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Bailey flows and Bose–Fermi identities for the conformal coset models $(A_1^{(1)})_N \times (A_1^{(1)})_{N'}/(A_1^{(1)})_{N+N'}$

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Abstract

We use the recently established higher-level Bailey lemma and Bose–Fermi polynomial identities for the minimal models $M(p,p')$ to demonstrate the existence of a Bailey flow from $M(p,p')$ to the coset models $(A_1^{(1)})_N \times (A_1^{(1)})_{N'}/(A_1^{(1)})_{N+N'}$ where N is a positive integer and N' is fractional, and to obtain Bose–Fermi identities for these models. The fermionic side of these identities is expressed in terms of the fractional-level Cartan matrix introduced in the study of $M(p,p')$. Possible relations between Bailey and renormalization group flow are discussed. © 1997 Elsevier Science B.V.

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1. Introduction

A decade ago Zamolodchikov [1] and Ludwig and Cardy [2] studied the phenomena of renormalization group (RG) flow from the minimal model $M(p,p+1)$ to the model $M(p-1,p)$ by means of perturbation theory when p is very large. These flows define a one parameter family of massless field theories. In 1991 Zamolodchikov [3] studied the ground state energy (or, equivalently, the effective central charge) for the one parameter flow from $M(4,5)$ to $M(3,4)$ in terms of the thermodynamic Bethe Ansatz equations and conjectured the generalization to all $M(p,p+1)$. Since then many other flows have been discovered using this perturbative method, such as the flow from $M(p,p')$ to $M(2p-p',p)$ [4] and flows between coset models [5,6].

All of these studies have been made using the techniques and philosophy of the renormalization group and all of them illustrate the famous c theorem [7] by flowing from a larger to a smaller effective central charge. For this reason it is often stated that renormalization group flow is irreversible.

It is thus most interesting that recently a non-perturbative construction from the mathematical literature of the Rogers-Ramanujan identities was used [8-10] to make connection between the minimal models in the opposite direction from the renormalization group flow. This construction is referred to as *Bailey flow*.

Bailey flow originates in the work of Bailey [11] and Slater [12,13] in their proofs of the many q-series identities of Rogers [14,15] and has been greatly extended by Andrews [16,17] and others [18-23]. The prototypical q-series identity is the original identity of Rogers and Ramanujan [14,24],

$$\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left(q^{j(10j+1+2a)}-q^{(2j+1)(5j+2-a)}\right)=\sum_{n=0}^{\infty}\frac{q^{n(n+a)}}{(q)_n}, \tag{1.1}$$

where $a=0,1$, and

$$(x;q)_n=(x)_n=(1-x)(1-xq)\dots(1-xq^{n-1}) \tag{1.2}$$

for $n\geqslant 1$ and $(x)_0=1$. The left-hand side is recognized as the character of the $M(2,5)$ minimal model in the form of Refs. [25,26] which is computed using the construction of Feigin and Fuchs [27]. This construction involves the elimination of states from a bosonic Fock space. Thus we refer to the left-hand side as a bosonic representation. Conversely, the right-hand side has an interpretation in terms of fermionic quasi-particles [28,29] and is called a fermionic representation.

Bailey's lemma is a constructive procedure which starts from a polynomial generalization of a Bose-Fermi identity for the characters/ branching functions of the initial conformal field theory (CFT) and produces a Bose-Fermi equality for a rational generalization of the characters/branching functions of some other CFT

$$\begin{array}{c} \text{BOSE} = \text{FERMI} \\ \Big\downarrow \quad \text{Bailey's lemma} \\ \text{BOSE}' = \text{FERMI}'. \end{array}$$

The Bailey flow from $M(p-1,p)$ to $M(p,p+1)$ is discussed in Refs. [8–10] and further flows to the unitary $N=1$ supersymmetric model $SM(p,p+2)$ and to the $N=2$ supersymmetric models with $c=3(1-2/p)$ are given in Ref. [10]. The Bailey flow from $M(p-1,p)$ to $M(p,p+1)$ is in the opposite direction from the RG flow and the remainder of the Bailey flows give relations between CFT's which have not previously been seen in the RG analysis. In a sense the Bailey construction is an "Aufbau Prinzip" and as such it promises to provide an alternative construction for most (possibly all) conformal field theories.

In this paper we extend these ideas to find a flow from the general minimal model $M(p,p')$ to the coset models

$$\frac{(A_1^{(1)})_N \times (A_1^{(1)})_{N'}}{(A_1^{(1)})_{N+N'}} \tag{1.3}$$

for integer level N and fractional level N'. For brevity these coset theories will be denoted by $(P,P')_N$, where

$$N' = \frac{NP}{P'-P} - 2 \quad \text{or} \quad N' = -2 - \frac{NP'}{P'-P},$$

with the restrictions $P<P'$, $P'-P \equiv 0 \pmod N$ and $\gcd(\frac{P'-P}{N}, P') = 1$. In this notation the minimal models $M(p,p')$ correspond to the coset model $(p,p')_1$ [30].

Using the method of Feigin and Fuchs [27] the bosonic form of the characters was given explicitly in Ref. [31] for the unitary minimal models $M(p,p+1)$ and in Refs. [25,26] for the non-unitary cases $M(p,p')$. The branching functions for the cosets (1.3) for integer levels was given in Refs. [32,33] by computing configuration sums of RSOS models and in Refs. [34–36] using the Feigin and Fuchs construction. Kac and Wakimoto [37] introduced admissible representations of affine Lie algebras which in general correspond to fractional levels and non-unitary CFTs. This paved the way for the study of the coset models (1.3) with one fractional level [38] and two fractional levels [39]. Further aspects of coset (1.3), including its spin content, have been studied in Ref. [40] in the context of unifying $\mathcal{W}$-algebras.

Our method to obtain the Bailey flow from $M(p,p')$ to the cosets (1.3) is to use the polynomial identities for $M(p,p')$ recently established in Refs. [41,42] with the new extension of the Bailey construction obtained in Refs. [22,23]. The branching function identities obtained from this flow have previously been found for the unitary case $(p,p+N)_N$ [43,44], for $(3,2N+3)_N$ [45] and the cosets (1.3) with $P' < (N+1)P$ [46] without the use of the Bailey flow.

The plan of the remainder of this paper is as follows. In Section 2 we summarize the results on Bailey flow which are needed for our construction. In Section 3 we construct the required Bailey pairs from the bosonic polynomial generalizations given in Refs. [47,48] of the characters $\chi^{(p,p')}_{r,s}$ of the minimal model $M(p,p')$ leaving the fermionic side still undetermined. We establish the Bailey flow $M(p,p') \to (p',p'+N(kp'+p))_N$ for k a non-negative integer by comparing the results of the bosonic side of the Bailey flow with the branching functions of the coset model (1.3) which are recalled in the appendix. To make the Bose–Fermi identities explicit we need the fermionic polynomials of the $M(p,p')$ models of Ref. [42]. Here we find it convenient to consider the regime $p' < 2p$ and state the results in Section 4. We present our final explicit results for the Bose–Fermi (or Rogers–Ramanujan) identities for the coset models $(P,P')_N$ in Section 5 where the case $P' < (N+1)P$ is treated in Section 5.1 and $P' > (N+1)P$ in Section 5.2. The fermionic form for the branching functions involves the same "fractional-level" Cartan matrix which arises in the study of the characters of $M(p,p')$ [41,42]. We conclude in Section 6 with a discussion of the relation between Bailey and RG flow.

2. The theory of Bailey flow

In this section we summarize Bailey's original lemma [11,17,49] and the recent results on the higher-level Bailey lemma [22,23] which we will use.

2.1. Bailey's lemma

Consider two sequences $\alpha = \{\alpha_L\}_{L\geqslant 0}$ and $\beta = \{\beta_L\}_{L\geqslant 0}$ which satisfy the relation

$$\beta_L = \sum_{i=0}^{L} \frac{\alpha_i}{(q)_{L-i}(aq)_{L+i}}. \tag{2.1}$$

A pair (α,β) satisfying (2.1) is called a Bailey pair relative to a.

Consider a second pair of sequences $\gamma = \{\gamma_L\}_{L\geqslant 0}$ and $\delta = \{\delta_L\}_{L\geqslant 0}$ which satisfies the relation

$$\gamma_L = \sum_{i=L}^{\infty} \frac{\delta_i}{(q)_{i-L}(aq)_{L+i}}, \tag{2.2}$$

which we call a conjugate Bailey pair. Bailey's lemma states that given a Bailey pair (α,β) and a conjugate Bailey pair (γ,δ), the following equation holds

$$\sum_{L=0}^{\infty} \alpha_L\gamma_L = \sum_{L=0}^{\infty} \beta_L\delta_L. \tag{2.3}$$

In Ref. [11], Bailey proved that the following (γ,δ) pair satisfies (2.2)

$$\gamma_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{M-L}(aq)_{M+L}}, \tag{2.4}$$

$$\delta_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L(aq/\rho_1\rho_2)_{M-L}}{(aq/\rho_1)_M(aq/\rho_2)_M(q)_{M-L}}. \tag{2.5}$$

Using this pair in (2.3) yields

$$\sum_{L=0}^{M} \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L} \frac{\alpha_L}{(q)_{M-L}(aq)_{M+L}}$$
$$= \sum_{L=0}^{M} \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L(aq/\rho_1\rho_2)_{M-L}}{(aq/\rho_1)_M(aq/\rho_2)_M(q)_{M-L}} \beta_L, \tag{2.6}$$

where the variables ρ_1 and ρ_2 can be chosen freely. When $\rho_1, \rho_2 \to \infty$, (2.6) simplifies to

$$\sum_{L=0}^{M} \frac{a^L q^{L^2}}{(q)_{M-L}(aq)_{M+L}} \alpha_L = \sum_{L=0}^{M} \frac{a^L q^{L^2}}{(q)_{M-L}} \beta_L, \tag{2.7}$$

where we used

$$\lim_{\rho\to\infty} \rho^{-L}(\rho)_L = (-1)^L q^{\frac{L(L-1)}{2}}. \tag{2.8}$$

For later use we will need several results of Andrews [17] concerning the Bailey pair (α, β).

