$$

for all $s=(i,j)\in\lambda$. Importantly, according to [**108**, Theorem 7.1], for generic $q,t,t_0,\dots,t_3$ (so as to avoid potential poles)

$$\begin{aligned}\tilde K_\lambda(x_1^{\pm},&\dots,x_n^{\pm};q,t,t^n;t_0,t_1,t_2,t_3)\\&=\begin{cases}K_\lambda(x_1,\dots,x_n;q,t;t_0,t_1,t_2,t_3) & \text{if } l(\lambda)\leqslant n,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}\tag{2.6.7}$$

where, for $f\in\Lambda$ (or $f\in\hat\Lambda$, see below), $f(x_1,\dots,x_k):=f(x_1,\dots,x_k,0,0,\dots)$.

Let $\hat\Lambda$ be the completion of the ring of symmetric functions with respect to the natural grading by degree, i.e., $\hat\Lambda$ is the inverse limit of Λ_n relative to the homomorphism $\rho_{m,n}:\Lambda_m\to\Lambda_n$ ($m\geqslant n$) which sends $m_\lambda(x_1,\dots,x_m)$ to $m_\lambda(x_1,\dots,x_n)$ for $l(\lambda)\leqslant n$ and to 0 otherwise. Then the virtual Koornwinder polynomials $\hat K_\lambda=\hat K_\lambda(q,t,Q;t_0,t_1,t_2,t_3)=\hat K_\lambda(x;q,t,Q;t_0,t_1,t_2,t_3)$ (which are again symmetric in the t_r) form a $\mathbb{Q}(q,t,Q,t_0,t_1,t_2,t_3)$-basis of $\hat\Lambda$, such that for $\lambda\subset m^n$,

$$\begin{aligned}\hat K_\lambda(x_1,&\dots,x_n;q,t,q^m;t_0,t_1,t_2,t_3)\\&=(x_1\cdots x_n)^mK_{m^n-\lambda}(x_1,\dots,x_n;q,t;t_0,t_1,t_2,t_3).\end{aligned}\tag{2.6.8}$$

When $Q=0$ the virtual Koornwinder polynomials can be expressed in terms of Macdonald polynomials as [**108**, Corollary 7.21]

$$\hat K_\lambda(x;q,t,0;t_0,t_1,t_2,t_3)=P_\lambda(x;q,t)\prod_{i\geqslant 1}\frac{\prod_{r=0}^{3}(t_rx_i;q)_\infty}{(x_i^2;q)_\infty}\prod_{i<j}\frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty},$$

from which it follows that

$$\begin{aligned}(2.6.9)\qquad &\lim_{m\to\infty}(x_1\cdots x_n)^m K_{m^n-\lambda}(x_1,\dots,x_n;q,t;t_0,t_1,t_2,t_3)\\ &\qquad = P_\lambda(x_1,\dots,x_n;q,t)\prod_{i=1}^n \frac{\prod_{r=0}^3 (t_r x_i;q)_\infty}{(x_i^2;q)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}.\end{aligned}$$

The lifted and virtual Koornwinder polynomials admit a lift of the Cauchy identity (2.6.5) to $\hat{\Lambda}_x\otimes\Lambda_y$ as follows, see [**108**, Theorem 7.14]:

$$\begin{aligned}(2.6.10)\qquad &\sum_\lambda (-1)^{|\lambda|}\hat{K}_\lambda(x;t,q,T;t_0,t_1,t_2,t_3)\tilde{K}_{\lambda'}(y;q,t,T;t_0,t_1,t_2,t_3)\\ &\qquad = \prod_{i,j\geqslant 1}(1-x_iy_j).\end{aligned}$$

This may be used to derive the following symmetry relation for virtual Koornwinder polynomials.

Proposition 2.2. *The virtual Koornwinder polynomials satisfy*

$$\begin{aligned}&\hat{K}_\lambda(x;q,t,Q;t_0,t_1,t_2,t_3)\\ &\qquad = \hat{K}_\lambda(x;q,t,Qt_0t_1/q;q/t_0,q/t_1,t_2,t_3)\prod_{i\geqslant 1}\frac{(t_0x_i,t_1x_i;q)_\infty}{(qx_i/t_0,qx_i/t_1;q)_\infty}.\end{aligned}$$

Proof. We start with (2.6.10) and identify the double product on the right as $\sigma_1[-xy]$. Carrying out the plethystic substitution

$$y\mapsto y+\frac{t_0-t/t_0}{1-t}+\frac{t_1-t/t_1}{1-t}=: y',$$

and applying the symmetry [**108**, Equation (7.2)]

$$\begin{aligned}(2.6.11)\qquad &\tilde{K}_\lambda\Big(\Big[y+\frac{t_0-t/t_0}{1-t}+\frac{t_1-t/t_1}{1-t}\Big];q,t,T;t_0,t_1,t_2,t_3\Big)\\ &\qquad = \tilde{K}_\lambda(y;q,t,Tt_0t_1/t;t/t_0,t/t_1,t_2,t_3),\end{aligned}$$

we obtain

$$\begin{aligned}(2.6.12)\qquad &\sum_\lambda (-1)^{|\lambda|}\hat{K}_\lambda(x;t,q,T;t_0,t_1,t_2,t_3)\tilde{K}_{\lambda'}(y;q,t,Tt_0t_1/t;t/t_0,t/t_1,t_2,t_3)\\ &\qquad = \prod_{i\geqslant 1}\frac{(t_0x_i,t_1x_i;t)_\infty}{(tx_i/t_0,tx_i/t_1;t)_\infty}\prod_{i,j\geqslant 1}(1-x_iy_j).\end{aligned}$$

Here the right-hand side follows from (2.4.2) and (2.4.5):

$$\begin{aligned}\sigma_1[-xy'] &= \sigma_1\Big[-x\Big(y+\frac{t_0-t/t_0}{1-t}+\frac{t_1-t/t_1}{1-t}\Big)\Big]\\ &= \sigma_1[-xy]\sigma_1\Big[x\,\frac{t/t_0-t_0}{1-t}\Big]\sigma_1\Big[x\,\frac{t/t_1-t_1}{1-t}\Big]\\ &= \prod_{i,j\geqslant 1}(1-x_iy_j)\prod_{i\geqslant 1}\frac{(t_0x_i,t_1x_i;t)_\infty}{(tx_i/t_0,tx_i/t_1;t)_\infty}.\end{aligned}$$

After replacing $(t_0, t_1, T) \mapsto (t/t_0, t/t_1, Tt_0t_1/t)$, the identity (2.6.12) takes the equivalent form

$$\sum_\lambda (-1)^{|\lambda|} \hat{K}_\lambda(x; t, q, Tt_0t_1/t; t/t_0, t/t_1, t_2, t_3) \tilde{K}_{\lambda'}(y; q, t, T; t_0, t_1, t_2, t_3) = \prod_{i \geqslant 1} \frac{(tx_i/t_0, tx_i/t_1; t)_\infty}{(t_0x_i, t_1x_i; t)_\infty} \prod_{i,j \geqslant 1} (1 - x_iy_j).$$

Expanding the double product on the right by the Cauchy identity (2.6.10), and extracting coefficients of $\tilde{K}_{\lambda'}(y; q, t, T; t_0, t_1, t_2, t_3)$, yields

$$\hat{K}_\lambda(x; t, q, Tt_0t_1/t; t/t_0, t/t_1, t_2, t_3) = \hat{K}_\lambda(x; t, q, T; t_0, t_1, t_2, t_3) \prod_{i \geqslant 1} \frac{(tx_i/t_0, tx_i/t_1; t)_\infty}{(t_0x_i, t_1x_i; t)_\infty}.$$

After the substitution $(t, q, T) \mapsto (q, t, Q)$ we are done. □

2.7. Macdonald–Koornwinder polynomials

2.7.1. Macdonald polynomials on root systems. Below we closely follow Macdonald's exposition in [**92**]. For basic definitions and facts pertaining to root systems we refer the reader to [**50**, Chapter III].

Let E be a Euclidean space with positive-definite symmetric bilinear form $\langle \cdot, \cdot \rangle$ and R a root system spanning E. The rank of R is the dimension of E. All root systems will be assumed to be irreducible, but not necessarily reduced. The root system dual to R, denoted $R^\vee$, is given by

$$R^\vee = \{\alpha^\vee : \ \alpha \in R\},$$

where $v^\vee := 2v/\langle v, v \rangle = 2v/\|v\|^2$ for $v \in E$.

A pair of root systems (R, S) in E is called admissible if R and S share the same Weyl group, W, and S is reduced. Given such an admissible pair and $\alpha \in R$, there exists a unique $u_\alpha > 0$ such that $u_\alpha^{-1}\alpha \in S$. Moreover, the map $\alpha \mapsto u_\alpha^{-1}\alpha$ from R to S is surjective (injective if R is reduced) and commutes with the action of W. Hence u_α is fixed along Weyl orbits, and $u_{2\alpha} = 2u_\alpha$ if $\alpha, 2\alpha$ are both in R. Two admissible pairs (R, S) and (R', S') are said to be similar if there exist positive real numbers a, b such that $R' = aR$ and $S' = bS$. Then $bu'_{\alpha'} = au_\alpha$ for $\alpha' \in R'$ and $\alpha \in R$ such that $\alpha' = a\alpha$. Since we only require the classification of admissible pairs of root systems up to similarity, and since roots of equal length are conjugate under the action of the Weyl group and hence in the same Weyl orbit, we may fix the value of u_α for roots of shortest length. The classification then breaks up into three cases.

(1) R is reduced and $S = R$ (and hence $u_\alpha = 1$).
(2) R is reduced but not simply laced, and $S = R^\vee$. Unlike Macdonald, who normalises the length of short roots in non-simply-laced reduced root systems as $\sqrt{2}$, we take the length of the short roots to be 1 when $R = \mathrm{B}_n$ and $\sqrt{2}$ in all other cases. In particular this implies that $\mathrm{B}_n^\vee = \mathrm{C}_n$. Writing u_{long} and u_{short} for u_α indexed by long and short roots respectively, we then have: (i) $u_{\mathrm{long}} = 1$ and $u_{\mathrm{short}} = 1/2$ for $(R, S) = (\mathrm{B}_n, \mathrm{C}_n)$, (ii) $u_{\mathrm{short}} = 1$ and $u_{\mathrm{long}} = 2$ for $(R, S) = (\mathrm{C}_n, \mathrm{B}_n)$ or $(R, S) = (\mathrm{F}_4, \mathrm{F}_4^\vee)$, (iii) $u_{\mathrm{short}} = 1$ and $u_{\mathrm{long}} = 3$ for $(R, S) = (\mathrm{G}_4, \mathrm{G}_4^\vee)$.

(3) R is the non-reduced root system BC_n and S is one of $\mathrm{B}_n, \mathrm{C}_n$. In both cases we fix $S \subset R$, so that $u_\alpha \in \{1,2\}$ for $S = \mathrm{B}_n$ and $u_\alpha \in \{1/2, 1\}$ for $S = \mathrm{C}_n$.

For (R,S) an admissible pair of root systems of rank r, we fix a basis of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$ of R and write $\alpha > 0$ if $\alpha \in R$ is a positive root with respect to Δ. The fundamental weights $\omega_1, \dots, \omega_r$ of R are given by $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. As usual, we denote the root, coroot and weight lattices of R by Q, $Q^\vee$ and P respectively. We also write $Q_+ = \sum_{i=1}^r \mathbb{Z}_{\geqslant 0} \alpha_i$ for the cone in Q spanned by the simple roots, and $P_+ = \sum_{i=1}^r \mathbb{Z}_{\geqslant 0} \omega_i$ for the set of dominant (integral) weights.

Let A be the group algebra over $\mathbb{R}$ of P, with elements e^λ, and A^W the algebra of W-invariant elements of A. A basis of A^W is given by the monomial symmetric functions

$$m_\lambda^W = \sum_\mu \mathrm{e}^\mu, \qquad \lambda \in P_+,$$

with μ summed over the W-orbit of λ.

The (R,S) Macdonald polynomials defined below depend on the variables q and t_α, $\alpha \in R$, such that t_α is constant along Weyl orbits. Hence there is only one t_α in case (1), two in case (2), and three in case (3). In each case we write this set of t_α's by $\underline{t}$. The generalised Macdonald density (compare with (2.5.2)) is then

$$\Delta(q, \underline{t}) := \prod_{\alpha \in R} \frac{(t_{2\alpha}^{1/2}\, \mathrm{e}^\alpha; q^{u_\alpha})_\infty}{(t_\alpha t_{2\alpha}^{1/2}\, \mathrm{e}^\alpha; q^{u_\alpha})_\infty}, \tag{2.7.1}$$

where $t_{2\alpha} := 1$ if $2\alpha \notin R$. Assuming $|q|, |t_\alpha| < 1$, this defines the following scalar product on A:

$$\langle f, g \rangle_{q,\underline{t}} := \frac{1}{|W|} \int_T f\, \bar{g}\, \Delta(q, \underline{t})\, \mathrm{d}T,$$

where integration is with respect to Haar measure on the torus $T = E/Q^\vee$ and, for $f = \sum_{\lambda \in P} f_\lambda\, \mathrm{e}^\lambda$, $\bar{f} := \sum_{\lambda \in P} f_\lambda\, \mathrm{e}^{-\lambda}$. The (R,S) Macdonald polynomials $P_\lambda(q, \underline{t})$, indexed by $\lambda \in P_+$, are the unique family of W-symmetric functions

$$P_\lambda(q, \underline{t}) = m_\lambda^W + \sum_{\mu < \lambda} u_{\lambda\mu}(q, \underline{t}) m_\mu^W \tag{2.7.2}$$

such that

$$\langle P_\lambda, P_\mu \rangle_{q,\underline{t}} = 0 \qquad \text{if} \quad \lambda \neq \mu.$$

The sum in (2.7.2) is with respect to the dominance (partial) order on P_+ defined by $\lambda \geqslant \mu$ if $\lambda - \mu \in Q_+$. Of course, when dealing with the $P_\lambda(q, \underline{t})$ as polynomials, the restrictions $|q|, |t_\alpha| < 1$ imposed above may be dropped, and typically we view q and the t_α as indeterminates.

Below we are interested in the generalised Macdonald polynomials for (R,S) one of the four admissible pairs $(\mathrm{B}_n, \mathrm{B}_n)$, $(\mathrm{B}_n, \mathrm{C}_n)$, $(\mathrm{C}_n, \mathrm{B}_n)$ and $(\mathrm{D}_n, \mathrm{D}_n)$. Moreover, in the Hall–Littlewood limit, $q \to 0$, (in which case the S-dependence drops out) we also consider $R = \mathrm{BC}_n$. In the following we assume the standard realisation in $\mathbb{R}^n$ of B_n, C_n and D_n, consistent with our normalisation of short roots in (2):

$$\Delta = \{\alpha_1, \dots, \alpha_n\} = \big\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\big\}, \qquad R = \mathrm{B}_n = \mathrm{C}_n^\vee, \tag{2.7.3a}$$

$$\Delta = \{\alpha_1, \dots, \alpha_n\} = \big\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\big\}, \quad R = \mathrm{D}_n. \tag{2.7.3b}$$

We further parametrise the set of dominant weights P_+ as (see e.g., [**88**, page 470])

$$\lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n, \tag{2.7.4}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition in the case of C_n, and a partition or half-partition in the case of B_n, D_n, with the exception that for D_n the part λ_n can be negative: $-\lambda_{n-1} \leqslant \lambda_n \leqslant \lambda_{n-1}$.[1] It is not difficult to check that (2.7.4) can be rewritten in terms of the fundamental weights as

$$(\lambda_1-\lambda_2)\omega_1 + \cdots + (\lambda_{n-1}-\lambda_n)\omega_{n-1} + 2\lambda_n\omega_n, \qquad R = \mathrm{B}_n, \tag{2.7.5a}$$

$$(\lambda_1-\lambda_2)\omega_1 + \cdots + (\lambda_{n-1}-\lambda_n)\omega_{n-1} + \lambda_n\omega_n, \qquad R = \mathrm{C}_n, \tag{2.7.5b}$$

$$(\lambda_1-\lambda_2)\omega_1 + \cdots + (\lambda_{n-1}-\lambda_n)\omega_{n-1} + (\lambda_{n-1}+\lambda_n)\omega_n, \qquad R = \mathrm{D}_n. \tag{2.7.5c}$$

Finally, writing $x_i = \mathrm{e}^{-\epsilon_i}$ (for $1 \leqslant i \leqslant n$), we will denote the four families of interest by

$$P_\lambda^{(\mathrm{B}_n,\mathrm{B}_n)}(x;q,t,t_2), \qquad P_\lambda^{(\mathrm{B}_n,\mathrm{C}_n)}(x;q,t,t_2),$$
$$P_\lambda^{(\mathrm{C}_n,\mathrm{B}_n)}(x;q,t,t_2), \qquad P_\lambda^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t),$$

where $x = (x_1, \dots, x_n)$ and

$$t = t_{\alpha_1}, \qquad t_2 = t_{\alpha_n}. \tag{2.7.6}$$

There are several relations between these polynomials. For example [**34**, Equation (5.60)],

$$P_\lambda^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t) = P_\lambda^{(\mathrm{B}_n,\mathrm{B}_n)}(x;q,t,1) \tag{2.7.7a}$$

if $l(\lambda) < n$, and

$$P_\lambda^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t) + P_{\bar\lambda}^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t) = P_\lambda^{(\mathrm{B}_n,\mathrm{B}_n)}(x;q,t,1), \tag{2.7.7b}$$

$$P_\lambda^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t) - P_{\bar\lambda}^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t) = P_{\lambda-(\frac12)^n}^{(\mathrm{B}_n,\mathrm{C}_n)}(x;q,t,q^{1/2}) \times \prod_{i=1}^n \big(x_i^{-1/2} - x_i^{1/2}\big) \tag{2.7.7c}$$

if λ is a partition or half-partition of length n. Here $\bar\lambda := (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$. Hence

$$P_{\bar\lambda}^{(\mathrm{D}_n,\mathrm{D}_n)}(x;q,t) = P_\lambda^{(\mathrm{D}_n,\mathrm{D}_n)}(\bar x;q,t), \qquad \bar x := (x_1, \dots, x_{n-1}, x_n^{-1}). \tag{2.7.8}$$

Similarly, comparing the Koornwinder density (2.6.1) with (2.7.1), it follows that

$$P_\lambda^{(\mathrm{C}_n,\mathrm{B}_n)}(x;q,t,t_2) = K_\lambda\big(x;q,t;\pm q^{1/2},\pm t_2^{1/2}\big) \tag{2.7.9}$$

(see also [**34**]). Although the $(\mathrm{B}_n,\mathrm{B}_n)$ and $(\mathrm{B}_n,\mathrm{C}_n)$ Macdonald polynomials are indexed by partitions or half-partitions, they too can be related to Koornwinder polynomials [**34**]. Before describing this relation, we briefly discuss another family of polynomials incorporating both B_n families.

[1]The map $\lambda_n \mapsto -\lambda_n$ corresponds to the Dynkin diagram automorphism interchanging ω_{n-1} and ω_n.

2.7.2. The Macdonald–Koornwinder polynomials $K_\lambda(x;q,t;t_2,t_3)$. Our description of Macdonald polynomials attached to root systems is by no means the most general and modern setup, see e.g., [**26**, **93**, **130**]. Beyond Macdonald's original approach we have already covered the Koornwinder polynomials, and in this section we discuss one further family of B_n-like polynomials, denoted by $K_\lambda(x;q,t;t_2,t_3)$. In the notation of [**130**, Definition 3.21] they correspond to the Macdonald–Koornwinder polynomials P_λ^+ with initial data given by the quintuple $D=(\mathrm{B}_n,\Delta,t,P,Q)$. Here Δ is the basis of simple roots of B_n given in (2.7.3a), P and Q again denote the weight and root lattices of B_n, and t stands for 'twisted'.

The polynomials $K_\lambda(x;q,t;t_2,t_3)$, where $x=(x_1,\dots,x_n)$ and $\lambda=(\lambda_1,\dots,\lambda_n)$ is a partition or half-partition, are B_n-symmetric functions in the sense of (2.7.2) such that

$$\big\langle K_\lambda(x;q,t;t_2,t_3),K_\mu(x;q,t;t_2,t_3)\big\rangle^{(n)}_{q,t;t_2,t_3}=0 \quad \text{for} \quad \lambda\neq\mu. \tag{2.7.10}$$

Here, for $f,g\in\Lambda^{\mathrm{BC}_n}$,

$$\langle f,g\rangle^{(n)}_{q,t;t_2,t_3}:=\int_{\mathbb{T}^n} f(x)g(x)\Delta(x;q,t;t_2,t_3)\,\mathrm{d}T(x) \tag{2.7.11}$$

with

$$\Delta(x;q,t;t_2,t_3):=\prod_{i=1}^n\frac{(x_i^{\pm};q^{1/2})_\infty}{(t_2x_i^{\pm},t_3x_i^{\pm};q)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{(x_i^{\pm}x_j^{\pm};q)_\infty}{(tx_i^{\pm}x_j^{\pm};q)_\infty}.$$

In the notation of the previous section this corresponds to $(R,S)=(\mathrm{B}_n,\mathrm{B}_n)$ and

$$\Delta(q,\underline{t})=\prod_{\alpha\text{ short}}\frac{(\mathrm{e}^\alpha,q^{1/2}\,\mathrm{e}^\alpha;q)_\infty}{(t_\alpha\,\mathrm{e}^\alpha,\bar{t}_\alpha\,\mathrm{e}^\alpha;q)_\infty}\prod_{\alpha\text{ long}}\frac{(\mathrm{e}^\alpha;q)_\infty}{(t_\alpha\,\mathrm{e}^\alpha;q)_\infty},$$

where $\underline{t}=(t_{\alpha_1},t_{\alpha_n},\bar{t}_{\alpha_n})=(t,t_2,t_3)$. It thus follows that

$$P_\lambda^{(\mathrm{B}_n,\mathrm{B}_n)}(x;q,t,t_2),=K_\lambda(x;q,t;t_2,q^{1/2}) \tag{2.7.12a}$$

$$P_\lambda^{(\mathrm{B}_n,\mathrm{C}_n)}(x;q,t,t_2)=K_\lambda(x;q,t;t_2,t_2q^{1/2}). \tag{2.7.12b}$$

The next lemma shows that the $K_\lambda(x;q,t;t_2,t_3)$ can be expressed in terms of Koornwinder polynomials, allowing us to prove results for the former using the latter.

Lemma 2.3. *For λ a partition or half-partition*

$$K_\lambda(x;q,t;t_2,t_3)=\begin{cases} K_\lambda(x;q,t;-1,-q^{1/2},t_2,t_3) & \lambda\text{ a partition},\\[2mm] K_{\lambda-(\frac12)^n}(x;q,t;-q,-q^{1/2},t_2,t_3)\displaystyle\prod_{i=1}^n\big(x_i^{1/2}+x_i^{-1/2}\big) & \text{otherwise}.\end{cases}$$

For $t_3=q^{1/2}$ or $t_3=t_2q^{1/2}$ this is equivalent to [**34**, Equations (5.50) & (5.51)].

Proof. The triangularity with respect to the monomial symmetric functions for B_n is clear, and in the following we show that $K_\lambda(x;q,t;t_2,t_3)$ as given by the lemma satisfies (2.7.10). Viewing the integral on the right of (2.7.11) as a constant term evaluation, it follows that (2.7.10) holds when λ is a partition and μ a half-partition. By (2.6.3) with $\{t_0,t_1\}=\{-1,-q^{1/2}\}$ it is also clear that it holds when λ and μ are both partitions. In the case of two half-partitions

$$\begin{aligned}&\big\langle K_\lambda(x;q,t;t_2,t_3),K_\mu(x;q,t;t_2,t_3)\big\rangle^{(n)}_{q,t;t_2,t_3}\\&\quad=\int_{\mathbb{T}^n}K_\nu(x;q,t;-q,-q^{1/2},t_2,t_3)K_\omega(x;q,t;-q,-q^{1/2},t_2,t_3)\\&\qquad\times\prod_{i=1}^n\big(x_i^{1/2}+x_i^{-1/2}\big)^2\frac{(x_i^\pm;q^{1/2})_\infty}{(t_2x_i^\pm,t_3x_i^\pm;q)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{(x_i^\pm x_j^\pm;q)_\infty}{(tx_i^\pm x_j^\pm;q)_\infty}\,\mathrm{d}T(x),\end{aligned}$$

where $\nu:=\lambda-(\frac{1}{2})^n$ and $\omega:=\mu-(\frac{1}{2})^n$. Since

$$\big(x_i^{1/2}+x_i^{-1/2}\big)^2=\frac{(-x_i^\pm;q^{1/2})_\infty}{(-qx_i^\pm,-q^{1/2}x_i^\pm;q)_\infty}$$

and

$$(-x_i^\pm,x_i^\pm;q^{1/2})_\infty=(x_i^{\pm2};q)_\infty,$$

the second line of the integrand is precisely the Koornwinder density

$$\Delta(x;q,t;-q,-q^{1/2},t_2,t_3).$$

Hence

$$\begin{aligned}&\big\langle K_\lambda(x;q,t;t_2,t_3),K_\mu(x;q,t;t_2,t_3)\big\rangle^{(n)}_{q,t;t_2,t_3}\\&\quad=\big\langle K_\nu(x;q,t;-q,-q^{1/2},t_2,t_3),K_\omega(x;q,t;-q,-q^{1/2},t_2,t_3)\big\rangle^{(n)}_{q,t;-q,-q^{1/2},t_2,t_3}.\end{aligned}$$

By (2.6.3) this vanishes unless $\nu=\omega$, i.e., unless $\lambda=\mu$. □

That

$$K_{\lambda-(\frac{1}{2})^n}(x;q,t;-q,-q^{1/2},t_2,t_3)\prod_i\big(x_i^{1/2}+x_i^{-1/2}\big)$$

is the natural extension of $K_\lambda\big(x;q,t;-1,-q^{1/2},t_2,t_3\big)$ to half-partitions λ may also be understood from the point of view of virtual Koornwinder polynomials. Taking

$$(x;Q,t_0,t_1)=(x_1,\dots,x_n;q^m,-1,-q^{1/2})$$

in Proposition (2.2) leads to

$$\begin{aligned}\text{(2.7.13)}\quad&\hat{K}_\lambda(x;q,t,q^m;-1,-q^{1/2},t_2,t_3)\\&\qquad=\hat{K}_\lambda(x;q,t,q^{m-1/2};-q,-q^{1/2},t_2,t_3)\prod_{i=1}^n(1+x_i).\end{aligned}$$

Since the left-hand side of (2.6.8) is well-defined for λ a partition and m a half-integer, we can use that equation to eliminate the virtual Koornwinder polynomials in (2.7.13). Also replacing $m^n-\lambda$ by λ, this results in

$$K_\lambda(x;q,t;-1,-q^{1/2},t_2,t_3)=K_{\lambda-(\frac{1}{2})^n}(x;q,t;-q,-q^{1/2},t_2,t_3)\prod_{i=1}^n\big(x_i^{1/2}+x_i^{-1/2}\big).$$

We conclude this section with the analogue of (2.6.9) for $K_\lambda(x;q,t;t_2,t_3)$.

LEMMA 2.4. *For m a nonnegative integer or half-integer and λ a partition,*

$$\lim_{m\to\infty}(x_1\cdots x_n)^m K_{m^n-\lambda}(x;q,t;t_2,t_3) = P_\lambda(x;q,t)\prod_{i=1}^n \frac{(t_2x_i,t_3x_i;q)_\infty}{(x_i;q^{1/2})_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}. \tag{2.7.14}$$

PROOF. For m an integer this is just (2.6.9) with $\{t_0,t_1\}=\{-1,-q^{1/2}\}$. For m a half-integer it follows from Lemma 2.3 that

$$(x_1\cdots x_n)^m K_{m^n-\lambda}(x;q,t;t_2,t_3) = (x_1\cdots x_n)^{m-\frac12}K_{(m-\frac12)^n-\lambda}(x;q,t;-q,-q^{1/2},t_2,t_3)\prod_{i=1}^n(1+x_i).$$

Letting m tend to infinity using (2.6.9) with $\{t_0,t_1\}=\{-q,-q^{1/2}\}$ results in the right-hand side of (2.7.14). □

2.8. Hall–Littlewood polynomials

2.8.1. Hall–Littlewood of type R. The ordinary (or A_{n-1}) Hall–Littlewood polynomials are defined as [**91**, page 208]

$$P_\lambda(x_1,\dots,x_n;t) := \frac{1}{v_\lambda(t)}\sum_{w\in\mathfrak{S}_n} w\biggl(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{1\leqslant i<j\leqslant n}\frac{x_i-tx_j}{x_i-x_j}\biggr), \tag{2.8.1}$$

where λ is a partition of length at most n, $v_\lambda(t):=\prod_{i\geqslant 0}(t;t)_{m_i(\lambda)}/(1-t)^{m_i(\lambda)}$ and $m_0(\lambda):=n-l(\lambda)$. They correspond to the Macdonald polynomials for $q=0$, i.e., $P_\lambda(x;t)=P_\lambda(x;0,t)$, and for $t=0$ reduce to the Schur functions.

Taking $q=0$ in the (R,S) Macdonald polynomials of Section 2.7.1 yields the more general Hall–Littlewood polynomials of type R. (The root system S no longer plays a role when $q=0$.) More simply, however, the Hall–Littlewood polynomials of type R can be explicitly computed from

$$P_\lambda(t)=\frac{1}{W_\lambda(t)}\sum_{w\in W} w\biggl(\mathrm{e}^\lambda\prod_{\alpha>0}\frac{1-t_\alpha t_{2\alpha}^{1/2}\,\mathrm{e}^{-\alpha}}{1-t_{2\alpha}^{1/2}\,\mathrm{e}^{-\alpha}}\biggr),\qquad \lambda\in P_+. \tag{2.8.2}$$

The normalising factor $W_\lambda(t)$ is the Poincaré polynomial of the stabilizer of λ in W.

For our purposes we need to consider (2.8.2) for R one of $\mathrm{BC}_n,\mathrm{B}_n,\mathrm{C}_n,\mathrm{D}_n$. As in Section 2.7.1, we express these four families using variables $x_1,\dots,x_n$ and partitions or half-partitions λ, rather than roots and dominant weights. Accordingly we write $P_\lambda^{(R)}(x_1,\dots,x_n;q,t,t_2,\dots,t_r)$, where $r=3$ in the case of BC_n, $r=2$ for B_n and C_n and $r=1$ for D_n. In each case we again assume (2.7.3) and (2.7.5), and identify $x_i=\exp(-\epsilon_i)$. As a basis of simple roots for the root system BC_n we take the B_n basis (2.7.3a) (as opposed to a C_n basis), and identify $(t_{\alpha_1},t_{\alpha_n},t_{2\alpha_n}^{1/2})=(t,-t_2/t_3,-t_3)$. Then (2.8.2) takes the equivalent form

$$P_\lambda^{(\mathrm{BC}_n)}(x;t,t_2,t_3)=\frac{1}{(t_2t_3;t)_{n-l(\lambda)}v_\lambda(t)} \times\sum_{w\in W} w\biggl(x^{-\lambda}\prod_{i=1}^n\frac{(1-t_2x_i)(1-t_3x_i)}{1-x_i^2}\prod_{1\leqslant i<j\leqslant n}\frac{(tx_i-x_j)(1-tx_ix_j)}{(x_i-x_j)(1-x_ix_j)}\biggr), \tag{2.8.3}$$

with W the hyperoctahedral group and λ a partition of length at most n. Alternatively (see e.g., [**134**]),

$$P_\lambda^{(\mathrm{BC}_n)}(x;t,t_2,t_3) = K_\lambda(x;0,t;0,0,t_2,t_3). \tag{2.8.4}$$

For $t=0$ this admits the determinantal form

$$\begin{aligned}P_\lambda^{(\mathrm{BC}_n)}(x;0,t_2,t_3) &= \frac{1}{\Delta_{\mathrm{C}}(x)} \\ &\times \det_{1\leqslant i,j\leqslant n}\big(x_i^{-\lambda_j+j-1}(1-t_2x_i)(1-t_3x_i) - x_i^{\lambda_j+2n-j-1}(x_i-t_2)(x_i-t_3)\big),\end{aligned} \tag{2.8.5}$$

where

$$\Delta_{\mathrm{C}}(x) := \prod_{i=1}^n (1-x_i^2) \prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1) = \det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}-x_i^{2n-j+1}\big) \tag{2.8.6}$$

is the C_n Vandermonde product.

Lemma 2.5. *Let*

$$\Phi(x;t,t_2,t_3) := \prod_{i=1}^n \frac{(1-t_2x_i)(1-t_3x_i)}{1-x_i^2} \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j}. \tag{2.8.7}$$

For m a positive integer,

$$P_{m^n}^{(\mathrm{BC}_n)}(x;t,t_2,t_3) = \sum_{\varepsilon\in\{\pm1\}^n} \Phi(x^\varepsilon;t,t_2,t_3)\prod_{i=1}^n x_i^{-\varepsilon_i m}. \tag{2.8.8}$$

Proof. By (2.8.1) and (2.8.7),

$$\begin{aligned}&\sum_{w\in\mathfrak{S}_n} w\bigg(x^{-\lambda}\prod_{i=1}^n \frac{(1-t_2x_i)(1-t_3x_i)}{1-x_i^2} \prod_{1\leqslant i<j\leqslant n}\frac{(tx_i-x_j)(1-tx_ix_j)}{(x_i-x_j)(1-x_ix_j)}\bigg) \\ &\quad = \Phi(x;t,t_2,t_3)\sum_{w\in\mathfrak{S}_n} w\bigg(x^{-\lambda}\prod_{1\leqslant i<j\leqslant n}\frac{tx_i-x_j}{x_i-x_j}\bigg) \\ &\quad = v_\lambda(t)\Phi(x;t,t_2,t_3)P_\lambda(x^{-1};t).\end{aligned}$$

Hence

$$P_\lambda^{(\mathrm{BC}_n)}(x;t,t_2,t_3) = \frac{1}{(t_2t_3;t)_{n-l(\lambda)}} \sum_{\varepsilon\in\{\pm1\}^n} \Phi(x^\varepsilon;t,t_2,t_3)P_\lambda(x^{-\varepsilon};t).$$

The claim now follows from $P_{m^n}(x;t) = \prod_{i=1}^n x_i^m$, and the fact that $n-l(\lambda)=0$ for $\lambda=m^n$ with $m\geqslant 1$. $\square$

In the case of C_n we can be brief. The identification of parameters (2.7.6) again applies, and from the $q=0$ case of (2.7.12b),

$$\begin{aligned}P_\lambda^{(\mathrm{C}_n)}(x;t,t_2) &= P_\lambda^{(\mathrm{BC}_n)}\big(x;t,\pm t_2^{1/2}\big) \\ &= P_\lambda^{(\mathrm{C}_n,\mathrm{B}_n)}(x;0,t,t_2) = K_\lambda\big(x;0,t;0,0,\pm t_2^{1/2}\big).\end{aligned} \tag{2.8.9}$$

Accordingly, (2.8.5) for $t_3=-t_2$ is a one-parameter deformation of the symplectic Schur function [**85**]

$$\mathrm{sp}_{2n,\lambda}(x) := \frac{1}{\Delta_{\mathrm{C}}}\det_{1\leqslant i,j\leqslant n}\big(x_i^{-\lambda_j+j-1} - x_i^{\lambda_j+2n-j+1}\big). \tag{2.8.10}$$

The Hall–Littlewood polynomials $P_\lambda^{(\mathrm{B}_n)}(x;t,t_2)$ are given by the (2.8.3) for $t_3=-1$, where now λ is a partition or half-partition. In the latter case $v_\lambda(t)$ is as defined on page 23 but with $\prod_{i\geqslant 0}$ a product over half-integers. By (2.7.12) we also have

$$P_\lambda^{(\mathrm{B}_n)}(x;t,t_2)=P_\lambda^{(\mathrm{B}_n,\mathrm{B}_n)}(x;0,t,t_2)=K_\lambda(x;0,t;t_2,0). \tag{2.8.11}$$

When $t=t_2=0$ the B_n Hall–Littlewood polynomials simplify to the odd orthogonal Schur functions [**85**]

$$\mathrm{so}_{2n+1,\lambda}(x):=\frac{1}{\Delta_{\mathrm{B}}(x)}\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{-\lambda_j+j-1}-x_i^{\lambda_j+2n-j}\bigr), \tag{2.8.12}$$

where

$$\Delta_{\mathrm{B}}(x):=\prod_{i=1}^n(1-x_i)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)=\det_{1\leqslant i,j\leqslant n}(x_i^{j-1}-x_i^{2n-j}). \tag{2.8.13}$$

The B_n analogue of Lemma 2.5 is given as follows.

Lemma 2.6. *For m a positive integer,*

$$P_{(\frac{m}{2})^n}^{(\mathrm{B}_n)}(x;t,t_2)=\sum_{\varepsilon\in\{\pm1\}^n}\Phi(x^\varepsilon;t,t_2,-1)\prod_{i=1}^n x_i^{-\varepsilon_i m/2}.$$

Finally, the D_n Hall–Littlewood polynomials $P_\lambda^{(\mathrm{D}_n)}(x;t)$ are given by (2.8.3) with $(t_2,t_3)=(-1,1)$, provided we multiply the right-hand side by 2 when $l(\lambda)<n$, and W is taken to be the group of even signed-permutations, i.e., $W=\mathfrak{S}_n\ltimes(\mathbb{Z}/2\mathbb{Z})^{n-1}$. As described on page 20, λ can be a partition or half-partition with λ_n an exceptional part that can take on negative integer or half-integer values as long as $|\lambda_n|\leqslant\lambda_{n-1}$. When $t=0$ this yields the even orthogonal Schur functions [**101**]

$$\begin{aligned}\mathrm{so}_{2n,\lambda}(x):=\frac{1}{2\Delta_{\mathrm{D}}(x)}\Bigl(&\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{\lambda_j+2n-j-1}+x_i^{-\lambda_j+j-1}\bigr)\\&-\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{\lambda_j+2n-j-1}-x_i^{-\lambda_j+j-1}\bigr)\Bigr),\end{aligned}$$

where

$$\Delta_{\mathrm{D}}(x):=\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)=\tfrac{1}{2}\det_{1\leqslant i,j\leqslant n}(x_i^{j-1}+x_i^{2n-j-1}). \tag{2.8.14}$$

Again we have a simple analogue of Lemma 2.5.

Lemma 2.7. *Let $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n$ and $\operatorname{sgn}(\varepsilon):=\prod_{i=1}^n\varepsilon_i$. Then, for m a positive integer,*

$$P_{(\frac{m}{2})^n}^{(\mathrm{D}_n)}(x;t)=\sum_{\substack{\varepsilon\in\{\pm1\}^n\\ \operatorname{sgn}(\varepsilon)=1}}\Phi(x^\varepsilon;t,1,-1)\prod_{i=1}^n x_i^{-\varepsilon_i m/2}.$$

2.8.2. Modified Hall–Littlewood polynomials. The simplest definition of the modified Hall–Littlewood polynomials is as a plethystically substituted ordinary Hall–Littlewood polynomial (see e.g., [**91**, page 234]):

$$P'_\lambda(x;t) := P_\lambda\Big(\Big[\frac{x}{1-t}\Big];t\Big) \quad \text{and} \quad Q'_\lambda(x;t) := Q_\lambda\Big(\Big[\frac{x}{1-t}\Big];t\Big), \tag{2.8.15}$$

where $Q_\lambda(x;t) := b_\lambda(t)P_\lambda(x;t)$ and

$$b_\lambda(t) := \prod_{i\geqslant 1}(t;t)_{m_i(\lambda)}.$$

This definition obscures the important fact that the modified Hall–Littlewood polynomials, and hence sums such as

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda\big(x_1^{\pm},\dots,x_n^{\pm};t\big) \tag{2.8.16}$$

(see (1.1.7)), are Schur positive. To exhibit the combinatorial nature of (2.8.16) and similar such expressions for the characters of affine Lie algebras given in Section 5.2, we need the Lascoux–Schützenberger description of the modified Hall–Littlewood polynomials in terms of the charge statistic on tableaux [**74**].

A filling of the Young diagram of a partition λ with positive integers such that rows are weakly increasing from left to right and columns are strictly increasing from top to bottom is called a semistandard Young tableau of shape λ, see e.g., [**40, 91, 126**]. The (weak) composition $\mu = (\mu_1, \mu_2, \dots)$ such that μ_i counts the number of squares filled with the number i is called the weight (or filling/content/type) of the tableau. If we denote the set of semistandard Young tableaux of shape λ and weight μ by $\mathrm{SSYT}(\lambda,\mu)$, then the Schur function s_λ may be expressed as

$$s_\lambda(x_1,x_2,\dots) = \sum_{T\in\mathrm{SSYT}(\lambda,\cdot)} x^T. \tag{2.8.17}$$

Here x^T is shorthand for $x^{\mathrm{weight}(T)}$, so that $x^T = x^\mu = x_1^{\mu_1}x_2^{\mu_2}\cdots$ if $T\in\mathrm{SSYT}(\cdot,\mu)$. Lascoux and Schützenberger equipped the set of semistandard Young tableaux with a statistic, called charge. For our purposes it suffices to consider tableaux in $\mathrm{SSYT}(\cdot,\mu)$ such that μ is a partition. To compute the charge of such a tableau, we first form the reverse reading word, $w = w(T)$, by consecutively reading the rows of T from right to left, starting with the top row and ending with the bottom row. For example, the reverse reading word of

1	1	1	2	3	4
2	2	3			
4	6				
5					

is 432111322645. In a clockwise manner, wrap the letters of $w = w_1\dots w_k$ around a circle, putting a marker between the first letter w_1 and last letter w_k. Repeatedly going round the circle in clockwise manner, read the letters of w starting with w_1. Label a total of μ'_1 letters as follows. If a letter i has just been labelled k, then the first letter $i+1$ read after that i is labelled k if it occurs before the marker and $k+1$ if it occurs after the marker. To get started, label the first 1 that is read by 0. For the reading word in our example this gives

$$
\begin{array}{ccccc}
 & & 5_1 \mid 4 & & \\
 & 4_1 & & 3_1 & \\
6_2 & & & & 2 \\
2 & & & & 1_0 \\
 & 2_0 & & 1 & \\
 & & 3 \quad 1 & &
\end{array}
$$

Keep repeating the above labelling procedure with the remaining unlabelled letters of w until all letter are labelled, in such a way that μ_i' letter are labelled in the ith step. The completed labelling of the word in our example is

$$
\begin{array}{ccccc}
 & & 5_1 \mid 4_2 & & \\
 & 4_1 & & 3_1 & \\
6_2 & & & & 2_1 \\
2_0 & & & & 1_0 \\
 & 2_0 & & 1_0 & \\
 & & 3_1 \quad 1_0 & &
\end{array}
$$

or, in one-line, notation $4_2 3_1 2_1 1_0 1_0 1_0 3_1 2_0 2_0 6_2 4_1 5_1$. The charge, $\mathrm{c}(T)$, of T is the sum of the labels of its reverse reading word. For the tableau in the example, $\mathrm{c}(T) = 2 + 1 + 1 + 0 + 0 + 0 + 1 + 0 + 0 + 2 + 1 + 1 = 9$.

