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Bethe-*Ansatz* Results for a Solvable O(*n*) Model on the Square Lattice

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The Bethe-*Ansatz* approach is used to solve a nineteen-vertex model equivalent to a spin model with O(n) symmetry on the square lattice. The bulk free energy is determined exactly on four branches of critical points. On two of these branches, the central charge is derived from finite-size corrections. On the two other branches, the nature of the Bethe-*Ansatz* solutions prevents us from computing the central change analytically. In this case, our approach still allows the exact numerical determination of the free energy on strips hundreds of sites wide.

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The critical point and low-temperature phase of O(n) models in two dimensions have been studied primarily by means of a specific soluble example on the honeycomb lattice.[1-5] Recently another soluble O(n) model, this time on the square lattice, was shown[6] to be equivalent to a nineteen-vertex model, previously studied in the framework of the quantum inverse scattering method by Izergin and Korepin.[7] For this same vertex model Reshetikhin[8] found the spectrum of the transfer matrix via a functional equation method.

Here we study this square O(n) model via the coordinate Bethe *Ansatz* to verify that the critical behavior is lattice independent, and in search of new universality classes of O(n) multicritical behavior. A reason to expect such behavior is the presence in the model of an interaction akin to dilution in discrete spin models and to chain attraction in polymer solutions, thus allowing for tricritical or θ behavior. We find and confirm[9] evidence for the existence in this model of critical and low-temperature behavior of the same type as in the honeycomb model. We present analytical results for the bulk free energy and the value of the conformal anomaly. Besides this we also find critical behavior of another type. Though we could also derive the free energy in this case, a novel structure of the solutions to the Bethe-*Ansatz* equations prevented us from deriving the central charge.

For the O(n) model we study here, the n-component spin variables reside on the edges of the square lattice. They interact via a Boltzmann weight which is the product over the vertices of

$$Q=u_0+u_1\{\mathbf{s}_1\cdot\mathbf{s}_2+\mathbf{s}_2\cdot\mathbf{s}_3+\mathbf{s}_3\cdot\mathbf{s}_4+\mathbf{s}_4\cdot\mathbf{s}_1\}+u_2\{\mathbf{s}_1\cdot\mathbf{s}_3+\mathbf{s}_2\cdot\mathbf{s}_4\}+u_3\{(\mathbf{s}_1\cdot\mathbf{s}_2)(\mathbf{s}_3\cdot\mathbf{s}_4)+(\mathbf{s}_2\cdot\mathbf{s}_3)(\mathbf{s}_4\cdot\mathbf{s}_1)\}\,, \tag{1}$$

where $\mathbf{s}_i$ are the four spin variables adjacent to the vertex, labeled anticlockwise, and u_j are four coupling constants. When the partition sum is expanded in powers of u_1, u_2, and u_3 it naturally turns into a gas of loops[9,10] in which each loop has a weight n. This loop gas can be mapped onto a nineteen-vertex model by placing arrows on the edges occupied by the loops, thus orienting each loop with a left- or right-handed sense of rotation. The weight n for each loop is recovered by giving a phase factor to each corner of the oriented loops, just like in the case of the Potts model[11] and the honeycomb O(n) model.[1] This results in nineteen allowed vertex configurations, each of which, along with the corresponding weight, is shown in Fig. 1. We find that the model is solvable by the Bethe-*Ansatz* method when

$$\begin{aligned} u_0&=1\,,\\ u_1&=u_3p(p^2-2)\,,\\ u_2&=u_3(p^2-1)\,,\\ u_3&=\{2-(3-p^2)(1-p^2)^2\}^{-1}\,,\\ r&=\exp[i(\tfrac{1}{2}\theta-\tfrac{1}{4}\pi)]\,, \end{aligned} \tag{2}$$

where $p=2\cos[\frac{1}{4}(\theta-\pi)]$, and the variable θ is defined by $n=-2\cos 2\theta$, with $\theta\in[-\pi,\pi]$. In this parametrization in θ there are, in the space of u_i, four branches for each value of $n\in[-2,2]$, apart from the sign of u_1. We will label these branches by the integral part of $2(\pi-\theta)/\pi$.