Dual Bailey pairs

Given a Bailey pair $(\alpha,\beta) = (\alpha(a,q), \beta(a,q))$ relative to a, we may replace q by $1/q$ to find that (A, B) defined by

$$A_L = a^L q^{L^2} \alpha_L(a^{-1}, q^{-1}) \quad \text{and} \quad B_L = a^{-L} q^{-L(L+1)} \beta_L(a^{-1}, q^{-1}) \tag{2.9}$$

is again a Bailey pair relative to a. The pair (A, B) is called the dual Bailey pair of (α, β).

Iterated Bailey pairs

Let (α, β) be a Bailey pair relative to a. Then from (2.6) the sequences (α', β') defined by

$$\alpha'_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L} \alpha_L \tag{2.10}$$

and

$$\beta'_L = \sum_{r=0}^{L} \frac{(\rho_1)_r(\rho_2)_r(aq/\rho_1\rho_2)^r(aq/\rho_1\rho_2)_{L-r}}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{L-r}} \beta_r \tag{2.11}$$

form again a Bailey pair relative to a. Eqs. (2.10) and (2.11) can of course be repeated an arbitrary number of times, leading to the Bailey chain

$$(\alpha, \beta) \to (\alpha', \beta') \to (\alpha'', \beta'') \to \dots \tag{2.12}$$

Iterating (2.10) and (2.11) k times with $\rho_1, \rho_2 \to \infty$ yields the Bailey pair

$$\alpha_L^{(k)} = a^{kL} q^{kL^2} \alpha_L,$$
$$\beta_L^{(k)} = \sum_{L \geqslant r_1 \geqslant \ldots \geqslant r_k \geqslant 0} \frac{a^{r_1+\ldots+r_k} q^{r_1^2+\ldots+r_k^2}}{(q)_{L-r_1} (q)_{r_1-r_2} \cdots (q)_{r_{k-1}-r_k}} \beta_{r_k}, \tag{2.13}$$

where $k \geqslant 1$.

2.2. Higher-level Bailey lemma

In this paper we will use the new conjugate Bailey pairs found in Refs. [22,23]. To state the result needed, we denote by $\mathcal{I}$ the incidence matrix of the Lie algebra A_{N-1}, $\mathcal{I}_{j,k} = \delta_{j,k-1} + \delta_{j,k+1}$, and by C the Cartan matrix $C = 2I - \mathcal{I}$ with I the identity matrix. The vectors $\boldsymbol{e}_i$ are the unit vectors $(\boldsymbol{e}_i)_j = \delta_{i,j}$ for $i = 1, \ldots, N-1$ and $\boldsymbol{e}_i = \mathbf{0}$ for $i \neq 1, \ldots, N-1$. Also define the q-binomial coefficient as

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{cases} \dfrac{(q)_{m+n}}{(q)_m (q)_n} & \text{for} \quad m, n \in \mathbb{Z}_+, \\ 0 & \text{otherwise}, \end{cases} \tag{2.14}$$

where $\mathbb{Z}_+$ denotes the set of non-negative integers.

Then for $M \geqslant 0$, $N \geqslant 1$, $\lambda \geqslant 0$, $0 \leqslant \ell < N$ and $\sigma = 0, 1$, the following conjugate Bailey pair relative to $a = q^\lambda$ is given in corollary 2.1 of Ref. [23]

$$\gamma_L = \frac{a^{L/N} q^{L^2/N}}{(q)_{M-L} (aq)_{M+L}} \sum_{\boldsymbol{\eta} \in \mathbb{Z}_+^{N-1}, \ \frac{L+(\lambda-\ell)/2}{N} - (C^{-1}\boldsymbol{\eta})_1 \in \mathbb{Z} + \frac{\sigma}{2}} q^{\boldsymbol{\eta} C^{-1} (\boldsymbol{\eta} - \boldsymbol{e}_\ell)} \prod_{j=1}^{N-1} \begin{bmatrix} \mu_j + \eta_j \\ \eta_j \end{bmatrix},$$
$$\delta_L = \frac{a^{L/N} q^{L^2/N}}{(q)_{M-L}} \sum_{\boldsymbol{n} \in \mathbb{Z}_+^{N-1}, \ \frac{L+(\lambda-\ell)/2}{N} - (C^{-1}\boldsymbol{n})_1 \in \mathbb{Z} + \frac{\sigma}{2}} q^{\boldsymbol{n} C^{-1} (\boldsymbol{n} - \boldsymbol{e}_\ell)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}, \tag{2.15}$$

where $\boldsymbol{\mu}, \boldsymbol{\eta}$ and $\boldsymbol{m}, \boldsymbol{n}$ are related by

$$\boldsymbol{\mu} + \boldsymbol{\eta} = \frac{1}{2}\Big(\mathcal{I}\boldsymbol{\mu} + (M-L)\,\boldsymbol{e}_1 + (M+L+\lambda)\,\boldsymbol{e}_{N-1} + \boldsymbol{e}_\ell\Big), \tag{2.16}$$
$$\boldsymbol{m} + \boldsymbol{n} = \frac{1}{2}\Big(\mathcal{I}\boldsymbol{m} + (2L+\lambda)\,\boldsymbol{e}_{N-1} + \boldsymbol{e}_\ell\Big). \tag{2.17}$$

The sums of the type

$$\sum_{\boldsymbol{n} \in \mathbb{Z}_+^{N-1}, \ \frac{m}{N} - (C^{-1}\boldsymbol{n})_1 \in \mathbb{Z} + \frac{\sigma}{2}}$$

are taken over the vector $\boldsymbol{n} \in \mathbb{Z}_+^{N-1}$ such that $\frac{m}{N} - (C^{-1}\boldsymbol{n})_1$ is an integer when $\sigma = 0$ and half an odd integer when $\sigma = 1$.

Before using the above conjugate Bailey pair to derive the higher-level Bailey lemma, we make some observations. Since the q-binomials are defined to be zero when its entries are non-integer or negative we only need to sum over those $\boldsymbol{\eta}, \boldsymbol{n} \in \mathbb{Z}_+^{N-1}$ in (2.15) which yield $\boldsymbol{\mu}, \boldsymbol{m} \in \mathbb{Z}_+^{N-1}$ from (2.16) and (2.17). Using further $C_{i,j}^{-1} = \min\{i,j\} - ij/N$ this implies that $(L - (\lambda - \ell)/2)/N - C^{-1}\boldsymbol{v}$ $(\boldsymbol{v} = \boldsymbol{\eta}, \boldsymbol{n})$ is half an integer, explaining the restrictions on the above sums. The fact that one can in fact independently choose this expression to be half an even integer $(\sigma = 0)$ or half an odd integer $(\sigma = 1)$, follows from other considerations [23]. We also note that since $C^{-1}\boldsymbol{v}$ is a multiple of $1/N$, we only obtain a non-trivial (non-zero) conjugate Bailey pair if we take $\lambda + \ell + N\sigma$ to be even.

We may now insert the conjugate Bailey pair (2.15) into (2.3). For later use we take $M \to \infty$ and eliminate $\boldsymbol{n}$ in favour of $\boldsymbol{m}$ via (2.17). This gives the following lemma:

Lemma 1 (higher-level Bailey lemma). Fix integers $N \geqslant 1$, $\lambda \geqslant 0$, $0 \leqslant \ell < N$ and $\sigma = 0, 1$, such that $\ell + \lambda + N\sigma$ is even, and let (α, β) form a Bailey pair relative to q^{λ}. Then the following identity holds:

$$\frac{1}{(q)_\infty} \sum_{L=0}^{\infty} q^{L(L+\lambda)/N} \alpha_L \sum_{\boldsymbol{\eta} \in \mathbb{Z}_+^{N-1},\ \frac{L+(\lambda-\ell)/2}{N} - (C^{-1}\boldsymbol{\eta})_1 \in \mathbb{Z} + \frac{\sigma}{2}} \frac{q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \boldsymbol{e}_\ell)}}{(q)_{\eta_1} \cdots (q)_{\eta_{N-1}}}$$
$$= \frac{q^{-\frac{\ell(N-\ell)+\lambda^2}{4N}}}{(q)_\lambda} \sum_{L=0}^{\infty} q^{\frac{1}{4}(2L+\lambda)^2} \beta_L \sum_{\boldsymbol{m} \in \mathbb{Z}_+^{N-1},\ \boldsymbol{m} \equiv \boldsymbol{Q} \ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m} - \frac{1}{2}(2L+\lambda)m_{N-1}} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix}, \tag{2.18}$$

where $\boldsymbol{m} \equiv \boldsymbol{Q}$ (mod 2) stands for m_j even when Q_j is even and m_j odd when Q_j is odd, and

$$\boldsymbol{Q} = (\boldsymbol{e}_{\ell+1} + \boldsymbol{e}_{\ell+3} + \ldots) + (\sigma + 1)(\boldsymbol{e}_1 + \boldsymbol{e}_3 + \ldots). \tag{2.19}$$

Note that for $N = 1$ lemma 1 reduces to (2.7) with $M \to \infty$ and $a = q^{\lambda}$.