Using the charge statistic, the modified Hall–Littlewood polynomial Q_μ' can be expressed as [**23**, **33**, **74**]

$$Q_\mu'(x;t) = \sum_{T \in \mathrm{SSYT}(\cdot,\mu)} t^{\mathrm{c}(T)} s_{\mathrm{shape}(T)}(x),$$

or, equivalently, as

$$Q_\mu'(x;t) = \sum_\lambda K_{\lambda\mu}(t) s_\lambda(x),$$

where $K_{\lambda\mu}(t)$ is the Kostka–Foulkes polynomial

$$K_{\lambda\mu}(t) := \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{c}(T)}.$$

The coefficients $K_{\lambda\mu}(t)/b_\mu(t)$ in the Schur expansion of the modified Hall–Littlewood polynomials $P_\mu'(x;t)$ are rational functions instead of polynomials, but, viewed as a formal power series in t,

$$\frac{K_{\lambda\mu}(t)}{b_\mu(t)} = \sum_{k \geqslant 0} a_k t^k, \quad a_k \in \mathbb{Z}_{\geqslant 0}.$$

Alternatively, we have the combinatorial expression [**63**, **144**]

$$Q_\lambda'(x_1, \dots, x_n; t) = \sum_{0 = \mu^{(n)} \subset \cdots \subset \mu^{(1)} \subset \mu^{(0)} = \lambda} \prod_{i=1}^n g_{\mu^{(i-1)}/\mu^{(i)}}(x_i; t), \tag{2.8.18}$$

where

$$g_{\lambda/\mu}(z;t) := Q_{\lambda/\mu}\Big(\Big[z\,\frac{1-q}{1-t}\Big]; t\Big) = z^{|\lambda/\mu|} t^{n(\lambda/\mu)} \prod_{i \geqslant 1} \begin{bmatrix} \lambda_i' - \mu_{i+1}' \\ \lambda_i' - \mu_i' \end{bmatrix}_t \tag{2.8.19}$$

is the h-Pieri coefficient for Hall–Littlewood polynomials

$$P_\mu(x;t)\prod_{i\geqslant 1}\frac{1}{1-zx_i}=\sum_{\lambda\supset\mu} g_{\mu/\nu}(z;t)P_\lambda(x;t),$$

see [**63**, **64**, **143**]. Note that for $n=1$ this trivialises to the principal specialisation formula [**91**, page 213]

$$Q'_\lambda(z;t)=z^{|\lambda|}Q_\lambda(1,q,q^2,\dots;q)=z^{|\lambda|}t^{n(\lambda)}. \tag{2.8.20}$$

In Section 5.2 we will show how (2.8.18) can be used to simplify some of our character formulas when restricted to the basic representation.

CHAPTER 3

Virtual Koornwinder integrals

3.1. Basic definitions

For $\{f_\lambda\}$ a basis of Λ_n, Λ or Λ^{BC_n} and g an arbitrary element of one of these spaces, we write $[f_\lambda]g$ for the coefficient c_λ in $g=\sum_\lambda c_\lambda f_\lambda$. Although typically $f_0=1$ we will still write $[f_0]g$ to avoid ambiguity as to the choice of basis.

Our approach to bounded Littlewood identities relies crucially on properties of two linear functionals, denoted I_K and $I_K^{(n)}$ and referred to as virtual Koornwinder integrals, acting on Λ and Λ^{BC_n} respectively. Let $f\in\Lambda$. Then the virtual Koornwinder integral I_K is defined as [**108**, page 110]

$$I_K\big(f;q,t,T;t_0,t_1,t_2,t_3\big):=\big[\tilde{K}_0(q,t,T;t_0,t_1,t_2,t_3)\big]f, \tag{3.1.1}$$

where $\tilde{K}_\lambda$ is the lifted Koornwinder polynomial. Similarly, for $f\in\Lambda^{\mathrm{BC}_n}$ and $x=(x_1,\dots,x_n)$ [**108**, page 95]

$$I_K^{(n)}\big(f;q,t;t_0,t_1,t_2,t_3\big):=\big[K_0(x;q,t;t_0,t_1,t_2,t_3)\big]f. \tag{3.1.2}$$

By (2.6.3) and $K_0=1$, it follows that

$$\begin{aligned}I_K^{(n)}\big(f;q,t;t_0,t_1,t_2,t_3\big)&=\frac{\langle 1,f(x)\rangle}{\langle 1,1\rangle}\\&=\frac{1}{\langle 1,1\rangle}\int_{\mathbb{T}^n}f(x)\Delta(x;q,t;t_0,t_1,t_2,t_3)\,\mathrm{d}T(x).\end{aligned} \tag{3.1.3}$$

Also, by (2.6.7), for $f\in\Lambda$ and generic q,t,t_0,t_1,t_2,t_3,

$$I_K^{(n)}\big(f(x_1^{\pm},\dots,x_n^{\pm};q,t;t_0,t_1,t_2,t_3\big)=I_K\big(f;q,t,t^n;t_0,t_1,t_2,t_3\big). \tag{3.1.4}$$

Remark 3.1. Invoking the ring homomorphism $\varphi:\Lambda_{2n}\to\Lambda^{\mathrm{BC}_n}$ given by

$$\varphi\big(m_\lambda(x_1,\dots,x_{2n})\big)=m_\lambda(x_1^{\pm},\dots,x_n^{\pm1})$$

(so that $\ker(\varphi)=\langle e_i-e_{2n-i}:\ 0\leqslant i<n\rangle$), it will be convenient to extend $I_K^{(n)}$ to also act on Λ_{2n} (or more simply $f\in\Lambda$). That is, we set

$$I_K^{(n)}\big(f;q,t;t_0,t_1,t_2,t_3\big):=\big[K_0(x;q,t;t_0,t_1,t_2,t_3)\big]f(x_1^{\pm},\dots,x_n^{\pm}),\qquad\text{for } f\in\Lambda_{2n}\text{, or } f\in\Lambda). \tag{3.1.5}$$

Since for such f

$$I_K^{(n)}\big(f(x_1,\dots,x_{2n});q,t;t_0,t_1,t_2,t_3\big)=I_K^{(n)}\big(f(x_1^{\pm},\dots,x_n^{\pm});q,t;t_0,t_1,t_2,t_3\big),$$

we will often not distinguish between (3.1.2) and (3.1.5) and simply write

$$I_K^{(n)}\big(f;q,t;t_0,t_1,t_2,t_3\big),$$

where f can be either a symmetric or BC_n-symmetric function.

The reader is warned that for specialisations that hit the poles (2.6.6) of the lifted Koornwinder polynomials[1], (3.1.4) should be treated with great caution as the right-hand side may not be well-defined. In such cases T needs to be specialised before the t_r (or at least before some of the t_r). For example, if $T = t^n$ and $\{t_0, t_1, t_2, t_3\} = \{1, -1, t^{1/2}, -t^{1/2}\} =: \{\pm 1, \pm t^{1/2}\}$, then the lifted Koornwinder polynomial $\tilde{K}_\lambda$ is ill-defined if

$$l(\lambda) - \tfrac{1}{2} m_1(\lambda) \leqslant n < l(\lambda).$$

Accordingly, for (3.1.4) to hold we must first specialise $T = t^n$ before specialising the t_r. For example, since

$$\tilde{K}_{1^2}(q, t, T; \pm 1, \pm t_2) = m_{1^2} - \frac{(1-T)(T/t+t)(t-t_2^2)}{(1-t)(1+t)(t-t_2^2(T/t)^2)}$$

(and $\tilde{K}_0 = 1$), we have

$$I_K(m_{1^2}; q, t, T; \pm 1, \pm t_2) = \frac{(1-T)(T/t+t)(t-t_2^2)}{(1-t)(1+t)(t-t_2^2(T/t)^2)}.$$

This gives $I_K(m_{1^2}(q, t, t; \pm 1, \pm t_2) = 1$, which trivially agrees with

$$I_K^{(1)}(m_{1^2}; q, t; \pm 1, \pm t_2) = I_K^{(1)}(1; q, t; \pm 1, \pm t_2) = 1. \tag{3.1.6}$$

However,

$$I_K(m_{1^2}; q, t, T; \pm 1, \pm t^{1/2}) = 0. \tag{3.1.7}$$

For specialising in the 'wrong' order we can use [**112**, Lemma 5.10].

LEMMA 3.2. *For fixed n, let $t_0 \cdots t_3 t^{n-2} = t^k$, where k is a nonnegative integer. Then*

$$\begin{aligned}
&\lim_{T \to t^n} I_K(f; q, t, T; t_0, t_1, t_2, t_3) \\
&\quad = \tfrac{1}{2} I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) \\
&\qquad + \tfrac{1}{2} I_K^{(k)}\Big(f\Big[x_1^{\pm} + \cdots + x_k^{\pm} + \sum_{r=0}^{3} \frac{t_r - t/t_r}{1-t}\Big]; q, t; t/t_0, t/t_1, t/t_2, t/t_3\Big).
\end{aligned}$$

REMARK 3.3. To be consistent with our convention that

$$f(x_1^{\pm}, \dots, x_k^{\pm}) = f(x_1, x_1^{-1}, \dots, x_k, x_k^{-1})$$

and $f(x_1 + \cdots + x_k) = f[x_1 + \cdots + x_k]$, we interpret $f[x_1^{\pm} + \cdots + x_k^{\pm}]$ as $f[x_1 + x_1^{-1} + \cdots + x_k + x_k^{-1}]$.

Continuing our previous example, for

$$\{t_0, t_1, t_2, t_3\} = \{\pm 1, \pm t^{1/2}\} = \{1, \varepsilon, t^{1/2}, \varepsilon t^{1/2}\} \tag{3.1.8}$$

we plethystically have

$$\sum_{r=0}^{3} \frac{t_r - t/t_r}{1-t} = \frac{1-t}{1-t} + \frac{\varepsilon - \varepsilon t}{1-t} + \frac{t^{1/2} - t^{1/2}}{1-t} + \frac{\varepsilon t^{1/2} - \varepsilon t^{1/2}}{1-t} = 1 + \varepsilon, \tag{3.1.9}$$

[1]Since $q^{2-\lambda_i-j}$ is a nonpositive integer power of q this can never happen when the product $t_0t_1t_2t_3$ contains a positive power of q.

so that

$$\begin{aligned}
&\lim_{T\to t} I_K\big(f;q,t,T;\pm 1,\pm t^{1/2}\big)\\
&\qquad = \tfrac{1}{2} I_K^{(1)}\big(f;q,t,\pm 1,\pm t^{1/2}\big) + \tfrac{1}{2} I_K^{(0)}\big(f[1+\varepsilon];q,t,\pm t,\pm t^{1/2}\big)\\
&\qquad = \tfrac{1}{2} I_K^{(1)}\big(f;q,t,\pm 1,\pm t^{1/2}\big) + \tfrac{1}{2} I_K^{(0)}\big(f(1,-1);q,t,\pm t,\pm t^{1/2}\big).
\end{aligned}$$

If $f = m_{1^2}$ then $f(1,-1) = -1$. By $I_K^{(0)}(f;q,t,t_0,t_1,t_2,t_3) = f$ and (3.1.6), this yields

$$\lim_{T\to t} I_K\big(m_{1^2};q,t,T;\pm 1,\pm t^{1/2}\big), = \tfrac{1}{2}(1-1) = 0,$$

in accordance with (3.1.7).

Lemma 3.4. *For μ a partition and generic q,t,t_2,t_3,*

$$I_K\big(f[x+\varepsilon];q,t,T;-t,-t^{1/2},t_2,t_3\big) = I_K\big(f;q,t,t^{1/2}T;-1,-t^{1/2},t_2,t_3\big). \tag{3.1.10}$$

Proof. Let $f \in \Lambda$. From definition (3.1.1) of the virtual Koornwinder integral and the symmetry (2.6.11) of the lifted Koornwinder polynomials, we infer that (see also [**108**, Equation (7.4)][2])

$$\begin{aligned}
I_K\Big(f\Big[x + \frac{t_0 - t/t_0}{1-t} + \frac{t_1 - t/t_1}{1-t}\Big];q,t,T;t/t_0,t/t_1,t_2,t_3\Big)&\\
= I_K\big(f;q,t,tT/t_0t_1;t_0,t_1,t_2,t_3\big)&.
\end{aligned} \tag{3.1.11}$$

If we specialise $\{t_0,t_1\} = \{-1,-t^{1/2}\} = \{\varepsilon,\varepsilon t^{1/2}\}$, so that

$$\frac{t_0 - t/t_0}{1-t} + \frac{t_1 - t/t_1}{1-t} = \frac{\varepsilon - \varepsilon t}{1-t} + \frac{\varepsilon t^{1/2} - \varepsilon t^{1/2}}{1-t} = \varepsilon,$$

equation (3.1.11) simplifies to (3.1.10). ☐

3.2. Closed-form evaluations—the Macdonald case

In this section we consider several closed-form evaluations of virtual Koornwinder integrals over Macdonald polynomials.

Theorem 3.5. *For μ a partition,*

$$\begin{aligned}
&I_K\big(P_\mu(q,t);q,t,T;\pm t^{1/2},\pm (qt)^{1/2}\big)\\
&\qquad = \chi(\mu' \text{ even})\,\frac{(T^2;q,t^2)_\nu}{(qT^2/t;q,t^2)_\nu}\cdot\frac{C^-_\nu(qt;q,t^2)}{C^-_\nu(t^2;q,t^2)},
\end{aligned} \tag{3.2.1}$$

where, for μ' an even partition, $\nu := (\mu'/2)' = (\mu_1,\mu_3,\dots)$.

As usual $(\pm t^{1/2},\pm (qt)^{1/2})$ in the above is shorthand for

$$\big(t^{1/2},-t^{1/2},(qt)^{1/2},-(qt)^{1/2}\big).$$

Theorem 3.5, which for $T = t^n$ is known as the $\mathrm{U}(2n)/\mathrm{Sp}(2n)$ vanishing integral, was conjectured in [**108**, Conjecture 1] and proven in [**113**, Theorem 4.1].

[2]In [**108**, Equation (7.4)] the denominator term $(1-t)$ should be corrected to $(1-t^k)$.

THEOREM 3.6. *For μ a partition,*

$$\begin{aligned}(3.2.2)\quad & I_K\big(P_\mu(q,t);q,t,T;-1,-q^{1/2},-t^{1/2},-(qt)^{1/2}\big)\\ &= (-1)^{|\mu|}\,\frac{(T;q^{1/2},t^{1/2})_\mu}{(-q^{1/2}T/t^{1/2};q^{1/2},t^{1/2})_\mu}\cdot\frac{C^-_\mu(-q^{1/2};q^{1/2},t^{1/2})}{C^-_\mu(t^{1/2};q^{1/2},t^{1/2})}.\end{aligned}$$

This theorem was first stated (up to a trivial sign-change) as the conjectural [**111**, Equation (5.79)]. By [**112**, Theorem 8.5], which implies [**111**, Conjecture Q6], it now also has been proven.

From a symmetry of the virtual Koornwinder polynomials, the virtual Koornwinder integral satisfies the duality [**108**, Corollary 7.6][3]

$$I_K\big(f;q,t,T;t_0,t_1,t_2,t_3\big) = I_K\Big(f\Big[-\varepsilon\Big(\frac{t}{q}\Big)^{1/2}\frac{1-q}{1-t}\,x\Big];t,q,1/T;s_0,s_1,s_2,s_3\Big),$$

where $s_r=-(qt)^{1/2}/t_r$. Applying this to (3.2.1), and then replacing (q,t,T,μ) by $(t,q,1/T,\mu')$ using (2.2.6), (2.2.7b), (2.4.6), (2.5.3) and (2.5.4), yields the following dual virtual Koornwinder integral.

COROLLARY 3.7 ([**113**, page 741]). *For μ a partition,*

$$(3.2.3)\quad I_K\big(P_\mu(q,t);q,t,T;\pm 1,\pm t^{1/2}\big) = \chi(\mu\ \textit{even})\,\frac{(T^2;q^2,t)_{\mu/2}}{(qT^2/t;q^2,t)_{\mu/2}}\cdot\frac{C^-_{\mu/2}(q;q^2,t)}{C^-_{\mu/2}(t;q^2,t)}.$$

We need two variants of this for $I_K^{(n)}$. For λ a partition of length at most n or a half-partition of length n, define

$$(3.2.4)\qquad A^{(n)}_\lambda(q,t) := \prod_{1\leqslant i<j\leqslant n}\frac{(qt^{j-i-1},t^{j-i+1};q^2)_{\lambda_i-\lambda_j}}{(qt^{j-i},t^{j-i};q^2)_{\lambda_i-\lambda_j}}.$$

It is important to note that $A^{(n)}_\lambda(q,t)$ depends on the relative differences between the λ_i, and that for λ a partition

$$(3.2.5)\qquad A^{(n)}_\lambda(q,t) = \frac{(t^n;q^2,t)_\lambda}{(qt^{n-1};q^2,t)_\lambda}\cdot\frac{C^-_\lambda(q;q^2,t)}{C^-_\lambda(t;q^2,t)}.$$

THEOREM 3.8. *For μ a partition of length at most $2n$, let*

$$\tilde\mu = (\mu_1-\mu_{2n},\dots,\mu_{2n-1}-\mu_{2n},0).$$

Then

$$\begin{aligned} & I_K^{(n)}\big(P_\mu(x_1^\pm,\dots,x_n^\pm;q,t);q,t;\pm 1,\pm t^{1/2}\big)\\ &\quad = (-1)^{\mu_{2n}}I_K^{(n-1)}\big(P_\mu(x_1^\pm,\dots,x_{n-1}^\pm,1,-1;q,t);q,t;\pm t,\pm t^{1/2}\big)\\ &\quad = \begin{cases} A^{(2n)}_{\mu/2}(q,t) & \textit{if } \tilde\mu \textit{ is even,}\\ 0 & \textit{otherwise,}\end{cases}\end{aligned}$$

where $\mu/2$ is a partition or half-partition given by $(\mu_1/2,\dots,\mu_{2n}/2)$.

[3]In [**108**, Corollary 7.6] $\tilde\omega_{q,t}$ should be corrected to $\tilde\omega_{t,q}$.

THEOREM 3.9. *For ν a partition of length at most $2n+1$, let*

$$\tilde{\nu} = (\nu_1 - \nu_{2n+1}, \dots, \nu_{2n} - \nu_{2n+1}, 0).$$

Then

$$\begin{aligned} &I_K^{(n)}\big(P_\nu(x_1^\pm, \dots, x_n^\pm, 1; q, t); q, t; -1, t, \pm t^{1/2}\big) \\ &\quad = (-1)^{\nu_{2n+1}} I_K^{(n)}\big(P_\nu(x_1^\pm, \dots, x_n^\pm, -1; q, t); q, t; 1, -t, \pm t^{1/2}\big) \\ &\quad = \begin{cases} A_{\nu/2}^{(2n+1)}(q,t) & \text{if } \tilde{\nu} \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

PROOF OF THEOREM 3.8. From (2.5.12) we have

$$P_\mu(x_1^\pm, \dots, x_n^\pm; q, t) = P_{\tilde{\mu}}(x_1^\pm, \dots, x_n^\pm; q, t)$$

and

$$P_\mu(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t) = (-1)^{\mu_{2n}} P_{\tilde{\mu}}(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t),$$

so that

$$\begin{aligned} \text{(3.2.6a)}\quad &I_K^{(n)}\big(P_\mu(x_1^\pm, \dots, x_n^\pm; q, t); q, t; t_0, t_1, t_2, t_3\big) \\ &\qquad = I_K^{(n)}\big(P_{\tilde{\mu}}(x_1^\pm, \dots, x_n^\pm; q, t); q, t; t_0, t_1, t_2, t_3\big) \end{aligned}$$

and

$$\begin{aligned} \text{(3.2.6b)}\quad &I_K^{(n-1)}\big(P_\mu(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t); q, t; t_0, t_1, t_2, t_3\big) \\ &\qquad = (-1)^{\mu_{2n}} I_K^{(n-1)}\big(P_{\tilde{\mu}}(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t); q, t; t_0, t_1, t_2, t_3\big). \end{aligned}$$

Since $A_{\mu/2}^{(2n)}(q,t) = A_{\tilde{\mu}/2}^{(2n)}(q,t)$, it thus suffices to prove that

$$\begin{aligned} \text{(3.2.7)}\quad &I_K^{(n)}\big(P_\omega(x_1^\pm, \dots, x_n^\pm; q, t); q, t; \pm 1, \pm t^{1/2}\big) \\ &\quad = I_K^{(n-1)}\big(P_\omega(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t); q, t; \pm 1, \pm t^{1/2}\big) \\ &\quad = \chi(\omega \text{ even}) A_{\omega/2}^{(2n)}(q,t), \end{aligned}$$

for ω a partition such that $l(\omega) < 2n$.

We now apply Lemma 3.2 with $\{t_0, t_1, t_2, t_3\} = \{\pm 1, \pm t^{1/2}\}$ and $f = P_\mu(q,t)$ for μ a partition of length at most $2n$. Using (3.1.8) and (3.1.9), this yields

$$\begin{aligned} &\lim_{T \to t^n} I_K\big(P_\mu(q,t), q, t, T; \pm 1, \pm t^{1/2}\big) \\ &\quad = \tfrac{1}{2} I_K^{(n)}\big(P_\mu(x_1^\pm, \dots, x_n^\pm; q, t); q, t; \pm 1, \pm t^{1/2}\big) \\ &\qquad + \tfrac{1}{2} I_K^{(n-1)}\big(P_\mu(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t); q, t; \pm t, \pm t^{1/2}\big). \end{aligned}$$

By (3.2.3) and (3.2.5) the left-hand side is equal to $\chi(\mu \text{ even})\, A_{\mu/2}^{(2n)}(q,t)$, resulting in

$$\begin{aligned} &\tfrac{1}{2} I_K^{(n)}\big(P_\mu(x_1^\pm, \dots, x_n^\pm; q, t); q, t; \pm 1, \pm t^{1/2}\big) \\ &\quad + \tfrac{1}{2} I_K^{(n-1)}\big(P_\mu(x_1^\pm, \dots, x_{n-1}^\pm, 1, -1; q, t); q, t; \pm t, \pm t^{1/2}\big) \\ &\qquad = \chi(\mu \text{ even})\, A_{\mu/2}^{(2n)}(q,t). \end{aligned}$$

Again using (3.2.6) as well as

$$\chi(\mu \text{ even})\, A^{(2n)}_{\mu/2}(q,t) = \chi(\mu_{2n} \text{ even})\chi(\tilde{\mu} \text{ even})\, A^{(2n)}_{\tilde{\mu}/2}(q,t),$$

and then renaming $\tilde{\mu}_i = \mu_i - \mu_{2n}$ as ω_i for $1 \leqslant i \leqslant 2n-1$, and μ_{2n} as k, it follows that

$$\begin{aligned}
&\tfrac{1}{2} I_K^{(n)}\bigl(P_\omega(x_1^\pm,\dots,x_n^\pm;q,t);q,t;\pm 1,\pm t^{1/2}\bigr)\\
&\quad + \tfrac{1}{2}(-1)^k I_K^{(n-1)}\bigl(P_\omega(x_1^\pm,\dots,x_{n-1}^\pm,1,-1;q,t);q,t;\pm t,\pm t^{1/2}\bigr)\\
&\qquad = \chi(k \text{ even})\chi\bigl(\omega \text{ even}\bigr)\, A^{(2n)}_{\omega/2}(q,t),
\end{aligned}$$

for ω a partition of length at most $2n-1$ and k an arbitrary integer. For odd k this implies the first equality in (3.2.7), so that the second equality follows from even k. □

Proof of Theorem 3.9. The proof if analogous to that of Theorem 3.8 except that we now need Lemma 3.4 on top of Lemma 3.2.

Let $z \in \{-1,1\}$. Since

$$\begin{aligned}
(3.2.8)\quad & I_K^{(n)}\bigl(P_\nu(x_1^\pm,\dots,x_n^\pm,z;q,t);q,t;t_0,t_1,t_2,t_3\bigr)\\
&\qquad = z^{\nu_{2n+1}} I_K^{(n)}\bigl(P_{\tilde{\nu}}(x_1^\pm,\dots,x_n^\pm,z;q,t);q,t;t_0,t_1,t_2,t_3\bigr),
\end{aligned}$$

and $A^{(2n+1)}_{\nu/2}(q,t) = A^{(2n+1)}_{\tilde{\nu}/2}(q,t)$, it is enough to show that

$$(3.2.9)\quad I_K^{(n)}\bigl(P_\tau(x_1^\pm,\dots,x_n^\pm,z;q,t);q,t;-z,zt,\pm t^{1/2}\bigr) = \chi(\tau \text{ even}) A^{(2n+1)}_{\tau/2}(q,t),$$

for τ a partition such that $l(\tau) \leqslant 2n$.

For $\{t_0,t_1,t_2,t_3\} = \{1,\varepsilon t, t^{1/2}, \varepsilon t^{1/2}\}$, we have $\sum_{r=0}^3 \frac{t_r - t/t_r}{1-t} = 1-\varepsilon$. Hence, by Lemma 3.2 with $f(x) = P_\nu\bigl([x+\varepsilon];q,t\bigr)$ and ν a partition of length at most $2n+1$,

$$\begin{aligned}
&\lim_{T\to t^n} I_K\bigl(P_\nu\bigl([x+\varepsilon];q,t\bigr),q,t,T;1,-t,\pm t^{1/2}\bigr)\\
&\qquad = \tfrac{1}{2} I_K^{(n)}\bigl(P_\nu(x_1^\pm,\dots,x_n^\pm,-1;q,t);q,t;1,-t,\pm t^{1/2}\bigr)\\
&\qquad\quad + \tfrac{1}{2} I_K^{(n)}\bigl(P_\nu(x_1^\pm,\dots,x_n^\pm,1;q,t);q,t;-1,t,\pm t^{1/2}\bigr)\\
&\qquad = \tfrac{1}{2} \sum_{z\in\{-1,1\}} I_K^{(n)}\bigl(P_\nu(x_1^\pm,\dots,x_n^\pm,z;q,t);q,t;-z,zt,\pm t^{1/2}\bigr).
\end{aligned}$$

On the other hand, by Lemma 3.4,

$$\begin{aligned}
&\lim_{T\to t^n} I_K\bigl(P_\nu\bigl([x+\varepsilon];q,t\bigr),q,t,T;1,-t,\pm t^{1/2}\bigr)\\
&\qquad = \lim_{T\to t^n} I_K\bigl(P_\nu(q,t),q,t,t^{1/2}T;\pm 1,\pm t^{1/2}\bigr)\\
&\qquad = \chi(\nu \text{ even})\, A^{(2n+1)}_{\nu/2}(q,t),
\end{aligned}$$

where the second equality follows from (3.2.3) and (3.2.5). Therefore,

$$\begin{aligned}
&\tfrac{1}{2} \sum_{z\in\{-1,1\}} I_K^{(n)}\bigl(P_\nu(x_1^\pm,\dots,x_n^\pm,z;q,t);q,t;-z,zt,\pm t^{1/2}\bigr)\\
&\qquad\qquad = \chi(\nu \text{ even})\, A^{(2n+1)}_{\nu/2}(q,t).
\end{aligned}$$

If we define $\tau_i := \nu_i - \nu_{2n+1}$ for $1 \leqslant i \leqslant 2n$ and $k := \nu_{2n+1}$, and then use (3.2.8), the above can also be written as

$$\tfrac{1}{2} \sum_{z\in\{-1,1\}} z^k I_K^{(n)}\big(P_\tau(x_1^\pm,\dots,x_n^\pm,z;q,t);q,t;-z,zt,\pm t^{1/2}\big) \\ = \chi(k \text{ even})\chi(\tau \text{ even})\, A_{\tau/2}^{(2n+1)}(q,t).$$

As before, by considering odd values of k this yields

$$I_K^{(n)}\big(P_\tau(x_1^\pm,\dots,x_n^\pm,1;q,t);q,t;-1,t,\pm t^{1/2}\big) \\ = I_K^{(n)}\big(P_\tau(x_1^\pm,\dots,x_n^\pm,-1;q,t);q,t;1,-t,\pm t^{1/2}\big).$$

Choosing k to be even completes the proof of (3.2.9). □

As our final evaluation of this section we claim the following.

THEOREM 3.10. *For μ a partition,*

$$I_K\big(P_\mu(q,t);q,t,T;-1,q,\pm t^{1/2}\big) = (-1)^{|\mu|}\,\frac{(T^2;q^2,t)_{\lceil\mu/2\rceil}}{(qT^2/t;q^2,t)_{\lceil\mu/2\rceil}} \cdot \frac{1}{b_\mu^{\mathrm{ea}}(q,t)}. \tag{3.2.10}$$

PROOF. From (3.1.3), definition (2.6.1) of the Koornwinder density and Gustafson's integral (2.6.4), it follows that

$$I_K^{(n)}\big(f;q,t;qt_0,t_1,t_2,t_3\big) \\ = I_K^{(n)}\Big(f(x_1^\pm,\dots,x_n^\pm)\prod_{i=1}^n(1-t_0x_i^\pm);q,t;t_0,t_1,t_2,t_3\Big)\prod_{i=1}^n\frac{1-t_0t_1t_2t_3t^{n+i-2}}{\prod_{r=1}^3(1-t_0t_rt^{i-1})}.$$

For $f = P_\mu(q,t)$ we can use the e-Pieri rule (2.5.8) to expand the integrand. Hence

$$I_K^{(n)}\big(P_\mu(q,t);q,t;qt_0,t_1,t_2,t_3\big) = \prod_{i=1}^n\frac{1-t_0t_1t_2t_3t^{n+i-2}}{\prod_{r=1}^3(1-t_0t_rt^{i-1})} \\ \times \sum_{\lambda\supset\mu}(-t_0)^{|\lambda/\mu|}\psi'_{\lambda/\mu}(q,t)I_K^{(n)}\big(P_\lambda(q,t);q,t;t_0,t_1,t_2,t_3\big).$$

For $(t_0,t_1,t_2,t_3) = (1,-1,t^{1/2},-t^{1/2})$ this yields

$$I_K^{(n)}\big(P_\mu(q,t);q,t;q,-1,\pm t^{1/2}\big) \\ = \tfrac{1}{2}\sum_{\lambda\supset\mu}(-1)^{|\lambda/\mu|}\psi'_{\lambda/\mu}(q,t)I_K^{(n)}\big(P_\lambda(q,t);q,t;\pm 1,\pm t^{1/2}\big).$$

The integral in the summand evaluates in closed form by Theorem 3.8. In particular it vanishes unless (i) λ is even or (ii) λ is odd and $l(\lambda) = 2n$. Since $\psi'_{\lambda/\mu}(q,t)$ is zero unless λ/μ is a vertical strip, this fixes λ as $\lambda = 2\lceil\mu/2\rceil =: \nu$ in case (i) and $\lambda = 2\lfloor\mu/2\rfloor + 1^{2n} =: \omega$ in case (ii). Noting the three congruences

$$|\nu| \equiv |\omega| \equiv 0 \pmod 2, \quad |\nu/\mu| = \mathrm{odd}(\mu) \equiv |\mu| \pmod 2,$$

and

$$|\omega/\mu| = 2n - \mathrm{odd}(\mu) \equiv |\mu| \pmod 2,$$

we obtain

$$I_K^{(n)}\big(P_\mu(q,t);q,t;q,-1,\pm t^{1/2}\big)$$
$$= \tfrac{1}{2}(-1)^{|\mu|}\Big(\psi'_{\nu/\mu}(q,t)A^{(2n)}_{\nu/2}(q,t)+\psi'_{\omega/\mu}(q,t)A^{(2n)}_{\omega/2}(q,t)\Big).$$

We will now show that the two terms on the right are equal, resulting in

$$I_K^{(n)}\big(P_\mu(q,t);q,t;q,-1,\pm t^{1/2}\big)=(-1)^{|\mu|}\psi'_{\nu/\mu}(q,t)A^{(2n)}_{\nu/2}(q,t). \tag{3.2.11}$$

First we note that since $\omega/2=\lfloor\mu/2\rfloor+(\frac{1}{2})^{2n}$ and $A^{(n)}_\mu(q,t)$ depends on the relative differences of the μ_i, we have

$$A^{(2n)}_{\omega/2}(q,t)=A^{(2n)}_{\lfloor\mu/2\rfloor}(q,t).$$

Moreover, by (3.2.4) and $\nu/2=\lceil\mu/2\rceil$,

$$A^{(2n)}_{\lfloor\mu/2\rfloor}(q,t)=A^{(2n)}_{\nu/2}(q,t)\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \mu_i\text{ odd},\ \mu_j\text{ even}}}\frac{1-q^{\mu_i-\mu_j}t^{j-i}}{1-q^{\mu_i-\mu_j}t^{j-i-1}}\cdot\frac{1-q^{\mu_i-\mu_j-1}t^{j-i}}{1-q^{\mu_i-\mu_j-1}t^{j-i+1}}$$
$$\times\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \mu_i\text{ even},\ \mu_j\text{ odd}}}\frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}\cdot\frac{1-q^{\mu_i-\mu_j-1}t^{j-i+1}}{1-q^{\mu_i-\mu_j-1}t^{j-i}}.$$

But from (2.5.9) it follows that

$$\psi'_{\nu/\mu}(q,t)=\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \mu_i\text{ even},\ \mu_j\text{ odd}}}\frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}\cdot\frac{1-q^{\mu_i-\mu_j-1}t^{j-i+1}}{1-q^{\mu_i-\mu_j-1}t^{j-i}}$$

and

$$\psi'_{\omega/\mu}(q,t)=\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \mu_i\text{ odd},\ \mu_j\text{ even}}}\frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}\cdot\frac{1-q^{\mu_i-\mu_j-1}t^{j-i+1}}{1-q^{\mu_i-\mu_j-1}t^{j-i}},$$

so that

$$\psi'_{\omega/\mu}(q,t)A^{(2n)}_{\omega/2}(q,t)=\psi'_{\nu/\mu}(q,t)A^{(2n)}_{\nu/2}(q,t),$$

establishing (3.2.11).

Since $\nu=2\lceil\mu/2\rceil$ is even, we can use (3.2.5) to write the right side of (3.2.11) as

$$(-1)^{|\mu|}\psi'_{\nu/\mu}(q,t)\,\frac{(t^{2n};q^2,t)_{\nu/2}}{(qt^{2n-1};q^2,t)_{\nu/2}}\cdot\frac{C^-_{\nu/2}(q;q^2,t)}{C^-_{\nu/2}(t;q^2,t)}.$$

By Lemma 2.1 this is also

$$(-1)^{|\mu|}\frac{(t^{2n};q^2,t)_{\nu/2}}{(qt^{2n-1};q^2,t)_{\nu/2}}\cdot\frac{1}{b^{\mathrm{ea}}_\mu(q,t)}.$$

Hence

$$I_K^{(n)}\big(P_\mu(q,t);q,t;-1,q,\pm t^{1/2}\big)=(-1)^{|\mu|}\,\frac{(t^{2n};q^2,t)_{\lceil\mu/2\rceil}}{(qt^{2n-1};q^2,t)_{\lceil\mu/2\rceil}}\cdot\frac{1}{b^{\mathrm{ea}}_\mu(q,t)}. \tag{3.2.12}$$

Since both sides vanish if $l(\mu)>2n$ this holds for all partitions μ.

For fixed μ

$$I_K\big(P_\mu(q,t);q,t,T;t_0,t_1,t_2,t_3\big)$$

is a rational function in T. By (3.1.4) and (3.2.12), equation (3.2.10) holds for $T = t^n$ for all nonnegative integers n. Hence it holds for arbitrary T. $\square$

3.3. Closed-form evaluations—the Hall–Littlewood case

We present one final virtual Koornwinder integral with Hall–Littlewood polynomial argument. It evaluates in terms of the generalised Rogers–Szegő polynomials (2.3.4), and does not appear to have a simple t-analogue for Macdonald polynomials.

Let

$$I_K^{(n)}(P_\mu(q,0); q, 0; t_0, t_1, t_2, t_3) := \lim_{t \to 0} I_K^{(n)}(P_\mu(q,t); q, t; t_0, t_1, t_2, t_3).$$

THEOREM 3.11. *For μ a partition of length at most $2n$,*

$$I_K^{(n)}(P_\mu(q,0); q, 0; 0, 0, t_2, t_3) = h_{\mu'}^{(2n)}(-t_2, -t_3; q). \tag{3.3.1}$$

PROOF. Let $x = (x_1, \dots, x_n)$. By (3.1.2), equation (3.3.1) may also be stated as the rational function identity

$$\big[K_0(x; q, 0; 0, 0, t_2, t_3)\big] P_\mu(x^\pm, q, 0) = h_{\mu'}^{(2n)}(-t_2, -t_3; q).$$

Without loss of generality we may thus assume that $|t_2|, |t_3| < 1$ in the following.

Noting that

$$h_\lambda^{(2n)}(0, 0; q) = \chi(\lambda \text{ even}),$$

the $t_2 = t_3 = 0$ case of (3.3.1), viz.

$$I_K^{(n)}(P_\mu(q,0); q, 0; 0, 0, 0, 0) = \chi(\mu' \text{ even}), \tag{3.3.2}$$

follows from (3.2.1) (with $T = t^n$) in the $t \to 0$ limit.

To include the parameter t_2 we use that

$$\langle 1, 1 \rangle_{q,0;0,0,t_2,0}^{(n)} = \langle 1, 1 \rangle_{q,0;0,0,0,0}^{(n)}$$

(see (2.6.4)) and

$$\Delta(x; q, 0; 0, 0, t_2, 0) = \Delta(x; q, 0; 0, 0, 0, 0) \prod_{i=1}^{n} \frac{1}{(t_2 x_i^\pm; q)_\infty}.$$

From (3.1.3) it thus follows that

$$\begin{aligned} f_\mu(t_2; q) &:= I_K^{(n)}\big(P_\mu(q,0); q, 0; 0, 0, t_2, 0\big) \\ &= I_K^{(n)}\bigg(P_\mu(x_1^\pm, \dots, x_n^\pm; q, 0) \prod_{i=1}^{2n} \frac{1}{(t_2 x_i^\pm; q)_\infty}; q, 0; 0, 0, 0, 0\bigg). \end{aligned} \tag{3.3.3}$$

By the g-Pieri rule (2.5.6) for $t = 0$, this yields

$$\begin{aligned} f_\mu(t_2; q) &= \sum_{\nu \succ \mu} t_2^{|\nu/\mu|} \varphi_{\nu/\mu}(q, 0) I_K^{(n)}\big(P_\nu(q,0); q, 0; 0, 0, 0, 0\big) \\ &= \sum_{\substack{\nu \succ \mu \\ \nu' \text{ even} \\ l(\nu) \leqslant 2n}} t_2^{|\nu/\mu|} \varphi_{\nu/\mu}(q, 0), \end{aligned}$$

where the second equality follows from (3.3.2). Since $\varphi_{\nu/\mu}(q, 0)$ is zero unless ν/μ is a horizontal strip and since ν' must be even, this fixes ν as $\nu_{2i-1} = \nu_{2i} = \mu_{2i-1}$

for $1 \leqslant i \leqslant n$. This is equivalent to $\nu_i' = \mu_i' + \chi(\mu_i' \text{ odd})$, so that $|\nu/\mu|$ is given by the number of odd parts of μ', i.e., by $\operatorname{odd}(\mu')$. Hence

$$f_\mu(t_2;q) = t_2^{\operatorname{odd}(\mu')} \varphi_{\nu/\mu}(q,0),$$

with ν fixed as above. From the expression for $\varphi_{\lambda/\mu}(q,t)$ as given in (2.5.7) it follows that[4]

$$\varphi_{\nu/\mu}(q,0) = \prod_{i\geqslant 1} \frac{(q;q)_{\mu_i-\mu_{i+1}}}{(q;q)_{\nu_i-\mu_i}(q;q)_{\mu_i-\nu_{i+1}}} = \frac{1}{(q;q)_{\nu_1-\mu_1}} \prod_{i\geqslant 1} \begin{bmatrix} \mu_i-\mu_{i+1} \\ \mu_i-\nu_{i+1} \end{bmatrix}_q. \tag{3.3.4}$$

When $\nu_{2i-1} = \nu_{2i} = \mu_{2i-1}$ this simplifies to $\varphi_{\nu/\mu}(q,0) = 1$, since either $\nu_{i+1} = \mu_i$ or $\nu_{i+1} = \mu_{i+1}$. Hence

$$f_\mu(t_2;q) = t_2^{\operatorname{odd}(\mu')}. \tag{3.3.5}$$

To also include the parameter t_3 we proceed in almost identical fashion. By

$$\langle 1,1\rangle^{(n)}_{q,0;0,0,t_2,t_3} = \frac{1}{(t_2t_3;q)_\infty} \langle 1,1\rangle^{(n)}_{q,0;0,0,t_2,0}$$

and

$$\Delta(x;0,0,t_2,t_3;q,0) = \Delta(x;0,0,t_2,0;q,0) \prod_{i=1}^n \frac{1}{(t_2x_i^{\pm};q)_\infty},$$

and following the previous steps, we obtain

$$\begin{aligned} f_\mu(t_2,t_3;q) &:= I_K^{(n)}\big(P_\mu(q,0);q,0;0,0,t_2,t_3\big) \\ &= (t_2t_3;q)_\infty \sum_{\nu\succ\mu} t_2^{|\nu/\mu|} f_\nu(t_2;q)\varphi_{\nu/\mu}(q,0) \\ &= (t_2t_3;q)_\infty \sum_{\substack{\nu\succ\mu \\ l(\nu)\leqslant 2n}} t_2^{\operatorname{odd}(\nu')} t_3^{|\nu/\mu|} \varphi_{\nu/\mu}(q,0). \end{aligned}$$

Here the second line uses the definition of $f_\nu(t_2;q)$ as given in (3.3.3), and the third line uses the evaluation (3.3.5). To complete the proof we write $\nu_i = \mu_i + k_i$ for $1 \leqslant i \leqslant 2n$, and note that (see [**139**, page 822])

$$\operatorname{odd}(\nu') = \operatorname{odd}(\mu') + \sum_{i=1}^{2n} (-1)^{i+1} k_i. \tag{3.3.6}$$

Once again using (3.3.4), we get

$$f_\mu(t_2,t_3;q) = (t_2t_3;q)_\infty\, t_2^{\operatorname{odd}(\mu')} \sum_{k_1,\dots,k_{2n}\geqslant 0} \frac{1}{(q;q)_{k_1}} \prod_{i\geqslant 1} t_2^{(-1)^{i+1}k_i} t_3^{k_i} \begin{bmatrix} \mu_i-\mu_{i+1} \\ k_{i+1} \end{bmatrix}_q.$$

Summing over k_1 by [**43**, Equation (II.1)]

$$\sum_{k\geqslant 0} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad \text{for } |z|<1,$$

[4] Alternatively, this follows from the Pieri coefficient $\varphi'_{\lambda/\mu}(t)$ for Hall–Littlewood polynomials, thanks to $\varphi_{\nu/\mu}(q,0) = \varphi'_{\nu'/\mu'}(0,q) = \varphi'_{\nu'/\mu'}(q)$.

and recalling definition (2.3.1), we finally obtain

$$\begin{aligned} f_\mu(t_2,t_3;q) &= t_2^{\mathrm{odd}(\mu')} \prod_{\substack{i=1 \\ i \text{ odd}}}^{2n-1} H_{m_i(\mu')}(t_3/t_2;q) \prod_{\substack{i=1 \\ i \text{ even}}}^{2n-1} H_{m_i(\mu')}(t_2 t_3;q) \\ &= h_{\mu'}^{(2n)}(-t_2,-t_3;q). \end{aligned}$$

$\square$

CHAPTER 4

Bounded Littlewood identities

In this section, which is at the heart of the paper, we use Macdonald–Koornwinder theory and virtual Koornwinder integrals in particular to prove bounded Littlewood identities for Macdonald and Hall–Littlewood polynomials.