We consider the square lattice wrapped on an infinitely long vertical cylinder with L sites around the circumference, and define the row-to-row transfer matrix T between two successive rows of vertical edges. Because each edge can be in three states, T is a 3^L by 3^L matrix. Since the number of upward minus the number of downward arrows in each row is a conserved quantity, T breaks up into $2L+1$ disjoint sectors. We follow closely Baxter's diagonalization of the transfer matrix of the honeycomb seven-vertex model.[2] This results in the following eigenvalues of the transfer matrix in the sector with net number l of upward arrows:

$$\Lambda\{z\}=u_3^L r^{-4}\prod_{j=1}^{l}M_-(z_j)+u_2^L\prod_{j=1}^{l}M_0(z_j)+u_3^L r^4\prod_{j=1}^{l}M_+(z_j)\,, \tag{3}$$

where the functions $M(z)$ are defined by

$$M_-(z)=\frac{u_2}{u_3}+\frac{u_1^2}{u_3^2z^{-1}-u_2u_3}\,,\qquad M_0(z)=\frac{u_0}{u_2}+\frac{u_1^2}{u_2^2z-u_2u_3}+\frac{u_1^2}{u_2^2z^{-1}-u_2u_3}\,,$$
$$M_+(z)=\frac{u_2}{u_3}+\frac{u_1^2}{u_3^2z-u_2u_3}\,, \tag{4}$$

and the parameters $z_1,\dots,z_l$ satisfy

$$z_j^L=(-1)^{l-1}r^4\prod_{k=1}^{l}\frac{S(z_k,z_j)}{S(z_j,z_k)} \tag{5}$$

for $j=1,\dots,l$ with

$$S(z,w)=\{1-z-w+zw+z(p^2-2)^2\}\{1-2w+zw+w(p^2-2)^2\}\,. \tag{6}$$

These equations are of the same form as for the honeycomb seven-vertex model,[2] apart from the powers of r explicitly appearing in Eqs. (3) and (5). This deviation from Baxter's derivation was introduced for the honeycomb model by Batchelor and Blöte[4] and Suzuki[5] for the following reason. The vertex weights in Fig. 1 and Eq. (1) are constructed such that each loop acquires a weight r^4 or r^{-4} from the accumulated phase factors of its corners. This is to get a total fugacity of $n=r^4+r^{-4}$ for each loop, after the arrows are summed out. However, the loops that wind the cylinder have a total phase factor 1, since they turn right and left equally often. This is corrected by the artificial introduction of a factor r^4 (r^{-4}) for each loop of arrows that wraps the cylinder to the left (right). These factors, that do not belong to the nineteen-vertex model per se, are the ones that appear in Eqs. (3) and (5). They ascertain that loops that wrap around the cylinder are treated exactly like those that do not, so that the partition sum correctly describes an O(n) model. One may modify the vertex model so as to justify these factors by giving an additional weight of $r^{4/L}$ ($r^{-4/L}$) to each arrow pointing left (right) and change the weights in Fig. 1 accordingly. Or alternatively one may introduce a seam running along the cylinder and give a weight r^4 (r^{-4}) to each arrow piercing the seam from the right (left).

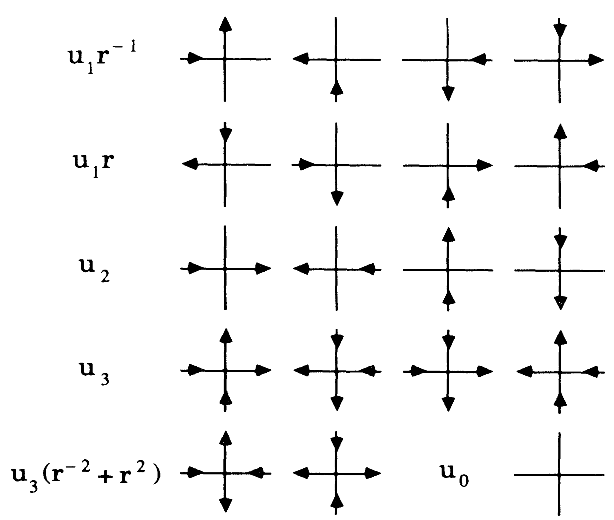

FIG. 1. The allowed vertex configurations. The corresponding Boltzmann weights are indicated to the left.