3. Bosonic polynomials for the minimal models $M(p, p')$ and Bailey flow

3.1. Bosonic polynomials

In order to make effective use of any of the forms of Bailey's lemma given in the preceding section it is necessary to find solutions of the defining Eq. (2.1) for Bailey pairs. Here we derive such pairs from polynomial identities for finitizations of the characters of the minimal models $M(p, p')$ given in Ref. [42],

$$B_{r,s}^{(p,p')}(L, b) = F_{r,s}^{(p,p')}(L, b), \tag{3.1}$$

where $1 \leqslant b, s \leqslant p' - 1$, $1 \leqslant r \leqslant p - 1$. The bosonic function $B_{r,s}^{(p,p')}(L, b)$ is given by

$$B_{r,s}^{(p,p')}(L, b) = \sum_{j=-\infty}^{\infty} \left(q^{j(jpp'+rp'-sp)} \begin{bmatrix} L \\ \frac{1}{2}(L + s - b) - jp' \end{bmatrix} \right.$$

$$-q^{(jp+r)(jp'+s)}\left[\begin{matrix}L\\ \frac{1}{2}(L-s-b)-jp'\end{matrix}\right]\Bigg), \tag{3.2}$$

which first appeared in the work of Andrews, Baxter and Forrester [47] for $p'=p+1$, and for general p,p' in the work of Forrester and Baxter [48]. Since the Bailey flows can be determined without knowledge of the explicit form of the fermionic side, we postpone giving $F^{(p,p')}_{r,s}(L,b)$ until Section 4.

The limit $L\to\infty$ of the polynomials (3.2) are the characters of the minimal models $M(p,p')$, namely $\lim_{L\to\infty}B^{(p,p')}_{r,s}(L,b)=\hat{\chi}^{(p,p')}_{r,s}(q)$ where [27,31,26]

$$\chi^{(p,p')}_{r,s}(q)=q^{\Delta_{r,s}-\frac{c}{24}}\hat{\chi}^{(p,p')}_{r,s}(q), \tag{3.3}$$

with

$$\hat{\chi}^{(p,p')}_{r,s}(q)=\hat{\chi}^{(p,p')}_{p-r,p'-s}(q)=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left(q^{j(jpp'+rp'-sp)}-q^{(jp+r)(jp'+s)}\right). \tag{3.4}$$

The central charge and conformal dimensions are given by

$$c=1-\frac{6(p'-p)^2}{pp'}\quad\text{and}\quad \Delta_{r,s}=\frac{(rp'-sp)^2-(p'-p)^2}{4pp'}, \tag{3.5}$$

respectively.

3.2. Bailey pairs

To obtain Bailey pairs from (3.1), we use the definition of the q-binomials (2.14) in (3.1) and replace L by $2L+\lambda$ where $\lambda=|b-s|$. Then multiplying by $(q)_\lambda/(q)_{2L+\lambda}$ we note that the resulting left-hand side is in the form of the right-hand side of (2.1) and we find the following Bailey pairs relative to $q^\lambda=q^{|b-s|}$.

Bailey pair arising from (3.1) and (3.2)

$$\alpha_L=\begin{cases}1 & \text{for } L=0,\\ q^{j(jpp'\pm rp'\mp sp)} & \text{for } L=jp'-(\lambda\mp b\pm s)/2,\quad (j\geqslant 1),\\ -q^{(jp+r)(jp'+s)} & \text{for } L=jp'-(\lambda-b-s)/2,\quad (j\geqslant 0),\\ -q^{(jp-r)(jp'-s)} & \text{for } L=jp'-(\lambda+b+s)/2,\quad (j\geqslant 1),\\ 0 & \text{otherwise,}\end{cases}$$

$$\beta_L=\frac{(q)_\lambda}{(q)_{2L+\lambda}}F^{(p,p')}_{r,s}(2L+\lambda,b). \tag{3.6}$$

Dual Bailey pair arising from (3.1) and (3.2)

$$
\alpha_L = \begin{cases} 1 & \text{for } L=0, \\ q^{j(j(p'-p)p' \pm (b-r)p' \mp s(p'-p))} & \text{for } L = jp' - (\lambda \mp b \pm s)/2, \quad (j \geqslant 1), \\ -q^{(j(p'-p)+b-r)(jp'+s)} & \text{for } L = jp' - (\lambda - b - s)/2, \quad (j \geqslant 0), \\ -q^{(j(p'-p)-b+r)(jp'-s)} & \text{for } L = jp' - (\lambda + b + s)/2, \quad (j \geqslant 1), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\beta_L = q^{L(L+\lambda)} \frac{(q)_\lambda}{(q)_{2L+\lambda}} F_{r,s}^{(p,p')}(2L+\lambda, b; 1/q). \tag{3.7}
$$

Iterated Bailey pair arising from (3.1) and (3.2)

$$
\alpha_L^{(k)} = \begin{cases} 1 & \text{for } L=0, \\ q^{j(jp'(p+kp') \pm p'(r+kb) \mp s(p+kp'))} & \text{for } L = jp' - (\lambda \mp b \pm s)/2, \quad (j \geqslant 1) \\ -q^{(jp'+s)(j(p+kp')+kb+r)} & \text{for } L = jp' - (\lambda - b - s)/2, \quad (j \geqslant 0), \\ -q^{(jp'-s)(j(p+kp')-kb-r)} & \text{for } L = jp' - (\lambda + b + s)/2, \quad (j \geqslant 1), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\beta_L^{(k)} = \sum_{L \geqslant r_1 \geqslant \ldots \geqslant r_k \geqslant 0} \frac{q^{r_1(r_1+\lambda)+\ldots+r_k(r_k+\lambda)}}{(q)_{L-r_1}(q)_{r_1-r_2}\cdots(q)_{r_{k-1}-r_k}} \frac{(q)_\lambda}{(q)_{2r_k+\lambda}} F_{r,s}^{(p,p')}(2r_k+\lambda, b). \tag{3.8}
$$

Iterated dual Bailey pair arising from (3.1) and (3.2)

$$
\alpha_L^{(k)} = \begin{cases} 1 & \text{for } L=0, \\ q^{j(jp'(kp'+p'-p) \pm p'(kb+b-r) \mp s(kp'+p'-p))} & \text{for } L = jp' - (\lambda \mp b \pm s)/2, \ (j \geqslant 1), \\ -q^{(jp'-s)(j(kp'+p'-p)-kb-b+r)} & \text{for } L = jp' - (\lambda - b - s)/2, \ (j \geqslant 0), \\ -q^{(jp'+s)(j(kp'+p'-p)+kb+b-r)} & \text{for } L = jp' - (\lambda + b + s)/2, \ (j \geqslant 1), \\ 0 & \text{otherwise}, \end{cases}
$$

$$
\beta_L^{(k)} = \sum_{L \geqslant r_1 \geqslant \ldots \geqslant r_k \geqslant 0} \frac{q^{r_1(r_1+\lambda)+\ldots+r_{k-1}(r_{k-1}+\lambda)+2r_k(r_k+\lambda)}}{(q)_{L-r_1}(q)_{r_1-r_2}\cdots(q)_{r_{k-1}-r_k}} \frac{(q)_\lambda}{(q)_{2r_k+\lambda}} F_{r,s}^{(p,p')}(2r_k+\lambda, b; 1/q). \tag{3.9}
$$

3.3. Bailey flow

We use the explicit expressions for α_L for the four sets of Bailey pairs derived previously to establish a Bailey flow from the minimal models $M(p,p')$ to the $A_1^{(1)}$ cosets. The normalized branching functions of these cosets, obtained from [39] and denoted $\hat{\chi}^{(P,P';N)}_{r,s;\ell}(q)$, are given in Appendix A in Eq. (A.2) for N integer and N' fractional.

Substituting α_L of Eqs. (3.6) and (3.8) into the left-hand side of the higher-level Bailey lemma (2.18) yields, after using the symmetry properties (A.7),

$$\hat{\chi}^{(p',p'+N(p+kp');N)}_{s,b+N(kb+r);\ell}(q) \tag{3.10}$$

for $k \geqslant 0$ and $\ell + \lambda + N(kb+r)$ even.

Similarly, from α_L in Eqs. (3.7) and (3.9) we find that the left-hand side of (2.18) becomes

$$\hat{\chi}^{(p',p'+N(kp'+p'-p);N)}_{s,b+N(kb+b-r);\ell}(q) \tag{3.11}$$

with $k \geqslant 0$ and $\ell + \lambda + N(kb+b-r)$ even.

Hence we have demonstrated the following Bailey flows

$$M(p,p') = (p,p')_1 \rightarrow \big(p',p'+N(kp'+p)\big)_N \quad (k \geqslant 0) \tag{3.12}$$

and

$$M(p,p') = (p,p')_1 \xrightarrow{d} \big(p',p'+N(kp'+p'-p)\big)_N \quad (k \geqslant 0). \tag{3.13}$$

The flows (3.12) and (3.13) show that the spectra of the cosets (1.3) with $N \geqslant 2$ can be expressed entirely in terms of truncated or finitized spectra of the $c<1$ theories, a property noted to hold for the unitary models by Nakayashiki and Yamada [50].

We note that if we start with $M(p'-p,p')$ instead of $M(p,p')$ the Bailey flow and dual Bailey flow interchange

$$\begin{aligned} &M(p'-p,p') \xrightarrow{d} \big(p',p'+N(kp'+p)\big)_N, \\ &M(p'-p,p') \rightarrow \big(p',p'+N(kp'+p'-p)\big)_N. \end{aligned} \tag{3.14}$$

Thus we see that to obtain fermionic representations for the $(P,P')_N$ theories using the right-hand side of the higher level Bailey lemma, it is sufficient to restrict our attention to $M(p,p')$ with $p < p' < 2p$.