4.1. Statement of results

4.1.1. q,t**-Identities.** There are five known Littlewood identities for Macdonald polynomials. By introducing an additional parameter a, the first four of these may easily be combined to form the pair of identities [**139**, Proposition 1.3]

(4.1.1)
$$\sum_{\lambda} a^{\mathrm{odd}(\lambda)} b_{\lambda}^{\mathrm{oa}}(q,t) P_{\lambda}(x;q,t) = \prod_{i=1}^{n} \frac{(1+ax_i)(qtx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}$$

and

$$\sum_{\lambda} a^{\mathrm{odd}(\lambda')} b_{\lambda}^{\mathrm{el}}(q,t) P_{\lambda}(x;q,t) = \prod_{i=1}^{n} \frac{(atx_i;q)_\infty}{(ax_i;q)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}. \tag{4.1.2}$$

Here

$$b_{\lambda}^{\mathrm{oa}}(q,t) := \prod_{\substack{s\in\lambda \\ a(s) \text{ odd}}} b_{\lambda}(s;q,t) \quad \text{and} \quad b_{\lambda}^{\mathrm{el}}(q,t) := \prod_{\substack{s\in\lambda \\ l(s) \text{ even}}} b_{\lambda}(s;q,t),$$

to be compared with (2.2.5) and (2.5.11). The cases $a=0$ and $a=1$ of (4.1.1) and (4.1.2) correspond to Macdonald's original four results, see [**91**, page 349]. The fifth identity was first conjectured by Kawanaka [**62**] and subsequently proven in [**71**] (see also [**111**]):

$$\sum_{\lambda} b_{\lambda}^{-}(q,t) P_{\lambda}(x;q^2,t^2) = \prod_{i=1}^{n} \frac{(-tx_i;q)_\infty}{(x_i;q)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{(t^2x_ix_j;q^2)_\infty}{(x_ix_j;q^2)_\infty}, \tag{4.1.3}$$

where

$$b_{\lambda}^{-}(q,t) := \prod_{s\in\lambda} \frac{1+q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}.$$

In the following we generalise all of (4.1.1)–(4.1.3).

For m a nonnegative integer and λ a partition, let

$$b_{\lambda;m}^{\mathrm{oa}}(q,t) := b_{\lambda}^{\mathrm{oa}}(q,t) \prod_{\substack{s\in\lambda \\ a'(s) \text{ odd}}} \frac{1-q^{2m-a'(s)+1}t^{l'(s)}}{1-q^{2m-a'(s)}t^{l'(s)+1}}.$$

Note that $b^{\mathrm{oa}}_{\lambda;m}(q,t)=0$ if $\lambda_1>2m+1$ and that for λ an even partition

$$b^{\mathrm{oa}}_{\lambda;m}(q,t)=b^{\mathrm{oa}}_{\lambda}(q,t)\prod_{\substack{s\in\lambda\\ a'(s)\text{ even}}}\frac{1-q^{2m-a'(s)}t^{l'(s)}}{1-q^{2m-a'(s)-1}t^{l'(s)+1}}. \tag{4.1.4}$$

Our first bounded Littlewood identity generalises (4.1.1).

THEOREM 4.1. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum_{\lambda}a^{\mathrm{odd}(\lambda)}b^{\mathrm{oa}}_{\lambda;m}(q,t)P_\lambda(x;q,t)=\bigg(\prod_{i=1}^{n}x_i^m(1+ax_i)\bigg)P^{(\mathrm{C}_n,\mathrm{B}_n)}_{m^n}(x;q,t,qt).$$

Using (2.7.9) to identify $P^{(\mathrm{C}_n,\mathrm{B}_n)}_{m^n}(x;q,t,qt)$ as a Koornwinder polynomial, and then using (2.6.9) for $\lambda=0$, it follows that the large-m limit of the right-hand side simplifies to the right-hand side of (4.1.1).

When $a=0$ the summand on the left vanishes unless λ is even, so that[1]

$$\sum_{\lambda\text{ even}}b^{\mathrm{oa}}_{\lambda;m}(q,t)P_\lambda(x;q,t)=(x_1\cdots x_n)^mP^{(\mathrm{C}_n,\mathrm{B}_n)}_{m^n}(x;q,t,qt). \tag{4.1.5}$$

This is a q,t-analogue of the Désarménien–Proctor–Stembridge determinant formula [**31**, **107**, **127**]

$$\sum_{\substack{\lambda\text{ even}\\ \lambda_1\leqslant 2m}}s_\lambda(x)=\frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}-x_i^{2m+2n-j+1}\big)}{\prod_{i=1}^{n}(1-x_i^2)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)}, \tag{4.1.6}$$

which expresses the symplectic Schur function $\mathrm{sp}_{2n,m^n}(x)$ (times $(x_1\cdots x_n)^m$) in terms of Schur functions. Equivalently, (4.1.6) is a branching formula for the character of the symplectic group $\mathrm{Sp}(n,\mathbb{C})$ indexed by $m\omega_n$ in terms of characters of the general linear group $\mathrm{GL}(n,\mathbb{C})$. As will be discussed in Section 5.1, like Macdonald's formula (1.1.2), the determinant (4.1.6) is important in the theory of plane partitions.

Another notable special case follows when $q=0$. For $s\in\lambda\subset(2m)^n$ such that $a'(s)$ is even we must have $2m-a'(s)\geqslant 2$, which implies that $b^{\mathrm{oa}}_{\lambda;m}(0,t)=1$. By (2.8.9) the $q=0$ specialisation of (4.1.5) is thus

$$\sum_{\substack{\lambda\text{ even}\\ \lambda_1\leqslant 2m}}P_\lambda(x;t)=(x_1\cdots x_n)^mP^{(\mathrm{C}_n)}_{m^n}(x;t,0). \tag{4.1.7}$$

For positive m the right-hand side can be expressed in terms of the function $\Phi(x;t,0,0)$ by Lemma 2.5. The resulting t-analogue of the Désarménien–Proctor–Stembridge determinant is due to Stembridge [**127**, Theorem 1.2] who used it to give new proofs of the Rogers–Ramanujan identities. We will see in Section 5.3 that Stembridge's method can be extended so that identities such as (4.1.7) yield Rogers–Ramanujan identities for certain affine Lie algebras $X_N^{(r)}$ for arbitrary N.

For m a nonnegative integer and λ a partition, let

$$b^{\mathrm{ol}}_{\lambda;m}(q,t):=\prod_{\substack{s\in\lambda\\ l'(s)\text{ odd}}}\frac{1-q^{m-a'(s)}t^{l'(s)-1}}{1-q^{m-a'(s)-1}t^{l'(s)}}\prod_{\substack{s\in\lambda\\ l(s)\text{ odd}}}\frac{1-q^{a(s)}t^{l(s)}}{1-q^{a(s)+1}t^{l(s)-1}}.$$

[1] By (2.5.12), the same result may be obtained in the $a\to\infty$ limit.

Note that $b^{\mathrm{ol}}_{\lambda;m}(q,t)=0$ if $\lambda_2>m$, which implies vanishing for $\lambda_1>m$ when λ' is even. Our next theorem contains the first of two bounded analogues of the $a=0$ case of (4.1.2).

Theorem 4.2. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum b^{\mathrm{ol}}_{\lambda;m}(q,t)P_\lambda(x;q,t)=(x_1\cdots x_n)^{\frac{m}{2}}P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;q,t,1), \tag{4.1.8}$$

where the sum is over partitions $\lambda\subset m^n$ such that $m_i(\lambda)$ is even for all $1\leqslant i\leqslant m-1$.

To see that this generalises (4.1.2) for $a=0$, we first note that in the large-m limit the right-hand side simplifies to the right-hand side of (4.1.2) for $a=0$ by (2.7.12a) and the $\lambda=0$ case of Lemma 2.4. Next, to simplify the left-hand side we note that there are two types of partitions contributing to the sum.

- **Type 1:** Partitions λ such that $m_i(\lambda)$ is even for all $1\leqslant i\leqslant m$, i.e., λ' is even.
- **Type 2:** Partitions λ such that $m_i(\lambda)$ is odd for $i=m$ and even for $1\leqslant i<m$, i.e., λ' is odd and $\lambda_1=m$.

Macdonald polynomials indexed by partitions of Type 2 have degree at least m, so that their contribution vanishes in the large-m limit. Hence we are left with a sum over partitions of Type 1, for which

$$\prod_{\substack{s\in\lambda\\ l(s)\text{ odd}}}\frac{1-q^{a(s)}t^{l(s)}}{1-q^{a(s)+1}t^{l(s)-1}}=\prod_{\substack{s\in\lambda\\ l(s)\text{ even}}}\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}=b^{\mathrm{el}}_\lambda(q,t),$$

resulting in the $a=0$ case of (4.1.2). In fact, (2.7.7) can be used to dissect (4.1.8), resulting in two bounded Littlewood identities for D_n, the first of which is our second bounded analogue of (4.1.2) for $a=0$.

Theorem 4.3. *For $x=(x_1,\dots,x_n)$, $\bar{x}=(x_1,\dots,x_{n-1},x_n^{-1})$ and m a nonnegative integer,*

$$\sum_{\lambda'\text{ even}} b^{\mathrm{ol}}_{\lambda;m}(q,t)P_\lambda(x;q,t)=(x_1\cdots x_n)^{\frac{m}{2}}P^{(\mathrm{D}_n,\mathrm{D}_n)}_{(\frac{m}{2})^n}(x;q,t) \tag{4.1.9a}$$

$$\sum_{\substack{\lambda'\text{ odd}\\ \lambda_1=m}} b^{\mathrm{ol}}_{\lambda;m}(q,t)P_\lambda(x;q,t)=(x_1\cdots x_n)^{\frac{m}{2}}P^{(\mathrm{D}_n,\mathrm{D}_n)}_{(\frac{m}{2})^n}(\bar{x};q,t). \tag{4.1.9b}$$

Taking $q=0$ in (4.1.9a) yields

$$\sum_{\substack{\lambda'\text{ even}\\ \lambda_1\leqslant m}} P_\lambda(x;t)\prod_{i=1}^{m-1}(t;t^2)_{m_i(\lambda)/2}=(x_1\cdots x_n)^{\frac{m}{2}}P^{(\mathrm{D}_n)}_{(\frac{m}{2})^n}(x;t). \tag{4.1.10}$$

By Lemma 2.7 this is equivalent to [**56**, Theorem 1; Eq. (7)] of Jouhet and Zeng, which itself is a t-analogue of Okada's determinant [**101**, Theorem 2.3 (3)]

$$\sum_{\substack{\lambda'\text{ even}\\ \lambda_1\leqslant m}} s_\lambda(x)=\frac{\sum_{\varepsilon\in\{\pm1\}}\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}+\varepsilon\, x_i^{m+2n-j-1}\big)}{2\prod_{i<j}(x_i-x_j)(x_ix_j-1)}.$$

For m a nonnegative integer and λ a partition, let

$$b^{\mathrm{el}}_{\lambda;m}(q,t) := b^{\mathrm{el}}_{\lambda}(q,t) \prod_{\substack{s\in\lambda \\ l'(s) \text{ even}}} \frac{1-q^{m-a'(s)}t^{l'(s)}}{1-q^{m-a'(s)-1}t^{l'(s)+1}}. \tag{4.1.11}$$

We note that $b^{\mathrm{el}}_{\lambda;m}(q,t)$ vanishes unless $\lambda_1 \leqslant m$. The next result is (1.1.6) from the introduction, which bounds (4.1.2) for $a=1$.

THEOREM 4.4. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum_{\lambda} b^{\mathrm{el}}_{\lambda;m}(q,t)P_\lambda(x;q,t) = (x_1\cdots x_n)^{\frac{m}{2}}\, P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;q,t,t). \tag{4.1.12}$$

The $q=0$ and $t=q$ specialisations of (4.1.12) correspond to (4.1.17) below for $t_2=t$, i.e.,

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant m}} \prod_{i=1}^{m-1}\Big((t;t^2)_{\lceil m_i(\lambda)/2\rceil}\Big)P_\lambda(x;t) = (x_1\cdots x_n)^{\frac{m}{2}}\, P^{(\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;t,t),$$

and Macdonald's determinant (1.1.2) respectively.

REMARK 4.5. Using (4.1.9a) it is not hard to prove an identity that generalises (4.1.2) in full:

$$\sum_{\lambda} a^{\mathrm{odd}(\lambda')}\hat{b}^{\mathrm{el}}_{\lambda;m}(q,t)P_\lambda(x;q,t) = \bigg(\prod_{i=1}^{n} x_i^{m/2}\frac{(atx_i;q)_\infty}{(ax_i;q)_\infty}\bigg)P^{(\mathrm{D}_n,\mathrm{D}_n)}_{(\frac{m}{2})^n}(x;q,t),$$

where

$$\hat{b}^{\mathrm{el}}_{\lambda;m}(q,t) := b^{\mathrm{el}}_{\lambda}(q,t) \prod_{\substack{s\in\lambda \\ l'(s) \text{ odd}}} \frac{1-q^{m-a'(s)}t^{l'(s)-1}}{1-q^{m-a'(s)-1}t^{l'(s)}}.$$

The largest part of λ in the sum on the left is not bounded, and unlike (4.1.8), (4.1.9a) or (4.1.12), this is not a polynomial identity.

For m a nonnegative integer and λ a partition such that $\lambda_1\leqslant m$, let

$$b^{-}_{\lambda;m}(q,t) := b^{-}_{\lambda}(q,t) \prod_{s\in\lambda} \frac{1-q^{m-a'(s)}t^{l'(s)}}{1+q^{m-a'(s)-1}t^{l'(s)+1}}.$$

Our final result for Macdonald polynomials is a bounded analogue of Kawanaka's conjecture (4.1.3).

THEOREM 4.6. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum_{\lambda} b^{-}_{\lambda;m}(q,t)P_\lambda(x;q^2,t^2) = (x_1\cdots x_n)^{\frac{m}{2}}\, P^{(\mathrm{B}_n,\mathrm{C}_n)}_{(\frac{m}{2})^n}(x;q^2,t^2,-t). \tag{4.1.13}$$

For $t=-q$ this simplifies to (1.1.2) and for $q=0$ it is (4.1.17) below with $(t,t_2)\mapsto(t^2,-t)$, viz.

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant m}} \prod_{i=1}^{m-1}\Big((-t;t)_{m_i(\lambda)}\Big)P_\lambda(x;t^2) = (x_1\cdots x_n)^{\frac{m}{2}}\, P^{(\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;t^2,-t). \tag{4.1.14}$$

Assuming m is positive and rewriting the right-hand side using Lemma 2.6 yields [**51**, Theorem 1] of Ishikawa et al.

4.1.2. t-Identities. Our final two theorems do not appear to have simple analogues for Macdonald polynomials.

Recall the generalised Rogers–Szegő polynomials (2.3.4).

THEOREM 4.7. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant 2m}} h_\lambda^{(2m)}(t_2,t_3;t)P_\lambda(x;t)=(x_1\cdots x_n)^m P_{m^n}^{(\mathrm{BC}_n)}(x;t,t_2,t_3). \tag{4.1.15}$$

This bounds [**139**, Theorem 1.1]

$$\sum_\lambda h_\lambda(t_2,t_3;t)P_\lambda(x;t)=\prod_{i=1}^n\frac{(1-t_2x_i)(1-t_3x_i)}{1-x_i^2}\prod_{1\leqslant i<j\leqslant n}\frac{1-tx_ix_j}{1-x_ix_j}, \tag{4.1.16}$$

where $h_\lambda(t_2,t_3;t)$ is the Rogers–Szegő polynomial (2.3.5). Moreover, if we replace $(t,t_2,t_3)\mapsto(0,-a,-b)$ and use (2.8.5) and $H_m(z;0)=1+z+\cdots+z^m$, we obtain the following two-parameter generalisation of the Désarménien–Proctor–Stembridge determinant (4.1.6):

$$\begin{aligned}\sum_{\substack{\lambda \\ \lambda_1\leqslant 2m}} s_\lambda(x)\prod_{\substack{i=1\\ i \text{ odd}}}^{2m-1}\frac{a^{m_i(\lambda)+1}-b^{m_i(\lambda)+1}}{a-b}\prod_{\substack{i=1\\ i \text{ even}}}^{2m-1}\frac{1-(ab)^{m_i(\lambda)+1}}{1-ab}\\ =\frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}(1+ax_i)(1+bx_i)-x_i^{2m+2n-j-1}(x_i+a)(x_i+b)\big)}{\prod_{i=1}^n(1-x_i^2)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)}.\end{aligned}$$

Recall (2.3.6). The $t_3=-1$ case of Theorem 4.7 extends as follows.

THEOREM 4.8. *For $x=(x_1,\dots,x_n)$ and m a nonnegative integer,*

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant m}} h_\lambda^{(m)}(t_2;t)P_\lambda(x;t)=(x_1\cdots x_n)^{\frac{m}{2}}P_{(\frac{m}{2})^n}^{(\mathrm{B}_n)}(x;t,t_2). \tag{4.1.17}$$

This is stated without proof in [**139**]. For $(t,t_2)\mapsto(0,-a)$ it simplifies to a one-parameter generalisation of Macdonald's determinant (1.1.2) from the introduction:

$$\sum_{\substack{\lambda \\ \lambda_1\leqslant m}} s_\lambda(x)\prod_{i=1}^{m-1}\frac{1-a^{m_i(\lambda)+1}}{1-a}=\frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}(1+ax_i)-x_i^{m+2n-j-1}(x_i+a)\big)}{\prod_{i=1}^n(1-x_i)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)}.$$

4.2. Proofs of Theorems 4.1–4.8

We begin by outlining the general strategy, which is to transform the problem of proving bounded Littlewood identities into that of evaluating virtual Koornwinder integrals.

Recall that if $g\in\Lambda_n$ and $\{f_\lambda\}$ is a basis of Λ_n, then $[f_\lambda]g$ is the coefficient of f_λ in the expansion of g. Working in full generality, we would like to find a closed-form expression for

$$f_\lambda^{(m)}(q,t;t_0,t_1,t_2,t_3):=\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m K_{m^n}(x;q,t;t_0,t_1,t_2,t_3), \tag{4.2.1}$$

where m is a nonnegative integer. Since

$$(x_1\cdots x_n)^m K_{m^n}(x;q,t;t_0,t_1,t_2,t_3)=\sum_{\lambda\subset(2m)^n}u_\lambda\, m_\lambda(x),$$

it follows that $f^{(m)}_\lambda(q,t;t_0,t_1,t_2,t_3)$ vanishes unless $\lambda\subset(2m)^n$.

PROPOSITION 4.9. *For m a nonnegative integer and $\lambda\subset(2m)^n$,*

$$f^{(m)}_\lambda(q,t;t_0,t_1,t_2,t_3)=(-1)^{|\lambda|}I^{(m)}_K\big(P_{\lambda'}(t,q);t,q;t_0,t_1,t_2,t_3\big). \tag{4.2.2}$$

PROOF. Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_m)$. According to the Cauchy identity for Koornwinder polynomials (2.6.5)

$$\sum_{\lambda\subset m^n}(-1)^{|\lambda|}(x_1\cdots x_n)^m K_{m^n-\lambda}(x;q,t;t_0,t_1,t_2,t_3)K_{\lambda'}(y;t,q;t_0,t_1,t_2,t_3) = \sigma_1\big[-xy^{\pm}\big]. \tag{4.2.3}$$

If we expand the right-hand side in terms of Macdonald polynomials using the Cauchy identity (2.5.5), this yields

$$\sum_{\lambda\subset m^n}(-1)^{|\lambda|}(x_1\cdots x_n)^m K_{m^n-\lambda}(x;q,t;t_0,t_1,t_2,t_3)K_{\lambda'}(y;t,q;t_0,t_1,t_2,t_3) = \sum_{\lambda\subset(2m)^n}(-1)^{|\lambda|}P_\lambda(x;q,t)P_{\lambda'}(y^{\pm};t,q).$$

Equating coefficients of $P_\lambda(x;q,t)K_0(y;t,q;t_0,t_1,t_2,t_3)$, we find

$$[P_\lambda(x;q,t)](x_1\cdots x_n)^m K_{m^n}(x;q,t;t_0,t_1,t_2,t_3) = (-1)^{|\lambda|}[K_0(y;t,q;t_0,t_1,t_2,t_3)]P_{\lambda'}(y^{\pm};t,q),$$

for $\lambda\subset(2m)^n$. Recalling (3.1.2) and (4.2.1) completes the proof. ☐

Next we consider the problem of computing

$$f^{(m)}_\lambda(q,t;t_2,t_3):=\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m K_{m^n}(x;q,t;t_2,t_3), \tag{4.2.4}$$

where m is a nonnegative integer or half-integer and $K_\lambda(x;q,t;t_2,t_3)$ is the Macdonald–Koornwinder polynomial of Section 2.7.2.

PROPOSITION 4.10. *For m a nonnegative integer or half-integer, $\lambda\subset(2m)^n$ and generic q,t,t_2,t_3*

$$f^{(m)}_\lambda(q,t;t_2,t_3)=(-1)^{|\lambda|}I_K\big(P_{\lambda'}(t,q);t,q,q^m;-1,-q^{1/2},t_2,t_3\big). \tag{4.2.5}$$

PROOF. When m is an integer we simply have

$$f^{(m)}_\lambda(q,t;t_2,t_3)=f^{(m)}_\lambda(q,t;-1,-q^{1/2},t_2,t_3).$$

By (4.2.2) this gives

$$f^{(m)}_\lambda(q,t;t_2,t_3)=(-1)^{|\lambda|}I^{(m)}_K\big(P_{\lambda'}(t,q);t,q;-1,-q^{1/2},t_2,t_3\big),$$

which may also be written as (4.2.5).

To deal with the half-integer case we set $k=m-1/2$ and replace m by k in (4.2.3), so that now $y=(y_1,\dots,y_k)$. Multiplying both sides by $\prod_{i=1}^n(1+x_i)=\sigma_1[-\varepsilon x]$, using that

$$\sigma_1[-\varepsilon x]\sigma_1[-xy^{\pm}]=\sigma_1[-\varepsilon x-xy^{\pm}]=\sigma_1[-x(y^{\pm}+\varepsilon)],$$

and finally expanding this by the Cauchy identity (2.5.5), we obtain

$$\begin{aligned}\prod_{i=1}^{n}&(1+x_i)\\ &\times\sum_{\lambda\subset k^n}(-1)^{|\lambda|}(x_1\cdots x_n)^k K_{k^n-\lambda}(x;q,t;t_0,t_1,t_2,t_3)K_{\lambda'}(y;t,q;t_0,t_1,t_2,t_3)\\ &\qquad=\sum_{\lambda\subset(2m)^n}(-1)^{|\lambda|}P_\lambda(x;q,t)P_{\lambda'}\big([y^{\pm}+\varepsilon];t,q\big).\end{aligned}$$

After specialising $\{t_0,t_1\}=\{-q,-q^{1/2}\}$ we can apply Lemma 2.3 to rewrite this as

$$\begin{aligned}\sum_{\lambda\subset k^n}&(-1)^{|\lambda|}(x_1\cdots x_n)^m K_{m^n-\lambda}(x;q,t;t_2,t_3)K_{\lambda'}(y;t,q;-q,-q^{1/2},t_2,t_3)\\ &\qquad=\sum_{\lambda\subset(2m)^n}(-1)^{|\lambda|}P_\lambda(x;q,t)P_{\lambda'}\big([y^{\pm}+\varepsilon];t,q\big).\end{aligned}$$

Equating coefficients of $P_\lambda(x;q,t)K_0(y;t,q;-q,-q^{1/2},t_2,t_3)$ yields

$$f_\lambda^{(m)}(q,t;t_2,t_3)=(-1)^{|\lambda|}I_K^{(k)}\big(P_{\lambda'}([y+\varepsilon];t,q);t,q;-q,-q^{1/2},t_2,t_3\big) \tag{4.2.6}$$

for $\lambda\subset(2m)^n$. For generic q,t,t_2,t_3 we can write the integral on the right as

$$I_K\big(P_{\lambda'}([y^{\pm}+\varepsilon];t,q);t,q,q^k;-q,-q^{1/2},t_2,t_3\big),$$

where now $y=(y_1,y_2,\dots)$. By Lemma 3.4 this is also

$$I_K\big(P_{\lambda'}(t,q);t,q,q^m;-1,-q^{1/2},t_2,t_3\big). \qquad\square$$

We are now ready to prove Theorems 4.1–4.8.

Proof of Theorem 4.1. We first prove the $a=0$ case, given in (4.1.5). By definition (4.2.1) and equation (2.7.9), this is equivalent to proving that for $\lambda\subset(2m)^n$

$$f_\lambda^{(m)}\big(q,t;\pm q^{1/2},\pm(qt)^{1/2}\big)=\begin{cases}b^{\mathrm{oa}}_{\lambda;m}(q,t) & \lambda\text{ is even,}\\ 0 & \text{otherwise.}\end{cases}$$

From (4.2.2) we have

$$f_\lambda^{(m)}\big(q,t;\pm q^{1/2},\pm(qt)^{1/2}\big)=(-1)^{|\lambda|}I_K^{(m)}\big(P_{\lambda'}(t,q);t,q;\pm q^{1/2},\pm(qt)^{1/2}\big).$$

Taking $(T,\mu)=(t^m,\lambda')$ in (3.2.1) and then interchanging q and t, it follows that the virtual Koornwinder integral on the right vanishes unless λ is even. Moreover, for even λ it evaluates in closed form to

$$\frac{(q^{2m};t,q^2)_{(\lambda/2)'}}{(q^{2m-1}t;t,q^2)_{(\lambda/2)'}}\cdot\frac{C^-_{(\lambda/2)'}(qt;t,q^2)}{C^-_{(\lambda/2)'}(q^2;t,q^2)}.$$

Using (2.2.6), we thus find

$$f_\lambda^{(m)}\big(q,t;\pm q^{1/2},\pm(qt)^{1/2}\big)=\Big(\frac{q}{t}\Big)^{|\lambda|/2}\frac{(q^{-2m};q^2,t)_{\lambda/2}}{(q^{1-2m}/t;q^2,t)_{\lambda/2}}\cdot\frac{C^-_{\lambda/2}(qt;q^2,t)}{C^-_{\lambda/2}(q^2;q^2,t)},$$

for λ even and zero otherwise. By (2.2.2) this can be also be written as

$$\begin{aligned}
(4.2.7)\qquad & f_\lambda^{(m)}\big(q,t;\pm q^{1/2},\pm(qt)^{1/2}\big) \\
&= \prod_{s\in\lambda/2}\bigg(\frac{1-q^{2m-2a'(s)}t^{l'(s)}}{1-q^{2m-2a'(s)-1}t^{l'(s)+1}}\cdot\frac{1-q^{2a(s)+1}t^{l(s)+1}}{1-q^{2a(s)+2}t^{l(s)}}\bigg) \\
&= \prod_{\substack{s\in\lambda\\ a(s)\text{ odd}}}\bigg(\frac{1-q^{2m-a'(s)}t^{l'(s)}}{1-q^{2m-a'(s)-1}t^{l'(s)+1}}\cdot\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}\bigg).
\end{aligned}$$

Since λ is even, odd arms-lengths correspond to even arm-colengths. The product on the right is thus $b^{\mathrm{oa}}_{\lambda;m}(q,t)$ in the representation given by (4.1.4), completing the proof of (4.1.5).

To obtain the full theorem we multiply both sides of (4.1.5) by $\prod_{i=1}^n(1+ax_i)$. By the e-Pieri rule (2.5.8) we must then show that

$$b^{\mathrm{oa}}_{\lambda;m}(q,t)=\sum_{\mu\text{ even}} a^{|\lambda/\mu|}\psi'_{\lambda/\mu}(q,t)\,b^{\mathrm{oa}}_{\mu;m}(q,t). \tag{4.2.8}$$

Because μ is even and $\psi'_{\lambda/\mu}(q,t)$ vanishes unless λ/μ is a vertical strip, μ is fixed as

$$\mu=2\lfloor\lambda/2\rfloor:=(2\lfloor\lambda_1/2\rfloor,2\lfloor\lambda_2/2\rfloor,\dots), \tag{4.2.9}$$

which implies that $|\lambda/\mu|=\mathrm{odd}(\lambda)$. We thus obtain

$$b^{\mathrm{oa}}_{\lambda;m}(q,t)=\psi'_{\lambda/\mu}(q,t)\,b^{\mathrm{oa}}_{\mu;m}(q,t)$$

with μ fixed as above. The m-dependent parts on both sides trivially agree since

$$\prod_{\substack{s\in\lambda\\ a'(s)\text{ odd}}} f_{a'(s),l'(s)}=\prod_{\substack{s\in\mu\\ a'(s)\text{ odd}}} f_{a'(s),l'(s)}.$$

It thus remains to show that

$$b^{\mathrm{oa}}_{\lambda}(q,t)=\psi'_{\lambda/\mu}(q,t)\,b^{\mathrm{oa}}_{\mu}(q,t).$$

Replacing (λ,μ,q,t) by (λ',μ',t,q), using

$$b^{\mathrm{oa}}_{\nu'}(t,q)=\frac{b^{\mathrm{el}}_{\nu}(q,t)}{b_\nu(q,t)}$$

on both sides, and finally appealing to [**91**, page 341]

$$\psi'_{\lambda'/\mu'}(t,q)=\varphi_{\lambda/\mu}(q,t)\,\frac{b_\mu(q,t)}{b_\lambda(q,t)},$$

we are left with

$$b^{\mathrm{el}}_{\lambda}(q,t)=\varphi_{\lambda/\mu}(q,t)b^{\mathrm{el}}_{\mu}(q,t)$$

for $\mu'=2\lfloor\lambda'/2\rfloor$. Since this is [**91**, p. 351], we are done. $\square$

Because they are simpler to prove than Theorems 4.2 and 4.3, we consider Theorems 4.4 and 4.6 first.

Proof of Theorem 4.4. It will be convenient to prove the claim with m replaced by $2m$. After this change m is a nonnegative integer or half-integer. It then follows from (4.2.4) and (2.7.12a) that we must prove for $\lambda \subset (2m)^n$ that

$$f_\lambda^{(m)}(q,t;t,q^{1/2}) = b^{\mathrm{el}}_{\lambda;2m}(q,t).$$

By Proposition 4.10,

$$f_\lambda^{(m)}(q,t;t,q^{1/2}) = (-1)^{|\lambda|} I_K\big(P_{\lambda'}(t,q);t,q,q^m;-1,t,\pm q^{1/2}\big).$$

The integral on the right can be computed by Theorem 3.10 with $(q,t,T,\mu)\mapsto(t,q,q^m,\lambda')$, resulting in

$$f_\lambda^{(m)}(q,t;t,q^{1/2}) = \frac{(q^{2m};t^2,q)_{\lceil\lambda'/2\rceil}}{(q^{2m-1}t;t^2,q)_{\lceil\lambda'/2\rceil}}\cdot\frac{1}{b^{\mathrm{ea}}_{\lambda'}(t,q)}.$$

Let $\nu := \lceil\lambda'/2\rceil' = (\lambda_1,\lambda_3,\dots)$. By (2.2.6a) we can write the first factor on the right as

$$\Big(\frac{q}{t}\Big)^{|\nu|}\frac{(q^{-2m};q,t^2)_\nu}{(q^{1-2m}/t;q,t^2)_\nu}.$$

By (2.2.2) this is also

$$\prod_{s\in\nu}\frac{1-q^{2m-a'(s)}t^{2l'(s)}}{1-q^{2m-a'(s)-1}t^{2l'(s)+1}} = \prod_{\substack{s\in\lambda\\ l'(s)\text{ even}}}\frac{1-q^{2m-a'(s)}t^{l'(s)}}{1-q^{2m-a'(s)-1}t^{l'(s)+1}}.$$

Since under conjugation legs become arms and arms become legs, we further have

$$b^{\mathrm{ea}}_{\lambda'}(t,q)\,b^{\mathrm{el}}_{\lambda}(q,t)=1.$$

Hence

$$f_\lambda^{(m)}(q,t;t,q^{1/2}) = b^{\mathrm{el}}_\lambda(q,t)\prod_{\substack{s\in\lambda\\ l'(s)\text{ even}}}\frac{1-q^{2m-a'(s)}t^{l'(s)}}{1-q^{2m-a'(s)-1}t^{l'(s)+1}} = b^{\mathrm{el}}_{\lambda;2m}(q,t)$$

as claimed. □

Proof of Theorem 4.6. We closely follow the previous proof and again replace m by $2m$. This time it follows from (4.2.4) and (2.7.12b) that we must prove

$$f_\lambda^{(m)}\big(q^2,t^2;-t,-qt\big) = b^{-}_{\lambda;2m}(q,t) \tag{4.2.10}$$

for $\lambda\subset(2m)^n$. By Proposition 4.10,

$$f_\lambda^{(m)}\big(q^2,t^2;-t,-qt\big) = (-1)^{|\lambda|} I_K\big(P_{\lambda'}(t^2,q^2);t^2,q^2,q^{2m};-1,-q,-t,-qt\big).$$

The integral on the right evaluates to

$$(-1)^{|\lambda|}\frac{(q^{2m};t,q)_{\lambda'}}{(-q^{2m-1}t;t,q)_{\lambda'}}\cdot\frac{C^-_{\lambda'}(-t;t,q)}{C^-_{\lambda'}(q;t,q)}.$$

by (3.2.2) with $(q,t,T,\mu)\mapsto(t^2,q^2,t^{2m},\lambda')$. Also using (2.2.6), we find

$$f_\lambda^{(m)}\big(q^2,t^2;-t,-qt\big) = \Big(-\frac{q}{t}\Big)^{|\lambda|}\frac{(q^{-2m};q,t)_\lambda}{(-q^{1-2m}/t;q,t)_\lambda}\cdot\frac{C^-_\lambda(-t;q,t)}{C^-_\lambda(q;q,t)}.$$

Equation (4.2.10) now follows by (2.2.2). □

Proof of Theorem 4.2. Again we prove the theorem with m replaced by $2m$. It then follows from (4.2.4) and (2.7.12a) that we must prove for $\lambda \subset (2m)^n$ that $f_\lambda^{(m)}\bigl(q,t;1,q^{1/2}\bigr)$ vanishes unless $m_i(\lambda)$ is even for all $1 \leqslant i \leqslant 2m-1$, in which case

$$f_\lambda^{(m)}\bigl(q,t;1,q^{1/2}\bigr) = b^{\mathrm{ol}}_{\lambda;2m}(q,t). \tag{4.2.11}$$

The problem with using Proposition 4.10 as in the proof of Theorem 4.6 is that the specialisation $\{t_2,t_3\} = \{1,q^{1/2}\}$ corresponds to one of the non-generic cases discussed on page 30. It would lead to

$$f_\lambda^{(m)}\bigl(q,t;1,q^{1/2}\bigr) = (-1)^{|\lambda|} I_K\bigl(P_{\lambda'}(t,q);t,q,q^m;\pm 1,\pm q^{1/2}\bigr), \tag{4.2.12}$$

where the integral on the right is not well-defined. It is still possible to use (4.2.12) by interpreting the right in an appropriate limiting sense, but instead we proceed slightly differently.

First, when m is an integer (4.2.5) simply says that

$$f_\lambda^{(m)}(q,t;t_2,t_3) = (-1)^{|\lambda|} I_K^{(m)}\bigl(P_{\lambda'}(t,q);t,q;-1,-q^{1/2},t_2,t_3\bigr).$$

In this equation there is no problem specialising $\{t_2,t_3\} = \{1,q^{1/2}\}$ so that

$$f_\lambda^{(m)}(q,t;1,q^{1/2}) = (-1)^{|\lambda|} I_K^{(m)}\bigl(P_{\lambda'}(t,q);t,q;\pm 1,\pm q^{1/2}\bigr).$$

The right-hand side can be computed by Theorem 3.8 with $(\mu,q,t,n) \mapsto (\lambda',t,q,m)$ so that

$$f_\lambda^{(m)}(q,t;1,q^{1/2}) = \begin{cases} A^{(2m)}_{\lambda'/2}(t,q) & \text{if } \widetilde{\lambda}' \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.13}$$

where we have also used that $|\lambda|$ is even if $\widetilde{\lambda}' = (\lambda'_1 - \lambda'_{2m},\dots,\lambda'_{2m-1}-\lambda'_{2m},0)$ is even.

When $m = k+1/2$ is a half-integer, we use (4.2.6) written as

$$f_\lambda^{(m)}(q,t;t_2,t_3) = (-1)^{|\lambda|} I_K^{(k)}\bigl(P_{\lambda'}(x_1,\dots,x_k,-1;t,q);t,q;-q,-q^{1/2},t_2,t_3\bigr) \tag{4.2.14}$$

instead of (4.2.5). Specialising $\{t_2,t_3\} = \{1,q^{1/2}\}$ this gives

$$f_\lambda^{(m)}(q,t;1,q^{1/2}) = (-1)^{|\lambda|} I_K^{(k)}\bigl(P_{\lambda'}(x_1,\dots,x_k,-1;t,q);t,q;1,-q,\pm q^{1/2}\bigr).$$

Now the right can be computed by Theorem 3.9 with $(\nu,q,t,n) \mapsto (\lambda',t,q,k)$. Since $2k+1 = 2m$ and $|\lambda| + \lambda_{2m}$ is even if $\widetilde{\lambda}'$ is even, this once again results in (4.2.13).