It is convenient to introduce new variables $w_1,\dots,w_l$ (Ref. 2) by

$$z_j=\frac{1+e^{i\theta}w_j}{e^{i\theta}+w_j}\,. \tag{7}$$

The Bethe-Ansatz equations (5) then read

$$\left(\frac{1+e^{i\theta}w_j}{e^{i\theta}+w_j}\right)^L=(-1)^l e^{2i\theta}\prod_{k=1}^{l}\frac{w_j+w_ke^{i\theta}}{w_k+w_je^{i\theta}}\frac{w_k-w_je^{2i\theta}}{w_j-w_ke^{2i\theta}}\,. \tag{8}$$

Bulk free energy.— A numerical investigation of the eigenvalues of the transfer matrix[9] has revealed that the critical behavior on branches 1 and 2, i.e., $(\pi/2)<\theta<\pi$ and $0<\theta<(\pi/2)$, is of the same type as found in the honeycomb model.[1,4,5] In this case the parameters z_k lie on the unit circle and w_k are real. The bulk free energy per site

$$f_\infty = \lim_{L\to\infty} \frac{1}{L} \ln \Lambda_{\max} \tag{9}$$

can be derived in the same manner as for the honeycomb model. When we rewrite Eq. (8) in terms of real $w_j = e^{2\lambda_j}$ and take the logarithms of both members of the equation, its imaginary part reads

$$2L \arctan[\tan\tfrac{1}{2}\theta \tanh\lambda_j] = 2\pi I_j + 2\theta - 2\sum_{k=1}^{l} \{\arctan[\tan\tfrac{1}{2}\theta \tanh(\lambda_j-\lambda_k)] + \arctan[\cot\theta \tanh(\lambda_j-\lambda_k)]\}\,, \tag{10}$$

where the (half) integer I_j accommodates the indefiniteness of the phase of the equation. The real part of the equation is automatically satisfied, which justifies the search for real-w solutions. The largest eigenvalue occurs in the $l=L$ sector and has $I_j = j-(L+1)/2$. In the limit $l=L\to\infty$ the solution to Eq. (10) turns into a continuous distribution of λ. Therefore, Eq. (10) can be written as an integral equation involving the density of roots $\sigma_\infty(\lambda)$,[2,4,5] and is solved by

$$\sigma_\infty(\lambda) = \frac{2}{\sqrt{3}(\pi-\theta)} \frac{\sinh\{4\pi\lambda/[3(\pi-\theta)]\}}{\sinh[2\pi\lambda/(\pi-\theta)]}\,. \tag{11}$$

The dominant term in Eq. (3) is the one involving M_0, which with the change of variables, Eq. (7), leads to

$$f_\infty(\theta) = \ln|u_2| + \int_{-\infty}^{\infty} d\lambda\, \sigma_\infty(\lambda) \ln \frac{\cosh 2\lambda + \sin\frac{3}{2}\theta}{\cosh 2\lambda - \sin\frac{1}{2}\theta}\,. \tag{12}$$

Taking Fourier transforms of $\sigma_\infty(\lambda)$ and $\ln M_0$, Eq. (12) can also be written as

$$f_\infty(\theta) = \ln|u_2| + 2\int_{-\infty}^{\infty} dx \frac{\sinh(\theta x)\sinh[\frac{1}{2}(\pi-\theta)x]}{x[2\cosh(\pi x-\theta x)-1]\sinh(\pi x)}\,. \tag{13}$$

This result, when evaluated numerically, is found to be in excellent agreement with the finite-size results of Blöte and Nienhuis[9] in the whole range of $\theta \in [0,\pi)$. This is in contrast with the honeycomb model where there is a cusp in the free energy as a function of θ at $\theta = 0.4279\ldots$.[2,4] At the particular point $\theta=\pi$, the substitution (7) is no longer valid, but should be replaced by $z=(1+i\lambda)/(1-i\lambda)$. In this case the free energy is given by

$$f_\infty(\pi) = \ln\tfrac{3}{11} + 4\int_0^\infty dx \frac{e^{-x}\sinh\frac{1}{2}x}{x(2\cosh x - 1)}\,. \tag{14}$$

This result again agrees with finite-size data.