4. The fermionic polynomials for $M(p,p')$ with $p < p' < 2p$

In (3.1) the Bose–Fermi polynomial identities for the minimal models $M(p,p')$ were given implicitly. In this section we give the explicit forms for $F^{(p,p')}_{r,s}(L,b)$ for $p < p' < 2p$ which is dual to regime $p' > 2p$ in the sense of $q \rightarrow 1/q$. Whilst the implicit identities (3.1) were stated for all $1 \leqslant b, s \leqslant p'-1$ and $1 \leqslant r \leqslant p-1$, the explicit form for $F_{r,s}^{(p,p')}(L,b)$ is so far only known for several special cases in b, r, s. In particular we cannot treat the variables b and r independently, and the choice of b fixes r. As a further restriction only certain values of s and b have been treated at present. Hence in the remainder of this paper we will only deal with a subset of all possible $F_{r,s}^{(p,p')}(L,b)$.

For the cases $M(2k-1,2k+1)$ and $M(p,p+1)$ the fermionic forms were proven in [51,52] and [53-55,9], respectively (for the first case by applying the duality $q \to 1/q$). For general p, p' the results were proven in Refs. [41,42].

The fermionic functions $F_{r,s}^{(p,p')}(L,b)$ are much more involved than the bosonic functions (3.2) and several preliminary sections are needed to introduce all the necessary notations and definitions. In Section 4.1 we review the continued fraction expansion of $p'/(p'-p)$ and the closely related Takahashi-Suzuki decomposition. Then, in Section 4.2, we define fundamental fermionic functions which are the building blocks of $F_{r,s}^{(p,p')}(L,b)$. In Section 4.3, $F_{r,s}^{(p,p')}(L,b)$ is given for four classes of values of b (and hence r), listed in Table 1.

4.1. Takahashi-Suzuki decomposition

Given p and p' such that $\gcd(p,p') = 1$ and such that $p < p' < 2p$, we define integers $\nu_0, \dots, \nu_n$ by the continued fraction expansion

$$\frac{p'}{p'-p} = 1 + \nu_0 + \cfrac{1}{\nu_1 + \cfrac{1}{\nu_2 + \dots + \cfrac{1}{\nu_n + 2}}}. \tag{4.1}$$

The number $n+1$ is referred to as the number of zones, ν_i being the size of the ith zone. Using the ν_i's we define another set of integers $t_1, \dots, t_{n+1}$ as

$$t_i = \sum_{j=0}^{i-1} \nu_j. \tag{4.2}$$

For convenience we also set $t_0 = -1$. Given these integers we define a generalized or fractional-level incidence matrix $\mathcal{I}_B$ with entries

$$(\mathcal{I}_B)_{j,k} = \begin{cases} \delta_{j,k+1} + \delta_{j,k-1} & \text{for } 1 \leqslant j < t_{n+1}, \quad j \neq t_i, \\ \delta_{j,k+1} + \delta_{j,k} - \delta_{j,k-1} & \text{for } j = t_i, \quad 1 \leqslant i \leqslant n - \delta_{\nu_n,0}, \\ \delta_{j,k+1} + \delta_{\nu_n,0}\delta_{j,k} & \text{for } j = t_{n+1}, \end{cases} \tag{4.3}$$

and a fractional-level Cartan matrix B as

$$B = 2I - \mathcal{I}_B. \tag{4.4}$$

To give the fermionic forms $F_{r,s}^{(p,p')}(L,b)$ we need to decompose r, s and b in terms of Takahashi-Suzuki (TS) lengths [56]. To describe this decomposition we first define the recursions

$$y_{m+1} = y_{m-1} + (\nu_m + \delta_{m,0} + 2\delta_{m,n}) y_m, \qquad y_{-1} = 0, \qquad y_0 = 1, \tag{4.5}$$

$$\bar{y}_{m+1} = \bar{y}_{m-1} + (\nu_m + \delta_{m,0} + 2\delta_{m,n}) \bar{y}_m, \qquad \bar{y}_{-1} = -1, \qquad \bar{y}_0 = 1. \tag{4.6}$$

For $j = 0, \ldots, t_{n+1} + 1$, the TS lengths l_{j+1} and truncated TS lengths $\bar{l}_{j+1}$ are then given by

$$\begin{aligned} l_{j+1} &= y_{m-1} + (j - t_m) y_m \\ \bar{l}_{j+1} &= \bar{y}_{m-1} + (j - t_m) \bar{y}_m \end{aligned} \qquad \text{for } t_m < j \leqslant t_{m+1} + \delta_{m,n} \text{ with } 0 \leqslant m \leqslant n. \tag{4.7}$$

An arbitrary integer b $(1 \leqslant b < p')$ may be uniquely decomposed into TS lengths as

$$b = \sum_{i=1}^{k} l_{\beta_i + 1} \tag{4.8}$$

provided that $t_{\xi_i} < \beta_i \leqslant t_{\xi_i+1} + \delta_{\xi_i,n}$ with $0 \leqslant \xi_1 < \xi_2 < \ldots < \xi_k \leqslant n$ with the additional restriction, $\xi_{i+1} \geqslant \xi_i + 2$ when $\beta_i = t_{\xi_i+1}$.

The following t_{n+1}-dimensional vectors $\boldsymbol{Q}^{(j)}$ $(j = 1, \ldots, t_{n+1} + 1)$ are needed to specify parities of summation variables. For $1 \leqslant i \leqslant t_{n+1}$ and $0 \leqslant m \leqslant n$ such that $t_m < j \leqslant t_{m+1} + \delta_{m,n}$ the components of $\boldsymbol{Q}^{(j)}$ are recursively defined as

$$Q_i^{(j)} = \begin{cases} 0 & \text{for } j \leqslant i \leqslant t_{n+1}, \\ j - i & \text{for } t_m \leqslant i < j, \\ Q_{i+1}^{(j)} + Q_{t_{m'}+1}^{(j)} & \text{for } t_{m'-1} \leqslant i < t_{m'},\ 1 \leqslant m' \leqslant m. \end{cases} \tag{4.9}$$

When $\nu_n = 0$, so that $t_{n+1} = t_n$, we need to set the initial condition $Q_{t_n+1}^{(t_n+1)} = 0$.

Finally, we define the projection operator $*$: $\boldsymbol{u} \to \boldsymbol{u}^*$ as $\boldsymbol{u}^* = (u_1, \ldots, u_{M-1})$ for an arbitrary M-dimensional vector $\boldsymbol{u} = (u_1, \ldots, u_M)$.

4.2. *The fundamental fermionic functions*

With the definitions of the previous subsection we now introduce fundamental fermionic functions f and $\tilde{f}$ used as building blocks for $F_{r,s}^{(p,p')}(L,b)$. First, for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^{t_{n+1}+1}$,

$$f(L, \boldsymbol{u}, \boldsymbol{v}) = \sum_{\boldsymbol{m} \in \mathbb{Z}_+^{t_{n+1}},\ \boldsymbol{m} \equiv \boldsymbol{Q}_{\boldsymbol{u}+\boldsymbol{v}} \ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\boldsymbol{m}B\boldsymbol{m} - \frac{1}{2}A_{\boldsymbol{u},\boldsymbol{v}}\boldsymbol{m}} \prod_{j=1}^{t_{n+1}} \begin{bmatrix} n_j + m_j \\ m_j' \end{bmatrix}, \tag{4.10}$$

with B as in (4.4), the $(\boldsymbol{m}, \boldsymbol{n})$-system given by

$$\boldsymbol{m} + \boldsymbol{n} = \frac{1}{2}(\mathcal{I}_B \boldsymbol{m} + \boldsymbol{u}^* + \boldsymbol{v}^* + L\boldsymbol{e}_1). \tag{4.11}$$

Furthermore

$$\boldsymbol{Q}_{\boldsymbol{u}} = \sum_{j=1}^{t_{n+1}+1} u_j \boldsymbol{Q}^{(j)},$$

and, for $t_i < j \leqslant t_{i+1}$,

$$(A_{u,v})_j = \begin{cases} u_j & \text{for } i \text{ odd,} \\ v_j & \text{for } i \text{ even.} \end{cases} \tag{4.12}$$

The notation $\boldsymbol{m} \equiv \boldsymbol{Q} \pmod 2$ stands for m_j even when Q_j is even and m_j odd when Q_j is odd. The q-binomials $\left[{n+m \atop m'}\right]$ in (4.10) differ slightly from those of (2.14),

$$\begin{bmatrix} n+m \\ m' \end{bmatrix} = \begin{cases} \dfrac{(q^{n+1})_m}{(q)_m} & \text{for } m \in \mathbb{Z}_+, n \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

Notice that for negative n, $\left[{n+m \atop m'}\right]$ can only be non-zero when $n+m<0$.

The second function $\tilde{f}$ is defined for the special vectors $\boldsymbol{u} = \boldsymbol{e}_{\nu_0-j-1} - \boldsymbol{e}_{\nu_0} + \boldsymbol{u}'$ with $0 \leqslant j < \nu_0$ and $(\boldsymbol{u}')_i = 0$ for $1 \leqslant i \leqslant \nu_0$,

$$\tilde{f}(L,\boldsymbol{u},\boldsymbol{v}) = \begin{cases} q^{\frac{L}{2}} f(L,\boldsymbol{u},\boldsymbol{v}) + (1-q^L) f(L-1, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-1} + \boldsymbol{e}_{\nu_0}, \boldsymbol{v}) \\ \qquad \text{for } j=0, \\ q^{\frac{j}{2}} \left[f(L+1, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j}, \boldsymbol{v}) - q^{\frac{L+1}{2}} f(L, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j+1}, \boldsymbol{v}) \right] \\ \qquad \text{for } 1 \leqslant j < \nu_0. \end{cases} \tag{4.14}$$

We make a final remark about the notation employed in this and subsequent sections. In Section 2.2 the vectors $\boldsymbol{e}_j$ were defined as $(N-1)$-dimensional unit vectors. In the above we use t_{n+1}-dimensional vectors $\boldsymbol{e}_j$ in (4.11) and $(t_{n+1}+1)$-dimensional vectors in (4.14). Indeed we will throughout use $\boldsymbol{e}_j$ for the jth unit vector, and assume that its dimension is clear from the context.