To complete the proof we first note that

$$A^{(2m)}_{\lambda'/2}(t,q) = A^{(2m)}_{\lfloor \lambda'/2 \rfloor}(t,q).$$

Indeed, either λ' is even, in which case $\lfloor \lambda'/2 \rfloor = \lambda'/2$ or λ' is odd and $\lambda_1 = 2m$, in which case $\lfloor \lambda'/2 \rfloor = \lambda'/2 - (1/2)^{2m}$. Since $A^{(2m)}_{\lambda'/2}(t,q)$ only depends on the relative differences between the parts of $\lambda'/2$ the change is justified. Denoting $\lfloor \lambda'/2 \rfloor$ by ν' we find that in the non-vanishing case, that is, when $m_i(\lambda)$ is even for all $1 \leqslant i \leqslant 2m-1$,

$$\begin{aligned} f_\lambda^{(m)}(q,t;1,q^{1/2}) &= A_{\nu'}^{(2m)}(t,q) \\ &= \frac{(q^{2m};t^2,q)_{\nu'}}{(q^{2m-1}t;t^2,q)_{\nu'}} \cdot \frac{C^-_{\nu'}(t;t^2,q)}{C^-_{\nu'}(q;t^2,q)} \\ &= \Big(\frac{q}{t}\Big)^{|\nu|} \frac{(q^{-2m};q,t^2)_\nu}{(q^{1-2m}/t;q,t^2)_\nu} \cdot \frac{C^-_\nu(t;q,t^2)}{C^-_\nu(q;q,t^2)}, \end{aligned}$$

where the second equality follows from (3.2.5) and the last equality from (2.2.6). Since $\nu' = \lfloor \lambda'/2 \rfloor$ we also have $\nu = \lfloor \lambda'/2 \rfloor'$, which can be simplified to $\nu = (\lambda_2, \lambda_4, \lambda_6, \dots)$. Recalling (2.2.2), we obtain

$$f_\lambda^{(m)}(q,t;1,q^{1/2}) = \prod_{s\in\nu} \bigg(\frac{1-q^{2m-a'(s)}t^{2l'(s)}}{1-q^{2m-a'(s)-1}t^{2l'(s)+1}} \cdot \frac{1-q^{a(s)}t^{2l(s)+1}}{1-q^{a(s)+1}t^{2l(s)}} \bigg).$$

To write this without reference to the partition ν we consider both factors in the product separately. The first factor is trivial:

$$\prod_{s\in\nu} \frac{1-q^{2m-a'(s)}t^{2l'(s)}}{1-q^{2m-a'(s)-1}t^{2l'(s)+1}} = \prod_{\substack{s\in\lambda \\ l'(s)\text{ odd}}} \frac{1-q^{2m-a'(s)}t^{l'(s)-1}}{1-q^{2m-a'(s)-1}t^{l'(s)}}. \tag{4.2.15}$$

For the second factor we use that for λ' even we must have $\lambda_{2i} = \lambda_{2i-1}$ for all i. We can therefore redefine ν as

$$\nu := \begin{cases} (\lambda_1, \lambda_3, \dots) & \text{if } \lambda' \text{ is even} \\ (\lambda_2, \lambda_4, \dots) & \text{if } \lambda' \text{ is odd.} \end{cases}$$

For such ν,

$$\prod_{s\in\nu} \frac{1-q^{a(s)}t^{2l(s)+1}}{1-q^{a(s)+1}t^{2l(s)}} = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)}}{1-q^{a(s)+1}t^{l(s)}} \tag{4.2.16}$$

in both cases. Combining (4.2.15) and (4.2.16) we obtain (4.2.11). □

Proof of Theorem 4.3. When m is odd the result is completely elementary. By (2.7.7b) and (2.7.8) we can write the right-hand side of (4.1.8) as

$$(x_1\cdots x_n)^{\frac{m}{2}} P^{(\mathrm{D}_n,\mathrm{D}_n)}_{(\frac{m}{2})^n}(x;q,t) + (x_1\cdots x_n)^{\frac{m}{2}} P^{(\mathrm{D}_n,\mathrm{D}_n)}_{(\frac{m}{2})^n}(\bar{x};q,t).$$

When m is odd the first term is a polynomial of even degree whereas the second term is a polynomial of odd degree. Since partitions λ of Type 1 have even size and partitions of Type 2 have size congruent to m modulo 2, it follows that for odd m we may dissect (4.1.8) as in Corollary 4.3.

To prove the theorem for even m we closely follow the proof of Proposition 4.10. In (4.2.3) we replace m by $m-1 =: k$ (we do not at this point assume that m is even) and multiply both sides by $\prod_{i=1}^n (1-x_i^2) = \sigma_1[-x(1+\varepsilon)]$. Then expanding the right-hand side in terms of Macdonald polynomials using (2.5.5) gives

$$\begin{aligned}\prod_{i=1}^n(1-x_i^2)&\\ \times\sum_{\lambda\subset k^n}(-1)^{|\lambda|}(x_1\cdots x_n)^k K_{k^n-\lambda}(x;q,t;t_0,t_1,t_2,t_3)K_{\lambda'}(y;t,q;t_0,t_1,t_2,t_3)&\\ =\sum_{\lambda\subset(2m)^n}(-1)^{|\lambda|}P_\lambda(x;q,t)P_{\lambda'}\big([y^\pm+1+\varepsilon];t,q\big),&\end{aligned}$$

where $y=(y_1,\dots,y_k)$. If we specialise $\{t_0,t_1,t_2,t_3\}=\{\pm q,\pm q^{1/2}\}$ and apply Lemma 2.3 followed by (2.7.12b), this leads to

$$\begin{aligned}\prod_{i=1}^n\big(x_i^{-1/2}-x_i^{1/2}\big)&\\ \times\sum_{\lambda\subset k^n}(-1)^{|\lambda|}(x_1\cdots x_n)^m P^{(\mathrm{B}_n,\mathrm{C}_n)}_{(m-\frac12)^n-\lambda}(x;q,t,q^{1/2})K_{\lambda'}(y;t,q;\pm q,\pm q^{1/2})&\\ =\sum_{\lambda\subset(2m)^n}(-1)^{|\lambda|}P_\lambda(x;q,t)P_{\lambda'}\big([y^\pm+1+\varepsilon];t,q\big).&\end{aligned}$$

Equating coefficients of $P_\lambda(x;q,t)K_0(y;t,q;\pm q,\pm q^{1/2})$ and then replacing y by x on the right yields

$$\begin{aligned}\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m P^{(\mathrm{B}_n,\mathrm{C}_n)}_{(m-\frac12)^n}(x;q,t,q^{1/2})\prod_{i=1}^n\big(x_i^{-1/2}-x_i^{1/2}\big)&\\ =(-1)^{|\lambda|}I_K^{(m-1)}\big(P_{\lambda'}(x_1^\pm,\dots,x_{m-1}^\pm,\pm1;t,q);t,q;\pm q,\pm q^{1/2}\big).&\end{aligned}\tag{4.2.17}$$

By the integer-m case of Proposition 4.10,

$$\begin{aligned}\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m K_{m^n}(x;q,t;t_2,t_3)&\\ =(-1)^{|\lambda|}I_K^{(m)}\big(P_{\lambda'}(t,q);t,q;-1,-q^{1/2},t_2,t_3\big).&\end{aligned}$$

For $\{t_2,t_3\}=\{1,q^{1/2}\}$ this can also be written as

$$\begin{aligned}\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m P^{(\mathrm{B}_n,\mathrm{B}_n)}_{m^n}(x;q,t,1)&\\ =(-1)^{|\lambda|}I_K^{(m)}\big(P_{\lambda'}(t,q);t,q;\pm1,\pm q^{1/2}\big)&\end{aligned}\tag{4.2.18}$$

thanks to (2.7.12a). Taking half the sum of (4.2.17) and (4.2.18) and recalling (2.7.7), it follows that

$$\begin{aligned}\big[P_\lambda(x;q,t)\big]&(x_1\cdots x_n)^m P^{(\mathrm{D}_n,\mathrm{D}_n)}_{m^n}(x;q,t)\\ &=\tfrac12(-1)^{|\lambda|}I_K^{(m)}\big(P_{\lambda'}(x_1^\pm,\dots,x_m^\pm;t,q);t,q;\pm1,\pm q^{1/2}\big)\\ &\quad+\tfrac12(-1)^{|\lambda|}I_K^{(m-1)}\big(P_{\lambda'}(x_1^\pm,\dots,x_{m-1}^\pm,\pm1;t,q);t,q;\pm q,\pm q^{1/2}\big).\end{aligned}\tag{4.2.19}$$

Both virtual Koornwinder integrals on the right can be computed by Theorem 3.8. Since

$$\chi\big(\widetilde{\lambda}'\text{ even}\big)\big(\tfrac12+\tfrac12(-1)^{\lambda'_{2m}}\big)=\chi(\lambda'\text{ even})$$

for $\widetilde{\lambda}'=(\lambda'_1-\lambda'_{2m},\dots,\lambda'_{2m-1}-\lambda'_{2m},0)$, we find

$$\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m P^{(\mathrm{D}_n,\mathrm{D}_n)}_{m^n}(x;q,t)=\chi(\lambda'\text{ even})A_{\lambda'/2}(t,q).$$

In the proof of Theorem 4.2 we have already shown that

$$A_{\lambda'/2}(t,q)=b^{\mathrm{ol}}_{\lambda;2m}(q,t)$$

for λ' even (or λ' odd and $\lambda_1=2m$). Hence

$$\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m P^{(\mathrm{D}_n,\mathrm{D}_n)}_{m^n}(x;q,t)=\chi(\lambda'\text{ even})b^{\mathrm{ol}}_{\lambda;2m}(q,t).$$

Replacing m by $m/2$, this proves (4.1.9a) for even m.

For completeness we remark that the analogue of (4.2.19) for half-integer m is easily shown to be

$$\begin{aligned}\big[P_\lambda(x;q,t)\big]&(x_1\cdots x_n)^m P^{(\mathrm{D}_n,\mathrm{D}_n)}_{m^n}(x;q,t)\\&=\tfrac{1}{2}(-1)^{|\lambda|}I^{(k)}_K\big(P_{\lambda'}(x_1^\pm,\dots,x_k^\pm,-1;t,q);t,q;1,-q,\pm q^{1/2}\big)\\&\quad+\tfrac{1}{2}(-1)^{|\lambda|}I^{(k)}_K\big(P_{\lambda'}(x_1^\pm,\dots,x_k^\pm,1;t,q);t,q;-1,q,\pm q^{1/2}\big),\end{aligned}$$

where $k=m-1/2$. By Theorem 3.9 this again implies that

$$\big[P_\lambda(x;q,t)\big](x_1\cdots x_n)^m P^{(\mathrm{D}_n,\mathrm{D}_n)}_{m^n}(x;q,t)=\chi(\lambda'\text{ even})b^{\mathrm{ol}}_{\lambda;2m}(q,t).$$

Of course, as noted above, this result follows more simply by a degree argument. □

Proof of Theorem 4.7. By (2.8.4) and (4.2.1) we must prove that

$$f^{(m)}_\lambda(0,t;0,0,t_2,t_3)=h^{(2m)}_\lambda(t_2,t_3;t) \tag{4.2.20}$$

for $\lambda\subset(2m)^n$. By (4.2.2),

$$f^{(m)}_\lambda(0,t;0,0,t_2,t_3)=(-1)^{|\lambda|}I^{(m)}_K\big(P_{\lambda'}(t,0);t,0;0,0,t_2,t_3\big). \tag{4.2.21}$$

The integral on the right can be evaluated thanks to Theorem 3.11 with (n,q,μ) replaced by (m,t,λ'). Hence

$$f^{(m)}_\lambda(0,t;0,0,t_2,t_3)=(-1)^{|\lambda|}h_\lambda(-t_2,-t_3;t)=h^{(2m)}_\lambda(t_2,t_3;t),$$

where the second equality follows from definition (2.3.4) and

$$(-1)^{|\lambda|}=\prod_{\substack{i\geqslant 1\\ i\text{ odd}}}(-1)^{m_i(\lambda)}.$$ □

A proof of (4.1.16) (the large-m limit of Theorem 4.7) using virtual Koornwinder integrals is due to Venkateswaran [**133**]. Her approach, however, is not a limiting version of ours. Crucial difference is that Venkateswaran stays within the t-world, whereas we have applied the virtual Koornwinder integral (3.3.1) over $P_\lambda(0,q)$.

Proof of Theorem 4.8. As in earlier proofs we replace m by $2m$. From (2.8.11) and (4.2.4) it follows that we must prove

$$f^{(m)}_\lambda(0,t;t_2,0)=h^{(2m)}_\lambda(t_2;t) \tag{4.2.22}$$

for $\lambda\subset(2m)^n$ and m a nonnegative integer or half-integer. For m an integer, (4.2.22) is the $t_3=-1$ case of (4.2.20), and in the remainder we assume m is a half-integer.

We will not apply Proposition 4.10 as it is not suitable for taking the $q\to 0$ limit. Instead we take that limit in (4.2.14). Then

$$f^{(m)}_\lambda(0,t;t_2,t_3)=(-1)^{|\lambda|}I^{(k)}_K\big(P_{\lambda'}(x_1^\pm,\dots,x_k^\pm,-1;t,0);t,0;0,0,t_2,t_3\big),$$

where $k=m-1/2$. By the branching rule (2.5.13) and relation (2.5.14) this becomes

$$f_\lambda^{(m)}(0,t;t_2,t_3)=\sum_{\mu\subset\lambda}(-1)^{|\mu|}\psi'_{\lambda/\mu}(0,t)I_K^{(k)}\big(P_{\mu'}(t,0);t,0;0,0,t_2,t_3\big).$$

The virtual Koornwinder integral on the right can be computed by Theorem 3.11 with (n,q,μ) replaced by (k,t,μ'). Hence

$$f_\lambda^{(m)}(0,t;t_2,t_3)=\sum_{\substack{\mu\subset\lambda\\ \mu_1\leqslant 2k}}(-1)^{|\mu|}\psi'_{\lambda/\mu}(0,t)\,h_\mu^{(2k)}(-t_2,-t_3;t).$$

We do not know how to evaluate this in closed form for arbitrary t_3, but for $t_3=0$ it follows from (2.3.4) that

$$h_\mu^{(2k)}(-t_2,0;t)=t_2^{\operatorname{odd}(\mu)}.$$

Also using that $(-1)^{|\mu|}=(-1)^{\operatorname{odd}(\mu)}$, we find

$$f_\lambda^{(m)}(0,t;t_2,0)=\sum_{\substack{\mu\subset\lambda\\ \mu_1\leqslant 2m-1}}(-t_2)^{\operatorname{odd}(\mu)}\psi'_{\lambda/\mu}(0,t).$$

Since $\psi'_{\lambda/\mu}(0,t)$ is the e-Pieri coefficient for ordinary Hall–Littlewood polynomials, we have [**91**, page 215]

$$\psi'_{\lambda/\mu}(0,t)=\prod_{i\geqslant 1}\begin{bmatrix}\lambda'_i-\lambda'_{i+1}\\ \lambda'_i-\mu'_i\end{bmatrix}_t.$$

Therefore

$$f_\lambda^{(m)}(0,t;t_2,0)=\sum_{\substack{\mu\subset\lambda\\ \mu_1\leqslant 2m-1}}(-t_2)^{\operatorname{odd}(\mu)}\prod_{i=1}^{2m-1}\begin{bmatrix}\lambda'_i-\lambda'_{i+1}\\ \lambda'_i-\mu'_i\end{bmatrix}_t.$$

Writing $\mu'_i=\lambda'_i-k_i$ and using (3.3.6) with $(\mu,\nu,n)\mapsto(\mu',\lambda',m)$, we finally obtain

$$\begin{aligned}f_\lambda^{(m)}(0,t;t_2,0)&=(-t_2)^{\operatorname{odd}(\lambda)}\prod_{i=1}^{2m-1}\sum_{k_i=0}^{m_i(\lambda)}(-t_2)^{(-1)^ik_i}\begin{bmatrix}m_i(\lambda)\\ k_i\end{bmatrix}_t\\&=(-t_2)^{\operatorname{odd}(\lambda)}\prod_{\substack{i=1\\ i\text{ odd}}}^{2m-1}H_{m_i(\lambda)}(-1/t_2;t)\prod_{\substack{i=1\\ i\text{ even}}}^{2m-1}H_{m_i(\lambda)}(-t_2;t)\\&=h_\lambda^{(2m)}(t_2,-1;t)=h_\lambda^{(2m)}(t_2;t).\end{aligned}$$

□

CHAPTER 5

Applications

5.1. Plane partitions

In this section we will show that our approach to bounded Littlewood identities provides an intimate connection between symmetric plane partitions and the theory of Gelfand pairs.

A plane partition π of shape λ is a filling of the Young diagram of λ with positive integers, called the parts of π, such that the resulting tableau is weakly decreasing along rows and columns, see e.g., [**16**,**68**,**124**,**125**] and references therein. The size of the plane partition π, denoted $|\pi|$, is the sum of the parts, that is

$$|\pi| = \sum_{(i,j)\in\lambda} \pi_{ij},$$

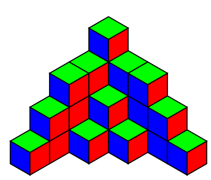
where the part π_{ij} is the filling of the square $(i,j) \in \lambda$. For example,

$$\begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 3 & 2 & 1 \\ \hline 3 & 2 & 1 \\ \cline{1-3} 3 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \tag{5.1.1a}$$

is a plane partition of shape $(5,3,2,1,1)$ and size 26.

A part of size r may be thought of as the stacking of r unit cubes, providing a geometric interpretation of plane partitions. Thus, the plane partition (5.1.1a) corresponds to

(5.1.1b)



A plane partition π of shape λ is symmetric if $\lambda = \lambda'$ and $\pi_{ij} = \pi_{ji}$ for all $(i,j) \in \lambda$. Clearly, the plane partition (5.1.1) is symmetric.

We write $\pi \subset \mathrm{B}(n,p,m)$ if the plane partition π fits in a box $\mathrm{B}(n,p,m)$ of size $n \times p \times m$, i.e., if the shape of π is contained in p^n and no part of π exceeds m. For symmetric plane partitions we may without loss of generality assume that $p = n$.

MacMahon's famous conjecture [**95**, **96**] for the generating function of symmetric plane partitions in $\mathrm{B}(n,n,m)$ is given by equation (1.1.5) of the introduction. The conjecture was proven, independently and almost simultaneously, by Andrews [**3**,**4**] and Macdonald [**90**], almost 80 years after it was first posed in 1898. Andrew's proof relied on earlier work of Bender and Knuth [**10**], who obtained two expressions for the generating function (1.1.3) as a determinant over q-binomial coefficients, one for even m and one for odd m. Using basic hypergeometric series and clever manipulations of determinants, Andrews evaluated both determinants, thereby confirming the conjecture. As already mentioned in the introduction, Macdonald's proof [**90**] (see also [**106**] by Proctor for similar ideas) relied first of all on the fact that the generating function for symmetric plane partitions can be expressed as a sum over specialised Schur functions as in (1.1.4). This was first noted by Gordon [**45**], who observed the equivalent fact that the generating function for symmetric plane partitions contained in $\mathrm{B}(n,n,m)$ is equal to the generating function for column-strict plane partitions contained in $B(\infty,m,2n-1)$, all of whose parts are odd. By the description (2.8.17) of the Schur function in terms of semistandard Young tableaux, this immediately implies (1.1.4). As shown by Okounkov and Reshetikhin, Gordon's observation is a simple consequence of the bijective correspondence between symmetric plane partitions of size k contained in $\mathrm{B}(n,n,m)$ and sequences of interlacing partitions

$$\mu^{(n-1)} \prec \cdots \prec \mu^{(1)} \prec \mu^{(0)} =: \lambda$$

such that $\lambda_1 \leqslant m$ and $|\lambda| + 2|\mu^{(1)}| + \cdots + 2|\mu^{(n-1)}| = k$. Here the '$i$th diagonal slice', $\mu^{(i)}$, is simply given by $\mu^{(i)} = (\pi_{1,i+1}, \pi_{2,i+2}, \pi_{3,i+3}, \dots)$. Stacking the slices with λ as base, and all other slices repeated once, results in a column-strict plane partition of shape λ all of whose parts are odd and at most $2n-1$, see [**103**] for details. The next step in Macdonald's proof was to recognise that

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} s_\lambda(q, q^3, \dots, q^{2n-1}) = \prod_{i=1}^{n} \frac{1-q^{m+2i-1}}{1-q^{2i-1}} \prod_{1\leqslant i<j\leqslant n} \frac{1-q^{2(m+i+j-1)}}{1-q^{2(i+j-1)}}$$

is a specialisation of the bounded Littlewood identity (1.1.2). To prove the latter, he developed a method based on partial fraction expansions [**91**, pages 232–234], which he used to prove the more general[1]

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} P_\lambda(x;t) = \sum_{\varepsilon\in\{\pm 1\}^n} \Phi(x^\varepsilon; t, 0, -1) \prod_{i=1}^{n} x_i^{(1-\varepsilon_i)m/2}. \tag{5.1.2}$$

What we will show in the remainder of this section is that the method developed in Section 4.2 implies that (1.1.2) and hence MacMahon's formula for symmetric plane partitions in a box is a consequence of the fact that $\big(\mathrm{GL}(n,\mathbb{R}), \mathrm{O}(n)\big)$ is a Gelfand pair. Because it is somewhat simpler to handle, we will first discuss a closely related theorem for symmetric plane partitions, due to Proctor [**107**, Theorem 1, (CYH)] and Stembridge [**127**, Corollary 4.3, (b)].

Theorem 5.1. *The generating function for symmetric plane partitions $\pi \subset \mathrm{B}(n,n,2m)$ such that the parts π_{ii} $(i \geqslant 1)$ along the main diagonal are even is given by*

$$\sum_{\substack{\pi\subset \mathrm{B}(n,n,2m) \\ \pi \text{ symmetric} \\ \pi_{ii} \text{ even}}} q^{|\pi|} = \prod_{i=1}^{n} \frac{1-q^{2m+2i}}{1-q^{2i}} \prod_{1\leqslant i<j\leqslant n} \frac{1-q^{2(2m+i+j)}}{1-q^{2(i+j)}}. \tag{5.1.3}$$

[1]By Lemma 2.6, this is equivalent to the $t_2 = 0$ case of Theorem 4.8.

The correspondence between symmetric plane partitions and interlacing partitions still holds, but now the parts of the zeroth slice, λ, must be even. Also, because m has been replaced by $2m$, $\lambda_1 \leqslant 2m$. It thus follows that the generating function on the left-hand side of (5.1.3) is given by

$$\sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2m}} s_\lambda(q, q^3, \dots, q^{2n-1}),$$

so that it remains to prove that

$$\sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2m}} s_\lambda(q, q^3, \dots, q^{2n-1}) = \prod_{i=1}^{n} \frac{1-q^{2m+2i}}{1-q^{2i}} \prod_{1 \leqslant i < j \leqslant n} \frac{1-q^{2(2m+i+j)}}{1-q^{2(i+j)}}.$$

Again there is a lift to a Littlewood identity, which in this case is the Désarménien–Proctor–Stembridge determinant formula (4.1.6). For our purposes we write this in terms of symplectic Schur functions as

$$\sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2m}} s_\lambda(x) = (x_1 \cdots x_n)^m \operatorname{sp}_{2n,m^n}(x). \tag{5.1.4}$$

If G is a Lie group and K a compact subgroup such that, for all (continuous and locally convex) irreducible representation ρ of G, the K-invariant subspace ρ^K has dimension at most 1, then (G, K) is called a Gelfand pair [**22**, **91**]. Hence, for (G, K) a Gelfand pair and χ a character of an irreducible representation of G,

$$\int_K \chi(g) \, \mathrm{d}g = 0 \text{ or } 1, \tag{5.1.5}$$

where the integration is with respect to normalised Haar measure on K. The Gelfand pair relevant to Theorem 5.1 is $G = \mathrm{GL}(n, \mathbb{H})$ and $K = U(n, \mathbb{H})$, the group of invertible $n \times n$ matrices over the division ring of quaternions and the quaternionic unitary group respectively, see [**91**, pages 446–456]. The group $U(n, \mathbb{H})$ is isomorphic to the compact symplectic group, $\mathrm{Sp}(n)$, and (5.1.5) becomes [**91**, page 451]

$$\int_{\mathrm{Sp}(n)} s_\lambda(g) \, \mathrm{d}g = \begin{cases} 1 & \text{if } \lambda' \text{ is even}, \\ 0 & \text{otherwise}, \end{cases} \tag{5.1.6}$$

where λ is a partition of length at most $2n$. By Weyl's integration formula [**41**, page 443], for G a compact Lie group and f a class function,

$$\int_G f(g) \, \mathrm{d}g = \frac{1}{|W|} \int_T f(t) \, |\Delta(t)|^2 \, \mathrm{d}t, \tag{5.1.7}$$

where T is a maximal torus in G, the integration on the right is with respect to normalised Haar measure on T and $\Delta(t)$ is the G-Vandermonde determinant

$$\Delta(\mathrm{e}^h) = \prod_{\alpha > 0} \left(\mathrm{e}^{\frac{1}{2}\alpha(h)} - \mathrm{e}^{-\frac{1}{2}\alpha(h)} \right).$$

Applying this to (5.1.6) yields (see e.g., [**113**])

$$\int_{\mathbb{T}^n} s_\lambda(x_1^\pm,\dots,x_n^\pm)\prod_{i=1}^n\left|x_i-x_i^{-1}\right|^2 \times \prod_{1\leqslant i<j\leqslant n}\left|x_i+x_i^{-1}-x_j-x_j^{-1}\right|^2\,\mathrm{d}T(x)=\chi(\lambda' \text{ even}), \tag{5.1.8}$$

with $\mathrm{d}T(x)$ as (2.6.2). This may also be written as

$$\int_{\mathbb{T}^n} s_\lambda(x_1^\pm,\dots,x_n^\pm)\prod_{i=1}^n(1-x_i^{\pm 2})\prod_{1\leqslant i<j\leqslant n}(1-x_i^\pm x_j^\pm)\,\mathrm{d}T(x)=\chi(\lambda' \text{ even}), \tag{5.1.9}$$

which is precisely the $q=t$ case of (3.2.1) for T^n, i.e., the $q=t$ case of the $\mathrm{U}(2n)/\mathrm{Sp}(2n)$ vanishing integral.[2] Since (5.1.4) is the $q=t$ case of Theorem 4.1 for $a=0$, and since we showed in Section 4.2 that the latter is a consequence of the full $\mathrm{U}(2n)/\mathrm{Sp}(2n)$ vanishing integral, it follows that (5.1.9) implies (5.1.4).

To use Gelfand pairs to prove (1.1.2), and hence MacMahon's conjecture, is slightly more involved. This time the prerequisite pair is $\big(\mathrm{GL}(n,\mathbb{R}),\mathrm{O}(n)\big)$, for which (5.1.5) takes the form

$$\int_{\mathrm{O}(n)} s_\lambda(g)\,\mathrm{d}g=\begin{cases}1 & \text{if } \lambda \text{ is even},\\ 0 & \text{otherwise},\end{cases} \tag{5.1.10}$$

where $l(\lambda)\leqslant n$, see [**91**, pages 414–424]. Again one can use (5.1.7) to integrate over a torus and obtain explicit integrals à la (5.1.9). Because of the distinction between $\mathrm{O}(2n)$ and $\mathrm{O}(2n+1)$, this leads to four integrals, see [**113**, Equations (1.1)–(1.4)]. Of these four we require

$$\int_{\mathbb{T}^n} s_\lambda(x_1^\pm,\dots,x_n^\pm,1)\prod_{i=1}^n|1-x_i|^2 \times \prod_{1\leqslant i<j\leqslant n}\left|x_i+x_i^{-1}-x_j-x_j^{-1}\right|^2\,\mathrm{d}T(x)=\chi(\tilde{\lambda} \text{ even}), \tag{5.1.11}$$

where $l(\lambda)\leqslant 2n+1$ and $\tilde{\lambda}:=(\lambda_1-\lambda_{2n+1},\dots,\lambda_{2n}-\lambda_{2n+1},0)$, as well as

$$\int_{\mathbb{T}^{n-1}} s_\mu(x_1^\pm,\dots,x_{n-1}^\pm,\pm 1)\prod_{i=1}^{n-1}|x_i-x_i^{-1}|^2 \times \prod_{1\leqslant i<j\leqslant n-1}\left|x_i+x_i^{-1}-x_j-x_j^{-1}\right|^2\,\mathrm{d}T(x)=\chi(\tilde{\mu} \text{ even}), \tag{5.1.12}$$

where $l(\mu)\leqslant 2n$ and $\tilde{\mu}:=(\mu_1-\mu_{2n},\dots,\mu_{2n}-\mu_{2n},0)$.

It is not difficult to show that (5.1.11) and (5.1.12) imply the following closely related integrals.

Lemma 5.2. *For μ a partition of length at most $2n$,*

$$\int_{\mathbb{T}^n} s_\mu(x_1^\pm,\dots,x_n^\pm)\prod_{i=1}^n(1-x_i^\pm)\prod_{1\leqslant i<j\leqslant n}(1-x_i^\pm x_j^\pm)\,\mathrm{d}T(x)=(-1)^{|\mu|}, \tag{5.1.13}$$

[2] The $\mathrm{Sp}(n)$ appearing in (5.1.6) versus the $\mathrm{Sp}(2n)$ used in the naming of the vanishing integral is due to a difference in conventions between [**91**] and [**108, 113**].

and for μ a partition of length at most $2n-1$,

$$(5.1.14)\quad \int_{\mathbb{T}^{n-1}} s_\mu(x_1^\pm,\dots,x_{n-1}^\pm,-1)\prod_{i=1}^{n-1}(1-x_i^{\pm 2}) \times \prod_{1\leqslant i<j\leqslant n-1}(1-x_i^\pm x_j^\pm)\,\mathrm{d}T(x)=(-1)^{|\mu|}.$$

Proof. Because the proofs are identical we only consider (5.1.13), and leave (5.1.14) to the reader.

Denote the integral in (5.1.13) by $I_\mu^{(n)}$. From specialising $x_{n+1}=1$ in the inverse of the branching rule for Schur functions (see e.g., [**7**, Equation (5.49)])

$$s_\mu(x_1,\dots,x_n)=\sum_{\nu'\prec\mu'}(-x_{n+1})^{|\mu/\nu|}s_\nu(x_1,\dots,x_{n+1}),$$

it follows that

$$I_\mu(n)=\sum_{\nu'\prec\mu'}(-1)^{|\mu/\nu|}\int_{\mathbb{T}^n} s_\nu(x_1^\pm,\dots,x_n^\pm,1)\prod_{i=1}^{n}(1-x_i^\pm) \times \prod_{1\leqslant i<j\leqslant n}(1-x_i^\pm x_j^\pm)\,\mathrm{d}T(x)$$

The left-hand side of (5.1.11) can be written as

$$\int_{\mathbb{T}^n} s_\lambda(x_1^\pm,\dots,x_n^\pm,1)\prod_{i=1}^{n}(1-x_i^\pm)\prod_{1\leqslant i<j\leqslant n}(1-x_i^\pm x_j^\pm)\,\mathrm{d}T(x),$$

so that we can compute each of the integrals in the sum. Thus

$$I_\mu(n)=\sum_{\nu'\prec\mu'}(-1)^{|\mu/\nu|}\chi(\tilde{\nu}\text{ even}).$$

Since $\nu\subset\mu$ and $l(\mu)\leqslant 2n$, we have $\nu_{2n+1}=0$, resulting in

$$I_\mu(n)=\sum_{\nu'\prec\mu'}(-1)^{|\mu/\nu|}\chi(\nu\text{ even}).$$

By a repeat of the argument following (4.2.8), it follows that ν is completely fixed as $\nu_i=2\lfloor\mu_i/2\rfloor$ (compare with (4.2.9)), completing the proof. □

To finally show that this implies the bounded Littlewood identity (1.1.2), and hence MacMahon's conjecture, we note that (1.1.2) is the $q=t$ specialisation of Theorem 4.4. This theorem was proved in Section 4.2 using the virtual Koornwinder integral (3.2.10) with $T=t^m$, where m is an integer or half-integer. Setting $q=t$ and assuming that m is an integer, say n, this is the integral

$$I_K^{(n)}(s_\mu;q,q;-1,q,\pm q^{1/2})=(-1)^{|\mu|},$$

which is nothing but (5.1.13) in disguise. On the other hand, setting $q=t$ and assuming that $m=n-1/2$ is a half-integer, we may use Lemma 3.4 to write this as the integral

$$I_K^{(n-1)}(s_\mu[x+\varepsilon];q,q;\pm q^{1/2},\pm q)=(-1)^{|\mu|}.$$

This time this may be may be recognised as (5.1.14).

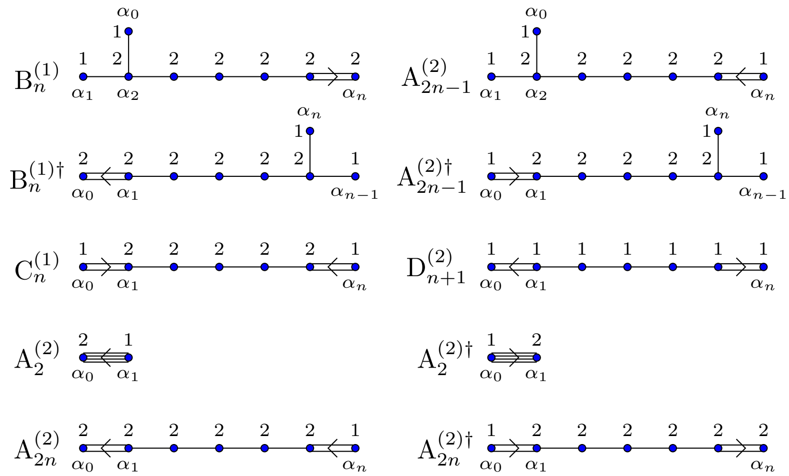


FIGURE 1. The Dynkin diagrams of the "BC$_n$-type" affine Lie algebras with labelling of vertices by simple roots $\alpha_0,\dots,\alpha_n$ and marks $a_0,\dots,a_n$.

REMARK 5.3. There are many parallels between the material presented in this section and the work of Baik and the first author on algebraic aspects of increasing subsequences [**7**]. In particular, [**7**, Theorem 5.2] contains the two Littlewood-type identities

$$\sum_{\substack{\lambda \text{ even}\\ l(\lambda)\leqslant n}} s_\lambda(x) = \int_{\mathrm{O}(n)} \det\bigl(\sigma_g(x)\bigr)\,\mathrm{d}g$$

$$\sum_{\substack{\lambda' \text{ even}\\ l(\lambda)\leqslant 2n}} s_\lambda(x) = \int_{\mathrm{Sp}(n)} \det\bigl(\sigma_g(x)\bigr)\,\mathrm{d}g,$$

where $x=(x_1,x_2,\dots)$ and $\sigma_z(x)$ is defined in (2.4.4), so that

$$\det\bigl(\sigma_g(x)\bigr) = \exp\biggl(\sum_{r\geqslant 1}\frac{p_r(x)}{r}\,\mathrm{Tr}(g^r)\biggr).$$

5.2. Character identities for affine Lie algebras

5.2.1. Main results. We will only define a minimum of notation needed to state our results, and for a more comprehensive introduction to the representation theory of affine Lie algebras we refer the reader to [**25**, **87**].

We will be concerned with affine Lie algebras $\mathfrak{g}$ of "BC$_n$ type", that is, $\mathfrak{g}=X_N^{(r)}$ with $X_N^{(r)}$ one of $\mathrm{B}_n^{(1)}$, $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n-1}^{(2)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$. Using standard labelling, for these affine Lie algebras the classical part is either B_n or C_n. The relevant Dynkin diagrams are shown in Figure 5.2.1. For $\mathrm{B}_n^{(1)}$, $\mathrm{A}_{2n-1}^{(2)}$ and $\mathrm{A}_{2n}^{(2)}$ we also use the nonstandard labelling of simple roots, indicated by the customary †, obtained by mapping $\alpha_i\mapsto\alpha_{n-i}$ for $0\leqslant i\leqslant n$. Apart from the simple roots $\alpha_0,\dots,\alpha_n$ and fundamental weights $\varpi_0,\dots,\varpi_n$ we need the null root δ given by $\delta=\sum_{i=0}^n a_i\alpha_i$, with the a_i the marks of $\mathfrak{g}$, see Figure 5.2.1. We are interested in representations of

$\mathfrak{g}$ known as integrable highest-weight modules. If P_+ is the set of dominant integral weights $P_+ = \sum_{i=0}^n \mathbb{Z}_{\geqslant 0}\varpi_i$ then these modules are indexed by $\Lambda \in P_+$, and will be denoted by $V(\Lambda)$ in the following. The character of $V(\Lambda)$ can be computed in closed form by the Weyl–Kac formula:

$$\operatorname{ch} V(\Lambda) = \frac{\sum_{w\in W} \operatorname{sgn}(w)\, \mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha})^{\operatorname{mult}(\alpha)}}. \tag{5.2.1}$$

Here W is the Weyl group of $\mathfrak{g}$, $\operatorname{sgn}(w)$ the signature of $w \in W$, $\rho = \varpi_0 + \cdots + \varpi_n$ the Weyl vector, and $\operatorname{mult}(\alpha)$ the multiplicity of α. In the denominator, the product runs over the positive roots of $\mathfrak{g}$.

Below we prove combinatorial character formulas for

$$\chi_m(\mathfrak{g}) := \mathrm{e}^{-m\varpi_0} \operatorname{ch} V(m\varpi_0) \tag{5.2.2}$$

for $\mathfrak{g}$ one of $\mathrm{B}_n^{(1)}, \mathrm{B}_n^{(1)\dagger}, \mathrm{C}_n^{(1)}, \mathrm{A}_{2n-1}^{(2)}, \mathrm{A}_{2n-1}^{(2)\dagger}, \mathrm{A}_{2n}^{(2)}, \mathrm{A}_{2n}^{(2)\dagger}, \mathrm{D}_{n+1}^{(2)}$. Since the diagrams of $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ are the same when read from left to right as from right to left, these two algebras occur only once in the above list. For $\mathrm{A}_{2n-1}^{(2)\dagger}, \mathrm{A}_{2n}^{(2)\dagger}$ and $\mathrm{D}_{n+1}^{(2)}$, however, we obtain two distinct formulas, making a total of eleven character formulas.

Recall that $P_\lambda^{(R)}$ denotes a Hall–Littlewood polynomial of type R. Also recall the definition of X in (1.1.8), which may be written plethystically as

$$X = (x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})\, \frac{1-t^N}{1-t}.$$

We complement this with

$$\begin{aligned}\bar{X} &:= (x_1^{\pm}, tx_1^{\pm}, \dots, t^{N-1}x_1^{\pm}, \dots\dots, x_{n-1}^{\pm}, tx_{n-1}^{\pm}, \dots, t^{N-1}x_{n-1}^{\pm}, 1, t, \dots, t^{N-1}) \\ &= (x_1 + x_1^{-1} + \cdots + x_{n-1} + x_{n-1}^{-1} + 1)\, \frac{1-t^N}{1-t}.\end{aligned} \tag{5.2.3}$$

Finally, in each of the formulas below $m_0(\lambda) := \infty$.

Theorem 5.4. *Let*

$$x_i := \mathrm{e}^{-\alpha_i - \cdots - \alpha_{n-1} - \alpha_n/2} \quad (1 \leqslant i \leqslant n), \qquad t := \mathrm{e}^{-\delta}, \tag{5.2.4}$$

and let m and n be positive integers. Then

$$\begin{aligned}\chi_m\big(\mathrm{C}_n^{(1)}\big) &= \lim_{N\to\infty} t^{mnN^2} P_{m^{2nN}}^{(\mathrm{C}_{2nN})}(t^{1/2}X; t, 0) \\ &= \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2m}} t^{|\lambda|/2} P'_\lambda(x_1^{\pm}, \dots, x_n^{\pm}; t)\end{aligned} \tag{5.2.5}$$

$$\begin{aligned}\chi_m\big(\mathrm{A}_{2n-1}^{(2)}\big) &= (t;t^2)_\infty \lim_{N\to\infty} t^{\frac{1}{2}mnN^2} P_{(\frac{m}{2})^{2nN}}^{(\mathrm{D}_{2nN})}(t^{1/2}X; t) \\ &= \sum_{\substack{\lambda' \text{ even} \\ \lambda_1 \leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^{\pm}, \dots, x_n^{\pm}; t) \prod_{i=0}^{m-1} (t;t^2)_{m_i(\lambda)/2}\end{aligned} \tag{5.2.6}$$

and

$$\chi_m\big(\mathrm{A}_{2n}^{(2)}\big) = \lim_{N\to\infty} t^{\frac{1}{2}mnN^2} P^{(\mathrm{B}_{2nN})}_{(\frac{m}{2})^{2nN}}(t^{1/2}X;t,0) = \sum_{\substack{\lambda \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^\pm,\dots,x_n^\pm;t). \tag{5.2.7}$$

THEOREM 5.5. *Let*

$$x_i := \mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}+(\alpha_{n-1}-\alpha_n)/2} \quad (1\leqslant i\leqslant n), \qquad t := \mathrm{e}^{-\delta},$$

and let m and n be positive integers. Then

$$\chi_m\big(\mathrm{A}_{2n-1}^{(2)\dagger}\big) = (t;t^2)_\infty \lim_{N\to\infty} t^{mnN^2} P^{(\mathrm{C}_{2nN})}_{m^{2nN}}(t^{1/2}X;t,t) = {\sum_{\substack{\lambda \\ \lambda_1\leqslant 2m}}}' t^{(|\lambda|+\mathrm{odd}(\lambda))/2} P'_\lambda(x_1^\pm,\dots,x_n^\pm;t) \prod_{i=0}^{2m-1} (t;t^2)_{\lceil m_i(\lambda)/2\rceil}, \tag{5.2.8}$$

where the prime in the sum over λ denotes the restriction that parts of odd size must have even multiplicity.

THEOREM 5.6. *Let*

$$x_i := \mathrm{e}^{-\alpha_i-\cdots-\alpha_n} \quad (1\leqslant i\leqslant n), \tag{5.2.9}$$

and let m and n be positive integers. Then

$$\chi_m\big(\mathrm{A}_{2n}^{(2)\dagger}\big) = \lim_{N\to\infty} t^{mnN^2} P^{(\mathrm{BC}_{2nN})}_{m^{2nN}}(t^{1/2}X;t,0,-t^{1/2}) = \sum_{\substack{\lambda \\ \lambda_1\leqslant 2m}} t^{(|\lambda|+\mathrm{odd}(\lambda))/2} P'_\lambda(x_1^\pm,\dots,x_n^\pm;t), \tag{5.2.10}$$

where $t := \mathrm{e}^{-\delta}$, and

$$\chi_m\big(\mathrm{D}_{n+1}^{(2)}\big) = (-t^{1/2};t^{1/2})_\infty \lim_{N\to\infty} t^{\frac{1}{2}mnN^2} P^{(\mathrm{B}_{2nN})}_{(\frac{m}{2})^{2nN}}(t^{1/2}X;t,-t^{1/2}) = \sum_{\substack{\lambda \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^\pm,\dots,x_n^\pm;t) \prod_{i=0}^{m-1} (-t^{1/2};t^{1/2})_{m_i(\lambda)}, \tag{5.2.11}$$

where $t^{1/2} := \mathrm{e}^{-\delta}$.