The situation is more complicated on the branches 3 and 4, i.e., $\theta \in [-\pi, 0]$. By solving the Bethe-*Ansatz* equations (5) numerically for finite but large systems, and keeping the solution with the largest eigenvalue, $l=L$, we made the following observations. The values of the w are no longer real but approach, as L increases, two straight half lines in the complex plane which can be parametrized by

$$w^\pm = \pm i \exp(2\lambda \mp \tfrac{1}{2} i\theta)\,. \tag{15}$$

Only in the limit as L approaches infinity do the values of these λ approach the real axis. The solution to the equations for even L consists of pairs of values w_j^+ and w_j^- located on each of the two radii in the w plane, and related by $w_j^- = (w_j^+)^*$. Therefore Eq. (8) now has two forms, one for each radius, and its right-hand side, expressed in λ, now consists of two products over the two radii. Because of the symmetry we need to investigate the equations for one radius only:

$$\left(\frac{\cosh[\lambda_j + \frac{1}{4}i(\theta+\pi)]}{\cosh[\lambda_j - \frac{1}{4}i(3\theta-\pi)]}\right)^L = e^{2i\theta} \prod_{k=1}^{L/2} \frac{\cosh(\lambda_j-\lambda_k-\frac{1}{2}i\theta)\sinh(\lambda_j-\lambda_k+i\theta)}{\cosh(\lambda_j-\lambda_k+\frac{1}{2}i\theta)\sinh(\lambda_j-\lambda_k-i\theta)} \times \frac{\cosh(\lambda_j-\lambda_k+\frac{1}{2}i\theta)\sinh(\lambda_j-\lambda_k-i\theta)}{\cosh(\lambda_j-\lambda_k-\frac{3}{2}i\theta)\sinh(\lambda_j-\lambda_k)}\,. \tag{16}$$

In the limit $L\to\infty$ this can be written as a complex integral equation which is solved by the root density

$$\sigma_\infty(\lambda) = \frac{1}{|\pi+3\theta|} \frac{1}{\cosh[2\pi\lambda/(\pi+3\theta)]}\,. \tag{17}$$

The fact that the solution is real confirms that we correctly guessed the asymptotic locus of the roots. To compute the eigenvalue of the transfer matrix this solution should be substituted into (3). We observed numerically that the three terms of (3) differ only by a finite factor even in the thermodynamic limit. Therefore for the bulk free energy it suffices to compute only one. Again taking the term M_0, we obtain

$$f_\infty(\theta)=\ln|u_2|+\int_{-\infty}^{\infty}d\lambda\,\sigma_\infty(\lambda)\left\{\ln\frac{\cosh[2\lambda-\frac{1}{2}i(\theta+\pi)]-\sin\frac{3}{2}\theta}{\cosh[2\lambda-\frac{1}{2}i(\theta+\pi)]+\sin\frac{1}{2}\theta}+\ln\frac{\cosh[2\lambda+\frac{1}{2}i(\theta+\pi)]-\sin\frac{3}{2}\theta}{\cosh[2\lambda+\frac{1}{2}i(\theta+\pi)]+\sin\frac{1}{2}\theta}\right\}. \tag{18}$$

As before we take a Fourier transform with the result

$$f_\infty(\theta)=\ln|u_2|-\int_{-\infty}^{\infty}dx\frac{\sinh(\theta x)\sinh[(\theta+\pi)x]}{x\cosh[\frac{1}{2}(\pi x+3\theta x)]\sinh(\pi x)}. \tag{19}$$

This expression is valid for all $\theta\in[-\pi,0]$ except for the special point $\theta=-\pi/3$. At this point the integral and the weight u_2 both diverge. The limit $\theta\to-\pi/3$ can, however, be taken, yielding

$$f_\infty(-\tfrac{1}{3}\pi)=-\ln 8+2\ln\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}. \tag{20}$$