4.3. Explicit fermionic functions

Using the functions f and $\tilde{f}$ as fundamental objects, we now give explicit forms for $F_{r,s}^{(p,p')}(L,b)$, with $p < p' < 2p$ such that $\nu_0, \nu_n \geqslant 1$ and $\nu_j > 1$ $(j = 1, \ldots, n-1)$. We consider here s being a single TS length

$$s = l_{\sigma+1} \quad \text{with} \quad t_\varsigma < \sigma \leqslant t_{\varsigma+1} + \delta_{\varsigma,n}$$

and we limit ourselves to b being one of the cases listed in Table 1. Case 1 corresponds to b being a single TS length and case 2 deals with values of b in the vicinity of a single TS length. Cases 3 and 4 are rather generic and deal with classes of b having a TS decomposition with TS lengths in adjacent zones (starting in zone 0 for case 3 and in a zone > 0 for case 4). The labels β_m are restricted to lie not too close to the upper boundary of the zone. The reason for having to distinguish between many different cases (of which we have only listed a few) is that $F_{r,s}^{(p,p')}(L,b)$ is a linear

Table 1
Cases for b considered

Case	b	r	Additional restrictions
1:	$b = l_{\beta+1}$	$r = \bar{l}_{\beta+1}$	$t_\xi < \beta \leqslant t_{\xi+1} + \delta_{\xi,n}$ $0 \leqslant \xi \leqslant n$
2a:	$b = l_{\beta+1} - j$	$r = \bar{l}_{\beta+1}$	$t_\xi + \delta_{\xi,1} < \beta < t_{\xi+1} + \delta_{\xi,n}$ $1 \leqslant \xi \leqslant n, \quad 1 \leqslant j < \nu_0$
2b:	$b = l_{\beta+1} + 1 + j$	$r = 1 + \bar{l}_{\beta+1}$	$t_\xi < \beta < t_{\xi+1} + \delta_{\xi,n}$ $1 \leqslant \xi \leqslant n, \quad 0 \leqslant j \leqslant \nu_0$
3:	$b = \sum_{m=0}^{\xi} l_{\beta_m+1}$	$r = \sum_{m=0}^{\xi} \bar{l}_{\beta_m+1}$	$0 \leqslant \beta_0 \leqslant \nu_0 - 1$ $t_m < \beta_m \leqslant t_{m+1} - 3 \quad (1 \leqslant m \leqslant \xi - 2)$ $t_{\xi-1} < \beta_{\xi-1} \leqslant t_\xi - 2$ $t_\xi < \beta_\xi \leqslant t_{\xi+1} - 1 + \delta_{\xi,n}$ $1 \leqslant \xi \leqslant n$
4:	$b = \sum_{m=\zeta}^{\xi} l_{\beta_m+1}$	$r = \sum_{m=\zeta}^{\xi} \bar{l}_{\beta_m+1}$	$t_m < \beta_m \leqslant t_{m+1} - 3 \quad (\zeta \leqslant m \leqslant \xi - 2)$ $t_{\xi-1} < \beta_{\xi-1} \leqslant t_\xi - 2$ $t_\xi < \beta_\xi \leqslant t_{\xi+1} - 1 + \delta_{\xi,n}$ $1 \leqslant \zeta < \xi \leqslant n$

combination of $f(L, \boldsymbol{u}, \boldsymbol{v})$ and $\tilde{f}(L, \boldsymbol{u}, \boldsymbol{v})$ where the vectors $\boldsymbol{u}$ in this linear combination can be determined by a recursive procedure [42]. To give explicit formulas for the vectors $\boldsymbol{u}$ as arising from this recursive procedure one needs to distinguish many cases.

As mentioned before, the choice of b fixes r and given b and its decomposition (4.8) in TS lengths, r reads

$$r(b) = \begin{cases} \displaystyle\sum_{i=1}^{k} \bar{l}_{\beta_i+1} & \text{for } 1 \leqslant b < p' - \nu_0, \\ \displaystyle\sum_{i=1}^{\beta} \bar{l}_{\beta_i+1} + 1 & \text{for } p' - \nu_0 \leqslant b \leqslant p' - 1. \end{cases} \tag{4.15}$$

In all four cases for b an overall normalization constant $k_{b,s}$ is fixed by the condition

$$F_{r,s}^{(p,p')}(L, b)\Big|_{q=0} = 1.$$

Furthermore, in all of the cases below, we use the abbreviation $\boldsymbol{u}_s$ for

$$\boldsymbol{u}_s = \boldsymbol{e}_\sigma - \boldsymbol{E}_{\varsigma+1,n}, \quad \text{where} \quad \boldsymbol{E}_{i,j} = \sum_{k=i}^{j} \boldsymbol{e}_{t_k}.$$

Case 1:

$$F_{r,s}^{(p,p')}(L, b) = q^{k_{b,s}} f(L, \boldsymbol{e}_\beta - \boldsymbol{E}_{\xi+1,n}, \boldsymbol{u}_s). \tag{4.16}$$

Case 2a:

$$F_{r,s}^{(p,p')}(L, b) = q^{k_{b,s}} \Big\{ q^{\frac{-2\nu_0 + \theta(\xi \text{ odd}) + 2j}{4}} \tilde{f}(L, \boldsymbol{e}_{j-1} + \boldsymbol{e}_{\beta-1} - \boldsymbol{E}_{1,n}, \boldsymbol{u}_s)$$

$$+\sum_{i=2}^{\xi} q^{\frac{-2\nu_0+\theta(i\ \mathrm{even})+2j}{4}} \tilde{f}(L, \boldsymbol{e}_{j-1} + \boldsymbol{e}_{t_i-1} + \boldsymbol{e}_\beta - \boldsymbol{E}_{1,i} - \boldsymbol{E}_{\xi+1,n}, \boldsymbol{u}_s)$$

$$+f(L, \boldsymbol{e}_{\nu_0-j} - \boldsymbol{e}_{\nu_0} + \boldsymbol{e}_\beta - \boldsymbol{E}_{\xi+1,n}, \boldsymbol{u}_s)\Big\}, \tag{4.17}$$

with $\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$.

Case 2b:

$$F_{r,s}^{(p,p')}(L,b) = q^{k_{b,s}} \Big\{ q^{\frac{\nu_0+1-3\theta(\xi\ \mathrm{odd})}{4}} f(L, \boldsymbol{e}_j + \boldsymbol{e}_{\beta+1} - \boldsymbol{E}_{1,n}, \boldsymbol{u}_s)$$

$$+\sum_{i=2}^{\xi} q^{\frac{\nu_0-\theta(i\ \mathrm{odd})}{4}} f(L, \boldsymbol{e}_j + \boldsymbol{e}_{t_i-1} + \boldsymbol{e}_\beta - \boldsymbol{E}_{1,i} - \boldsymbol{E}_{\xi+1,n}, \boldsymbol{u}_s)$$

$$+\theta(j<\nu_0) q^{\frac{\nu_0-2j-1}{4}} \tilde{f}(L, \boldsymbol{e}_{\nu_0-j-1} - \boldsymbol{e}_{\nu_0} + \boldsymbol{e}_\beta - \boldsymbol{E}_{\xi+1,n}, \boldsymbol{u}_s)\Big\}. \tag{4.18}$$

For the remaining two cases some more notation is needed. Let

$$b = \sum_{m=\zeta}^{\xi} l_{\beta_m+1}, \qquad \text{with } t_m < \beta_m \leqslant t_{m+1} + \delta_{m,n} \text{ and } 0 \leqslant \zeta \leqslant m \leqslant \xi \leqslant n. \tag{4.19}$$

Then we define two vectors

$$\boldsymbol{\beta} = (\beta_\zeta, \beta_{\zeta+1}, \ldots, \beta_{\xi-1}, \beta_\xi), \tag{4.20}$$

$$\boldsymbol{i} = (i_\zeta, i_{\zeta+1}, \ldots, i_{\xi-1}, 0), \tag{4.21}$$

where the β_m ($\zeta \leqslant m \leqslant \xi$) are defined from (4.19) and $i_m \in \{0,1\}$ ($\zeta \leqslant m < \xi$). Since the components of $\boldsymbol{i}$ are 0 or 1, the vector $\boldsymbol{i}$ may also be represented by a binary word

$$\boldsymbol{i} = 1^{b_1}0^{a_1}1^{b_2}0^{a_2}\ldots 1^{b_\ell}0^{a_\ell}, \quad \text{with} \begin{cases} a_k \geqslant 1, & 1 \leqslant k \leqslant \ell, \\ b_k \geqslant 1, & 2 \leqslant k \leqslant \ell, \qquad b_1 \geqslant 0. \end{cases} \tag{4.22}$$

The numbers a_k and b_k give the length of the kth substring of 0's and 1's (starting with a string of 1's of possibly zero length). Finally let us define for $t_m < j \leqslant t_{m+1} + \delta_{m,n}$

$$R(j+1) = t_{m+1} + t_m - j - 1,$$

and in addition $R_0 = Id$ and $R_1 = R$. This prepares us for the cases 3 and 4.