The identity (5.2.5), without the limiting expression in the middle, was first obtained in [**9**, Theorem 1.1; (1.4a)]. Equation (5.2.7), which is (1.1.7) from the introduction, extends [**9**, Theorem 1.1; (1.4b)] from integer to half-integer values of m. In the same manner, (5.2.11) extends [**9**, Theorem 5.4]. The identity 5.2.10, again without the limiting expression on the right, is [**9**, Theorem 5.3].

In each of the remaining formulas $\mathrm{e}^{-\alpha_n}$ is specialised.

THEOREM 5.7. *Let*

$$x_i := -\,\mathrm{e}^{-\alpha_i-\cdots-\alpha_n} \quad (1\leqslant i\leqslant n), \qquad t := \mathrm{e}^{-\delta}, \tag{5.2.12}$$

and specialise $\mathrm{e}^{-\alpha_n} \mapsto -1$*. Then, for* m *and* n *positive integers,*

$$\begin{aligned}\chi_m\bigl(\mathrm{A}_{2n}^{(2)\dagger}\bigr) &= \lim_{N\to\infty} t^{\frac{1}{2}m(2n-1)N^2} P_{m^{(2n-1)N}}^{(\mathrm{C}_{(2n-1)N})}(t^{1/2}\bar{X};t,0) \\ &= \sum_{\substack{\lambda \text{ even} \\ \lambda_1\leqslant 2m}} t^{|\lambda|/2} P'_\lambda(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t)\end{aligned} \tag{5.2.13}$$

and

$$\begin{aligned}\chi_m\bigl(\mathrm{B}_{n}^{(1)}\bigr) &= (t;t^2)_\infty \lim_{N\to\infty} t^{\frac{1}{4}m(2n-1)N^2} P_{(\frac{m}{2})^{(2n-1)N}}^{(\mathrm{D}_{(2n-1)N})}(t^{1/2}\bar{X};t) \\ &= \sum_{\substack{\lambda' \text{ even} \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t) \prod_{i=0}^{m-1} (t;t^2)_{m_i(\lambda)/2}.\end{aligned} \tag{5.2.14}$$

Theorem 5.8. *Let*

$$x_i := -\,\mathrm{e}^{-\alpha_i-\cdots-\alpha_n} \quad (1\leqslant i\leqslant n), \qquad t^{1/2} := -\,\mathrm{e}^{-\delta}, \tag{5.2.15}$$

and specialise $\mathrm{e}^{-\alpha_n} \mapsto -1$*. Then, for* m *and* n *positive integers,*

$$\begin{aligned}\chi_m\bigl(\mathrm{D}_{n+1}^{(2)}\bigr) &= \lim_{N\to\infty} t^{\frac{1}{4}m(2n-1)N^2} P_{(\frac{m}{2})^{(2n-1)N}}^{(\mathrm{B}_{(2n-1)N})}(t^{1/2}\bar{X};t,0) \\ &= \sum_{\substack{\lambda \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t).\end{aligned} \tag{5.2.16}$$

Theorem 5.9. *Let*

$$x_i := \mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}+(\alpha_{n-1}-\alpha_n)/2} \quad (1\leqslant i\leqslant n), \qquad t := \mathrm{e}^{-\delta}, \tag{5.2.17}$$

and specialise $\mathrm{e}^{-\alpha_n} \mapsto \mathrm{e}^{-\alpha_{n-1}}$*. Then, for* m *and* n *positive integers,*

(5.2.18)

$$\begin{aligned}\chi_m\bigl(\mathrm{B}_{n}^{(1)\dagger}\bigr) &= (-t^{1/2};t^{1/2})_\infty \lim_{N\to\infty} t^{\frac{1}{4}m(2n-1)N^2} P_{(\frac{m}{2})^{(2n-1)N}}^{(\mathrm{B}_{(2n-1)N})}(t^{1/2}\bar{X};t,-t^{1/2}) \\ &= \sum_{\substack{\lambda \\ \lambda_1\leqslant m}} t^{|\lambda|/2} P'_\lambda(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t) \prod_{i=0}^{m-1} (-t^{1/2};t^{1/2})_{m_i(\lambda)}\end{aligned}$$

and

$$\begin{aligned}\chi_m\bigl(\mathrm{A}_{2n-1}^{(2)\dagger}\bigr) &= \lim_{N\to\infty} t^{m(n-1/2)N^2} P_{m^{(2n-1)N}}^{(\mathrm{BC}_{(2n-1)N})}(t^{1/2}\bar{X};t,0,-t^{1/2}) \\ &= \sum_{\substack{\lambda \\ \lambda_1\leqslant 2m}} t^{(|\lambda|+\mathrm{odd}(\lambda))/2} P'_\lambda(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t).\end{aligned} \tag{5.2.19}$$

Thanks to the Macdonald identities, the characters of certain one-parameter subfamilies of representations admit product forms. For example, by Corollary A.4 of Appendix A, it follows that for $\mathfrak{g} = \mathrm{B}_n^{(1)\dagger}$ and weights

$$\Lambda = (k-1)\rho + k\omega_0, \qquad k\in\mathbb{Z}_{\geqslant 1}, \tag{5.2.20}$$

we have

$$\mathrm{e}^{-\Lambda}\operatorname{ch} V(\Lambda) = \frac{(t^k;t^k)_\infty^{n-1}(t^{2k};t^{2k})_\infty}{(t;t)_\infty^n} \prod_{i=1}^n \frac{\theta(t^k x_i^{2k};t^{2k})}{\theta(t^{1/2}x_i;t)} \prod_{1\leqslant i<j\leqslant n} \frac{\theta(x_i^k x_j^{\pm k};t^k)}{\theta(x_i x_j^{\pm};t)},$$

where $x_1,\dots,x_n$ and t are as in (5.2.17). In much the same way, it follows from the Macdonald identity for $\mathrm{C}_n^{(1)}$ that for $\mathrm{A}_{2n}^{(2)}$ and weights (5.2.20),

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{(t^k;t^k)_\infty^n}{(t;t)_\infty^n}\prod_{i=1}^n\frac{\theta(x_i^{2k};t^k)}{\theta(t^{1/2}x_i;t)\theta(x_i^2;t^2)}\prod_{1\leqslant i<j\leqslant n}\frac{\theta(x_i^kx_j^{\pm k};t^k)}{\theta(x_ix_j^{\pm};t)},$$

where $x_1,\dots,x_n$ and t are given by (5.2.4). Similarly, by the Macdonald identity for $\mathrm{A}_{2n}^{(2)\dagger}$ we have the $\mathrm{D}_{n+1}^{(2)}$ identity

$$\begin{aligned}\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)&=\frac{(t^k;t^k)_\infty^n}{(t;t)_\infty^{n-1}(t^{1/2};t^{1/2})_\infty}\\&\qquad\times\prod_{i=1}^n\frac{\theta(x_i^k;t^k)\theta(t^kx_i^{2k};t^{2k})}{\theta(x_i;t^{1/2})}\prod_{1\leqslant i<j\leqslant n}\frac{\theta(x_i^kx_j^{\pm k};t^k)}{\theta(x_ix_j^{\pm};t)},\end{aligned}$$

where Λ is again given by (5.2.20), and $x_1,\dots,x_n$ and t are as in (5.2.9).

All three families include the basic representation, $V(\varpi_0)$, obtained by taking $k=1$:

$$\mathrm{e}^{-\varpi_0}\operatorname{ch}V(\varpi_0)=\kappa_{\mathfrak{g}}(t)\prod_{i=1}^n\theta(-t^{1/2}x_i;t),\tag{5.2.21}$$

where

$$\kappa_{\mathfrak{g}}(t):=\begin{cases}(-t;t)_\infty & \text{for } \mathfrak{g}=\mathrm{B}_n^{(1)\dagger},\\ 1 & \text{for } \mathfrak{g}=\mathrm{A}_{2n}^{(2)},\\ (-t^{1/2};t^{1/2})_\infty & \text{for } \mathfrak{g}=\mathrm{D}_{n+1}^{(2)}.\end{cases}$$

Using the representation (2.8.18) for the modified Hall–Littlewood polynomials, each of the character formulas in Theorems 5.4–5.6 can be written as a multiple basic hypergeometric series. For $\mathrm{B}_n^{(1)\dagger},\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ we can then restrict to the basic representation and, as a consistency check, compare with (5.2.21). To this end we define f by

$$\begin{aligned}f(x_1,\dots,x_n;t)&:=\sum_{\substack{\lambda\\ \lambda_1\leqslant 1}}t^{|\lambda|/2}P'_\lambda(x_1,\dots,x_n;t)\\&=\sum_{r=0}^\infty\frac{t^{r/2}}{(t;t)_r}\,Q'_{1^r}(x_1,\dots,x_n;t).\end{aligned}$$

Replacing r by r_1, and using (2.8.18) and (2.8.19), this yields

$$f(x_1,\dots,x_n;t)=\sum_{r_1\geqslant\cdots\geqslant r_n\geqslant 0}\frac{t^{r_1/2}}{(t;t)_{r_1}}\prod_{i=1}^n x_i^{r_i-r_{i+1}}t^{\binom{r_i-r_{i+1}}{2}}\begin{bmatrix}r_i\\ r_{i+1}\end{bmatrix}_t,$$

where $r_{n+1}:=0$. Introducing new summation indices $k_1,\dots,k_n$ by $k_i=r_i-r_{i+1}$, the n-fold sum factors as[3]

$$f(x_1,\dots,x_n;t)=\prod_{i=1}^n\biggl(\sum_{k_i=0}^\infty\frac{(t^{1/2}x_i)^{k_i}t^{\binom{k_i}{2}}}{(t;t)_{k_i}}\biggr)=\prod_{i=1}^n(-t^{1/2}x_i;t)_\infty,\tag{5.2.22}$$

[3]This may also be proved by taking $m=1$ and $q=0$ in (2.5.5), and by carrying out the plethystic substitution $x\mapsto x/(1-t)$.

where the second equality follows from [**43**, Equation (II.2)]

$$\sum_{k\geqslant 0}\frac{z^k q^{\binom{k}{2}}}{(q;q)_k}=(-z;q)_\infty.$$

We now observe that for $m=1$ the identities (5.2.7), (5.2.11) and (5.2.18) simplify to

$$\mathrm{e}^{-\varpi_0}\operatorname{ch}V(\varpi_0)=\begin{cases}(-t^{1/2};t^{1/2})_\infty f(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t) & \text{for } \mathfrak{g}=\mathrm{B}_n^{(1)\dagger}\\ f(x_1^{\pm},\dots,x_n^{\pm};t) & \text{for } \mathfrak{g}=\mathrm{A}_{2n}^{(2)},\\ (-t^{1/2};t^{1/2})_\infty f(x_1^{\pm},\dots,x_n^{\pm};t) & \text{for } \mathfrak{g}=\mathrm{D}_{n+1}^{(2)}.\end{cases}$$

By (5.2.22) this indeed gives (5.2.21) for $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$. For $\mathrm{B}_n^{(1)\dagger}$ it leads to

$$\begin{aligned}\mathrm{e}^{-\varpi_0}\operatorname{ch}V(\varpi_0)\big|_{\mathrm{e}^{-\alpha_n}\mapsto\mathrm{e}^{-\alpha_{n-1}}}&=(-t^{1/2};t^{1/2})_\infty(-t^{1/2};t)_\infty\prod_{i=1}^{n-1}\theta(-t^{1/2}x_i;t)\\&=(-t;t)_\infty\prod_{i=1}^{n}\theta(-t^{1/2}x_i;t)|_{x_n=1},\end{aligned}$$

again in agreement with (5.2.21). To show that the $m=1$ case of the second $\mathrm{D}_{n+1}^{(2)}$ identity (5.2.16) is also in accordance with (5.2.21), we replace $x_i\mapsto -x_i$ and $t^{1/2}\mapsto -t^{1/2}$ on the right-hand side of (5.2.21) and then specialise $x_n=1$. This yields the expression

$$\begin{aligned}(-t^{1/2};t)_\infty\prod_{i=1}^{n-1}\theta(-t^{1/2}x_i;t)&=f(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t)\\&=\sum_{\substack{\lambda\\ \lambda_1\leqslant 1}}t^{|\lambda|/2}P'_\lambda(x_1^{\pm},\dots,x_{n-1}^{\pm},1;t),\end{aligned}$$

as required.

5.2.2. Proof of Theorems 5.4–5.9. Below we present proofs of the eleven character formulas of the previous section.

Recall the Vandermonde determinants of type $\mathrm{B}_n,\mathrm{C}_n$ and D_n given in (2.8.6), (2.8.13) and (2.8.14).

Proposition 5.10. *For $x=(x_1,\dots,x_n)$, m a positive integer and N a non-negative integer,*

$$\begin{aligned}&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}}t^{|\lambda|/2}h_\lambda^{(2m)}(t_2,t_3;t)P_\lambda\bigg(\bigg[x\,\frac{1-t^N}{1-t}\bigg];t\bigg)\\&\quad=(x_1\cdots x_n)^{mN}t^{mnN^2/2}P_{m^{nN}}^{(\mathrm{BC}_{nN})}\bigg(\bigg[t^{1/2}x\,\frac{1-t^N}{1-t}\bigg];t,t_2,t_3\bigg)\\&\quad=\frac{\prod_{i=1}^n(t^{1/2}t_2x_i,t^{1/2}t_3x_i;t)_N}{\prod_{i,j=1}^n(tx_ix_j;t)_N}\prod_{1\leqslant i<j\leqslant n}(tx_ix_j;t)_{2N}\\&\qquad\times\sum_{r_1,\dots,r_n\geqslant 0}\frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\frac{(t^{1/2}t_2^{-1}x_i,t^{1/2}t_3^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i,t^{1/2}t_3x_i;t)_{r_i}}(t_2t_3)^{r_i}(x_i^2t^{r_i})^{mr_i}\end{aligned}\tag{5.2.23}$$

$$\times \prod_{i,j=1}^{n} \frac{(t^{-N}x_i/x_j, x_ix_j;t)_{r_i}}{(tx_i/x_j, t^{N+1}x_ix_j;t)_{r_i}}\, t^{Nr_i}.$$

REMARK 5.11. A more general hypergeometric identity than (5.2.23) holds, obtained by replacing

$$x \mapsto t^{1/2}\biggl(x_1\frac{1-t^{N_1}}{1-t} + x_2\frac{1-t^{N_2}}{1-t} + \cdots + x_n\frac{1-t^{N_n}}{1-t}\biggr)$$

in (4.1.15). From a hypergeometric point of view this more general identity, which on the right features the C_n hypergeometric series

$$\sum_{r_1,\dots,r_n\geqslant 0} \frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \frac{(t^{1/2}t_2^{-1}x_i, t^{1/2}t_3^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i, t^{1/2}t_3x_i;t)_{r_i}}\,(t_2t_3)^{r_i}(x_i^2t^{r_i})^{mr_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(t^{-N_j}x_i/x_j, x_ix_j;t)_{r_i}}{(tx_i/x_j, t^{N_j+1}x_ix_j;t)_{r_i}}\, t^{N_jr_i},$$

is more natural. For our purposes, however, we do not require this greater degree of generality.

PROOF. Identity (5.2.23) follows from (4.1.15) by the substitution

$$\begin{aligned} x &\mapsto t^{1/2}(x_1, x_1t, \dots, x_1t^{N-1}, \dots\dots, x_n, x_nt, \dots, x_nt^{N-1}) \\ &= t^{1/2}x\,\frac{1-t^N}{1-t} \end{aligned} \tag{5.2.24}$$

(so that, implicitly, $n \mapsto nN$). The two left-most expressions immediately follow from (5.2.23), but to show equality with the hypergeometric sum on the right some work is required.

First we use Lemma 2.5 to trade the right-hand side of (4.1.15) for

$$\sum_{\varepsilon\in\{\pm 1\}^n} \Phi(x^\varepsilon;t_2,t_3;t) \prod_{i=1}^{n} x_i^{(1-\varepsilon_i)m}.$$

Next we observe that $\Phi(x^\varepsilon;t_2,t_3;t)$ contains the factor

$$\prod_{1\leqslant i<j\leqslant n} (1 - tx_i^{\varepsilon_i}x_j^{\varepsilon_j})$$

which vanishes if there exists an i $(1 \leqslant i \leqslant n-1)$ such that

$$tx_i^{\varepsilon_i}x_{i+1}^{\varepsilon_{i+1}} = 1.$$

Therefore, by the substitution (5.2.24), the summand vanishes if for some i,u,p,

$$(\varepsilon_i,\varepsilon_{i+1}) = (1,-1) \quad\text{and}\quad (x_i,x_{i+1}) \mapsto (x_ut^p, x_ut^{p+1}).$$

Consequently, the only sequences ε that yield a non-vanishing summand are of the form

$$\varepsilon = \bigl(\underbrace{-1,\dots,-1}_{r_1\text{ times}}, \underbrace{1,\dots,1}_{N-r_1\text{ times}}, \underbrace{-1,\dots,-1}_{r_2\text{ times}}, \underbrace{1,\dots,1}_{N-r_2\text{ times}}, \dots, \underbrace{-1,\dots,-1}_{r_n\text{ times}}, \underbrace{1,\dots,1}_{N-r_n\text{ times}}\bigr).$$

The r_i are exactly the summation indices of (5.2.23). The rest of the proof is tedious but elementary and left to the reader. □

Replacing

$$x \mapsto (x_1, y_1, \dots, x_n, y_n) \tag{5.2.25}$$

in (5.2.23) (so that $n \mapsto 2n$), and then using [**9**, Proposition 5.1] to take the limit $y_i \mapsto x_i^{-1}$ for all $1 \leqslant i \leqslant n$, we obtain the following corollary of Proposition 5.10.

COROLLARY 5.12. *Let m a positive integer, N a nonnegative integer and X the alphabet* (1.1.8)*. Then*

$$\begin{aligned}
&\sum_{\substack{\lambda \\ \lambda_1 \leqslant 2m}} t^{|\lambda|/2} h_\lambda^{(2m)}(t_2, t_3; t) P_\lambda(X; t) \\
&\qquad = t^{mnN^2} P_{m^{2nN}}^{(\mathrm{BC}_{2nN})}(t^{1/2}X; t, t_2, t_3) \\
&\qquad = \prod_{i=1}^n \frac{(t^{1/2}t_2 x_i^{\pm}, t^{1/2}t_3 x_i^{\pm}; t)_N}{(t x_i^{\pm 2}; t)_N} \begin{bmatrix} 2N \\ N \end{bmatrix}_t \prod_{1 \leqslant i < j \leqslant n} \frac{(t x_i^{\pm} x_j^{\pm}; t)_{2N}}{(t x_i^{\pm} x_j^{\pm}; t)_N^2} \\
&\qquad\quad \times \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n \frac{(t^{1/2}t_2^{-1}x_i, t^{1/2}t_3^{-1}x_i; t)_{r_i}}{(t^{1/2}t_2 x_i, t^{1/2}t_3 x_i; t)_{r_i}} (t_2t_3)^{r_i} (x_i^2 t^{r_i})^{mr_i} \\
&\qquad\qquad \times \prod_{i,j=1}^n \frac{(t^{-N} x_i x_j^{\pm}; t)_{r_i}}{(t^{N+1} x_i x_j^{\pm}; t)_{r_i}} t^{2Nr_i}.
\end{aligned}$$

Since

$$\lim_{N \to \infty} P_\lambda(X; t) = P_\lambda\Big(\Big[\frac{x_1^{\pm} + \dots + x_n^{\pm}}{1-t}\Big]; t\Big) = P'_\lambda(x_1^{\pm}, \cdots x_n^{\pm}; t),$$

the above corollary is a bounded analogue of [**9**, Theorem 5.2; (5.6a)], which states (without the second line) that

(5.2.26)

$$\begin{aligned}
&\sum_{\substack{\lambda \\ \lambda_1 \leqslant 2m}} t^{|\lambda|/2} h_\lambda^{(2m)}(t_2, t_3; t) P'_\lambda(x_1^{\pm}, \dots, x_n^{\pm}; t) \\
&\quad = \lim_{N \to \infty} t^{mnN^2} P_{m^{2nN}}^{(\mathrm{BC}_{2nN})}(t^{1/2}X; t, t_2, t_3) \\
&\quad = \frac{1}{(t;t)_\infty^n} \prod_{i=1}^n \frac{(t^{1/2}t_2 x_i^{\pm}, t^{1/2}t_3 x_i^{\pm}; t)_\infty}{(t x_i^{\pm 2}; t)_\infty} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{(t x_i^{\pm} x_j^{\pm}; t)_\infty} \\
&\quad\quad \times \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n \frac{(t^{1/2}t_2^{-1}x_i, t^{1/2}t_3^{-1}x_i; t)_{r_i}}{(t^{1/2}t_2 x_i, t^{1/2}t_3 x_i; t)_{r_i}} (t_2t_3t^{-n})^{r_i} (x_i^2 t^{r_i})^{(m+n)r_i},
\end{aligned}$$

for m a positive integer.

If instead of (5.2.25) we make the substitution

$$x \mapsto (x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n) \tag{5.2.27}$$

in (5.2.23) (so that $n \mapsto 2n-1$) and then take the limit $y_i \mapsto x_i^{-1}$ for all $1 \leqslant i \leqslant n-1$ and $x_n \to 1$ using [**9**, Proposition 5.1], we obtain a bounded version of [**9**, Theorem

5.2; (5.6b)][4]

(5.2.28)

$$
\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}} t^{|\lambda|/2} h_\lambda^{(2m)}(t_2,t_3;t) P'_\lambda(x_1^\pm,\dots,x_{n-1}^\pm,1;t)\\
&\quad=\lim_{N\to\infty} t^{m(n-1/2)N^2} P^{(\mathrm{BC}_{(2n-1)N})}_{m^{(2n-1)N}}(t^{1/2}\bar{X};t,t_2,t_3)\\
&\quad=\frac{1}{(t;t)_\infty^n (t^{1/2}t_2,t^{1/2}t_3;t)_\infty}\prod_{i=1}^n \frac{(t^{1/2}t_2x_i^\pm,t^{1/2}t_3x_i^\pm;t)_\infty}{(-tx_i^\pm;t)_\infty(tx_i^{\pm 2};t^2)_\infty}\\
&\qquad\times\prod_{1\leqslant i<j\leqslant n}\frac{1}{(tx_i^\pm x_j^\pm;t)_\infty}\sum_{r\in\mathbb{Z}^n}\biggl(\frac{\Delta_{\mathrm{B}}(-xt^r)}{\Delta_{\mathrm{B}}(-x)}\\
&\qquad\times\prod_{i=1}^n\frac{(t^{1/2}t_2^{-1}x_i,t^{1/2}t_3^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i,t^{1/2}t_3x_i;t)_{r_i}}\bigl(-t_2t_3t^{1/2-n}\bigr)^{r_i}\bigl(x_i^2t^{r_i}\bigr)^{(m+n-1/2)r_i}\biggr),
\end{aligned}
$$

where m is a positive integer and $x_n := 1$.

We are now ready to prove Theorems 5.4–5.9. Our first two character formulas follow from (5.2.26) and (5.2.28) by letting t_2 and t_3 tend to zero. The hypergeometric sums on the right can then be identified with $\chi_m(\mathrm{C}_n^{(1)})$ and $\chi_m(\mathrm{A}_{2n}^{(2)\dagger})$ respectively, by [**9**, Lemmas 2.1 & 2.3] and [**9**, Lemma 2.3] (which are simple rewritings of the Weyl–Kac formula for $\mathrm{C}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)\dagger}$). In the latter case this identification requires the specialisation $\mathrm{e}^{-\alpha_n}\mapsto -1$, corresponding to the condition $x_n := 1$ in (5.2.28). In the $t_2,t_3\to 0$ limit the left-hand sides simplify since $h_\lambda^{(2m)}(0,0;t)=\chi(\lambda \text{ even})$. We thus obtain (5.2.5) and (5.2.13).

Next we specialise $t_3=-t^{1/2}$ in (5.2.26) and (5.2.28). Using

(5.2.29a)
$$\frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\frac{(-x_i;t)_{r_i}}{(-tx_i;t)_{r_i}}=\frac{\Delta_{\mathrm{B}}(xt^r)}{\Delta_{\mathrm{B}}(x)}$$

(5.2.29b)
$$\frac{\Delta_{\mathrm{B}}(-xt^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^n\frac{(-x_i;t)_{r_i}}{(-tx_i;t)_{r_i}}=\frac{\Delta_{\mathrm{D}}(xt^r)}{\Delta_{\mathrm{D}}(x)}$$

respectively, this yields

(5.2.30)

$$
\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}} t^{|\lambda|/2} h_\lambda^{(2m)}(t_2,-t^{1/2};t) P'_\lambda(x_1^\pm,\dots,x_n^\pm;t)\\
&\quad=\lim_{N\to\infty} t^{mnN^2} P^{(\mathrm{BC}_{2nN})}_{m^{2nN}}(t^{1/2}X;t,t_2,-t^{1/2})\\
&\quad=\frac{1}{(t;t)_\infty^n}\prod_{i=1}^n\frac{(t^{1/2}t_2x_i^\pm;t)_\infty}{(tx_i^\pm;t)_\infty(tx_i^{\pm 2};t^2)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(tx_i^\pm x_j^\pm;t)_\infty}\\
&\qquad\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(xt^r)}{\Delta_{\mathrm{B}}(x)}\prod_{i=1}^n\frac{(t^{1/2}t_2^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i;t)_{r_i}}\,(-t_2t^{1/2-n})^{r_i}(x_i^2t^{r_i})^{(m+n)r_i}
\end{aligned}
$$

[4]Taking $x_n\to t^{1/2}$ instead of $x_n\to 1$ yields additional character identities to those of Theorems 5.4 and 5.9.

and

(5.2.31)

$$\begin{aligned}
&\sum_{\substack{\lambda \\ \lambda_1\leqslant 2m}} t^{|\lambda|/2} h_\lambda^{(2m)}(t_2,-t^{1/2};t) P'_\lambda(x_1^\pm,\dots,x_{n-1}^\pm,1;t) \\
&\quad = \lim_{N\to\infty} t^{m(n-1/2)N^2} P^{(\mathrm{BC}_{(2n-1)N})}_{m^{(2n-1)N}}(t^{1/2}\bar{X};t,t_2,-t^{1/2}) \\
&\quad = \frac{1}{(t;t)_\infty^{n-1}(t^2;t^2)_\infty(t^{1/2}t_2;t)_\infty} \prod_{i=1}^n \frac{(t^{1/2}t_2x_i^\pm;t)_\infty}{(tx_i^{\pm 2};t^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{(tx_i^\pm x_j^\pm;t)_\infty} \\
&\qquad \times \sum_{r\in\mathbb{Z}^n} \bigg(\frac{\Delta_{\mathrm{D}}(xt^r)}{\Delta_{\mathrm{D}}(x)} \prod_{i=1}^n \frac{(t^{1/2}t_2^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i;t)_{r_i}} \big(t_2t^{1-n}\big)^{r_i} (x_i^2 t^{r_i})^{(m+n-1/2)r_i} \bigg),
\end{aligned}$$

where $x_n := 1$ in the second identity. Taking the $t_2 \to 0$ limit using

$$h_\lambda^{(m)}(0,b;t) = (-b)^{\mathrm{odd}(\lambda)},$$

and identifying the respective right-hand sides as $\chi_m(\mathrm{A}_{2n}^{(2)\dagger})$ and $\chi_m(\mathrm{A}_{2n-1}^{(2)\dagger})$ by [**9**, Lemma 2.3] and Lemma A.3, results in (5.2.10) and (5.2.19). In particular we note that x_n being equal to 1 in (5.2.31) implies the specialisation $\mathrm{e}^{-\alpha_n} \mapsto \mathrm{e}^{-\alpha_{n-1}}$, see (A.1.6). In similar manner we specialise $t_2 = t^{1/2}$ in (5.2.30) (considering (5.2.31) does not lead to a character identity). Using (5.2.29b) with $x \mapsto -x$, it follows from Lemma A.3 that the right-hand side simplifies to

$$(-t;t)_\infty\, \chi_m\big(\mathrm{A}_{2n-1}^{(2)\dagger}\big).$$

By (2.3.4), the Rogers–Szegő polynomial in the summand on the left becomes

$$h_\lambda^{(2m)}(-t^{1/2},t^{1/2};t) = t^{\mathrm{odd}(\lambda)/2} \prod_{\substack{i=1 \\ i \text{ odd}}}^{m-1} H_{m_i(\lambda)}(-1;t) \prod_{\substack{i=1 \\ i \text{ even}}}^{m-1} H_{m_i(\lambda)}(-t;t).$$

Using (2.3.7b) and (2.3.7c) this yields

$$h_\lambda^{(2m)}(-t^{1/2},t^{1/2};t) = \prod_{i=1}^{2m-1} (t;t^2)_{\lceil m/2 \rceil}$$

if $m_i(\lambda)$ is even for all $i = 1,3,\dots,2m-1$, and zero otherwise. Finally noting that $(-t;t)_\infty(t;t^2)_\infty = 1$, Theorem 5.5 follows. There is one further specialisation of (5.2.30) and (5.2.31) that leads to character identities, namely $t_2 = -1$. We will consider this case as part of a more general treatment of (5.2.26) and (5.2.28) for $t_3 = -1$.

Recalling that Theorem 4.8 extends the $t_3=-1$ case of Theorem 4.7 to half-integer values of m, the $t_3=-1$ specialisations of (5.2.26) and (5.2.28) lead to

$$
\begin{aligned}
(5.2.32)\qquad &\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} t^{|\lambda|/2} h_\lambda^{(m)}(t_2;t) P'_\lambda(x_1^\pm,\dots,x_n^\pm;t)\\
&=\lim_{N\to\infty} t^{mnN^2/2} P^{(\mathrm{B}_{2nN})}_{(m/2)^{2nN}}(t^{1/2}X;t,t_2)\\
&=\frac{1}{(t;t)_\infty^n}\prod_{i=1}^n \frac{(t^{1/2}t_2x_i^\pm;t)_\infty}{(t^{1/2}x_i^\pm;t)_\infty (t^2x_i^{\pm 2};t^2)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(tx_i^\pm x_j^\pm;t)_\infty}\\
&\quad\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n\frac{(t^{1/2}t_2^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i;t)_{r_i}}(-t_2t^{-n})^{r_i}(x_i^2t^{r_i})^{(m+2n)r_i/2}
\end{aligned}
$$

and

(5.2.33)

$$
\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} t^{|\lambda|/2} h_\lambda^{(m)}(t_2;t) P'_\lambda(x_1^\pm,\dots,x_{n-1}^\pm,1;t)\\
&=\lim_{N\to\infty} t^{m(2n-1)N^2/4} P^{(\mathrm{B}_{(2n-1)N})}_{(m/2)^{(2n-1)N}}(t^{1/2}\bar{X};t,t_2)\\
&=\frac{1}{(t;t)_\infty^n(t^{1/2}t_2,-t^{1/2};t)_\infty}\prod_{i=1}^n \frac{(t^{1/2}t_2x_i^\pm;t)_\infty}{(t^{1/2}x_i^\pm,-tx_i^\pm;t)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(tx_i^\pm x_j^\pm;t)_\infty}\\
&\quad\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(-xt^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^n\frac{(t^{1/2}t_2^{-1}x_i;t)_{r_i}}{(t^{1/2}t_2x_i;t)_{r_i}}(t_2t^{1/2-n})^{r_i}(x_i^2t^{r_i})^{(m+2n-1)r_i/2},
\end{aligned}
$$

where $x_n:=1$. If we now let t_2 tend to zero, use that $h_\lambda^{(m)}(0;t)=1$, and further use [**9**, Lemmas 2.2 & 2.4] (see also (5.3.4) for the former) to identify the right-hand sides as $\chi_m(\mathrm{A}_{2n}^{(2)})$ and $\chi_m(\mathrm{D}_{n+1}^{(2)})$, we obtain (5.2.7) and (5.2.16). We again note that the condition $x_n:=1$ in (5.2.33) implies that in the $\mathrm{D}_{n+1}^{(2)}$ case we must specialise $\mathrm{e}^{-\alpha_n}\mapsto -1$. Two further cases, already mentioned in relation with (5.2.30) and (5.2.31), arise from (5.2.32) and (5.2.33) by specialising $t_2=-t^{1/2}$. On the right we can once again use (5.2.29) as well as [**9**, Lemma 2.4] and Lemma A.3 to recognise the hypergeometric sums as

$$(-t^{1/2};t^{1/2})_\infty\,\chi_m(\mathfrak{g})$$

for $\mathfrak{g}=\mathrm{D}_{n+1}^{(2)}$ and $\mathrm{B}_n^{(1)\dagger}$ respectively. In the latter case we must assume the specialisation $\mathrm{e}^{-\alpha_n}\mapsto\mathrm{e}^{-\alpha_{n-1}}$. On the left we use (2.3.6) and (2.3.7d) to find

$$h_\lambda^{(m)}(t^{1/2};t)=\prod_{i=1}^{m-1}(-t^{1/2};t^{1/2})_{m_i(\lambda)},$$

completing the proofs of (5.2.11) and (5.2.18). To prove our final two results we consider (5.2.32) and (5.2.33) for $t_2=1$. Then, by (4.1.10), we can add the additional restriction "λ' is even" to the sum on the left and $|r|\equiv 0 \pmod 2$ to the sum on the right. Using Lemmas A.2 and A.1 this proves (5.2.6) and (5.2.14).

5.3. Rogers–Ramanujan identities

Starting with the pioneering series of papers [**79–84**], the link between affine Lie algebras and vertex operator algebras on the one hand and Rogers–Ramanujan identities on the other is by now well established, see also [**5**,**18**,**24**,**27**,**30**,**38**,**39**,**57**, **99**, **144**]. Nonetheless, examples of q-series identities (as opposed to combinatorial identities) that lift classical Rogers–Ramanujan-type identities to affine Lie algebra at arbitrary rank and level while still permitting a product form, are rare. Recently Griffin et al. [**47**] showed how to use combinatorial character formulas of the type proven in Section 5.2 to obtain doubly-infinite families of Rogers–Ramanujan identities, including a generalisation of the Rogers–Ramanujan [**114**] and Andrews–Gordon [**1**, **44**] identities to the affine Lie algebra $\mathrm{A}_{2n}^{(2)}$. Following the approach of [**47**], we prove several new doubly-infinite families of Rogers–Ramanujan identities, including a $\mathrm{B}_n^{(1)}$ generalisation of Bressoud's Rogers–Ramanujan identities for even moduli [**14**, **15**].

Theorem 5.13 ($\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identities). *Let m, n be positive integers. Then*

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} q^{|\lambda|/2} P_\lambda(1, q, q^2, \dots; q^{2n}) = \frac{(q^\kappa; q^\kappa)_\infty^{n-1} (q^{\kappa/2}; q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1} (q^{1/2}; q^{1/2})_\infty} \prod_{i=1}^n \theta(q^i; q^{\kappa/2}) \prod_{1 \leqslant i < j \leqslant n} \theta(q^{j-i}, q^{i+j}; q^\kappa) \tag{5.3.1}$$

for $\kappa := m + 2n + 1$, and

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant 2m}} t^{|\lambda|/2 + n\, \mathrm{odd}(\lambda)} P_\lambda(1, q, q^2, \dots; q^{2n}) = \frac{(q^\kappa; q^\kappa)_\infty^{n-1} (q^{\kappa/2}; q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1} (q^{1/2}; q^{1/2})_\infty} \prod_{i=1}^n \theta(q^{i+m}; q^{\kappa/2}) \prod_{1 \leqslant i < j \leqslant n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa) \tag{5.3.2}$$

for $\kappa := 2m + 2n + 1$.

Proof. To prove (5.3.1) we apply the specialisation

$$F: \ \mathbb{C}[[\mathrm{e}^{-\alpha_0}, \dots, \mathrm{e}^{-\alpha_n}]] \to \mathbb{C}[[q^{1/2}]]$$

given by

$$F(\mathrm{e}^{-\alpha_0}) = q^{1/2} \quad \text{and} \quad F(\mathrm{e}^{-\alpha_i}) = q \quad \text{for } 1 \leqslant i \leqslant n \tag{5.3.3}$$

to the $\mathrm{A}_{2n}^{(2)}$ character identity (5.2.7). Since the the null root for $\mathrm{A}_{2n}^{(2)}$ is given by $\delta = 2\alpha_0 + \dots + 2\alpha_{n-1} + \alpha_n$, it follows from (5.2.4) that

$$F(x_i) = q^{n-i+1/2} \quad (1 \leqslant i \leqslant n) \quad \text{and} \quad F(t) = q^{2n}.$$

Hence

$$\begin{aligned}F\Bigg(\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} t^{|\lambda|/2}P'_\lambda(x_1^{\pm},\dots,x_n^{\pm};t)\Bigg)\\
&=\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} q^{n|\lambda|}P'_\lambda(q^{-n+1/2},q^{-n+3/2},\dots,q^{n-1/2};q^{2n})\\
&=\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} q^{|\lambda|/2}P'_\lambda(1,q,\dots,q^{2n-1};q^{2n})\\
&=\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} q^{|\lambda|/2}P_\lambda(1,q,q^2,\dots;q^{2n}).\end{aligned}$$

Here the second equality uses the homogeneity of the modified Hall–Littlewood polynomial and the the third equality follows from (2.8.15) and

$$f\Big[\frac{1+q+\dots+q^{2n-1}}{1-q^{2n}}\Big]=f\Big[\frac{1}{1-q}\Big]=f[1+q+q^2+\cdots]\qquad\text{for } f\in\Lambda.$$

To apply F to the left-hand side of (5.2.7) we use that for arbitrary $\Lambda\in P_+$ parametrised as (compare with (2.7.5b))

$$\Lambda=c_0\varpi_0+(\lambda_1-\lambda_2)\varpi_1+\dots+(\lambda_{n-1}-\lambda_n)\varpi_{n-1}+\lambda_n\varpi_n$$

with $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition and c_0 a nonnegative integer, we can rewrite the Weyl–Kac formula for $\mathrm{A}_{2n}^{(2)}$ as [**9**, Lemma 2.2],

(5.3.4)

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{1}{(t;t)_\infty^n\prod_{i=1}^n\theta(t^{1/2}x_i;t)\theta(x_i^2;t^2)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_i/x_j,x_ix_j;t)}\\
\times\sum_{r\in\mathbb{Z}^n}\widetilde{\mathrm{sp}}_{2n,\lambda}(xt^r)\prod_{i=1}^n x_i^{\kappa r_i+\lambda_i}t^{\kappa r_i^2/2-nr_i}.$$

Here $\kappa:=2n+c_0+2\lambda_1+1$, $\widetilde{\mathrm{sp}}_{2n,\lambda}$ is the normalised symplectic Schur function (A.1.1) and $x_1,\dots,x_n$ and t are defined by (5.2.4). Using

$$F\Bigg((t;t)_\infty^n\prod_{i=1}^n\theta(t^{1/2}x_i;t)\theta(x_i^2;t^2)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_i/x_j,x_ix_j;t)\Bigg)\\
=(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty\,q^{\sum_{i<j}(n-j+1/2)}$$

as well as the equations (2.8.10), (A.1.1), and appealing to multilinearity, yields

$$\begin{aligned}F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)&=\frac{q^{-\sum_{i<j}(n-j+1/2)}}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\\
&\quad\times\det_{1\leqslant i,j\leqslant n}\Bigg(\sum_{r\in\mathbb{Z}}q^{(\kappa r+\lambda_i-\lambda_j+j-1)(n-i+1/2)+n\kappa r^2-2nr(\lambda_j+n-j+1)}\\
&\qquad-\sum_{r\in\mathbb{Z}}q^{(\kappa r+\lambda_i+\lambda_j+2n-j+1)(n-i+1/2)+n\kappa r^2+2nr(\lambda_j+n-j+1)}\Bigg).\end{aligned}$$

Replacing $(i,j)\mapsto(n-j+1,n-i+1)$ in the determinant and then changing $r\mapsto -r-1$ in the second sum, we get

$$\begin{aligned}(5.3.5)\quad F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)&=\frac{1}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\\&\quad\times\det_{1\leqslant i,j\leqslant n}\bigg(\sum_{r\in\mathbb{Z}}x_i^{2nr-i+1}q^{2n\kappa\binom{r}{2}+\kappa r/2}\Big((x_iq^{\kappa r})^{j-1}-(x_iq^{\kappa r})^{2n-j}\Big)\bigg),\end{aligned}$$

where $x_i:=q^{\kappa/2-i-\lambda_{n-i+1}}$. Again using multilinearity and recalling the B_n Vandermonde determinant (2.8.13), this may be written as

$$\begin{aligned}F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)&=\frac{1}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\\&\qquad\times\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(xq^{\kappa r})\prod_{i=1}^n x_i^{2nr_i-i+1}q^{2n\kappa\binom{r_i}{2}+\kappa r_i/2}.\end{aligned}$$

By the $\mathrm{D}_{n+1}^{(2)}$ Macdonald identity [**89**] in the form given by [**117**, Corollary 6.2], i.e.,

$$\begin{aligned}&\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{B}}(xq^r)\prod_{i=1}^n x_i^{2nr_i-i+1}q^{2n\binom{r_i}{2}+r_i/2}\\&\qquad=(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty\prod_{i=1}^n\theta(x_i;q^{1/2})\prod_{1\leqslant i<j\leqslant n}\theta(x_ix_j^{\pm};q),\end{aligned}$$

we obtain the product formula

$$\begin{aligned}F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)&=\frac{(q^\kappa;q^\kappa)_\infty^{n-1}(q^{\kappa/2};q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\prod_{i=1}^n\theta(q^{\kappa/2-i-\lambda_{n-i+1}};q^{\kappa/2})\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\theta(q^{\lambda_{n-j+1}-\lambda_{n-i+1}+j-i},q^{\kappa-i-j-\lambda_{n-i+1}-\lambda_{n-j+1}};q^\kappa).\end{aligned}$$

By $\theta(x;q)=\theta(q/x;q)$ and a reversal of the products, this simplifies to

$$\begin{aligned}(5.3.6)\quad F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)&=\frac{(q^\kappa;q^\kappa)_\infty^{n-1}(q^{\kappa/2};q^{\kappa/2})_\infty}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\prod_{i=1}^n\theta(q^{\lambda_i+n-i+1};q^{\kappa/2})\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\theta(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};q^\kappa).\end{aligned}$$

For $\lambda=0$ and $c_0=m$ this gives the claimed right-hand side of (5.3.1).