Central charge.—The derivation of size-dependent corrections to the eigenvalues of exactly solvable models is currently receiving considerable attention.[4,12] One motivation is that the dominant correction to the largest eigenvalue has been identified with the central charge c of the associated conformal (Virasoro) algebra.[13] In particular, for an isotropic system in the limit of large L (Refs. 14 and 15),

$$f_L=\frac{\ln\Lambda_{\max}}{L}=f_\infty+\frac{\pi c}{6L^2}+o(L^{-2}). \tag{21}$$

Given that the Bethe-*Ansatz* equations (5) are the same as for the honeycomb $O(n)$ model, the calculation of the central charge on branches 1 and 2 is a straightforward reworking of the honeycomb derivation.[4,5] The result,

$$c(\theta)=1-\frac{3(\pi-2\theta)^2}{\pi\theta}, \tag{22}$$

is in accordance with both the analytic result for the honeycomb lattice[4,5] and the numerical results for the square lattice.[9] Although the continuation of the same solution of the Bethe-*Ansatz* equations for $\Lambda_{\max}$ to branches 3 and 4 does correspond to an eigenvalue of the transfer matrix, it is no longer the largest. As remarked above, the root distribution for $\Lambda_{\max}$ on these branches is entirely different. Here our knowledge of these roots is as yet confined to the infinite-lattice limit, and consequently we have not been able to derive the value of the central charge. A similar situation prevents an analytic derivation of c from the Bethe-*Ansatz* solution of the Takhatajan-Babujian models.[16] However, the Bethe-*Ansatz*-equations approach is still useful for the calculation of eigenvalues of finite systems.[17] In this case it is still possible to solve the Bethe-*Ansatz* equations numerically for systems several hundred sites wide. In contrast, direct diagonalization of the transfer matrix is limited to systems at least one order of magnitude smaller. Numerical estimates of c, based on Bethe-*Ansatz* computations of $\Lambda_{\max}$ for large but finite systems, are currently in progress and will be published separately.

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[1]B. Nienhuis, Phys. Rev. Lett. **49**, 1062 (1982).

[2]R. J. Baxter, J. Phys. A **19**, 2821 (1986); J. Phys. A **20**, 5241 (1987).

[3]P. di Francesco, H. Saleur, and J. B. Zuber, J. Stat. Phys. **49**, 57 (1987).

[4]M. T. Batchelor and H. W. J. Blöte, Phys. Rev. Lett. **61**, 138 (1988); Phys. Rev. B **39**, 2391 (1989).

[5]J. Suzuki, J. Phys. Soc. Jpn. **57**, 2966 (1988).

[6]B. Nienhuis (to be published).

[7]A. G. Izergin and V. E. Korepin, Commun. Math. Phys. **79**, 303 (1981).

[8]N. Yu. Reshetikhin, Zh. Eksp. Teor. Fiz. **84**, 1190 (1983) [Sov. Phys. JETP **57**, 691 (1983)]; V. I. Vichirko and N. Yu. Reshetikhin, Teor. Mat. Fiz. **56**, 260 (1983) [Theor. Math. Phys. **56**, 805 (1983)].

[9]H. W. J. Blöte and B. Nienhuis, J. Phys. A (to be published).

[10]E. Domany, D. Mukamel, B. Nienhuis, and A. Schwimmer, Nucl. Phys. **B190 [FS3]**, 279 (1981).

[11]R. J. Baxter, S. B. Kelland, and F. Y. Wu, J. Phys. A **9**, 397 (1976).

[12]M. Karowski, Nucl. Phys. **B300 [FS22]**, 473 (1988).

[13]J. L. Cardy, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1987), Vol. 11.

[14]H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).

[15]I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).

[16]F. C. Alcaraz and M. J. Martins, J. Phys. A **21**, 4397 (1988).

[17]M. T. Batchelor, M. N. Barber, and P. A. Pearce, J. Stat. Phys. **49**, 1117 (1987).