Case 3:

$$F_{r,s}^{(p,p')}(L,b) = q^{k_{b,s}} \Big\{ \sum_{i_1,\ldots,i_{\xi-1}=0,1;\ i_0=0} q^{-\varphi_i} f(L, \boldsymbol{e}_{\beta_0+i_1} - \boldsymbol{e}_{\nu_0} + \boldsymbol{u}_{\boldsymbol{i},\boldsymbol{\beta}}, \boldsymbol{u}_s)$$

$$+ \sum_{i_1,\ldots,i_{\xi-1}=0,1;\ i_0=1} q^{-\psi_i} \tilde{f}(L, \boldsymbol{e}_{\nu_0-\beta_0-i_1-1} - \boldsymbol{e}_{\nu_0} + \boldsymbol{u}_{i,\beta}, \boldsymbol{u}_s)\Big\}, \tag{4.23}$$

where

$$\boldsymbol{u}_{i,\beta} = \sum_{m=1}^{\xi-1} \boldsymbol{e}_{R_{i_m}(\beta_m+1)+|i_{m+1}-i_m|-|i_m-i_{m-1}|} + \boldsymbol{e}_{1+\beta_\xi-i_{\xi-1}} - \boldsymbol{E}_{2,n}$$

$$\varphi_i = \frac{1}{2}\sum_{j=2}^{\ell}(-1)^{a_1+\sum_{k=2}^{j-1}(a_k+b_k)}\theta(b_j \text{ even})$$

$$\psi_i = \frac{\beta_0 + i_1 - 1}{2} + \frac{1}{2}\sum_{j=1}^{\ell}(-1)^{\sum_{k=2}^{j-1}(a_k+b_k)}\theta(b_j \text{ even}) \tag{4.24}$$

and a_k, b_k and ℓ follow from the representation (4.22) of $\boldsymbol{i}$.

Case 4:

$$F_{r,s}^{(p,p')}(L,b) = q^{k_{b,s}}\Big[\sum_{i_{\zeta+1},\ldots,i_{\xi-1}=0,1;\ i_\zeta=0} q^{-\varphi_i} f(L, \boldsymbol{u}_{i,\beta}^{(1)}, \boldsymbol{u}_s)$$
$$+ \sum_{i_{\zeta+1},\ldots,i_{\xi-1}=0,1;\ i_\zeta=1} \Big\{q^{-\psi_i} f(L, \boldsymbol{u}_{i,\beta}^{(2)}, \boldsymbol{u}_s) + q^{-\psi_i+\frac{(-1)^\zeta}{4}} f(L, \boldsymbol{u}_{i,\beta}^{(3)}, \boldsymbol{u}_s)\Big\}\Big], \tag{4.25}$$

where

$$\boldsymbol{u}_{i,\beta}^{(1)} = \boldsymbol{e}_{\beta_\zeta+i_{\zeta+1}} + \boldsymbol{u}_{i,\beta},$$
$$\boldsymbol{u}_{i,\beta}^{(2)} = -\boldsymbol{e}_{t_\zeta} + \boldsymbol{e}_{t_{\zeta+1}+t_\zeta-\beta_\zeta-i_{\zeta+1}-1} + \boldsymbol{u}_{i,\beta},$$
$$\boldsymbol{u}_{i,\beta}^{(3)} = \boldsymbol{e}_{t_\zeta-1} - \boldsymbol{e}_{t_\zeta} + \boldsymbol{e}_{t_{\zeta+1}+t_\zeta-\beta_\zeta-i_{\zeta+1}} + \boldsymbol{u}_{i,\beta},$$
$$\boldsymbol{u}_{i,\beta} = \sum_{m=\zeta+1}^{\xi-1} \boldsymbol{e}_{R_{i_m}(\beta_m+1)+|i_{m+1}-i_m|-|i_m-i_{m-1}|} + \boldsymbol{e}_{\beta_\xi+1-i_{\xi-1}} - \boldsymbol{E}_{\zeta+1,n}, \tag{4.26}$$

and

$$\varphi_i = \frac{1}{2}(-1)^\zeta \sum_{j=2}^{\ell}(-1)^{a_1+\sum_{k=2}^{j-1}(a_k+b_k)}\theta(b_j \text{ even}),$$

$$\psi_i = (-1)^\zeta\Big(\frac{1}{4}(-1)^{b_1} + \frac{1}{2}\sum_{j=2}^{\ell}(-1)^{\sum_{k=1}^{j-1}(a_k+b_k)}\theta(b_j \text{ even})\Big), \tag{4.27}$$

with a_k, b_k and ℓ as defined in (4.22).

5. The explicit Bose–Fermi identities for the $A_1^{(1)}$ cosets

We are finally in the position to give explicit Bose–Fermi identities for the $A_1^{(1)}$ cosets (1.3). Corresponding to the four classes of Bailey pairs given in Section 3.2 we treat the cases (i) $(N/2+1)P < P' < (N+1)P$, (ii) $P < P' < (N/2+1)P$, (iii) $(Nk+N/2+1)P < P' < (Nk+N+1)P$ and (iv) $(Nk+1)P < P' < (Nk+N/2+1)P$ with $k \geqslant 1$ separately.

A principle result of this section will be that in all four cases the fermionic branching functions for $(P, P')_N$ involve an $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$-system determined from the continued fraction expansion

$$\frac{NP'}{P'-P} = 1 + \hat{\nu}_0 + \cfrac{1}{\hat{\nu}_1 + \cfrac{1}{\hat{\nu}_2 + \ldots + \cfrac{1}{\hat{\nu}_{\hat{n}}+2}}} \tag{5.1}$$

as observed for the cases (i) and (ii) in Ref. [46]. In order to present the results of this section as compactly as possible we introduce a notation which distinguishes between variables that occur in the $M(p,p')$ polynomials – the input in the Bailey lemma – and the corresponding quantities for the branching functions of $(P, P')_N$. To this end we adopt the notation that all quantities in the $(P, P')_N$ branching functions which have a counterpart x for $M(p,p')$ will be denoted by $\hat{x}$. In particular, using the continued fraction expansion (5.1) we define $\hat{t}_i, \mathcal{I}_{\hat{B}}, \hat{B}, \hat{y}_m, \hat{\bar{y}}_m, \hat{l}_{j+1}, \hat{\bar{l}}_{j+1}$ and $\hat{\boldsymbol{Q}}^{(j)}$ by taking the corresponding definitions for $t_i, \mathcal{I}_B, B, y_m, \bar{y}_m, l_{j+1}, \bar{l}_{j+1}$ and $\mathbf{Q}^{(j)}$ of Section 4.1 and by replacing all variables therein by their hatted counterparts.

5.1. Fermionic branching functions for $(P, P')_N$ with $P' < (N+1)P$

As in Section 4, we define a fundamental fermionic function $\hat{f}$ for the $A_1^{(1)}$ cosets (1.3), which will be the building block of the fermionic branching functions. For $\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}} \in \mathbb{Z}^{\hat{t}_{\hat{n}+1}}$ we set

$$\hat{f}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}}) = \sum_{\hat{\boldsymbol{m}} \in \mathbb{Z}_+^{\hat{t}_{\hat{n}+1}},\ \hat{\boldsymbol{m}} \equiv \hat{\boldsymbol{Q}} \ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\hat{\boldsymbol{m}}\hat{B}\hat{\boldsymbol{m}} - \frac{1}{2}\hat{\boldsymbol{A}}\hat{\boldsymbol{m}}} \frac{1}{(q)_{\hat{m}_N}} \prod_{j=1,\ j\neq N}^{\hat{t}_{\hat{n}+1}} \begin{bmatrix} \hat{m}_j + \hat{n}_j \\ \hat{m}_j' \end{bmatrix}, \tag{5.2}$$

with $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$-system given by

$$\hat{\boldsymbol{m}} + \hat{\boldsymbol{n}} = \frac{1}{2}\left(\mathcal{I}_{\hat{B}}\hat{\boldsymbol{m}} + \hat{\boldsymbol{u}}\right) \tag{5.3}$$

and $\mathcal{I}_{\hat{B}}$ and $\hat{B}$ based on the continued fraction expansion (5.1). Notice that when $\hat{m}_j + \hat{n}_j \notin \mathbb{Z}$, then, according to (4.13), $\hat{f}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}})$ is zero.

5.1.1. The case $(N/2+1)P < P' < (N+1)P$

The fermionic representation of $\hat{\chi}^{(p',p'+Np;N)}_{s,b+Nr;\ell}(q)$, with $s+b+Nr+\ell$ even, follows from substituting β_L of equation (3.6) into the right-hand side of the higher-level Bailey lemma (2.18), using the explicit form of $F^{(p,p')}_{r,s}(2L+\lambda, b)$.

As a first step we substitute just β_L leaving $F_{r,s}^{(p,p')}$ unspecified. We extend the $(N-1)$-dimensional $(\boldsymbol{m},\boldsymbol{n})$-system of Eq. (2.17) to an N-dimensional one by setting $m_N = 2L+\lambda$ and by defining

$$\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(\mathcal{I}_{T^{(N)}}\boldsymbol{m}+\boldsymbol{e}_\ell), \tag{5.4}$$

with $\mathcal{I}_{T^{(N)}}$ the incidence matrix of the tadpole graph with N nodes, $(\mathcal{I}_{T^{(N)}})_{i,j} = \delta_{|i-j|,1} + \delta_{i,j}\delta_{i,N}$. We also define the matrix $T^{(N)}$ as the corresponding Cartan-type matrix $T^{(N)} = 2I - \mathcal{I}_{T^{(N)}}$.