To prove (5.3.2) we apply the specialisation

$$F^\dagger:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q^{1/2}]]$$

given by

$$(5.3.7)\qquad F^\dagger(\mathrm{e}^{-\alpha_n})=q^{1/2}\quad\text{and}\quad F^\dagger(\mathrm{e}^{-\alpha_i})=q\quad\text{for }0\leqslant i\leqslant n-1$$

to the $\mathrm{A}_{2n}^{(2)\dagger}$ character identity (5.2.10). This implies the same specialisation of $x_1,\dots,x_n$ and q as before, i.e.,

$$F^\dagger(x_i)=q^{n-i+1/2}\quad(1\leqslant i\leqslant n)\quad\text{and}\quad F^\dagger(t)=q^{2n},$$

so that

$$F^{\dagger}\Bigg(\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}} t^{(|\lambda|+\mathrm{odd}(\lambda))/2}P'_{\lambda}(x_1^{\pm},\dots,x_n^{\pm};t)\Bigg) = \sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}} q^{|\lambda|/2+n\,\mathrm{odd}(\lambda)}P_{\lambda}(1,q,q^2,\dots;q^{2n}).$$

Moreover, since (5.3.3) and (5.3.7) are compatible with the map from $\mathrm{A}_{2n}^{(2)}$ to $\mathrm{A}_{2n}^{(2)\dagger}$ (corresponding to a reversal of the labelling of simple roots) we can again use (5.3.6):

$$\begin{aligned}
&F^{\dagger}\big(\mathrm{e}^{-m\varpi_0}\,\mathrm{ch}\,V(m\varpi_0)\big)\big|_{\mathfrak{g}=\mathrm{A}_{2n}^{(2)\dagger}}\\
&\quad= F\big(\mathrm{e}^{-m\varpi_n}\,\mathrm{ch}\,V(m\varpi_n)\big)\big|_{\mathfrak{g}=\mathrm{A}_{2n}^{(2)}}\\
&\quad= \frac{(q^{\kappa};q^{\kappa})_{\infty}^{n-1}(q^{\kappa/2};q^{\kappa/2})_{\infty}}{(q;q)_{\infty}^{n-1}(q^{1/2};q^{1/2})_{\infty}}\prod_{i=1}^{n}\theta(q^{i+m};q^{\kappa/2})\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-1};q^{\kappa}),
\end{aligned}$$

where $\kappa=2m+2n+1$. □

Theorem 5.14 ($\mathrm{D}_{n+1}^{(2)}$ Rogers–Ramanujan identities). *For m,n positive integers and $\kappa:=m+2n$,*

(5.3.8)
$$\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} q^{|\lambda|/2}P_{\lambda}(1,q,q^2,\dots;q^{2n-1}) = \frac{(q^{\kappa};q^{\kappa})_{\infty}^{n}}{(q;q)_{\infty}^{n-1}(q^{1/2};q)_{\infty}(q^2;q^2)_{\infty}}\prod_{i=1}^{n}\theta(q^{i+(m-1)/2};q^{\kappa})\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-1};q^{\kappa}),$$

and

(5.3.9)
$$\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} q^{|\lambda|/2}\Bigg(\prod_{i=1}^{m-1}(-q^n;q^n)_{m_i(\lambda)}\Bigg)P_{\lambda}(1,q,q^2,\dots;q^{2n}) = \frac{(q^{\kappa};q^{\kappa})_{\infty}^{n-1}(q^{\kappa/2};q^{\kappa/2})_{\infty}}{(q;q)_{\infty}^{n-1}(q^{1/2};q)_{\infty}^{2}(q^2;q^2)_{\infty}}\prod_{i=1}^{n}\theta(q^{i-1/2};q^{\kappa/2})\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-1};q^{\kappa}).$$

For $n=1$ the first of these identities is the second equation on page 235 of [**137**].

Sketch of the proof. In the character identity (5.2.16) we carry out the specialisation

$$F:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q^{1/2}]]$$

given by

$$F(\mathrm{e}^{-\alpha_0})=q^{1/2},\quad F(\mathrm{e}^{-\alpha_n})=-1\quad\text{and}\quad F(\mathrm{e}^{-\alpha_i})=q\quad\text{for } 1\leqslant i\leqslant n-1.$$

Noting that F applied to (5.2.15) yields

$$F(x_i)=q^{n-i}\quad(1\leqslant i\leqslant n-1),\qquad F(t^{1/2})=q^{n-1/2},$$

and following the proof of (5.3.1), it follows that the right-hand side of (5.2.16) maps to the left-hand side of (5.3.8). If we parametrise $\Lambda \in P_+$ as (compare with (2.7.5a))

$$\Lambda = c_0\varpi_0 + (\lambda_1 - \lambda_2)\varpi_1 + \cdots + (\lambda_{n-1} - \lambda_n)\varpi_{n-1} + 2\lambda_n\varpi_n, \tag{5.3.10}$$

with $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition or half-partition and c_0 a nonnegative, and again follow the previous proof, we find

$$\begin{aligned} F\big(\mathrm{e}^{-\Lambda}\operatorname{ch} V(\Lambda)\big) = \frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^{n-1}(q^{1/2};q)_\infty(q^2;q^2)_\infty} &\prod_{i=1}^n \theta(q^{\lambda_i+n-i+(\kappa+1)/2};q^\kappa) \\ \times &\prod_{1\leqslant i<j\leqslant n} \theta(q^{\lambda_i-\lambda_j-i+j}, q^{\lambda_i+\lambda_j+2n-i-j+1};q^\kappa), \end{aligned}$$

where $\kappa := 2n + c_0 + 2\lambda_1$. The only change compared to the proof of (5.3.1) is that we have used the $\mathrm{B}_n^{(1)}$ instead of $\mathrm{D}_{n+1}^{(2)}$ Macdonald identity. For $\Lambda = m\varpi_0$, i.e., $\lambda = 0$ and $c_0 = m$ the above product gives the right-hand side of (5.3.8), completing the proof.

Similarly, to prove (5.3.9) we apply the specialisation F to (5.2.11), where this time

$$\begin{gathered} F:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]] \to \mathbb{C}[[q^{1/2}]] \\ F(\mathrm{e}^{-\alpha_0}) = F(\mathrm{e}^{-\alpha_n}) = q^{1/2} \quad \text{and} \quad F(\mathrm{e}^{-\alpha_i}) = q \quad \text{for } 1 \leqslant i \leqslant n-1. \end{gathered}$$

Applied to (5.2.9) this gives

$$F(x_i) = q^{n-i+1/2} \quad (1 \leqslant i \leqslant n), \qquad F(t^{1/2}) = q^n,$$

so that the left side of (5.2.11) specialises to the the right side of (5.3.9). With the same parametrisation of Λ as in (5.3.10) and once more using the $\mathrm{D}_{n+1}^{(2)}$ Macdonald identity, it follows that

$$\begin{aligned} F\big(\mathrm{e}^{-\Lambda}\operatorname{ch} V(\Lambda)\big) = \frac{(q^\kappa;q^\kappa)_\infty^{n-1}(q^{\kappa/2};q^{\kappa/2})_\infty(-q^n;q^n)_\infty}{(q;q)_\infty^{n-1}(q^{1/2};q)_\infty^2(q^2;q^2)_\infty} &\prod_{i=1}^n \theta(q^{\lambda_i+n-i+1/2};q^{\kappa/2}) \\ \times &\prod_{1\leqslant i<j\leqslant n} \theta(q^{\lambda_i-\lambda_j-i+j}, q^{\lambda_i+\lambda_j+2n-i-j+1};q^\kappa) \end{aligned}$$

with κ as before. For $\Lambda = m\varpi_0$, i.e., $\lambda = 0$ and $c_0 = m$ this gives the right-hand side of (5.3.9). $\square$

Theorem 5.15 ($\mathrm{B}_n^{(1)}$ Rogers–Ramanujan identity). *Let m, n be a positive integers and $\kappa := m + 2n - 1$. Then*

$$\begin{aligned} &\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} q^{|\lambda|/2} \bigg(\prod_{i=1}^{m-1} (-q^{n-1/2};q^{n-1/2})_{m_i(\lambda)}\bigg) P_\lambda(1,q,q^2,\dots;q^{2n-1}) \\ &\quad= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty} \prod_{i=1}^n \theta(q^{i+m/2-1/2};q^\kappa) \prod_{1\leqslant i<j\leqslant n} \theta(q^{j-i},q^{i+j-2};q^\kappa). \end{aligned} \tag{5.3.11}$$

By (2.8.20),

$$P_\lambda(1,q,q^2,\dots;q)=\frac{q^{n(\lambda)}}{b_\lambda(q)}\overset{\lambda_1\leqq m}{=}\prod_{i=1}^{m}\frac{q^{\binom{\lambda_i'}{2}}}{(q;q)_{m_i(\lambda)}}.$$

This shows that if we replace $q\mapsto q^2$, $\lambda_i'\mapsto N_i$ and $m_i(\lambda)\mapsto n_i$ (so that $N_i=n_i+\cdots+n_m$) in the $n=1$ case of Theorem 5.15 we obtain Bressoud's even modulus identity [**14**, **15**]

$$\sum_{n_1,\cdots,n_m\geqslant 0}\frac{q^{N_1^2+\cdots+N_m^2}}{(q;q)_{n_1}\cdots(q;q)_{n_{m-1}}(q^2;q^2)_{n_m}}=\frac{(q^{m+1},q^{m+1},q^{2m+2};q^{2m+2})_\infty}{(q;q)_\infty}.$$

Remark 5.16. If after the substitution (5.2.27) we let x_n tend to $t^{1/2}$ instead of 1, and then specialise $(a,b)=(-t^{1/2},-1)$, we obtain an identity for $\chi_m(\mathrm{B}_n^{(1)})$ not included in Section 5.2. Upon specialisation this yields a companion to (5.3.11) as follows:

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leqslant m}}q^{|\lambda|}\biggl(\prod_{i=1}^{m-1}(-q^{n-1/2};q^{n-1/2})_{m_i(\lambda)}\biggr)P_\lambda(1,q,q^2,\dots;q^{2n-1})\qquad\qquad&\\ =\frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\prod_{i=1}^{n}\theta(q^{i-1/2};q^\kappa)\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-1};q^\kappa).&\end{aligned}$$

For $n=1$ this corresponds to

$$\sum_{n_1,\cdots,n_m\geqslant 0}\frac{q^{N_1^2+\cdots+N_m^2+N_1+\cdots+N_m}}{(q;q)_{n_1}\cdots(q;q)_{n_{m-1}}(q^2;q^2)_{n_m}}=\frac{(q,q^{2m+1},q^{2m+2};q^{2m+2})_\infty}{(q;q)_\infty},$$

again due to Bressoud [**14**, **15**].

Sketch of the proof of Theorem 5.15. We start with the $\mathrm{B}_n^{(1)\dagger}$ formula (5.2.18) and make the specialisation

$$F^\dagger:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q^{1/2}]] \tag{5.3.12a}$$

$$F^\dagger(\mathrm{e}^{-\alpha_0})=q^{1/2}\quad\text{and}\quad F^\dagger(\mathrm{e}^{-\alpha_i})=q\quad\text{for } 1\leqslant i\leqslant n. \tag{5.3.12b}$$

Applied to (5.2.17) this yields

$$F^\dagger(x_i)=q^{n-i}\quad(1\leqslant i\leqslant n-1),\qquad\qquad F^\dagger(t)=q^{2n-1},$$

so that, up to a factor $(-q^{n-1/2};q^{n-1/2})_\infty$, the left-hand side of (5.3.11) follows by application of $F^\dagger$.

To obtain the product-form on the right we first consider the more general $\mathrm{B}_n^{(1)}$ specialisation formula

$$\begin{aligned}F\bigl(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\bigr)=\frac{(q^\kappa;q^\kappa)_\infty^n(-q^{n-1/2};q^{n-1/2})_\infty}{(q;q)_\infty^{n-1}(q^{1/2};q^{1/2})_\infty}\prod_{i=1}^{n}\theta(q^{\lambda_i+n-i+1/2};q^\kappa)&\\ \times\prod_{1\leqslant i<j\leqslant n}\theta(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};q^\kappa),&\end{aligned}$$

where Λ and κ are as in Lemma A.1, and where F is the specialisation

$$F(\mathrm{e}^{-\alpha_n})=q^{1/2}\quad\text{and}\quad F(\mathrm{e}^{-\alpha_i})=q\quad\text{for } 0\leqslant i\leqslant n-1$$

of $\mathrm{B}_n^{(1)}$. Proof of this result follows from the $\mathrm{B}_n^{(1)}$ Macdonald identity. Again $F^\dagger$ and F are compatible so that

$$\begin{aligned}
&F^\dagger\big(\mathrm{e}^{-m\varpi_0}\,\mathrm{ch}\,V(m\varpi_0)\big)\big|_{\mathfrak{g}=\mathrm{B}_n^{(1)\dagger}} \\
&\quad= F\big(\mathrm{e}^{-m\varpi_n}\,\mathrm{ch}\,V(m\varpi_n)\big)\big|_{\mathfrak{g}=\mathrm{B}_n^{(1)}} \\
&\quad= \frac{(q^\kappa;q^\kappa)_\infty^n(-q^{n-1/2};q^{n-1/2})_\infty}{(q;q)_\infty^n(q^{1/2};q)_\infty}\prod_{i=1}^n\theta(q^{i+m/2-1/2};q^\kappa) \\
&\qquad\qquad\times\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-2};q^\kappa),
\end{aligned}$$

where $\kappa=m+2n-1$. Up to the factor $(-q^{n-1/2};q^{n-1/2})_\infty$ this is the right-hand side of (5.3.11). ∎

Theorem 5.17 ($\mathrm{A}_{2n-1}^{(2)}$ Rogers–Ramanujan identities). *Let m,n be a positive integers and $\kappa:=2m+2n$. Then*

$$\begin{aligned}
&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}} q^{|\lambda|/2+(n-1/2)\,\mathrm{odd}(\lambda)}P_\lambda(1,q,q^2,\dots;q^{2n-1}) \\
&\qquad= \frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^n(q;q^2)_\infty}\prod_{i=1}^n\theta(q^{i+\kappa/2-1};q^\kappa)\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-2};q^\kappa)
\end{aligned}\tag{5.3.13}$$

and

$$\begin{aligned}
&{\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m}}}' q^{|\lambda|/2+n\,\mathrm{odd}(\lambda)}\bigg(\prod_{i=1}^{2m-1}(q^{2n};q^{4n})_{\lceil m_i(\lambda)/2\rceil}\bigg)P_\lambda(1,q,q^2,\dots;q^{2n}) \\
&\qquad= \frac{(q^\kappa;q^\kappa)_\infty^n(-q^{\kappa/2};q^\kappa)_\infty}{2(q;q)_\infty^n} \\
&\qquad\quad\times\prod_{i=1}^n\theta(-q^{i-1},q^{i+\kappa/2-1};q^\kappa)\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-2};q^\kappa),
\end{aligned}\tag{5.3.14}$$

where the prime denotes the restriction $m_i(\lambda)\equiv 0 \pmod 2$ for $i=1,3,\dots,2m-1$.

In the rank-1 case (5.3.13) can also be written as

$$\sum_{n_1\geqslant\cdots\geqslant n_{2m}\geqslant 0}\frac{q^{\frac{1}{2}(N_1^2+\cdots+N_{2m}^2+n_1+n_3+\cdots+n_{2m-1})}}{(q;q)_{n_1}\cdots(q;q)_{n_{2m}}}=\frac{(q^{m+1},q^{m+1},q^{2m+2};q^{2m+2})_\infty}{(q;q)_\infty(q;q^2)_\infty},$$

where $N_i=n_i+\cdots+n_{2m}$. Since [**139**, Lemma A.1]

$$\begin{aligned}
&\sum_{n_1,\dots,n_{2m}\geqslant 0}\frac{a_1^{n_1}a_2^{n_2}\cdots a_{2m}^{n_{2m}}q^{\frac{1}{2}(N_1^2+\cdots+N_{2m}^2)}}{(q;q)_{n_1}\cdots(q;q)_{n_{2m}}} \\
&\qquad= \sum_{n_1,\dots,n_m\geqslant 0}\frac{a_2^{n_1}a_4^{n_2}\cdots a_{2m}^{n_m}q^{N_1^2+\cdots+N_m^2}(-q^{1/2-N_1}a_1/a_2;q)_{N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_m}},
\end{aligned}$$

provided that $a_{2i}/a_{2i-1}=a_2/a_1$ for all $2\leqslant i\leqslant m$, this may also be written as

$$\sum_{n_1,\dots,n_m\geqslant 0}\frac{q^{N_1^2+\cdots+N_m^2}(-q^{1-N_1};q)_{N_1}}{(q;q)_{n_1}\cdots(q;q)_{n_m}}=\frac{(q^{m+1},q^{m+1},q^{2m+2};q^{2m+2})_\infty}{(q;q)_\infty(q;q^2)_\infty}.$$

For $m=1$ this is identity (12) in Slater's list of Rogers–Ramanujan-type identities [**122**].

Sketch of the proof. The first result follows from the principal specialisation [**77**, **78**, **86**]

$$\begin{gathered}F:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q]]\\ F(\mathrm{e}^{-\alpha_i})=q\quad\text{for }0\leqslant i\leqslant n\end{gathered}$$

applied to the $\mathrm{A}_{2n-1}^{(2)\dagger}$ formula (5.2.19). Since this specialisation does not distinguish between $\mathrm{A}_{2n-1}^{(2)\dagger}$ and $\mathrm{A}_{2n-1}^{(2)}$, we can use the general $\mathrm{A}_{2n-1}^{(2)}$ principal specialisation formula [**78**, **86**]

$$\begin{aligned}F\big(\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)\big)=\frac{(q^\kappa;q^\kappa)_\infty^n}{(q;q)_\infty^n(q;q^2)_\infty}&\prod_{i=1}^n\theta(q^{\lambda_i+n-i+1};q^\kappa)\\ &\times\prod_{1\leqslant i<j\leqslant n}\theta(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};q^\kappa),\end{aligned}$$

where $\kappa=2n+c_0+\lambda_1+\lambda_2$ and

$$\Lambda=c_0\varpi_0+(\lambda_1-\lambda_2)\varpi_1+\cdots+(\lambda_{n-1}-\lambda_n)\varpi_{n-1}+\lambda_n\varpi_n$$

for c_0 a nonnegative integer and $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition. Taking $c_0=0$ and $\lambda=m^n$ gives the right-hand side of (5.3.13). The left-hand side follows in the usual way, noting that

$$F(x_i)=q^{n-i+1}\quad(1\leqslant i\leqslant n-1)\quad\text{and}\quad F(t)=q^{2n-1}.$$

The identity (5.3.14) follows from the specialisation F applied to the $\mathrm{A}_{2n-1}^{(2)\dagger}$ formula (5.2.8), where now F stands for

$$F:\ \mathbb{C}[[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}]]\to\mathbb{C}[[q]]\tag{5.3.15a}$$

$$F(\mathrm{e}^{-\alpha_i})=q\quad\text{for }0\leqslant i\leqslant n-1\quad\text{and}\quad F(\mathrm{e}^{-\alpha_n})=q^2.\tag{5.3.15b}$$

According to (A.1.6) this yields

$$F(x_i)=q^{n-i+1/2}\quad(1\leqslant i\leqslant n)\quad\text{and}\quad F(t)=q^{2n},\tag{5.3.16}$$

which we should apply to the $\mathrm{A}_{2n-1}^{(2)\dagger}$ character given in (A.1.8). Unlike the previous cases, the steps required to obtain the product form are slightly different to those in the proof of (5.3.1), and below we outline the key steps in the derivation.

From (5.3.16), the D_n Vandermonde determinant and multilinearity, it follows that

$$F\biggl(2\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(xt^r)\prod_{i=1}^n(-1)^{r_i}x_i^{\kappa r_i-i+1}t^{\frac{1}{2}\kappa r_i^2-(n-1)r_i}\biggr)$$
$$=\det_{1\leqslant i,j\leqslant n}\biggl(\sum_{r\in\mathbb{Z}}(-1)^r q^{(\kappa r-i+j)(n-i+1/2)+n\kappa r^2-2nr(n-j)}$$
$$+\sum_{r\in\mathbb{Z}}(-1)^r q^{(\kappa r-i-j+2n)(n-i+1/2)+n\kappa r^2+2nr(n-j)}\biggr).$$

After interchanging i and j and negating r in the first sum, the right-hand side becomes

$$\det_{1\leqslant i,j\leqslant n}\biggl(\sum_{r\in\mathbb{Z}}(-1)^r q^{2n\kappa\binom{r}{2}+\kappa r/2+(2nr-i+1)(n-i)}$$
$$\times\Bigl(q^{(\kappa r+n-i)(j-1)}+q^{(\kappa r+n-i)(2n-j)}\Bigr)\biggr)$$
$$=\det_{1\leqslant i,j\leqslant n}\biggl(\sum_{r\in\mathbb{Z}}(-1)^r x_i^{2nr-i+1}q^{2n\kappa\binom{r}{2}+\kappa r/2}\Bigl((x_iq^{\kappa r})^{j-1}-(x_iq^{\kappa r})^{2n-j}\Bigr)\biggr),$$

where $x_i:=-q^{n-i}$. Up to the change $q^{\kappa/2}\mapsto q^{-\kappa/2}$, the above determinant is the same as the one on the right of (5.3.5). From here on we can thus follow the previous computations to find

$$F\biggl(2\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(xt^r)\prod_{i=1}^n(-1)^{r_i}x_i^{\kappa r_i-i+1}t^{\frac{1}{2}\kappa r_i^2-(n-1)r_i}\biggr)$$
$$=(q^\kappa;q^\kappa)_\infty^n(-q^{\kappa/2};q^\kappa)_\infty\prod_{i=1}^n\theta(-q^{i-1},q^{i+\kappa/2-1};q^\kappa)$$
$$\times\prod_{1\leqslant i<j\leqslant n}\theta(q^{j-i},q^{i+j-2};q^\kappa).\quad\square$$

We conclude this section with two remarks. First of all, we have not considered the specialisations of (5.2.5) and (5.2.13) as the resulting $\mathrm{C}_n^{(1)}$ and $\mathrm{A}_{2n}^{(2)}$ identities were already obtained in [**47**], the $\mathrm{A}_{2n}^{(2)}$ case corresponding to a generalisation of the Rogers–Ramanujan and Andrews–Gordon identities for odd moduli. We have also omitted the specialisation of (5.2.6) and (5.2.14), but for different reasons. The most natural substitutions on the combinatorial sides would be $x_i\mapsto q^{n-i+1/2}$, $(1\leqslant i\leqslant n)$, $t\mapsto t^{2n}$ and $x_i\mapsto q^{n-i+1}$ $(1\leqslant i\leqslant n-1)$, $t\mapsto t^{2n-1}$ respectively. This corresponds to the specialisations

$$F(\mathrm{e}^{-\alpha_0})=q^2\quad\text{and}\quad F(\mathrm{e}^{-\alpha_i})=q\quad\text{for } 1\leqslant i\leqslant n$$

for $\mathrm{A}_{2n-1}^{(2)}$, and

$$F(\mathrm{e}^{-\alpha_i})=q\quad\text{for } 1\leqslant i\leqslant n-1\quad\text{and}\quad F(\mathrm{e}^{-\alpha_0})=q^2,\ F(\mathrm{e}^{-\alpha_n})=-1$$

for $\mathrm{B}_n^{(2)}$. However, $F(\mathrm{e}^{-m\varpi_0}\operatorname{ch}V(m\varpi_0))$ does not factor for such F.

5.4. Quadratic transformations for Kaneko–Macdonald-type basic hypergeometric series

5.4.1. Kaneko–Macdonald-type basic hypergeometric series. Basic hypergeometric series of Kaneko–Macdonald type, which were first introduced in [**58**, **94**], are an important generalisation of ordinary basic hypergeometric series to multiple series with Macdonald polynomial argument. They have been extensively studied in [**8**, **58**, **60**, **75**, **76**, **94**, **108**, **109**] and applied to problems in enumerative and algebraic combinatorics, such as the enumeration of Lozenge tilings [**116**], the computation of the major index generating function of standard Young tableaux [**69**] and the evaluation of Selberg integrals and Dyson-like constant terms identities [**58**–**60**, **138**, **140**–**142**].

For $x=(x_1,\dots,x_n)$, the Kaneko–Macdonald basic hypergeometric series ${}_r\Phi_s$ is defined as

$$\begin{aligned}&{}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,t;x\right]\\&\qquad:=\sum_{\lambda}\frac{(a_1,\dots,a_r;q,t)_\lambda}{(b_1,\dots,b_s;q,t)_\lambda}\Big((-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\Big)^{s-r+1}\,\frac{t^{n(\lambda)}P_\lambda(x;q,t)}{C^-_\lambda(q;q,t)}.\end{aligned}$$

For our purposes it suffices to consider the principal specialisation

$$\begin{aligned}(5.4.1)\quad &{}_r\Phi_s^{(n)}\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,t;z\right]\\&:={}_r\Phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,t;z(1,t,\dots,t^{n-1})\right]\\&=\sum_{\substack{\lambda\\ l(\lambda)\leqslant n}}\frac{(t^n,a_1,\dots,a_r;q,t)_\lambda}{(b_1,\dots,b_s;q,t)_\lambda}\Big((-1)^{|\lambda|}q^{n(\lambda')}t^{-n(\lambda)}\Big)^{s-r+1}\,\frac{z^{|\lambda|}t^{2n(\lambda)}}{C^-_\lambda(q,t;q,t)}.\end{aligned}$$

Since $C^-_r(q,t;q,t)=(q,t;q)_r$ we have ${}_r\Phi_s^{(1)}={}_r\phi_s$, with on the right an ordinary basic hypergeometric series, see [**43**]. Due to the factor $(t^n;q,t)_\lambda$, the summand vanishes unless $l(\lambda)\leqslant n$. For generic $b_1,\dots,b_r$ the restriction in the sum over λ may thus be dropped. If $a_r=q^{-m}$ the series terminates, with support given by $\lambda\subset m^n$. A ${}_{r+1}\Phi_r^{(n)}$ series is said to be balanced if

$$t^{n-1}a_1\cdots a_{r+1}=b_1\cdots b_r\quad\text{and}\quad z=q.$$

Assume that $r\geqslant s$. Replacing λ by its complement with respect to m^n, and using (2.2.8) as well as

$$(5.4.2)\qquad (a;q,t)_{m^n}=(-a)^{mn}q^{n\binom{m}{2}}t^{-m\binom{n}{2}}(q^{1-m}t^{n-1}/a;q,t)_{m^n},$$

gives

(5.4.3)

$$\begin{aligned}&{}_{r+1}\Phi_s^{(n)}\left[\begin{matrix}a_1,\dots,a_r,q^{-m}\\ b_1,\dots,b_s\end{matrix};q,t;z\right]\\&=\Big(\frac{z}{q}\Big)^{mn}\Big((-1)^{mn}q^{n\binom{m}{2}}t^{-m\binom{n}{2}}\Big)^{s-r-1}\,\frac{(a_1,\dots,a_r;q,t)_{m^n}}{(b_1,\dots,b_s;q,t)_{m^n}}\\&\quad\times{}_{r+1}\Phi_r^{(n)}\left[\begin{matrix}0^{r-s},q^{1-m}t^{n-1}/b_1,\dots,q^{1-m}t^{n-1}/b_s,q^{-m}\\ q^{1-m}t^{n-1}/a_1,\dots,q^{1-m}t^{n-1}/a_r\end{matrix};q,t;\frac{b_1\cdots b_sq^{m+1}}{a_1\cdots a_rzt^{n-1}}\right].\end{aligned}$$

Here 0^{r-s} represents $r-s$ numerator parameters equal to 0. Similarly, replacing λ by its conjugate and applying (2.2.6) yields the duality relation

$$
\begin{aligned}
(5.4.4)\quad {}_{r+1}\Phi_s^{(n)}&\left[\begin{matrix} a_1,\dots,a_r,q^{-m} \\ b_1,\dots,b_s \end{matrix};q,t;z\right] \\
&= {}_{r+1}\Phi_r^{(m)}\left[\begin{matrix} 1/a_1,\dots,1/a_r,t^{-n} \\ 0^{r-s},1/b_1,\dots,1/b_s \end{matrix};t,q;\frac{a_1\cdots a_r z t^n}{b_1\cdots b_s q^m}\right],
\end{aligned}
$$

where it is again assumed that $r \geqslant s$.

The expression of the (monic) Askey–Wilson polynomials as a balanced ${}_4\phi_3$ series [**6**] has an analogue for principally specialised Koornwinder polynomials indexed by the rectangular partition m^n.

Lemma 5.18. *For m a nonnegative integer,*

$$
\begin{aligned}
(5.4.5)\quad & K_{m^n}\big(z(1,t,\dots,t^{n-1});q,t;t_0,t_1,t_2,t_3\big) \\
&= t_0^{-mn} t^{-m\binom{n}{2}} \frac{(t_0t_1t^{n-1},t_0t_2t^{n-1},t_0t_3t^{n-1};q,t)_{m^n}}{(t_0t_1t_2t_3q^{m-1}t^{n-1};q,t)_{m^n}} \\
&\quad\times {}_4\Phi_3^{(n)}\left[\begin{matrix} zt_0t^{n-1},t_0/z,t_0t_1t_2t_3q^{m-1}t^{n-1},q^{-m} \\ t_0t_1t^{n-1},t_0t_2t^{n-1},t_0t_3t^{n-1} \end{matrix};q,t;q\right].
\end{aligned}
$$

For later use we note that by (5.4.2) and (5.4.3) we may rewrite this as

$$
\begin{aligned}
(5.4.6)\quad & K_{m^n}\big(z(1,t,\dots,t^{n-1});q,t;t_0,t_1,t_2,t_3\big) \\
&= z^{-mn} t^{-m\binom{n}{2}} (zt_0t^{n-1},zq^{1-m}t^{n-1}/t_0;q,t)_{m^n} \\
&\quad\times {}_4\Phi_3^{(n)}\left[\begin{matrix} q^{1-m}/t_0t_1,q^{1-m}/t_0t_2,q^{1-m}/t_0t_3,q^{-m} \\ zq^{1-m}t^{n-1}/t_0,q^{1-m}/zt_0,q^{2-2m}/t_0t_1t_2t_3 \end{matrix};q,t;q\right].
\end{aligned}
$$

We further note that the symmetry of the left-hand side of (5.4.5) under permutation of the t_i implies the multiple Sears transformation [**8**, Eq. (5.10)]

$$
\begin{aligned}
(5.4.7)\quad {}_4\Phi_3^{(n)}&\left[\begin{matrix} a,b,c,q^{-m} \\ d,e,f \end{matrix};q,t;q\right] \\
&= \frac{(e/a,f/a;q,t)_{m^n}}{(e,f;q,t)_{m^n}}\, a^{mn}\, {}_4\Phi_3^{(n)}\left[\begin{matrix} a,d/b,d/c,q^{-m} \\ d,de/bc,df/bc \end{matrix};q,t;q\right],
\end{aligned}
$$

where $t^{n-1}abc = q^{m-1}def$.

Proof of Lemma 5.18. In [**102**, Theorem 7.10] Okounkov gives an expansion of the Koornwinder polynomials in terms of BC_n interpolation polynomials $\bar{P}^*_\mu(x;q,t,s)$ [**102, 108**]. The coefficients in this expansion are BC_n q-binomial coefficients $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s}$ (see e.g., [**108**, page 64]) times a ratio of principally specialised Koornwinder polynomials:

$$
\begin{aligned}
(5.4.8)\quad & K_\lambda(x;q,t;t_0,t_1,t_2,t_3) \\
&= \sum_{\mu\subset\lambda} \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} \frac{K_\lambda\big(t_0(1,t,\dots,t^{n-1});q,t;t_0,t_1,t_2,t_3\big)}{K_\mu\big(t_0(1,t,\dots,t^{n-1});q,t;t_0,t_1,t_2,t_3\big)}\, \bar{P}^*_\mu(x;q,t,t_0),
\end{aligned}
$$

where $s = t^{n-1}\sqrt{t_0t_1t_2t_3/q}$.

Specialising $x = z(1, t, \dots, t^{n-1})$ and $\lambda = m^n$ in (5.4.8), and using [**108**, Proposition 4.1]

$$\begin{bmatrix} m^n \\ \mu \end{bmatrix}_{q,t,s} = (-q)^{|\mu|} t^{n(\mu)} q^{n(\mu')} \, \frac{(t^n, q^{-m}, s^2 q^m t^{1-n}; q, t)_\mu}{C^-_\mu(q, t; q, t) C^+_\mu(s^2; q, t)}$$

as well as [**35**, Theorem 3], [**120**]

$$\begin{aligned} &K_\lambda\bigl(t_0(1, t, \dots, t^{n-1}); q, t;\, t_0, t_1, t_2, t_3\bigr) \\ &\qquad = \frac{t^{n(\lambda)}}{(t_0 t^{n-1})^{|\lambda|}} \cdot \frac{(t^n, t_0 t_1 t^{n-1}, t_0 t_2 t^{n-1}, t_0 t_3 t^{n-1}; q, t)_\lambda}{C^-_\lambda(t; q, t) C^+_\lambda(t_0 t_1 t_2 t_3 t^{2n-2}/q; q, t)} \end{aligned}$$

and [**108**, Corollary 3.11]

$$\bar{P}^*_\mu\bigl(z(1, t, \dots, t^{n-1}); q, t, s\bigr) = \frac{t^{2n(\mu)} q^{-n(\mu')}}{(-s t^{n-1})^{|\mu|}} \cdot \frac{(t^n, s/z, szt^{n-1}; q, t)_\mu}{C^-_\mu(t; q, t)},$$

we obtain (5.4.6). □

Lemma 2.3 implies an analogue of (5.4.6) for the Macdonald–Koornwinder polynomial $K_{m^n}(q, t; t_2, t_3)$.

Lemma 5.19. *For m a nonnegative integer or half-integer,*

$$\begin{aligned} &K_{m^n}\bigl(z(1, t, \dots, t^{n-1}); q, t; t_2, t_3\bigr) \\ &\quad = z^{-mn} t^{-m\binom{n}{2}} (-zq^{1/2-m} t^{n-1}; q, t)_{(2m)^n} \\ &\qquad \times {}_4\Phi_3^{(n)}\left[\begin{matrix} -q^{1/2-m}/t_2, -q^{1/2-m}/t_3, q^{1/2-m}, q^{-m} \\ -zq^{1/2-m} t^{n-1}, -q^{1/2-m}/z, q^{3/2-2m}/t_2 t_3 \end{matrix}; q, t; q\right]. \end{aligned} \tag{5.4.9}$$

Proof. When m is an integer the claim follows from (5.4.6) by specialising $\{t_0, t_1\} = \{-q^{1/2}, -1\}$ and applying

$$(a, aq^{-m}; q, t)_{m^n} = (aq^{-m}; q, t)_{(2m)^n}. \tag{5.4.10}$$

When m is a half-integer we set $m = k + 1/2$, where k a nonnegative integer. By Lemma 2.3, we then need to show that

$$\begin{aligned} &K_{k^n}\bigl(z(1, t, \dots, t^{n-1}); q, t; -q, -q^{1/2}, t_2, t_3\bigr) \\ &\quad = z^{-kn} t^{-k\binom{n}{2}} (-zq^{-k} t^{n-1}, -zqt^{n-1}; q, t)_{k^n} \\ &\qquad \times {}_4\Phi_3^{(n)}\left[\begin{matrix} -q^{-k}/t_2, q^{-k}/t_3, q^{-k-1/2}, q^{-k} \\ -zq^{-k} t^{n-1}, -q^{-k}/z, q^{1/2-2k}/t_2 t_3 \end{matrix}; q, t; q\right]. \end{aligned} \tag{5.4.11}$$

Here we have also used

$$\prod_{i=1}^n \bigl(x_i^{1/2} + x_i^{-1/2}\bigr)\big|_{x_i = zt^{i-1}} = z^{-n/2} t^{-\binom{n}{2}/2} (-z; t)_n$$

and

$$\frac{(-zq^{-k} t^{n-1}; q, t)_{(2k+1)^n}}{(-z; t)_n} = (-zq^{-k} t^{n-1}, zqt^{n-1}; q, t)_{k^n}.$$

Since (5.4.11) is (5.4.6) with $(t_0, t_1, m) \mapsto (-q, -q^{1/2}, k)$ we are done. □

Equipped with the above lemmas we can specialise Theorems 4.1 and 4.6 to obtain quadratic transformation formulas for Kaneko–Macdonald-type basic hypergeometric series.

5.4.2. The specialisation of Theorem 4.1. Our first result arises by specialising $x \mapsto z(1, t, \dots, t^{n-1})$ in the bounded Littlewood identity (4.1.5).

PROPOSITION 5.20. *For m a nonnegative integer,*

$$\begin{aligned}(5.4.12)\quad & {}_4\Phi_3^{(n)}\left[\begin{matrix} a, -a, -t^n, q^{-m} \\ aq^{1/2}t^n, -aq^{1/2}t^n, -q^{-m}/t \end{matrix}; q, t; q\right] \\ &\qquad = \frac{(a^2t)^{mn} q^{m^2n} (qt^{2n}; q, t)_{m^n}}{(a^2qt^{2n}; q^2, t^2)_{m^n} (-qt^n; q, t)_{m^n}} \, {}_2\Phi_1^{(n)}\left[\begin{matrix} q^{-m}/t, q^{-m} \\ qt^{2n-1} \end{matrix}; q, t^2; \frac{q}{a^2}\right].\end{aligned}$$

Before giving the details of the proof, we first present two equivalent identities which allow us to connect with quadratic transformation formulas of Cohl et al. [**28**] and Gasper and Rahman [**42**]. To this end we need the very-well poised multiple basic hypergeometric series [**29**, **108**, **136**]

$$\begin{aligned}(5.4.13)\quad & {}_{r+1}W_r^{(n)}(a_1; a_4, \dots, a_{r+1}; q, t; z) \\ &:= \sum_{\substack{\lambda \\ l(\lambda)\leqslant n}} \left(\frac{(a_1q, a_1q^2, a_1q/t, a_1q^2/t; q^2, t^2)_\lambda}{C_\lambda^+(a_1, a_1q/t; q, t)} \right. \\ &\qquad \left. \times \frac{(t^n, a_4, \dots, a_{r+1}; q, t)_\lambda}{(a_1q/t^n, a_1q/a_4, \dots, a_1q/a_{r+1}; q, t)_\lambda} \cdot \frac{z^{|\lambda|} t^{2n(\lambda)}}{C_\lambda^-(q, t; q, t)} \right),\end{aligned}$$

where $|q|, |z| < 1$ if the series does not terminate. For $n = 1$ this definition simplifies to that of the very-well poised basic hypergeometric series

$${}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) := \sum_{k=0}^\infty \frac{1 - a_1q^{2k}}{1 - a_1} \cdot \frac{(a_1, a_4, \dots, a_{r+1}; q)_k \, z^k}{(q, a_1q/a_4, \dots, a_1q/a_{r+1}; q)_k},$$

see [**43**]. For nonnegative integers n and m it follows from (2.2.6) that

$$\begin{aligned}(5.4.14)\quad & {}_{r+1}W_r^{(n)}(a_1; a_4, \dots, a_r, q^{-m}; q, t; z) \\ &= {}_{r+1}W_r^{(m)}\left((a_1qt)^{-1}; a_4^{-1}, \dots, a_r^{-1}, t^{-n}; t, q; \frac{(a_4 \cdots a_r)^2 z q^{1-2m} t^{2n-1}}{(a_1q)^{r-3}}\right).\end{aligned}$$

Watson's transformation between a very-well poised ${}_8W_7$ series and a balanced ${}_4\phi_3$ series (see [**43**, Equation (III.18)]) has the following multiple analogue [**29**, **108**, **136**]:

$$\begin{aligned}(5.4.15)\quad & {}_8W_7^{(n)}\big(a; b, c, d, e, q^{-m}; q, t; a^2q^{m+2}t^{1-n}/bcde\big) \\ &\qquad = \frac{(aq, aq/de; q, t)_{m^n}}{(aq/d, aq/e; q, t)_{m^n}} \, {}_4\Phi_3^{(n)}\left[\begin{matrix} aq/bc, d, e, q^{-m} \\ aq/b, aq/c, deq^{-m}t^{n-1}/a \end{matrix}; q, t; q\right].\end{aligned}$$

Transforming the right-hand side using the multiple Sears transformation (5.4.7) with

$$(a, b, c, d, e, f) \mapsto (e, aq/bc, d, deq^{-m}t^{n-1}/a, aq/b, aq/c)$$

yields the iterated Watson transformation

$$
\begin{aligned}
(5.4.16)\quad {}_8W_7^{(n)}&\big(a;b,c,d,e,q^{-m};q,t;a^2q^{m+2}t^{1-n}/bcde\big)\\
&=\frac{(aq,aq/be,aq/ce,aq/de;q,t)_{m^n}}{(aq/b,aq/c,aq/d,aq/e;q,t)_{m^n}}\,e^{mn}\\
&\quad\times {}_4\Phi_3^{(n)}\left[\begin{matrix} eq^{-m}t^{n-1}/a,bcdeq^{-1-m}t^{n-1}/a^2,e,q^{-m}\\ beq^{-m}t^{n-1}/a,ceq^{-m}t^{n-1}/a,deq^{-m}t^{n-1}/a\end{matrix};q,t;q\right].
\end{aligned}
$$

Applying (5.4.16) with

$$(a,b,c,d,e)\mapsto\big(q^{-m}t^{2n-1}/a,-q^{1/2}t^n,q^{-m}/at,q^{1/2}t^n,-t^n\big)$$

to the ${}_4\Phi_3^{(n)}$ series in (5.4.12), then replacing a by $1/c$, and finally simplifying some of the q,t-shifted factorials using (5.4.2),

$$(a,-a,aq^{1/2},-aq^{1/2};q,t)_{m^n}=(a;q,t^2)_{(2m)^n}$$

and

$$(a,at^n;q,t)_{m^n}=(at^n;q,t)_{m^{2n}},$$

we obtain the following quadratic transformation formula.

Corollary 5.21. *For m a nonnegative integer,*

$$
\begin{aligned}
(5.4.17)\quad {}_2\Phi_1^{(n)}\left[\begin{matrix} q^{-m}/t,q^{-m}\\ qt^{2n-1}\end{matrix};q,t^2;c^2q\right]&=\frac{(c^2q^{1-2m}t^{2n-2};q,t^2)_{(2m)^n}}{(cq^{1-m}t^{2n-1};q,t)_{m^{2n}}}\\
&\quad\times {}_8W_7^{(n)}\big(cq^{-m}t^{2n-1};cq^{-m}/t,-t^n,q^{1/2}t^n,-q^{1/2}t^n,q^{-m};q,t;cq\big).
\end{aligned}
$$

For $n=1$ this is a terminating version of [**28**, Theorem 9] by Cohl et al., suggesting the following nonterminating analogue.

Let

$$(a;q,t)_{\infty^n}:=\prod_{i=1}^n(at^{1-i};q)_\infty.$$

Theorem 5.22. *For $|q|,|cq|,|c^2q|<1$,*

$$
\begin{aligned}
(5.4.18)\quad {}_2\Phi_1^{(n)}\left[\begin{matrix} a,a/t\\ qt^{2n-1}\end{matrix};q,t^2;c^2q\right]&=\frac{(a^2c^2qt^{2n-2};q,t^2)_{\infty^n}}{(c^2qt^{2n-2};q,t^2)_{\infty^n}}\cdot\frac{(cqt^{2n-1};q,t)_{\infty^{2n}}}{(acqt^{2n-1};q,t)_{\infty^{2n}}}\\
&\quad\times {}_8W_7^{(n)}\big(act^{2n-1};ac/t,a,-t^n,q^{1/2}t^n,-q^{1/2}t^n;q,t;cq\big).
\end{aligned}
$$

An easy consistency check is provided by the $ac=t$ case. Then the ${}_8W_7^{(n)}$ series trivialises to 1 and the ${}_2\Phi_1^{(n)}$ series can be summed by the multiple Gauss sum [**58**, Proposition 5.4]

$$(5.4.19)\qquad {}_2\Phi_1^{(n)}\left[\begin{matrix} a,b\\ c\end{matrix};q,t;\frac{ct^{1-n}}{ab}\right]=\frac{(c/a,c/b;q,t)_{\infty^n}}{(c,c/ab;q,t)_{\infty^n}},$$

for $|q|,|ct^{1-n}/ab|<1$. A full proof of Theorem 5.22 based on a quadratic transformation formula for elliptic Selberg integrals is given in Appendix B.

Applying the dualities (5.4.4) and (5.4.14) to a terminating summation or transformation formula for Kaneko–Macdonald-type basic hypergeometric series, interchanges the roles of m and n (and q and t). Since, by lack of a parameter m, no analogues of these dualities exist in the nonterminating setting, a second inequivalent nonterminating analogue of (5.4.17) can be obtain by first dualising, then interchanging m and n as well as q and t, and finally, after writing the resulting transformation in a suitable form, by replacing q^{-m} by a. To be more precise, the dual of (5.4.17) (after making the substitutions $m \leftrightarrow n$ and $q \leftrightarrow t$) is

$$
{}_2\Phi_1^{(n)}\left[\begin{matrix} qt^n, q^{-2m} \\ q^{1-2m}/t \end{matrix}; q^2, t; c^2t^{-2n}\right] = \frac{(c^2; q^2, t)_{m^{2n}}}{(c; q, t)_{(2m)^n}} \\
\times {}_8W_7^{(n)}\big(q^{-2m}t^{n-1}/c; qt^n/c, q^{-m}t^{-1/2}, -q^{-m}t^{-1/2}, q^{-m}, -q^{-m}; q, t; q/c\big).
$$

Using the generalised Watson and Sears transformations (5.4.15) and (5.4.7), we can transform the ${}_8W_7^{(n)}$ on the right. Also replacing $c \mapsto ct^{1/2}$ this gives our second corollary.[5]

Corollary 5.23. *For m a nonnegative integer,*

$$
\tag{5.4.20} {}_2\Phi_1^{(n)}\left[\begin{matrix} qt^n, q^{-2m} \\ q^{1-2m}/t \end{matrix}; q^2, t; c^2t^{1-2n}\right] = \frac{(c^2q^{-2m}; q^2, t)_{m^{2n}}}{(c^2q^{-m}, q^{-2m}/t; q, t)_{m^n}} \\
\times {}_8W_7^{(n)}\big(c^2q^{-m-1}; c, -c, ct^{1/2}, -ct^{1/2}, q^{-m}; q, t; q^{-m}t^{-n}\big).
$$

For $n = 1$ this is a terminating analogue of a transformation formula of Gasper and Rahman [**42**, Equation (1.4)] (see also [**43**, Equation (3.5.4)]). This suggests a second nonterminating transformation formula.

Theorem 5.24. *For $|q|, |c^2t^{1-2n}|, |at^{-n}| < 1$,*

$$
\tag{5.4.21} {}_2\Phi_1^{(n)}\left[\begin{matrix} a^2, qt^n \\ a^2q/t \end{matrix}; q^2, t; c^2t^{1-2n}\right] = \frac{(a^2c^2; q^2, t)_{\infty^{2n}}}{(c^2; q^2, t)_{\infty^{2n}}} \cdot \frac{(a/t, c^2; q, t)_{\infty^n}}{(a^2/t, ac^2; q, t)_{\infty^n}} \\
\times {}_8W_7^{(n)}\big(ac^2/q; a, c, -c, ct^{1/2}, -ct^{1/2}; q, t; at^{-n}\big).
$$

For a proof of this result we again refer to Appendix B.

Proof of Proposition 5.20. In (4.1.5) we specialise $x \mapsto z(1, t, \dots, t^{n-1})$ and replace the summation index λ by 2λ. On the left side we then use

$$
b^{\mathrm{oa}}_{2\lambda;m}(q,t) = \Big(\frac{q}{t}\Big)^{|\lambda|} \frac{(q^{-2m}; q^2, t)_\lambda}{(q^{1-2m}/t; q^2, t)_\lambda} \cdot \frac{C^-_\lambda(qt; q^2, t)}{C^-_\lambda(q^2; q^2, t)}
$$

(see the proof of Theorem 4.1 on page 47) and

$$
\begin{aligned}
P_{2\lambda}\big(z(1, t, \dots, t^{n-1}); q, t\big) &\overset{(2.5.15)}{=} z^{2|\lambda|}t^{2n(\lambda)}\frac{(t^n; q, t)_{2\lambda}}{C^-_{2\lambda}(t; q, t)} \\
&= z^{2|\lambda|}t^{2n(\lambda)}\frac{(t^n, qt^n; q^2, t)_\lambda}{C^-_\lambda(t, qt; q^2, t)}.
\end{aligned}
$$

As a result we obtain the ${}_2\Phi_1^{(n)}$ series

$$
\tag{5.4.22} {}_2\Phi_1^{(n)}\left[\begin{matrix} qt^n, q^{-2m} \\ q^{1-2m}/t \end{matrix}; q^2, t; \frac{z^2q}{t}\right].
$$

[5] Alternatively, (5.4.20) may be obtained by equating (5.4.22) and (5.4.23) below, and using (5.4.7) and (5.4.15).

On the right we use (5.4.6) with

$$(t_0,t_1,t_2,t_3)=\big(q^{1/2},-q^{1/2},(qt)^{1/2},-(qt)^{1/2}\big)$$

and the elementary relation (5.4.10) to find

(5.4.23)
$$(zq^{1/2-m}t^{n-1};q,t)_{(2m)^n}\;{}_4\Phi_3^{(n)}\biggl[\begin{matrix}q^{-m}t^{-1/2},-q^{-m}t^{-1/2},-q^{-m},q^{-m}\\ zq^{1/2-m}t^{n-1},q^{1/2-m}/z,q^{-2m}t^{-1}\end{matrix};q,t;q\biggr].$$

Next we equate (5.4.22) and (5.4.23)[6], apply the duality (5.4.4) to both sides, and interchange m and n as well as q and t. By

$$(a;t,q)_{n^m}=(aq^{1-m}t^{n-1};q,t)_{m^n}$$

(which follows from (2.2.6a) and (2.2.8a) for $\lambda=m^n$) this yields

$$\begin{aligned}(5.4.24)\quad {}_2\Phi_1^{(n)}&\biggl[\begin{matrix}q^{-m}/t,q^{-m}\\ qt^{2n-1}\end{matrix};q,t^2;z^2q^{2m}t\biggr]\\ &=(zt^{n-1/2};q,t)_{m^{2n}}\;{}_4\Phi_3^{(n)}\biggl[\begin{matrix}q^{1/2}t^n,-q^{1/2}t^n,-t^n,q^{-m}\\ zt^{n-1/2},q^{1-m}t^{n-1/2}/z,qt^{2n}\end{matrix};q,t;q\biggr].\end{aligned}$$

We finally rewrite the right-hand side using the multiple Sears transformation (5.4.7) with

$$(a,b,c,d,e)\mapsto\big(-t^n,q^{1/2}t^n,-q^{1/2}t^n,zt^{n-1/2},q^{1-m}t^{n-1/2}/z\big),$$

and use

$$\begin{aligned}(zt^{n-1/2};q,t)_{m^{2n}}\,&\frac{(-zt^{-1/2},-q^{-m}/t;q,t)_{m^n}}{(zt^{n-1/2},q^{-m}t^{-1-n};q,t)_{m^n}}\\ &=\frac{(z^2t^{-1};q^2,t^2)_{m^n}(-q^{-m}t^{-1};q,t)_{m^n}}{(q^{-m}t^{-1-n};q,t)_{m^n}}\end{aligned}$$

to clean up the prefactor. By the substitution $z\mapsto q^{1/2-m}t^{-1/2}/a$ and application of (5.4.2) the claim follows. □

5.4.3. The specialisation of Theorem 4.6. In this section we consider the principal specialisation of the bounded Littlewood identity (4.1.13).

PROPOSITION 5.25. *For m a nonnegative integer,*

$$\begin{aligned}(5.4.25)\quad {}_4\Phi_3^{(n)}&\biggl[\begin{matrix}a,aq,q^{-m},q^{1-m}\\ aq^{1-m}/t,aq^{2-m}/t,qt^{2n}\end{matrix};q^2,t^2;q^2\biggr]\\ &=\frac{(t^{2n};q^2,t^2)_{m^n}}{(t^{2n},t^{2n-1}/a;q,t^2)_{m^n}}\;{}_2\Phi_1^{(n)}\biggl[\begin{matrix}-t^n,q^{-m}\\ -q^{1-m}/t\end{matrix};q,t;\frac{q}{a}\biggr].\end{aligned}$$

For $n=1$, and up to the change $t\mapsto b$, this is a transformation stated on page 2310 of [**11**].

Again we derive some related results before giving a proof. If we let $c,f\to 0$ and $b,d\to\infty$ in the multiple Sears transformation (5.4.7), such that b/d and f/c are fixed as z/q and $azq^{-m}t^{n-1}/e$ respectively, we find

$${}_2\Phi_1^{(n)}\biggl[\begin{matrix}a,q^{-m}\\ e\end{matrix};q,t;z\biggr]=\frac{(e/a;q,t)_{m^n}}{(e;q,t)_{m^n}}\,a^{mn}\;{}_3\Phi_1^{(n)}\biggl[\begin{matrix}a,q/z,q^{-m}\\ aq^{1-m}t^{n-1}/e\end{matrix};q,t;\frac{z}{e}\biggr].$$

[6]The transformation obtained by equating (5.4.22) and (5.4.23) generalises Verma's quadratic transformation [**135**, Equation (2.5)].

This can be used to transform the right-hand side of (5.4.25), so that an equivalent form of that identity is given by

$$
\begin{aligned}
&{}_4\Phi_3^{(n)}\left[\begin{matrix} a, aq, q^{-m}, q^{1-m} \\ aq^{1-m}/t, aq^{2-m}/t, qt^{2n} \end{matrix}; q^2, t^2; q^2\right] \\
&\qquad = \frac{(q^{1-m}/t; q, t^2)_{m^n}}{(aq^{1-m}/t; q, t^2)_{m^n}}\, a^{mn}\, {}_3\Phi_1^{(n)}\left[\begin{matrix} a, -t^n, q^{-m} \\ t^{2n} \end{matrix}; q, t; -\frac{q^m t}{a}\right].
\end{aligned}
$$

This generalises the $c = aq^{1-m}/t$ case of Jain's quadratic transformation [**55**, Equation (3.6)]:

$$
{}_4\phi_3\left[\begin{matrix} a, aq, q^{-m}, q^{1-m} \\ c, cq, qt^2 \end{matrix}; q^2, q^2\right] = \frac{(c/a; q)_m}{(c; q)_m}\, a^n\, {}_4\phi_2\left[\begin{matrix} a, t, -t, q^{-m} \\ t^2, aq^{1-m}/c \end{matrix}; q, -\frac{q}{c}\right].
$$

Unfortunately, the obvious guess

$$
\begin{aligned}
&{}_4\Phi_3^{(n)}\left[\begin{matrix} a, aq, q^{-m}, q^{1-m} \\ c, cq, qt^{2n} \end{matrix}; q^2, t^2; q^2\right] \\
&\qquad = \frac{(c/a; q, t^2)_{m^n}}{(c; q, t^2)_{m^n}}\, a^{mn}\, {}_4\Phi_2^{(n)}\left[\begin{matrix} a, t^n, -t^n, q^{-m} \\ t^{2n}, aq^{1-m}t^{n-1}/c \end{matrix}; q, t; -\frac{q}{c}\right],
\end{aligned}
$$

is false for all $n \geqslant 2$.

Another rewriting of (5.4.25) arises by expressing the left-hand side as a very well-poised series. First we note that if we make the simultaneous substitutions

$$
(m, e, q, t) \mapsto \bigl(\lfloor m/2\rfloor, q^{1-2\lceil m/2\rceil}, q^2, t^2\bigr)
$$

in (5.4.16), and use

$$
\frac{(aq^2; q^2, t^2)_{\lfloor m/2\rfloor^n}}{(aq^{1+2\lceil m/2\rceil}; q^2, t^2)_{\lfloor m/2\rfloor^n}} = \frac{(aq; q, t^2)_{m^n}}{(aq; q^2t^2)_{m^n}},
$$

it follows that

$$
\begin{aligned}
&{}_8W_7^{(n)}\bigl(a; b, c, d, q^{-m}, q^{1-m}; q^2, t^2; a^2q^{2m+3}t^{2-2n}/bcd\bigr) \\
&\quad = \frac{(aq/b, aq/c, aq/d; q^2, t^2)_{m^n}\,(aq; q, t^2)_{m^n}}{(aq/b, aq/c, aq/d; q, t^2)_{m^n}\,(aq; q^2, t^2)_{m^n}}\, q^{-n\binom{m}{2}} \\
&\qquad \times {}_4\Phi_3^{(n)}\left[\begin{matrix} q^{1-2m}t^{2n-2}/a, bcdq^{-1-2m}t^{2n-2}/a^2, q^{-m}, q^{1-m} \\ bq^{1-2m}t^{2n-2}/a, cq^{1-2m}t^{2n-2}/a, dq^{1-2m}t^{2n-2}/a \end{matrix}; q^2, t^2; q^2\right].
\end{aligned}
$$

Taking

$$
(a, b, c, d) \mapsto \bigl(q^{1-2m}t^{2n-2}/a, qt^{2n}/a, q^{1-m}/t, q^{2-m}/t\bigr),
$$

this can be applied to transform the left-hand side of (5.4.25). Then replacing $a \mapsto q/c$ and simplifying the generalised q-shifted factorials using (5.4.2), (5.4.10) and

$$
(a, aq; q^2, t)_\lambda = (a; q, t)_{2\lambda},
$$

we obtain the following companion to Corollaries 5.21 and 5.23.

Corollary 5.26. *For m a nonnegative integer,*

$$
\begin{aligned}
&{}_2\Phi_1^{(n)}\left[\begin{matrix} -t^n, q^{-m} \\ -q^{1-m}/t \end{matrix}; q, t; c\right] = \frac{(cq^{-m}t^{2n-1}; q, t^2)_{m^n}\,(cq^{1-2m}t^{2n-2}; q^2, t^2)_{m^n}}{(cq^{1-2m}t^{2n-2}; q, t^2)_{m^n}} \\
&\qquad \times {}_8W_7^{(n)}\bigl(cq^{-2m}t^{2n-2}; ct^{2n}, q^{1-m}/t, q^{2-m}/t, q^{-m}, q^{1-m}; q^2, t^2; c\bigr).
\end{aligned}
$$

Again this permits a nonterminating analogue.

THEOREM 5.27. *For* $|q|,|c|<1$,

$$ {}_2\Phi_1^{(n)}\left[\begin{matrix}a,-t^n\\-aq/t\end{matrix};q,t;c\right]=\frac{(act^{2n-1},acqt^{2n-2};q,t^2)_{\infty^n}}{(ct^{2n-1};q,t^2)_{\infty^n}(cqt^{2n-2},a^2cq^2t^{2n-2};q^2,t^2)_{\infty^n}} \times {}_8W_7^{(n)}\bigl(a^2ct^{2n-2};ct^{2n},a,aq,aq/t,aq^2/t;q^2,t^2;c\bigr). \tag{5.4.26}$$

This time $c=qt^{-2n}$ provides a consistency check. Assuming $|q|,|q/t^{2n}|<1$ for convergence, the left-hand side can be summed by (5.4.19) and the right-hand side, which simplifies to a ${}_6W_5^{(n)}$ series, can be summed by

$$ {}_6W_5^{(n)}\bigl(a;b,c,d;q,t;aqt^{1-n}/bcd\bigr)=\frac{(aq,aq/bc,aq/bd,aq/cd;q,t)_{\infty^n}}{(aq/b,aq/c,aq/d,aq/bcd;q,t)_{\infty^n}}, $$

see e.g., [**29**, **36**, **108**, **115**, **136**]. We also note that for $c=t^{-2n}$ the ${}_8W_7^{(n)}$ series trivialises to 1. For such c the left-hand side is, however, not summable by the multiple Gauss sum (5.4.19), and we obtain the curious summation

$$ {}_2\Phi_1^{(n)}\left[\begin{matrix}a,-t^n\\-aq/t\end{matrix};q,t;t^{-2n}\right]=\frac{(a/t,aq/t^2;q,t^2)_{\infty^n}}{(1/t;q,t^2)_{\infty^n}(q/t^2,a^2q^2/t^2;q^2,t^2)_{\infty^n}}, \tag{5.4.27}$$

where $|q|,|1/t|<1$. For $n=1$ this is a consequence of the $n=1$ case of (5.4.19), together with Heine's contiguous relation [**43**, page 26]

$$ {}_2\phi_1\left[\begin{matrix}a,b\\cq\end{matrix};q,z\right]={}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]-cz\frac{(1-a)(1-b)}{(1-c)(1-cq)}\,{}_2\phi_1\left[\begin{matrix}aq,bq\\cq^2\end{matrix};q,z\right]. $$

For $n>1$ there does not appear to be a simple analogue of this relation which, combined with (5.4.19), would imply (5.4.27).

PROOF OF PROPOSITION 5.25. This time we specialise $x=z(1,t^2,\dots,t^{2n-2})$ in (4.1.13). On the left we use (2.5.15) and

$$ b^-_{\lambda;m}(q,t)=\Bigl(-\frac{q}{t}\Bigr)^{|\lambda|}\frac{(q^{-m};q,t)_\lambda}{(-q^{1-m}/t;q,t)_\lambda}\cdot\frac{C^-_\lambda(-t;q,t)}{C^-_\lambda(q;q,t)}, $$

(see the proof of Theorem 4.6 on page 49), as well as

$$ (a^2;q^2,t^2)_\lambda=(a,-a;q,t)_\lambda \quad\text{and}\quad C_\lambda(a^2;q^2,t^2)=C_\lambda(a,-a;q,t). $$

On the right we first write

$$ P^{(\mathrm{B}_n,\mathrm{C}_n)}_{(\frac{m}{2})^n}(x;q^2,t^2,-t)=K_{(\frac{m}{2})^n}(x;q^2,t^2;-t,-qt) $$

using (2.7.12b), then specialise x, and finally apply (5.4.9). As a result,

$$ {}_2\Phi_1^{(n)}\left[\begin{matrix}-t^n,q^{-m}\\-q^{1-m}/t\end{matrix};q,t;-\frac{zq}{t}\right]=(-zq^{1-m}t^{2n-2};q^2,t^2)_{m^n} \times {}_4\Phi_3^{(n)}\left[\begin{matrix}q^{-m}/t,q^{1-m}/t,q^{-m},q^{1-m}\\-zq^{1-m}t^{2n-2},-q^{1-m}/z,q^{2-m}/t^2\end{matrix};q^2,t^2;q^2\right]. $$

Again we apply the Sears transformation (5.4.7), this time with

$$ (a,b,c,d,e,m,q,t) \mapsto (q^{1-2\lceil m/2\rceil},q^{-m}/t,q^{1-m}/t,-q^{1-m}/z,-zq^{1-m}t^{2n-2},\lfloor m/2\rfloor,q^2,t^2). $$

Noting that

$$\begin{aligned}(-zq^{1-m}t^{2n-2};q^2,t^2)_{m^n} &= \frac{(-zq^{-m+2\lceil m/2\rceil}t^{2n-2},qt^{2n};q^2,t^2)_{\lfloor m/2\rfloor^n}}{(-zq^{1-m}t^{2n-2},q^{2\lceil m/2\rceil}t^{2n};q^2,t^2)_{\lfloor m/2\rfloor^n}}\\ &= \frac{(t^{2n},-zt^{2n-2};q,t^2)_{m^n}}{(t^{2n};q^2,t^2)_{m^n}}\end{aligned}$$

and replacing $z \mapsto -t/a$ completes the proof. $\square$

CHAPTER 6

Open problems

We conclude this paper with a list of open problems.

6.1. Missing q-analogues

If we specialise $\{t_2, t_3\} = \{\pm t^{1/2}\}$ or $\{t_2, t_3\} = \{\pm \mathrm{i} t^{1/4}\}$ in Theorem 4.7 then the Rogers–Szegő polynomials in the summand factorise by (2.3.7).

Open problem 1. *Find q-analogues of*

$$\sum t^{\mathrm{odd}(\lambda)/2}\bigg(\prod_{i=1}^{2m-1}(t;t^2)_{\lceil m_i(\lambda)/2\rceil}\bigg)P_\lambda(x;t) = (x_1\cdots x_n)^m P_{m^n}^{(\mathrm{C}_n)}(x;t,t)$$

and

$$\sum t^{\mathrm{odd}(\lambda)/4}\bigg(\prod_{\substack{i=1\\ i \text{ even}}}^{2m-1}(t^{1/2};t)_{\lceil m_i(\lambda)/2\rceil}(-t;t)_{\lfloor m_i(\lambda)/2\rfloor}\bigg) \\ \times\bigg(\prod_{\substack{i=1\\ i \text{ odd}}}^{2m-1}(t;t^2)_{m_i(\lambda)/2}\bigg)P_\lambda(x;t) = (x_1\cdots x_n)^m P_{m^n}^{(\mathrm{C}_n)}\big(x;t,t^{1/2}\big).$$

In both cases the sum is over partitions λ such that $\lambda_1 \leqslant 2m$ and such that parts of odd size have even multiplicity.

Similarly, if we specialise $t_2 = 0$ in Theorem 4.8 then the Rogers–Szegő polynomial in the summand trivialises to 1.

Open problem 2. *Find a q-analogue of*

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant m}} P_\lambda(x;t) = (x_1\cdots x_n)^{\frac{m}{2}} P_{(\frac{m}{2})^n}^{(\mathrm{B}_n)}(x;t,0).$$

As remarked previously, the above is equivalent to (5.1.2), which was key in Macdonald's proof of the MacMahon conjecture. This makes finding a q-analogue particularly desirable.

6.2. Littlewood identities for near-rectangular partitions

The bounded Littlewood identities proven in this paper correspond to decompositions of (R, S) Macdonald polynomials (or R Hall–Littlewood polynomials) indexed by rectangular partitions or half-partitions of maximal length. In the Schur case more general shapes have been considered in the literature. For example, Goulden and Krattenthaler [**46**, **66**, **67**] proved the following result for the character of the irreducible $\mathrm{Sp}(2n,\mathbb{C})$-module of highest weight $\omega_r+(m-1)\omega_n$, generalising the Désarménien–Proctor–Stembridge formula (4.1.6):

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant 2m\\ \mathrm{odd}(\lambda)=r}} s_\lambda(x)=(x_1\cdots x_n)^m\,\mathrm{sp}_{2n,m^{n-r}(m-1)^r}(x).$$

Here $0\leqslant r\leqslant n$ ($\omega_0:=0$) and $m^{n-r}(m-1)^r$ is shorthand for the near-rectangular partition

$$m^{n-r}(m-1)^r=(\underbrace{m,\dots,m}_{n-r\text{ times}},\underbrace{m-1,\dots,m-1}_{r\text{ times}}).$$

Open problem 3. (*i*) *For positive integers m,n and r an integer such that $0\leqslant r\leqslant n$, prove that*[1]

$$\sum_{\substack{\lambda\\ \mathrm{odd}(\lambda)=r}} b^{\mathrm{oa}}_{\lambda;m,r}(q,t)P_\lambda(x;q,t)=(x_1\cdots x_n)^m P^{(\mathrm{C}_n,\mathrm{B}_n)}_{m^{n-r}(m-1)^r}(x;q,t,qt), \tag{6.2.1}$$

where

$$b^{\mathrm{oa}}_{\lambda;m,r}(q,t)=b^{\mathrm{oa}}_\lambda(q,t)\prod_{\substack{s\in\lambda/1^r\\ a'_\lambda(s)\text{ even}}}\frac{1-q^{2m-a'_\lambda(s)}t^{l'_\lambda(s)}}{1-q^{2m-a'_\lambda(s)-1}t^{l'_\lambda(s)+1}}.$$

(*ii*) *Prove similar such near-rectangular identities for other admissible pairs (R,S).*

6.3. Littlewood identities of Pfaffian type

In Chapter 4 we obtained bounded analogues of most of the known Littlewood identities in the literature. There are however a number of extensions of the (unbounded) Littlewood identities discussed in that section where the product forms on the right are replaced by Pfaffians. Two characteristic examples are [**52**, Theorem 4.1][2]

$$\sum_\lambda a^{\mathrm{odd}(\lambda)}b^{\mathrm{odd}(\lambda')}s_\lambda(x)=\prod_{1\leqslant i<j\leqslant n}\frac{1}{x_i-x_j}\prod_{i=1}^n\frac{1}{1-b^2x_i^2} \times \operatorname*{Pf}_{1\leqslant i,j\leqslant n}\left(\frac{(x_i-x_j)\big((1+b^2x_ix_j)(1+a^2x_ix_j)+ab(1+x_ix_j)(x_i+x_j)\big)}{1-x_i^2x_j^2}\right) \tag{6.3.1}$$

---

[1] The identity (6.2.1) has recently been proved by Nguyen [**100**].

[2] There is an unfortunate error in the right-hand side of [**52**, Theorem 4.1], in that the factor $(1+stx_ix_j)$ should have been $(1+stvx_ix_j)$. With this correction, it readily follows that the theorem in question only depends on two variables instead of four. Scaling $x_i\mapsto x_i/(stv)^{1/2}$ and using that

$$s^{\sum_{i\geqslant 1}\lambda_{2i}-|\lambda|/2}t^{\sum_{i\geqslant 1}\lambda_{2i-1}-|\lambda|/2}=(t/s)^{\mathrm{odd}(\lambda')/2},$$

equation (6.3.1) follows after replacing $u/v^{1/2}$ by a and $(t/s)^{1/2}$ by b.

and [**112**, Corollary 6.26] (see also [**13**, Conjecture 1] and [**145**, Theorem 5])

$$\sum_{\lambda' \text{ even}} b^{\mathrm{el}}_{\lambda}(q,t) P_{\lambda}(x;q,t) \prod_{\substack{i=1 \\ i \text{ even}}}^{n} \big(1-uq^{\lambda_i}t^{n-i}\big) \tag{6.3.2}$$
$$= \prod_{1\leqslant i<j\leqslant n} \frac{(tx_ix_j;q)_{\infty}}{(x_i-x_j)(qx_ix_j;q)_{\infty}} \operatorname*{Pf}_{1\leqslant i,j\leqslant n}\bigg(\frac{(x_i-x_j)(1-u+(u-t)x_ix_j)}{(1-x_ix_j)(1-tx_ix_j)}\bigg),$$

where in both cases n is assumed to be even. By [**70**, **128**]

$$\operatorname*{Pf}_{1\leqslant i,j\leqslant n}\bigg(\frac{x_i-x_j}{1-x_ix_j}\bigg) = \prod_{1\leqslant i<j\leqslant n} \frac{x_i-x_j}{1-x_ix_j}, \qquad n \text{ even},$$

and $\operatorname{Pf}(a_ia_jA_{ij}) = \operatorname{Pf}(A_{ij})\prod_i a_i$, the $b=1$ and $a=1$ specialisations of (6.3.1) simplify to the $q=t$ cases of (4.1.1) and (4.1.2) respectively. Similarly, for $u=0$ the identity (6.3.2) yields the $a=0$ case of (4.1.2).

Open problem 4. *Find bounded analogues of* (6.3.1) *and* (6.3.2).

6.4. Elliptic Littlewood identities

Most of the virtual Koornwinder integrals of Section 3.2 have elliptic analogues, see [**111**, **112**].

Open problem 5. *Prove elliptic analogues of the bounded Littlewood identities of Theorems 4.1–4.6.*

For elliptic analogues of (4.1.1)–(4.1.3) we refer the reader to [**111**].

6.5. q,t-Littlewood–Richardson coefficients

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $V(\lambda)$ an irreducible $\mathfrak{g}$-module of highest weight $\lambda \in P_+$. The Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ of type $\mathfrak{g}$ is defined as the multiplicity of $V(\lambda)$ in the tensor product $V(\mu)\otimes V(\mu)$:

$$c^{\lambda}_{\mu\nu} = \dim \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda), V(\mu)\otimes V(\nu)\big).$$

If $\chi_\lambda = \sum_{\mu\in P} K_{\lambda\mu}\,\mathrm{e}^{\mu}$ (with $K_{\lambda\mu}$ the multiplicity of μ in $V(\lambda)$) denotes the formal character of $V(\lambda)$, then

$$\chi_\mu\chi_\nu = \sum_{\lambda\in P_+} c^{\lambda}_{\mu\nu}\chi_\lambda.$$

In [**129**], Stembridge gave a classification of all multiplicity-free tensor products, i.e., of all pairs of dominant weight μ,ν such that $c^{\lambda}_{\mu\nu}\leqslant 1$ for all $\lambda\in P_+$. For such a pair it is not generally known for which λ the Littlewood–Richardson coefficient is actually non-vanishing, except for a number of classical Lie algebras and specially chosen weights (typically, for μ and ν multiples of miniscule weights). For example, using his minor summation formula, Okada proved that for $\mathfrak{g}=\mathrm{B}_n$ and

$$(\mu,\nu) = (r\omega_n, s\omega_n) \tag{6.5.1}$$

with r,s nonnegative integers, $c^{\lambda}_{\mu\nu}=1$ for all dominant weights λ of the form (2.7.5a), where $(\lambda_1,\dots,\lambda_n)$ is a partition/half-partition if $r+s$ is even/odd, and $\lambda_1\leqslant (r+s)/2$, $\lambda_n \geqslant |r-s|/2$. For weights λ not of this form $c^{\lambda}_{\mu\nu}=0$. In terms of odd-orthogonal Schur functions, this result may be expressed as the identity [**101**, Theorem 2.5 (1)]

$$\mathrm{so}_{2n+1,(\frac{r}{2})^n}(x)\,\mathrm{so}_{2n+1,(\frac{s}{2})^n}(x)=\sum \mathrm{so}_{2n+1,(\frac{r+s}{2})^n-\lambda}(x).$$

where the sum is over partitions (as opposed to half-partitions) λ of such that $\lambda_1 \leqslant \min\{r,s\}$.

For r,s nonnegative integers and λ a partition, let

$$\begin{aligned} b^{\mathrm{el}}_{\lambda;m_1,m_2}(q,t) &:= b^{\mathrm{el}}_{\lambda}(q,t) \prod_{\substack{s\in\lambda \\ l'(s)\text{ even}}} \frac{1-q^{m_1-a'(s)}t^{l'(s)}}{1-q^{m_1-a'(s)-1}t^{l'(s)+1}} \cdot \frac{1-q^{m_2-a'(s)}t^{l'(s)}}{1-q^{m_2-a'(s)-1}t^{l'(s)+1}} \\ &\quad \times \prod_{\substack{s\in\lambda \\ l(s)\text{ even}}} \frac{1-q^{m_1+m_2-\hat{a}(s)-1}t^{\hat{l}(s)+1}}{1-q^{m_1+m_2-\hat{a}(s)}t^{\hat{l}(s)}}. \end{aligned}$$

We note that $b^{\mathrm{el}}_{\lambda;m_1,m_2}(q,t)=0$ unless $\lambda_1 \leqslant \min\{m_1,m_2\}$, and

$$\lim_{m_2\to\infty} b^{\mathrm{el}}_{\lambda;m,m_2}(q,t)=b^{\mathrm{el}}_{\lambda;m}(q,t), \tag{6.5.2}$$

where $b^{\mathrm{el}}_{\lambda;m}(q,t)$ is defined in (4.1.11).

Open problem 6. *For $x=(x_1,\dots,x_n)$ and r,s nonnegative integers, prove that*

$$\begin{aligned} P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{r}{2})^n}(x;q,t,t)P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{s}{2})^n}(x;q,t,t) \\ = \sum_{\lambda} b^{\mathrm{el}}_{\lambda;r,s}(q,t)P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{r+s}{2})^n-\lambda}(x;q,t,t). \end{aligned} \tag{6.5.3}$$

Dividing both sides by

$$(x_1\cdots x_n)^{-\frac{1}{2}r}P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{s}{2})^n}(x;q,t,t),$$

and then using that

$$\begin{aligned} (x_1\cdots x_n)^{\frac{1}{2}r} \lim_{s\to\infty} \frac{P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{r+s}{2})^n-\lambda}(x;q,t,t)}{P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{s}{2})^n}(x;q,t,t)} \\ &= \frac{\lim_{s\to\infty}(x_1\cdots x_n)^{\frac{1}{2}(r+s)}P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{r+s}{2})^n-\lambda}(x;q,t,t)}{\lim_{s\to\infty}(x_1\cdots x_n)^{\frac{1}{2}s}P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{s}{2})^n}(x;q,t,t)} \\ &= P_\lambda(x;q,t), \end{aligned}$$

by (2.7.12a) and (2.7.14), it follows that in the large-s limit we recover the bounded Littlewood identity of Theorem 1.1 with m replaced by r.

There is an analogous result for $(\mathrm{C}_n, \mathrm{B}_n)$. For m_1, m_2 nonnegative integers and λ an even partition, let

$$b^{\mathrm{oa}}_{\lambda;m_1,m_2}(q,t) := b^{\mathrm{oa}}_{\lambda}(q,t) \prod_{\substack{s\in\lambda \\ a'(s)\text{ even}}} \frac{1-q^{2m_1-a'(s)}t^{l'(s)}}{1-q^{2m_1-a'(s)-1}t^{l'(s)+1}} \cdot \frac{1-q^{2m_2-a'(s)}t^{l'(s)}}{1-q^{2m_2-a'(s)-1}t^{l'(s)+1}}$$
$$\times \prod_{\substack{s\in\lambda \\ a(s)\text{ even}}} \frac{1-q^{2m_1+2m_2-\hat{a}(s)-1}t^{\hat{l}(s)+1}}{1-q^{2m_1+2m_2-\hat{a}(s)}t^{\hat{l}(s)}}.$$

This time $b^{\mathrm{oa}}_{\lambda;m_1,m_2}(q,t)=0$ unless $\lambda_1 \leqslant \min\{2m_1, 2m_2\}$, and

$$\lim_{m_2\to\infty} b^{\mathrm{oa}}_{\lambda;m,m_2}(q,t) = b^{\mathrm{oa}}_{\lambda;m}(q,t),$$

where $b^{\mathrm{oa}}_{\lambda;m}(q,t)$ for even partitions λ is given by (4.1.11).

Open problem 7. *For $x=(x_1,\dots,x_n)$ and r,s nonnegative integers, prove that*

$$P^{(\mathrm{C}_n,\mathrm{B}_n)}_{r^n}(x;q,t,qt)P^{(\mathrm{C}_n,\mathrm{B}_n)}_{s^n}(x;q,t,qt) = \sum_{\lambda\text{ even}} b^{\mathrm{oa}}_{\lambda;r,s}(q,t) P^{(\mathrm{C}_n,\mathrm{B}_n)}_{(r+s)^n-\lambda}(x;q,t,t).$$

Let $x=(x_1,\dots,x_n)$ and r,s integers such that $-r\leqslant s\leqslant r$.

Taking the large-s limit using (2.6.9) and (2.7.9) yields the Littlewood identity (4.1.5). Moreover, in the classical limit we recover Okada's formula for the C_n Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ with μ and ν given by (6.5.1), see [**101**, Theorem 2.5 (1)]. For such μ,ν, $c^{\lambda}_{\mu\nu}=1$ if λ is a weight of the form (2.7.5b), where $(\lambda_1,\dots,\lambda_n)$ is a partition such that and $\lambda_1\leqslant r+s$, $\lambda_n\geqslant |r-s|$ and λ_1+r+s is even. For all other λ, $c^{\lambda}_{\mu\nu}=0$.

6.6. Dyson–Macdonald-type identities

Let $\mathfrak{g}$ be an affine Lie algebra with simple roots $\{\alpha_0,\dots,\alpha_n\}$. The map $\mathrm{e}^{-\alpha_1},\dots,\mathrm{e}^{-\alpha_n}\mapsto 1$ (so that $\mathrm{e}^{-\delta}\mapsto \mathrm{e}^{-a_0\alpha_0}$) is known as the basic specialisation [**87**]. When applied to character formulas for affine Lie algebras, the basic specialisation results in (generalised) Dyson–Macdonald type expansions for powers of the Dedekind eta-function, see e.g., [**9**, **37**, **87**, **97**, **131**, **144**]. For example, taking the basic specialisation of the $\mathrm{B}_n^{(1)\dagger}$ identity (5.2.18) (and replacing t by q) yields the following generalisation of [**89**, p. 135, (6c)]:

$$\frac{1}{\eta(\tau/2)^{2n}\eta(\tau)^{2n^2-3n}} \sum (-1)^{\frac{|v|-|\rho|}{m+2n-1}} \chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{2(m+2n-1)}+\frac{\|\rho\|^2}{2(2n-1)}} = \sum_{\substack{\lambda \\ \lambda_1\leqslant m}} q^{|\lambda|/2} P'_{\lambda}(\underbrace{1,\dots,1}_{2n-1\text{ times}};q) \prod_{i=0}^{m-1} (-q^{1/2};q^{1/2})_{m_i(\lambda)}. \tag{6.6.1}$$

Here $q=\exp(2\pi\mathrm{i}\tau)$, $\rho=(n-1,\dots,1,0)$,

$$\chi_{\mathrm{D}}(v/w) := \prod_{i<j}(v_i^2-v_j^2)/(w_i^2-w_j^2),$$

and the sum is over $v\in\mathbb{Z}^n$ such that $v_i\equiv\rho_i \pmod{m+2n-1}$.

Let $C = C_n$ be the Cartan matrix of the Lie algebra A_n, i.e., $(C^{-1})_{ab} = \min\{a,b\} - ab/(n+1)$, and for $\{r_i^{(a)}\}_{1\leqslant a\leqslant n;\, 1\leqslant i\leqslant k}$ a set of nonnegative integers, let

$$R_i^{(a)} := r_i^{(a)} + \cdots + r_k^{(a)}.$$

Following [**9**, **144**] we define

$$F_{k,n}(q) := \begin{cases} (-q^{1/2};q)_\infty^{2n-1} & \text{for } k=0 \\[2mm] (-q^{1/2};q^{1/2})_\infty^{2n-1} \displaystyle\sum_{\{r_i^{(a)}\}} \frac{q^{\frac{1}{2}\sum_{a,b=1}^n \sum_{i=1}^k C_{ab} R_i^{(a)} R_i^{(b)}}}{\prod_{a=1}^n \prod_{i=1}^{k-1} \big((q;q)_{r_i^{(a)}}\big)(q^2;q^2)_{r_k^{(a)}}} & \text{for } k \geqslant 1 \end{cases}$$

and

$$G_{k,n}(q) := (-q;q)_\infty^n \sum_{\{r_i^{(a)}\}} \frac{q^{\frac{1}{2}\sum_{a,b=1}^n \sum_{i=1}^k C_{ab} R_i^{(a)} R_i^{(b)}} \big(-q^{1/2-R_1^{(1)}};q\big)_{R_1^{(1)}}}{\prod_{a=1}^n \prod_{i=1}^{k-1} \big((q;q)_{r_i^{(a)}}\big)(q^2;q^2)_{r_k^{(a)}}}$$

for $k \geqslant 1$.