With this we obtain

$$\hat{\chi}^{(p',p'+Np;N)}_{s,b+Nr;\ell}(q) = q^{-\frac{\ell(N-\ell)+\lambda^2}{4N}} \sum_{\boldsymbol{m}\in\mathbb{Z}_+^N,\ \boldsymbol{m}\equiv\boldsymbol{Q}\ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\boldsymbol{m}T^{(N)}\boldsymbol{m}} \prod_{j=1}^{N-1} \begin{bmatrix} m_j+n_j \\ m_j \end{bmatrix} \frac{F_{r,s}^{(p,p')}(m_N,b)}{(q)_{m_N}}, \tag{5.5}$$

where

$$\boldsymbol{Q} = (\boldsymbol{e}_{\ell+1}+\boldsymbol{e}_{\ell+3}+\ldots) + (r+1)(\boldsymbol{e}_1+\boldsymbol{e}_3+\ldots). \tag{5.6}$$

Notice that it follows from (5.4) that $m_j+n_j > 0$ if $\boldsymbol{m}\in\mathbb{Z}_+^N$. Hence the binomials $\begin{bmatrix} m_j+n_j \\ m_j \end{bmatrix}$ in (5.5) can be replaced by $\begin{bmatrix} m_j+n_j \\ m_j \end{bmatrix}'$ of (4.13). Similar arguments hold in all three cases to follow.

Next we need to substitute the fermionic polynomials. Since $F_{r,s}^{(p,p')}$ is a linear combination of the elementary functions f and $\tilde{f}$, we first determine the resulting expressions when f and $\tilde{f}$ are used instead of $F_{r,s}^{(p,p')}$.

The formulas simplify if we define a new $(N+t_{n+1})$-dimensional $(\hat{\boldsymbol{m}},\hat{\boldsymbol{n}})$-system, combining the $(\boldsymbol{m},\boldsymbol{n})$-system (5.4) with the t_{n+1}-dimensional $(\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}})$-system of $f(m_N,\boldsymbol{u},\boldsymbol{v})$ obtained from (4.11)

$$\tilde{\boldsymbol{m}}+\tilde{\boldsymbol{n}} = \frac{1}{2}\Big(\mathcal{I}_B\tilde{\boldsymbol{m}}+\boldsymbol{u}^*+\boldsymbol{v}^*+m_N\boldsymbol{e}_1\Big). \tag{5.7}$$

Specifically, we define $\hat{\boldsymbol{m}}$ as

$$\hat{m}_j = \begin{cases} m_j & \text{for } 1\leqslant j\leqslant N, \\ \tilde{m}_{j-N} & \text{for } N<j\leqslant t_{n+1}+N, \end{cases} \tag{5.8}$$

and a corresponding vector $\hat{\boldsymbol{n}}$ through the $(\hat{\boldsymbol{m}},\hat{\boldsymbol{n}})$-system (5.3), where the vector $\hat{\boldsymbol{u}}$ is given by

$$\hat{u}_j = \begin{cases} \delta_{j,\ell} & \text{for } 1\leqslant j\leqslant N, \\ (\boldsymbol{u}^*+\boldsymbol{v}^*)_{j-N} & \text{for } N<j\leqslant \hat{t}_{\hat{n}+1}, \end{cases} \tag{5.9}$$

and $\mathcal{I}_{\hat{B}}$ is based on the continued fraction expansion (5.1) with $P=p'$ and $P'=p'+Np$. Using $\hat{B}=2I-\mathcal{I}_{\hat{B}}$ this yields

$$\hat{B} = \left(\begin{array}{c|c} T^{(N)} & \\ & 1 \\ \hline -1 & \\ & B \end{array} \right) \tag{5.10}$$

which is in fact also the matrix one obtains for the quadratic exponent (up to the antisymmetric part of $\hat{B}$ which is not fixed by $\hat{\boldsymbol{m}}\hat{B}\hat{\boldsymbol{m}}$).

From (5.10) we see that the continued fraction expansion (5.1) with $P = p'$ and $P' = p' + Np$ is related to that of $M(p, p')$ -used as input- by

$$\hat{\nu}_0 = N \qquad \text{and} \qquad \hat{\nu}_i = \nu_{i-1} \quad (1 \leqslant i \leqslant \hat{n}) \tag{5.11}$$

where $\hat{n} = n + 1$. Hence one additional zone (of size N) is added, and the genuine quasi-particle (corresponding to the $1/(q)_{m_N}$ in (5.2)) sits at the last entry of this new zeroth zone.

If we further set

$$\hat{A}_j = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant N, \\ (\boldsymbol{A}_{u,v})_{j-N} & \text{for } N < j \leqslant \hat{t}_{\hat{n}+1}, \end{cases} \tag{5.12}$$

and

$$\hat{Q}_j = \begin{cases} Q_j & \text{for } 1 \leqslant j \leqslant N, \\ (\boldsymbol{Q}_{u+v})_{j-N} & \text{for } N < j \leqslant \hat{t}_{\hat{n}+1}, \end{cases} \tag{5.13}$$

with $\boldsymbol{Q}$ given by (5.6), we find that

$$f(m_N, \boldsymbol{u}, \boldsymbol{v}) \xrightarrow{\text{BF}} q^{-\frac{\ell(N-\ell)+\lambda^2}{4N}} \hat{f}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}}), \tag{5.14}$$

where the arrow denotes the Bailey flow obtained by inserting $f(m_N, \boldsymbol{u}, \boldsymbol{v})$ into (5.5) instead of $F^{(p,p')}_{r,s}(m_N, b)$.

Analogously, one finds for $\boldsymbol{u} = \boldsymbol{e}_{\nu_0-j-1} - \boldsymbol{e}_{\nu_0} + \boldsymbol{u}'$ with $0 \leqslant j < \nu_0$ and $(\boldsymbol{u}')_i = 0$ for $1 \leqslant i \leqslant \nu_0$

$$\tilde{f}(m_N, \boldsymbol{u}, \boldsymbol{v}) \xrightarrow{\text{BF}} q^{-\frac{\ell(N-\ell)+\lambda^2}{4N}}$$
$$\times \begin{cases} \hat{f}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}_1 - \boldsymbol{e}_N, \hat{\boldsymbol{Q}}_1) + \hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{N+1} - \boldsymbol{e}_{\hat{t}_2-1} + \boldsymbol{e}_{\hat{t}_2}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) & \\ \quad -\hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{N+1} - \boldsymbol{e}_{\hat{t}_2-1} + \boldsymbol{e}_{\hat{t}_2}, \hat{\boldsymbol{A}}_2 - 2\boldsymbol{e}_N, \hat{\boldsymbol{Q}}_2) & \text{for } j = 0, \\ q^{\frac{j}{2}} \left[\hat{f}(\hat{\boldsymbol{u}} + \boldsymbol{e}_{N+1} - \boldsymbol{e}_{\hat{t}_2-j-1} + \boldsymbol{e}_{\hat{t}_2-j}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) \right. & \\ \quad \left. -q^{\frac{1}{2}} \hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_2-j-1} + \boldsymbol{e}_{\hat{t}_2-j+1}, \hat{\boldsymbol{A}}_3 - \boldsymbol{e}_N, \hat{\boldsymbol{Q}}_3) \right] & \text{for } 1 \leqslant j < \nu_0, \end{cases} \tag{5.15}$$

where $\hat{\boldsymbol{u}}$ as in (5.9), $\hat{\boldsymbol{A}}_1, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{A}}_3$ as in (5.12) and $\hat{\boldsymbol{Q}}_1, \hat{\boldsymbol{Q}}_2, \hat{\boldsymbol{Q}}_3$ as in (5.13) with $\boldsymbol{u}$ replaced by $\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j+1}$, respectively.

Making the replacements (5.14) and (5.15) in (4.16)-(4.18), (4.23) and (4.25) gives the fermionic representation of the branching function $\hat{\chi}^{(p',p'+Np;N)}_{s,b+Nr;\ell}(q)$.

5.1.2. The case $P < P' < (1 + N/2)P$

The fermionic representation of $\hat{\chi}^{(p',p'+N(p'-p);N)}_{s,b+N(b-r);\ell}(q)$, with $s+b+N(b-r)+\ell$ even, follows from substituting β_L of equation (3.7) into the right-hand side of the higher-level Bailey lemma (2.18) again using the explicit expressions for $F^{(p,p')}_{r,s}(2L+\lambda, b; 1/q)$.

We follow the same strategy as before, and as an intermediate step we leave the fermionic polynomials yet undetermined. We make again the variable change $m_N = 2L+\lambda$ and define the N-dimensional $(\boldsymbol{m}, \boldsymbol{n})$-system $\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(\mathcal{I}_{C^{(N)}}\, \boldsymbol{m} + \boldsymbol{e}_\ell)$, where the matrices $\mathcal{I}_{C^{(N)}}$ and $C^{(N)}$ are the incidence and Cartan matrix of A_N. Then we obtain

$$\hat{\chi}^{(p',p'+N(p'-p);N)}_{s,b+N(b-r);\ell}(q)$$

$$= q^{-\frac{\ell(N-\ell)+(N+1)\lambda^2}{4N}} \sum_{\boldsymbol{m}\in\mathbb{Z}^N_+,\ \boldsymbol{m}\equiv\boldsymbol{Q}\ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\boldsymbol{m}C^{(N)}\boldsymbol{m}} \prod_{j=1}^{N-1} \begin{bmatrix} m_j+n_j \\ m_j \end{bmatrix} \frac{F^{(p,p')}_{r,s}(m_N, b; 1/q)}{(q)_{m_N}}, \tag{5.16}$$

where

$$\boldsymbol{Q} = (\boldsymbol{e}_{\ell+1} + \boldsymbol{e}_{\ell+3} + \ldots) + (b-r)(\boldsymbol{e}_1 + \boldsymbol{e}_3 + \ldots). \tag{5.17}$$

Similar to the previous manipulations, we now determine the result when $F^{(p,p')}_{r,s}$ in (5.16) in replaced by f and $\tilde{f}$.