Open problem 8. *For m, n positive integers, prove that*

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant m}} q^{|\lambda|/2} P'_\lambda(\underbrace{1,\dots,1}_{2n-1 \text{ times}};q) \prod_{i=1}^{m-1} (-q^{1/2};q^{1/2})_{m_i(\lambda)} = \begin{cases} F_{k,2n-1}(q) & \text{if } m = 2k+1 \\ G_{k,2n-1}(q) & \text{if } m = 2k. \end{cases}$$

For $n = 1$ this follows from a minor modification of [**139**, Lemma A.1], for $m = 1$ it follows from $P'_{(1^r)}(x;t) = e_r(x)$ and

$$\sum_{r=0}^{\infty} z^r e_r\Big[\frac{n}{1-q}\Big] = (-z;q)_\infty^n,$$

and for $m = 2$ a proof is given in [**9**, Theorem 3.7]. By (6.6.1), the above problem for even m is equivalent to [**144**, Conjecture 2.4; (2.6a)].

APPENDIX A

The Weyl–Kac formula

In this first of two appendices we state some simple consequences of the Weyl–Kac formula, needed in the proofs of our combinatorial character formulas in Section 5.2.

Recall the symplectic and odd-orthogonal Schur functions (2.8.10) and (2.8.12). It will be convenient to also define the normalised functions

$$\widetilde{\mathrm{so}}_{2n+1,\lambda}(x)=\Delta_{\mathrm{B}}(x)\,\mathrm{so}_{2n+1,\lambda}(x) \quad\text{and}\quad \widetilde{\mathrm{sp}}_{2n,\lambda}(x)=\Delta_{\mathrm{C}}(x)\,\mathrm{so}_{2n+1,\lambda}(x), \tag{A.1.1}$$

so that $\widetilde{\mathrm{so}}_{2n+1,0}(x)=\Delta_{\mathrm{B}}(x)$ and $\widetilde{\mathrm{sp}}_{2n,0}(x)=\Delta_{\mathrm{C}}(x)$.

Mimicking the proofs of [**9**, Lemmas 2.1–2.4] yields expressions for the characters of $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n-1}^{(2)}$ in terms of the symplectic and odd orthogonal Schur functions as follows.

Lemma A.1 ($\mathrm{B}_n^{(1)}$ character formula). *Let*

$$x_i:=\mathrm{e}^{-\alpha_i-\cdots-\alpha_n}\quad(1\leqslant i\leqslant n),\qquad t:=\mathrm{e}^{-\delta}, \tag{A.1.2}$$

and parametrise $\Lambda\in P_+$, *as*

$$\Lambda=c_0\varpi_0+(\lambda_1-\lambda_2)\varpi_1+\cdots+(\lambda_{n-1}-\lambda_n)\varpi_{n-1}+2\lambda_n\varpi_n,$$

where c_0 *is a nonnegative integer and* $\lambda=(\lambda_1,\dots,\lambda_n)$ *a partition or half-partition. Then*

$$\begin{aligned}\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda)&=\frac{1}{(t;t)_\infty^n\prod_{i=1}^n\theta(x_i;t)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_ix_j^{\pm};t)}\\&\qquad\times\sum_{\substack{r\in\mathbb{Z}^n\\|r|\equiv 0\ (2)}}\widetilde{\mathrm{so}}_{2n+1,\lambda}(xt^r)\prod_{i=1}^n x_i^{\kappa r_i+\lambda_i}t^{\frac12\kappa r_i^2-(n-\frac12)r_i},\end{aligned} \tag{A.1.3}$$

where $\kappa=2n-1+c_0+\lambda_1+\lambda_2$.

Lemma A.2 ($\mathrm{A}_{2n-1}^{(2)}$ character formula). *Let*

$$x_i:=\mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}-\alpha_n/2}\quad(1\leqslant i\leqslant n),\qquad t:=\mathrm{e}^{-\delta}, \tag{A.1.4}$$

and parametrise $\Lambda\in P_+$, *as*

$$\Lambda=c_0\varpi_0+(\lambda_1-\lambda_2)\varpi_1+\cdots+(\lambda_{n-1}-\lambda_n)\varpi_{n-1}+\lambda_n\varpi_n,$$

where c_0 *is a nonnegative integer and* $\lambda=(\lambda_1,\dots,\lambda_n)$ *a partition. Then*

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{1}{(t;t)_\infty^{n-1}(t^2;t^2)_\infty\prod_{i=1}^n\theta(x_i^2;t^2)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_ix_j^{\pm};t)} \times\sum_{\substack{r\in\mathbb{Z}^n\\|r|\equiv 0\,(2)}}\widetilde{\mathrm{sp}}_{2n,\lambda}(xt^r)\prod_{i=1}^n x_i^{\kappa r_i+\lambda_i}t^{\frac12\kappa r_i^2-nr_i}, \tag{A.1.5}$$

where $\kappa=2n+c_0+\lambda_1+\lambda_2$.

For a two-parameter subset of weights, the above two formulas may be rewritten as a pair of character formulas for $\mathrm{B}_n^{(1)\dagger}$ and $\mathrm{A}_{2n-1}^{(2)\dagger}$ where the sum is over the full $\mathbb{Z}^n$-lattice. For κ,k positive integers and $x=(x_1,\dots,x_n)$, define

$$\mathscr{N}_{\kappa,k}(x;t):=\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(x^kt^{kr})\prod_{i=1}^n(-1)^{r_i}x_i^{\kappa r_i-(k-1)(i-1)}t^{\frac12\kappa r_i^2-k(n-1)r_i},$$

where $\Delta_{\mathrm{D}}(x)$ is the D_n Vandermonde product (2.8.14) and $x^k:=(x_1^k,\dots,x_n^k)$. Further set

$$\mathscr{D}_{\mathfrak{g}}(x;t):=\begin{cases}(t;t)_\infty^n\prod_{i=1}^n\theta(t^{1/2}x_i;t)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_ix_j^{\pm};t) & \text{for } \mathfrak{g}=\mathrm{B}_n^{(1)\dagger},\\ (t;t)_\infty^{n-1}(t^2;t^2)_\infty\prod_{i=1}^n\theta(tx_i^2;t^2)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_ix_j^{\pm};t) & \text{for } \mathfrak{g}=\mathrm{A}_{2n-1}^{(2)\dagger}.\end{cases}$$

Lemma A.3 ($\mathrm{B}_n^{(1)\dagger}$ and $\mathrm{A}_{2n-1}^{(2)\dagger}$ character formulas). *Let*

$$x_i:=\mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}+(\alpha_{n-1}-\alpha_n)/2}\quad(1\leqslant i\leqslant n),\qquad t:=\mathrm{e}^{-\delta} \tag{A.1.6}$$

and, for k *a positive integer and* m *a nonnegative integer,*

$$\Lambda=(k-1)\rho+m\varpi_0\in P_+. \tag{A.1.7}$$

Then

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{1}{\mathscr{D}_{\mathfrak{g}}(x;t)}\times\begin{cases}\mathscr{N}_{m+k(2n-1),k}(x;t) & \textit{for } \mathfrak{g}=\mathrm{B}_n^{(1)\dagger},\\ \mathscr{N}_{m+2kn,k}(x;t) & \textit{for } \mathfrak{g}=\mathrm{A}_{2n-1}^{(2)\dagger}.\end{cases} \tag{A.1.8}$$

For $m=0$ and $k=1$ this gives what may be viewed as $\mathrm{B}_n^{(1)\dagger}$ and $\mathrm{A}_{2n-1}^{(2)\dagger}$ Macdonald identities:

$$\mathscr{N}_{2n-1,1}(x;t)=\mathscr{D}_{\mathrm{B}_n^{(1)\dagger}}(x;t)$$

and

$$\mathscr{N}_{2n,1}(x;t)=\mathscr{D}_{\mathrm{A}_{2n-1}^{(2)\dagger}}(x;t). \tag{A.1.9}$$

Corollary A.4 ($\mathrm{B}_n^{(1)\dagger}$ product formula). *Let* k *be a positive integer and*

$$\Lambda=(k-1)\rho+k\varpi_0\in P_+. \tag{A.1.10}$$

Then

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{\mathscr{D}_{\mathrm{A}_{2n-1}^{(2)\dagger}}(x;t)}{\mathscr{D}_{\mathrm{B}_n^{(1)\dagger}}(x;t)}\prod_{i=1}^n x_i^{-(k-1)(i-1)}.$$

PROOF. If we set $m=k$ in the $\mathrm{B}_n^{(1)\dagger}$ case of (A.1.8) then

$$\mathrm{e}^{-\Lambda}\operatorname{ch}V(\Lambda)=\frac{\mathscr{N}_{2kn,k}(x;t)}{\mathscr{D}_{\mathrm{B}_n^{(1)\dagger}}(x;t)}$$

with Λ as in (A.1.10). By

$$\mathscr{N}_{2kn,k}(x;t)=\mathscr{N}_{2n,1}(x^k;t^k)\prod_{i=1}^n x_i^{-(k-1)(i-1)}$$

and (A.1.9) the result follows. $\square$

We still need to prove Lemma A.3. For this we first prepare a determinantal identity. For $\sigma=0,1$ and κ,k positive integers, let

$$\begin{aligned}\mathscr{N}_{\kappa,k;\sigma}(x;t):=(-1)^\sigma\sum_{\substack{r\in\mathbb{Z}^n\\|r|\equiv\sigma\ (2)}}\det_{1\leqslant i,j\leqslant n}\Big(&x_i^{-\kappa r_i-k(i-j)+i-1}t^{\frac12\kappa r_i^2+k(n-j)r_i}\\&-x_i^{-\kappa(r_i+1)+k(2n-i-j)+i-1}t^{\frac12\kappa(r_i+1)^2-k(n-j)(r_i+1)}\Big).\end{aligned}$$

LEMMA A.5. *We have*

$$\mathscr{N}_{\kappa,k;\sigma}(x;t)=\mathscr{N}_{\kappa,k}(x;t). \tag{A.1.11}$$

PROOF. Let $\varepsilon\in\{-1,1\}$. Then

$$\begin{aligned}&\mathscr{N}_{\kappa,k;0}(x;t)+\varepsilon\mathscr{N}_{\kappa,k;1}(x;t)\\&=\sum_{r\in\mathbb{Z}^n}\det_{1\leqslant i,j\leqslant n}\Big((-1)^{\frac12(1+\varepsilon)r_i}x_i^{-\kappa r_i-k(i-j)+i-1}t^{\frac12\kappa r_i^2+k(n-j)r_i}\\&\qquad-(-1)^{\frac12(1+\varepsilon)r_i}x_i^{-\kappa(r_i+1)+k(2n-i-j)+i-1}t^{\frac12\kappa(r_i+1)^2-k(n-j)(r_i+1)}\Big)\\&=\det_{1\leqslant i,j\leqslant n}\bigg(\sum_{r\in\mathbb{Z}}(-1)^{\frac12(1+\varepsilon)r}x_i^{-\kappa r-k(i-j)+i-1}t^{\frac12\kappa r^2+k(n-j)r}\\&\qquad-\sum_{r\in\mathbb{Z}}(-1)^{\frac12(1+\varepsilon)r}x_i^{-\kappa(r+1)+k(2n-i-j)+i-1}t^{\frac12\kappa(r+1)^2-k(n-j)(r+1)}\bigg)\\&=\det_{1\leqslant i,j\leqslant n}\bigg(\sum_{r\in\mathbb{Z}}(-1)^{\frac12(1+\varepsilon)r}x_i^{\kappa r-k(i-j)+i-1}t^{\frac12\kappa r^2-k(n-j)r}\\&\qquad+\varepsilon\sum_{r\in\mathbb{Z}}(-1)^{\frac12(1+\varepsilon)r}x_i^{\kappa r+k(2n-i-j)+i-1}t^{\frac12\kappa r^2+k(n-j)r}\bigg)\\&=\sum_{r\in\mathbb{Z}^n}\prod_{i=1}^n(-1)^{\frac12(1+\varepsilon)r_i}x_i^{\kappa r_i-(k-1)(i-1)}t^{\frac12\kappa r_i^2-k(n-1)r_i}\\&\qquad\times\det_{1\leqslant i,j\leqslant n}\Big(\big(x_it^{r_i}\big)^{k(j-1)}+\varepsilon\big(x_it^{r_i}\big)^{k(2n-j-1)}\Big).\end{aligned}$$

Here the second and last equality use multilinearity and the third equality follows from a shift of $r\mapsto r-1$ in the second sum over r. When $j=n$ the final line reads $(1+\varepsilon)\big(x_it^{r_i}\big)^{k(n-1)}$, so that

$$\mathscr{N}_{\kappa,k;0}(x;t)-\mathscr{N}_{\kappa,k;1}(x;t)=0. \tag{A.1.12}$$

Also, by the D_n Vandermonde determinant (2.8.14),

$$\begin{aligned}\mathscr{N}_{\kappa,k;0}(x;t)&+\mathscr{N}_{\kappa,k;1}(x;t)\\&=2\sum_{r\in\mathbb{Z}^n}\Delta_{\mathrm{D}}(x^kt^{kr})\prod_{i=1}^n(-1)^{r_i}x_i^{\kappa r_i-(k-1)(i-1)}t^{\frac{1}{2}\kappa r_i^2-k(n-1)r_i}\\&=2\mathscr{N}_{\kappa,k}(x;t).\end{aligned}$$

Together with (A.1.12) this implies (A.1.11). $\square$

Proof of Lemma A.3. For κ,k positive integers, define

$$\begin{aligned}\mathscr{M}_{\kappa,k}(x;t):=\sum_{\substack{r\in\mathbb{Z}^n\\|r|\equiv 0\ (2)}}\det_{1\leqslant i,j\leqslant n}\Big(&x_i^{\kappa r_i+k(j-1)-(k-1)(i-1)}q^{\kappa\binom{r_i}{2}+k(j-1)r_i}\\&-x_i^{\kappa(r_i+1)-k(j-1)-(k-1)(i-1)}q^{\kappa\binom{r_i+1}{2}-k(j-1)r_i}\Big).\end{aligned}$$

Taking $c_0=k-1$, $\lambda_i=\frac{1}{2}m+(k-1)(n-i+\frac{1}{2})$ $(1\leqslant i\leqslant n)$ in A.1.3, and using (2.8.12) and (A.1.1), we obtain

$$\begin{aligned}\text{(A.1.13)}\quad &\mathrm{e}^{-(k-1)\rho-m\varpi_n}\operatorname{ch}V\big((k-1)\rho+m\varpi_n\big)\\&=\frac{\mathscr{M}_{m+k(2n-1),k}(x;t)}{(t;t)_\infty^n\prod_{i=1}^n\theta(x_i;t)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_ix_j^{\pm};t)},\end{aligned}$$

where $\mathfrak{g}=\mathrm{B}_n^{(1)}$ and $x_1,\dots,x_n$ are given by A.1.2. Similarly, taking $c_0=k-1$, $\lambda_i=\frac{1}{2}m+(k-1)(n-i+1)$ $(1\leqslant i\leqslant n)$ in A.1.5, and using (2.8.10) and (A.1.1), we get

$$\begin{aligned}\text{(A.1.14)}\quad &\mathrm{e}^{-(k-1)\rho-m\varpi_n}\operatorname{ch}V\big((k-1)\rho+m\varpi_n\big)\\&=\frac{\mathscr{M}_{m+2kn,k}(x)}{(t;t)_\infty^{n-1}(t^2;t^2)_\infty\prod_{i=1}^n\theta(x_i^2;t^2)\prod_{1\leqslant i<j\leqslant n}x_j\theta(x_ix_j^{\pm};t)},\end{aligned}$$

where $\mathfrak{g}=\mathrm{A}_{2n-1}^{(2)}$ and $x_1,\dots,x_n$ are given by A.1.4.

Next we replace $\alpha_i\mapsto\alpha_{n-i}$ for $0\leqslant i\leqslant n$. This maps maps ϖ_n to ϖ_0 but leaves the Weyl vector ρ unchanged. On the right it has the effect of replacing (A.1.2) by

$$x_{n-i+1}=\mathrm{e}^{-\alpha_0-\cdots-\alpha_{i-1}}\quad(1\leqslant i\leqslant n)$$

in the case of (A.1.13), and by

$$x_{n-i+1}=\mathrm{e}^{-\alpha_0/2-\alpha_1-\cdots-\alpha_{i-1}}\quad(1\leqslant i\leqslant n).$$

in the case of $\mathrm{A}_{2n-1}^{(2)}$. Moreover, in the first case t is now given by

$$t=\mathrm{e}^{-2\alpha_0-\cdots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n}$$

in accordance with the interpretation of δ as the null root of $\mathrm{B}_n^{(1)\dagger}$, and in the second case by

$$t=\mathrm{e}^{-\alpha_0-2\alpha_1-\cdots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n}$$

in accordance with $\mathrm{A}_{2n-1}^{(2)\dagger}$. We now replace $x_i\mapsto t^{1/2}/x_{n-i+1}$—so that the transformed x_i is given by (A.1.6) in both cases—and $r_i\mapsto r_{n-i+1}$ for $1\leqslant i\leqslant n$.

Also reversing the order of the rows and columns in the determinant and using $\theta(x;t)=\theta(t/x;t)$, we get

$$\mathrm{e}^{-(k-1)\rho-m\varpi_0}\,\mathrm{ch}\,V\big((k-1)\rho+m\varpi_0\big)
= \frac{1}{\mathscr{D}_{\mathfrak{g}}(x;t)}\times\begin{cases}\mathscr{N}_{m+k(2n-1),k;0}(x) & \text{for } \mathrm{B}_n^{(1)\dagger},\\[2ex] \mathscr{N}_{m+2kn,k;0}(x) & \text{for } \mathrm{A}_{2n-1}^{(2)\dagger}.\end{cases}$$

By Lemma A.5 the claim follows. $\square$

APPENDIX B

Limits of elliptic hypergeometric integrals

We review some results of van de Bult and the first author [**20**, **21**] regarding limits of elliptic beta integrals. We then prove the quadratic transformation formulas of Theorems 5.22, 5.24 and 5.27 by taking limits in three quadratic transformation formulas for elliptic beta integrals, originally conjectured in [**111**] in the context of elliptic Littlewood identities, and subsequently proved in [**19**, **112**].

For complex p, q such that $|p|, |q| < 1$ and $z \in \mathbb{C}^*$, let $\Gamma(z; p, q)$ be the elliptic gamma function [**119**]

$$\Gamma(z; p, q) := \prod_{i,j=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^i q^j},$$

which satisfies the functional equation

$$\Gamma(z; p, q)\Gamma(pq/z; p, q) = 1. \tag{B.1.1}$$

For $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ and $t, t_0, \dots, t_{2m+5} \in \mathbb{C}^*$ (m a nonnegative integer) such that

$$t^{2n-2} t_0 \cdots t_{2m+5} = (pq)^{m+1},$$

define the density

$$\Delta(z; t_0, \dots, t_{2m+5}; t; p, q) := \prod_{i=1}^{n} \frac{\prod_{r=0}^{2m+5} \Gamma(t_r z_i^{\pm}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le i < j \le n} \frac{\Gamma(t z_i^{\pm} z_j^{\pm}; p, q)}{\Gamma(z_i^{\pm} z_j^{\pm}; p, q)}.$$

Note that for $0 < \alpha < 1$,

$$\lim_{p \to 0} \Delta(z; t_0, t_1, t_2, t_3, p^{\alpha} t_4, p^{1-\alpha} q / t_0 t_1 t_2 t_3 t_4; t; p, q) = \Delta(z; q, t; t_0, t_1, t_2, t_3),$$

with on the right the Koornwinder density (2.6.1). The elliptic density may be used to define the higher-order elliptic Selberg integral (also known as a type-II C_n beta integral) as [**110**, **118**, **123**]

$$\begin{aligned} &I\!I_m^{(n)}(t_0, \dots, t_{2m+5}; t; p, q) \\ &\qquad := \kappa_n \int_{C^n} \Delta(z; t_0, \dots, t_{2m+5}; t; p, q) \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n}, \end{aligned} \tag{B.1.2}$$

where

$$\kappa_n := \frac{1}{2^n n!} \left(\frac{(p;p)_\infty (q;q)_\infty \Gamma(t; p, q)}{2\pi \mathrm{i}} \right)^n$$

and C is positively oriented, star-shaped Jordan curve around the origin such that $C = C^{-1}$ and such that the points $\{t_r p^i q^j\}_{0 \leqslant r \leqslant 2m+5; i,j \geqslant 0}$ all lie in the interior of C.

For $\alpha = (\alpha_0, \ldots, \alpha_7) \in \mathbb{R}^7$ and $u = (u_0, \ldots, u_7) \in (\mathbb{C}^*)^7$ such that

$$t^{2n-2} u_0 \cdots u_7 = q^2, \tag{B.1.3}$$

let

$$B_\alpha^{(n)}(u; t; p, q) := I\!I_1^{(n)}(u_0 p^{\alpha_0}, \ldots, u_7 p^{\alpha_7}; t; p, q).$$

From [**20**, Proposition 4.3] and [**21**, Proposition 6.5] we may infer the following $p \to 0$ limit of $B_\alpha^{(n)}(u; t; p, q)$.

Proposition B.1. *Let $q, t, u_0, \ldots, u_7 \in \mathbb{C}^*$ such that $|q|, |u_1 u_2| < 1$ and such that* (B.1.3) *holds, and let*

$$\alpha = (-\gamma, -\beta, \beta, \gamma, \gamma, \delta, 1-\delta, 1-\gamma)$$

for $0 \leqslant \beta < \gamma < \delta \leqslant 1/2$. Then

$$\lim_{p\to 0} (u_0 u_1 p^{-\gamma-\beta} t^{n-1}, u_0 u_2 p^{-\gamma+\beta} t^{n-1}; q, t)_{\infty^n} \, B_\alpha^{(n)}(u; t, p, q)$$
$$= \frac{(q t^{n-1} u_0/u_7; q, t)_{\infty^n}}{(t^n, t^{n-1} u_0 u_3, t^{n-1} u_0 u_4; q, t)_{\infty^n}} \, {}_2\Phi_1^{(n)}\left[\begin{matrix} t^{n-1} u_0 u_3, t^{n-1} u_0 u_4 \\ q t^{n-1} u_0/u_7 \end{matrix}; q, t; u_1 u_2\right].$$

To shorten some of our subsequent calculations we restate this in a form that hides the symmetry in u_3 and u_4, and which is obtained by applying the multiple analogue of Heine transformation [**8**, Equation (2.2)]

$${}_2\Phi_1^{(n)}\left[\begin{matrix} a, b \\ c \end{matrix}; q, t; z\right] = \frac{(b, azt^{n-1}; q, t)_{\infty^n}}{(c, zt^{n-1}; q, t)_{\infty^n}} \, {}_2\Phi_1^{(n)}\left[\begin{matrix} c/b, zt^{n-1} \\ azt^{n-1} \end{matrix}; q, t; bt^{1-n}\right], \tag{B.1.4}$$

for $|z|, |bt^{1-n}| < 1$.

Corollary B.2. *Let $q, t, u_0, \ldots, u_7 \in \mathbb{C}^*$ such that $|q|, |u_0 u_4| < 1$ and such that* (B.1.3) *holds, and let*

$$\alpha = (-\gamma, -\beta, \beta, \gamma, \gamma, \delta, 1-\delta, 1-\gamma)$$

for $0 \leqslant \beta < \gamma < \delta \leqslant 1/2$. Then

$$\lim_{p\to 0} (u_0 u_1 p^{-\gamma-\beta} t^{n-1}, u_0 u_2 p^{-\gamma+\beta} t^{n-1}; q, t)_{\infty^n} \, B_\alpha^{(n)}(u; t, p, q)$$
$$= \frac{(q^2/u_4 u_5 u_6 u_7; q, t)_{\infty^n}}{(t^n, t^{n-1} u_1 u_2, t^{n-1} u_0 u_3; q, t)_{\infty^n}} \, {}_2\Phi_1^{(n)}\left[\begin{matrix} t^{n-1} u_1 u_2, q/u_4 u_7 \\ q^2/u_4 u_5 u_6 u_7 \end{matrix}; q, t; u_0 u_4\right].$$

The second $p \to 0$ limit of $B_\alpha^{(n)}(u; t; p, q)$ we need is as follows.

Proposition B.3. *Let $q, t, u_0, \ldots, u_7 \in \mathbb{C}^*$ such that $|q|, |u_1 u_5| < 1$ and such that* (B.1.3) *holds, and let*

$$\alpha = (-\gamma, -\gamma, \gamma, \gamma, \gamma, \gamma, 1-\gamma, 1-\gamma) \tag{B.1.5}$$

for $0 < \gamma < 1/2$. Then

$$\lim_{p\to 0} (u_0 u_1 p^{-2\gamma} t^{n-1}; q, t)_{\infty^n} \, B_\alpha^{(n)}(u; t; p, q) \tag{B.1.6}$$
$$= \frac{(q t^{n-1} u_0/u_6, q t^{n-1} u_0/u_7; q, t)_{\infty^n}}{(t^n, t^{n-1} u_0 u_5, q^2 t^{n-1} u_0/u_5 u_6 u_7; q, t)_{\infty^n}} \prod_{r=2}^{4} \frac{(q^2/u_r u_5 u_6 u_7; q, t)_{\infty^n}}{(t^{n-1} u_0 u_r, t^{n-1} u_1 u_r; q, t)_{\infty^n}}$$
$$\times \, {}_8W_7^{(n)}\Big(\frac{q t^{n-1} u_0}{u_5 u_6 u_7}; t^{n-1} u_0 u_2, t^{n-1} u_0 u_3, t^{n-1} u_0 u_4, \frac{q}{u_5 u_6}, \frac{q}{u_5 u_7}; q, t; u_1 u_5\Big).$$

This result follows from a special case ($m=1$ and $(\alpha_0,\dots,\alpha_7)$ given by the sequence (B.1.5)) of [**21**, Proposition 6.3]. After some symmetrisation, this expresses the left-hand side of (B.1.6) as a virtual Koornwinder integral. By [**108**, Theorem 5.15] this integral can be expressed as the ${}_8W_7^{(n)}$ series on given on the right of (B.1.6).

Remark B.4. There is some redundancy in the expression on the right, and by the substitution

$$(u_1,u_2,\dots,u_7)\mapsto(u_0u_1,u_2/u_0,u_3/u_0,u_4/u_0,u_5/u_0,u_0u_6,u_0u_7),$$

so that the constraint (B.1.3) simplifies to $t^{2n-2}u_1\cdots u_7=q^2$, the u_0-dependence drops out.

We now have the tools to prove the nonterminating quadratic transformation formulas of Section 5.4. Because it is much simpler than Theorem 5.22, we first consider Theorem 5.24.

Proof of Theorem 5.24. Our starting point is the following quadratic transformation formula for elliptic Selberg integrals.

Theorem B.5. *For $p,q,t,t_0,\dots,t_4\in\mathbb{C}^*$ such that $|p|,|q|<1$ and*

$$t^{n-1}t_0t_1t_2t_3=pq, \tag{B.1.7}$$

we have

$$\begin{aligned} &I\!I_1^{(n)}\big(t^{-1/2}t_0^2,t^{-1/2}t_1^2,t^{-1/2}t_2^2,t^{-1/2}t_3^2,t^{1/2},pt^{1/2},qt^{1/2},pqt^{1/2};t;p^2,q^2\big)\\ &\qquad=\prod_{i=1}^{2n}\prod_{0\leqslant r<s\leqslant 2}\frac{\Gamma(-t^{i/2-1}t_rt_s;p,q)}{\Gamma(t^{i/2-1/2}t_rt_s;p,q)}\\ &\qquad\quad\times I\!I_1^{(n)}\big(t^{\pm 1/4}t_0,t^{\pm 1/4}t_1,t^{\pm 1/4}t_2,-t^{\pm 1/4}t_3;t;p,q\big). \end{aligned}\tag{B.1.8}$$

This theorem is the special case

$$\boldsymbol{\lambda}=\mathbf{0}\quad\text{and}\quad(n,t,t_0,t_1,t_2,u_0)\mapsto(2n,t^{1/2},t^{-1/4}t_0,t^{-1/4}t_1,t^{-1/4}t_2,-t^{-1/4}t_3)$$

of [**111**, Conjecture Q1], which was proved in [**19**, Corollary 7.6] and [**112**, Section 8].

If we make the substitutions

$$(t_0,t_1,t_2,t_3)\mapsto(p^{-1/4},ap^{1/4}t^{1/2-n},cp^{1/4}t^{1-n},p^{3/4}qt^{n-1/2}/ac),$$

consistent with the constraint (B.1.7), the transformation (B.1.8) takes the form

$$\begin{aligned} &B_\alpha^{(n)}\bigg(t^{-1/2},t^{1/2},qt^{1/2},a^2t^{1/2-2n},c^2t^{3/2-2n},t^{1/2},qt^{1/2},\frac{q^2t^{2n-3/2}}{a^2c^2};t;p^2,q^2\bigg)\\ &=\prod_{i=1}^{2n}\frac{\Gamma(-ct^{(1-i)/2},-at^{-i/2},-acp^{1/2}t^{i/2+1/2-2n};p,q)}{\Gamma(ct^{1-i/2},at^{(1-i)/2},acp^{1/2}t^{i/2+1-2n};p,q)}\\ &\times B_{\alpha'}^{(n)}\bigg(t^{1/4},t^{-1/4},ct^{5/4-n},ct^{3/4-n},at^{3/4-n},at^{1/4-n},-\frac{qt^{n-1/4}}{ac},-\frac{qt^{n-3/4}}{ac};t;p,q\bigg), \end{aligned}$$

where

$$\alpha=\big(-\tfrac14,0,0,\tfrac14,\tfrac14,\tfrac12,\tfrac12,\tfrac34\big)\quad\text{and}\quad\alpha'=\big(-\tfrac14,-\tfrac14,\tfrac14,\tfrac14,\tfrac14,\tfrac14,\tfrac34,\tfrac34\big). \tag{B.1.9}$$

Multiplying both sides by

$$(p^{-1/2}t^{n-1};q,t)_{\infty^n} = (p^{-1/2}t^{n-1},p^{-1/2}qt^{n-1};q^2,t)_{\infty^n},$$

we can take the $p \to 0$ limit using Corollary B.2 and Proposition B.3. After some simplifications the claim follows. $\square$

Proof of Theorem 5.22. This time the proof is more involved, and as a first step we need to carry out a non-trivial rewriting of the theorem.

Appealing to analytic continuation, it suffices to prove (5.4.18) for $|t| < 1$. As a function of a, both sides of (5.4.18) are analytic for $|a| < |1/cqt^n|$. Hence it is enough to prove the identity for the sequence of a-values $\{t^N\}_{N\geqslant 0}$ with accumulation point 0. If we set $a = t^N$ in (5.4.18) we obtain

(B.1.10)
$$
{}_2\Phi_1^{(n)}\left[\begin{matrix} t^N, t^{N-1} \\ qt^{2n-1}\end{matrix}; q, t^2; c^2q\right] = \frac{(c^2qt^{2N+2n-2};q,t^2)_{\infty^n}}{(c^2qt^{2n-2};q,t^2)_{\infty^n}} \cdot \frac{(cqt^{2n-1};q,t)_{\infty^{2n}}}{(cqt^{N+2n-1};q,t)_{\infty^{2n}}}
\times {}_8W_7^{(n)}\big(ct^{N+2n-1};ct^{N-1},t^N,-t^n,q^{1/2}t^n,-q^{1/2}t^n;q,t;cq\big).
$$

Now define integers m_1, m_2 as

$$m_1 := \left\lfloor \tfrac{N}{2} \right\rfloor \quad \text{and} \quad m_2 := 2\left\lceil \tfrac{N}{2} \right\rceil - 1 = 2(N-m_1)-1.$$

Since $\{N-1,N\} = \{2m_1, m_2\}$, it follows from (5.4.1) that

$$
{}_r\Phi_s^{(n)}\left[\begin{matrix} t^N, t^{N-1}, a_3,\dots,a_r \\ b_1,\dots,b_s\end{matrix}; q, t^2; z\right] = {}_2\Phi_1^{(m_1)}\left[\begin{matrix} t^{2n}, t^{m_2}, a_3,\dots,a_r \\ b_1,\dots,b_s\end{matrix}; q, t^2; z\right].
$$

Also, by (5.4.13),

$$
{}_{r+1}W_r^{(n)}(a_1;t^N,a_5,\dots,a_{r+1};q,t;z) = {}_{r+1}W_r^{(N)}(a_1;t^n,a_5,\dots,a_{r+1};q,t;z).
$$

Applying these two results to (B.1.10), and using

$$
\frac{(c^2qt^{2N+2n-2},cqt^{2n-1},cqt^{2n-2};q,t^2)_{\infty^n}}{(c^2qt^{2n-2},cqt^{N+2n-1},cqt^{N+2n-2};q,t^2)_{\infty^n}}
= \frac{(c^2qt^{2N+2n-2};q,t^2)_{\infty^N}}{(c^2qt^{2N-2};q,t^2)_{\infty^N}} \cdot \frac{(cqt^{N-1};q,t)_{\infty^N}}{(cqt^{N+2n-1};q,t)_{\infty^N}},
$$

yields

$$
{}_2\Phi_1^{(m_1)}\left[\begin{matrix} b^2, t^{m_2} \\ b^2q/t\end{matrix}; q, t^2; c^2q\right] = \frac{(b^2c^2qt^{2N-2};q,t^2)_{\infty^N}}{(c^2qt^{2N-2};q,t^2)_{\infty^N}} \cdot \frac{(cqt^{N-1};q,t)_{\infty^N}}{(b^2cqt^{N-1};q,t)_{\infty^N}}
\times {}_8W_7^{(N)}\big(b^2ct^{N-1};ct^{N-1},b,-b,bq^{1/2},-bq^{1/2};q,t;cq\big),
$$

where $b = t^n$. In the following we will prove this transformation formula for arbitrary b. We may then rename N as n so that the identity to be proved takes the form

$$
{}_2\Phi_1^{(n_1)}\left[\begin{matrix} b^2, t^{n_2} \\ b^2q/t\end{matrix}; q, t^2; c^2q\right] = \frac{(b^2c^2qt^{2n-2};q,t^2)_{\infty^n}}{(c^2qt^{2n-2};q,t^2)_{\infty^n}} \cdot \frac{(cqt^{n-1};q,t)_{\infty^n}}{(b^2cqt^{n-1};q,t)_{\infty^n}}
\times {}_8W_7^{(n)}\big(b^2ct^{n-1};ct^{n-1},b,-b,bq^{1/2},-bq^{1/2};q,t;cq\big), \tag{B.1.11}
$$

where $|q|, |cq|, |c^2q| < 1$ and

$$n_1 := \lfloor \tfrac{n}{2} \rfloor, \qquad n_2 := 2\lceil \tfrac{n}{2} \rceil - 1 = 2(n-n_1) - 1. \tag{B.1.12}$$

The prerequisite integral transformation is as follows.

Theorem B.6. *Let $p, q, t, t_0, \dots, t_4 \in \mathbb{C}^*$ such that $|p|, |q| < 1$ and such that* (B.1.7) *holds. Let n_1 and n_2 be given by* (B.1.12) *and τ by*

$$\tau = \begin{cases} 1 & \text{if } n \text{ is even} \\ \dfrac{1}{\Gamma(t_0^2, t_1^2, t_2^2, t_3^2, p, t, pt; p^2, q)} & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\begin{aligned} &II_1^{(n_1)}\bigl(t_0^2, t_1^2, t_2^2, t_3^2, t^{n_2-2n_1+1}, p, t, pt; t^2; p^2, q\bigr) \\ &\qquad = \tau \prod_{i=1}^{n} \prod_{0\leqslant r<s\leqslant 2} \frac{\Gamma(-t^{i-1}t_rt_s; p, q^{1/2})}{\Gamma(q^{-1/2}t^{i-1}t_rt_s; p, q^{1/2})} \\ &\qquad\qquad \times II_1^{(n)}\bigl(q^{\pm 1/4}t_0, q^{\pm 1/4}t_1, q^{\pm 1/4}t_2, -q^{\pm 1/4}t_3; t; p, q\bigr). \end{aligned}$$

Noting that $t^{n_2-2n_1+1}$ is 1 for n even and t^2 for n odd, the above theorem is [**111**, Conjecture Q4] with

$$\boldsymbol{\lambda} = \mathbf{0} \quad \text{and} \quad (q, u_0) \mapsto (q^{1/2}, -t_3),$$

which was proved in [**112**, Section 8].

If we make the substitutions

$$(t_0, t_1, t_2, t_3) = (p^{-1/4}, bp^{1/4}q^{1/2}t^{1-n}, cp^{1/4}q^{1/2}, p^{3/4}/bc)$$

then (B.1.7) is automatically satisfied. Again defining α and α' as in (B.1.9), we thus obtain

$$\begin{aligned} &B_\alpha^{(n_1)}\Bigl(1, t^{n_2-2n_1+1}, t, b^2qt^{2-2n}, c^2q, 1, t, \frac{1}{b^2c^2}; t^2; p^2, q\Bigr) \\ &\quad = \tau' \prod_{i=1}^{n} \frac{\Gamma(-bq^{1/2}t^{1-i}, -cq^{1/2}t^{n-i}, -bcp^{1/2}qt^{1-i}; p, q^{1/2})}{\Gamma(bt^{1-i}, ct^{n-i}, bcp^{1/2}q^{1/2}t^{1-i}; p, q^{1/2})} \\ &\quad\quad \times B_{\alpha'}^{(n)}\Bigl(q^{-1/4}, q^{1/4}, bq^{1/4}t^{1-n}, bq^{3/4}t^{1-n}, cq^{1/4}, cq^{3/4}, -\frac{q^{-1/4}}{bc}, -\frac{q^{1/4}}{bc}; t; p, q\Bigr), \end{aligned}$$

where $\tau' = 1$ if n is even and

$$\tau' = \frac{1}{\Gamma(p^{-1/2}, b^2p^{1/2}qt^{2-2n}, c^2p^{1/2}q, p^{3/2}/b^2c^2, p, t, pt; p^2, q)}.$$

After multiplying both sides by

$$\begin{aligned} &(p^{-1/2}t^{n_2-1}, p^{-1/2}t^{2n_1-1}; q, t^2)_{\infty^{n_1}} \\ &\qquad = (p^{-1/2}t^{n-1}; q, t)_{\infty^n} \times \begin{cases} 1 & \text{if } n \text{ is even} \\ \dfrac{1}{(p^{-1/2}; q)_\infty} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

we can use Corollary B.2 and Proposition B.3 to take the $p \to 0$ limit, resulting in (B.1.11). $\square$

Proof of Theorem 5.27. Our final proof is very similar to that of Theorem 5.24. We begin with the $\boldsymbol{\lambda}=0$ case of [**111**, Conjecture Q3], proved in [**19**, Theorem 6.1] and [**112**, Section 8].

Theorem B.7. *For $p,q,t,u,t_0,\dots,t_3\in\mathbb{C}^*$ such that $|p|,|q|<1$ and*

$$t^{2n-2}t_0t_1t_2t_3=pq^2, \tag{B.1.13}$$

we have

$$\begin{aligned}
&I\!I_2^{(n)}\bigl(t^{-1/2}t_0,t^{-1/2}t_1,t^{-1/2}t_2,t^{1/2}t_3,t^{-1/2}u,pqt^{-1/2}/u,\pm t^{1/2},\pm(pt)^{1/2};t;p,q\bigr)\\
&\quad=\prod_{i=1}^n\prod_{0\leqslant r<s\leqslant 2}\frac{\Gamma(t^{2i-3}t_rt_s;p,q)}{\Gamma(t^{2i-2}t_rt_s/q;p,q)}\\
&\qquad\times I\!I_2^{(n)}\bigl(q^{\pm1/2}t_0,q^{\pm1/2}t_1,q^{\pm1/2}t_2,q^{\pm1/2}tt_3,q^{1/2}u/t,pq^{3/2}/tu;t^2;p,q^2\bigr).
\end{aligned}$$

For $u=t_3$ both sides reduce to a $I\!I_1^{(n)}$ integral by (B.1.1), so that

$$\begin{aligned}
&I\!I_1^{(n)}\bigl(t^{-1/2}t_0,t^{-1/2}t_1,t^{-1/2}t_2,t^{-1/2}t_3,\pm t^{1/2},\pm(pt)^{1/2};t;p,q\bigr)\\
&\qquad=\prod_{i=1}^n\prod_{0\leqslant r<s\leqslant 2}\frac{\Gamma(t^{2i-3}t_rt_s;p,q)}{\Gamma(t^{2i-2}t_rt_s/q;p,q)}\\
&\qquad\qquad\times I\!I_1^{(n)}\bigl(q^{\pm1/2}t_0,q^{\pm1/2}t_1,q^{\pm1/2}t_2,(q^{1/2}/t)^{\pm}t_3;t^2;p,q^2\bigr).
\end{aligned}$$

After carrying out the substitutions

$$(t_0,t_1,t_2,t_3)\mapsto(p^{-1/4},ap^{1/4}qt^{1-2n},p^{3/4}q/ac,cp^{1/4}t),$$

consistent with (B.1.13), we then obtain

$$\begin{aligned}
&B_\alpha^{(n)}\biggl(t^{-1/2},t^{1/2},-t^{1/2},aqt^{1/2-2n},ct^{1/2},t^{1/2},-t^{1/2},\frac{qt^{-1/2}}{ac};t;p,q\biggr)\\
&=\prod_{i=1}^n\frac{\Gamma(aqt^{-2i},p^{1/2}qt^{2n-2i-1}/ac,pq^2t^{-2i}/c;p,q)}{\Gamma(at^{1-2i},p^{1/2}t^{2n-2i}/ac,pqt^{1-2i}/c;p,q)}\\
&\times B_{\alpha'}^{(n)}\biggl(q^{1/2},q^{-1/2},aq^{3/2}t^{1-2n},aq^{1/2}t^{1-2n},cq^{-1/2}t^2,cq^{1/2},\frac{q^{3/2}}{ac},\frac{q^{1/2}}{ac};t^2;p,q^2\biggr),
\end{aligned}$$

with α and α' once again given by (B.1.9). Multiplying both sides by

$$(p^{-1/4}t^{n-1},-p^{1/4}t^{n-1};q,t)_{\infty^n}=(p^{-1/2}t^{2n-2};q^2,t^2)_{\infty^n}$$

and then letting p tend to 0 yields (5.4.26). $\square$

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