Again combining the different $(\boldsymbol{m}, \boldsymbol{n})$-systems, making the same variable change as in (5.8), we find a new $(N+t_{n+1})$-dimensional $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$-system, given by (5.3), with $\mathcal{I}_{\hat{B}}$ given by the continued fraction expansion (5.1) with $P = p'$ and $P' = p' + N(p'-p)$. The vector $\hat{\boldsymbol{u}}$ is still given by (5.9). Using $\hat{B} = 2I - \mathcal{I}_{\hat{B}}$ this yields

$$\hat{B} = \left(\begin{array}{c|c} C^{(N)} & \begin{matrix} & \\ -1 & \end{matrix} \\ \hline \begin{matrix} & -1 \\ & \end{matrix} & B \end{array}\right) \tag{5.18}$$

which is also the matrix that one obtains for the quadratic exponent. This time no antisymmetric part had to be added to the quadratic form matrix to ensure the relation $\hat{B} = 2I - \mathcal{I}_{\hat{B}}$.

Hence the continued fraction expansion (5.1) is this time related to that of the $M(p,p')$ fermionic polynomials as follows

$$\hat{\nu}_0 = N + \nu_0 \qquad \text{and} \qquad \hat{\nu}_i = \nu_i \quad (1 \leqslant i \leqslant \hat{n}), \tag{5.19}$$

with $\hat{n} = n$. This means that no additional zone is added, but instead the size of the zeroth zone has increased by N so that the genuine quasi-particle sits in the interior of zone zero.

Further taking

$$\hat{A}_j = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant N, \\ (\boldsymbol{u}^* + \boldsymbol{v}^* - \boldsymbol{A}_{\boldsymbol{u},\boldsymbol{v}})_{j-N} & \text{for } N < j \leqslant \hat{t}_{\hat{n}+1}, \end{cases} \tag{5.20}$$

and $\hat{\boldsymbol{Q}}$ as in (5.13) (with $\boldsymbol{Q}$ therein being (5.17) instead of (5.6)) we find that

$$f(m_N, \boldsymbol{u}, \boldsymbol{v}; 1/q) \xrightarrow{\mathrm{BF}} q^{-\frac{\ell(N-\ell)+(N+1)\lambda^2}{4N}} \hat{f}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}}). \tag{5.21}$$

The analogous transformation for $\tilde{f}$ is given by

$$\begin{aligned} &\tilde{f}(m_N, \boldsymbol{u}, \boldsymbol{v}; 1/q) \xrightarrow{\mathrm{BF}} q^{-\frac{\ell(N-\ell)+(N+1)\lambda^2}{4N}} \\ &\times \begin{cases} \hat{f}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}_1 + \boldsymbol{e}_N, \hat{\boldsymbol{Q}}_1) + \hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_1-1} + \boldsymbol{e}_{\hat{t}_1} - \boldsymbol{e}_{N+1}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) & \\ \quad -\hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_1-1} + \boldsymbol{e}_{\hat{t}_1} - \boldsymbol{e}_{N+1}, \hat{\boldsymbol{A}}_2 + 2\boldsymbol{e}_N, \hat{\boldsymbol{Q}}_2) & \text{for } j = 0, \\ q^{-\frac{j}{2}} \big[\hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_1-j-1} + \boldsymbol{e}_{\hat{t}_1-j} + \boldsymbol{e}_{N+1}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) & \\ \quad -q^{-\frac{1}{2}} \hat{f}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_1-j-1} + \boldsymbol{e}_{\hat{t}_1-j+1}, \hat{\boldsymbol{A}}_3 + \boldsymbol{e}_N, \hat{\boldsymbol{Q}}_3)\big] & \text{for } 1 \leqslant j < \nu_0, \end{cases} \end{aligned} \tag{5.22}$$

where $\hat{\boldsymbol{u}}$ as in (5.9), $\hat{\boldsymbol{A}}_1, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{A}}_3$ as in (5.20) and $\hat{\boldsymbol{Q}}_1, \hat{\boldsymbol{Q}}_2, \hat{\boldsymbol{Q}}_3$ as in (5.13) with $\boldsymbol{u}$ replaced by $\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j+1}$, respectively.

Making the replacements (5.21) and (5.22) in (4.16)–(4.18), (4.23) and (4.25) gives the fermionic form of the branching function $\hat{\chi}^{(p',p'+N(p'-p);N)}_{s,b+N(b-r);\ell}(q)$.

5.2. Fermionic branching functions for $(P, P')_N$ with $P' > (N+1)P$

To proceed in a fashion similar to Section 5.1 we now introduce the elementary fermionic function

$$\hat{g}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}}) = \sum_{\hat{\boldsymbol{m}} \in \mathbb{Z}_+^{\hat{t}_{\hat{n}+1}},\ \hat{\boldsymbol{m}} \equiv \hat{\boldsymbol{Q}} \ (\mathrm{mod}\, 2)} \frac{q^{\frac{1}{4}\hat{\boldsymbol{m}}\hat{B}\hat{\boldsymbol{m}} - \frac{1}{2}\hat{\boldsymbol{A}}\hat{\boldsymbol{m}}}}{(q)_{\hat{n}_{\hat{t}_1+1}} \cdots (q)_{\hat{n}_{\hat{t}_2}} (q)_{\hat{m}_{\hat{t}_2+1}}} \prod_{j=1,\ j \neq \hat{t}_1+1,\ldots,\hat{t}_2+1}^{\hat{t}_{\hat{n}+1}} \begin{bmatrix} \hat{m}_j + \hat{n}_j \\ \hat{m}'_j \end{bmatrix}, \tag{5.23}$$

with $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$-system given by (5.3), where $\mathcal{I}_{\hat{B}}$ and $\hat{B} = 2I - \mathcal{I}_{\hat{B}}$ follow from the continued fraction expansion (5.1) as before. Again, $\hat{g}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}})$ is zero if $\hat{m}_j + \hat{n}_j \notin \mathbb{Z}$. In contrast to (5.2), however, some of the genuine quasi-particles corresponding to the factors $1/(q)_a$ are labelled by $a = \hat{n}_j$ instead of $a = \hat{m}_j$. Note that $\hat{n}_j$, determined from (5.3), can take negative values, and we adopt the convention here that $1/(q)_{-n} = 0$ for $n > 0$.

5.2.1. The case $(Nk + N/2 + 1)P < P' < (Nk + N + 1)P$ with $k \geqslant 1$

The fermionic representation of $\hat{\chi}^{(p',p'+N(p+kp');N)}_{s,b+N(kb+r);\ell}(q)$, with $s + b + N(kb + r) + \ell$ even, follows from substituting $\beta^{(k)}_L$ of equation (3.8) into the right-hand side of the higher-level Bailey lemma (2.18), using the explicit form of $F^{(p,p')}_{r,s}(2r_k + \lambda, b)$.

As before, we first give the results when $F_{r,s}^{(p,p')}$ is still unspecified. We extend the $(N-1)$-dimensional $(\boldsymbol{m},\boldsymbol{n})$-system of the Bailey lemma by defining

$$
\begin{aligned}
n_j &= \begin{cases} L - r_1 & \text{for } j = N, \\ r_{j-N} - r_{j-N+1} & \text{for } N < j < N+k, \end{cases} \\
m_{N+k} &= 2r_k + \lambda
\end{aligned}
\tag{5.24}
$$

and setting the $(N+k)$-dimensional $(\boldsymbol{m},\boldsymbol{n})$-system

$$
\boldsymbol{m} + \boldsymbol{n} = \frac{1}{2}\big(\mathcal{I}_{B^{(N+k)}}\boldsymbol{m} + \boldsymbol{e}_\ell\big). \tag{5.25}
$$

The matrix $\mathcal{I}_{B^{(N+k)}}$ is defined by (4.3) with $\nu_0 = N-1$, $\nu_1 = k$, $\nu_2 = 1$ and $\nu_3 = 0$ and $B^{(N+k)} = 2I - \mathcal{I}_{B^{(N+k)}}$. The reason for labelling some of the variables in (5.24) by n_j is that these do not have any parity restrictions. This way we also ensure that the $(\boldsymbol{m},\boldsymbol{n})$-system (5.25) is based on a fractional-level incidence matrix of the form (4.3).

With these variable changes we obtain

$$
\begin{aligned}
&\hat{\chi}_{s,b+N(kb+r);\ell}^{(p',p'+N(p+kp');N)}(q) \\
&\quad = q^{-\frac{\ell(N-\ell)+(Nk+1)\lambda^2}{4N}} \sum_{\boldsymbol{m}\in\mathbb{Z}_+^{N+k},\ \boldsymbol{m}\equiv\boldsymbol{Q}\ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\boldsymbol{m}B^{(N+k)}\boldsymbol{m}} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix} \frac{F_{r,s}^{(p,p')}(m_{N+k},b)}{(q)_{n_N}\dots(q)_{n_{N+k-1}}(q)_{m_{N+k}}},
\end{aligned}
\tag{5.26}
$$

where

$$
Q_j = \begin{cases} (\delta_{\ell+1,j} + \delta_{\ell+3,j} + \dots) + (kb+r+1)(\delta_{j,1} + \delta_{j,3} + \dots) & \text{for } 1 \leqslant j < N, \\ 0 & \text{for } N \leqslant j < N+k, \\ \lambda & \text{for } j = N+k. \end{cases}
\tag{5.27}
$$

Next we again substitute f and $\tilde{f}$ for $F_{r,s}^{(p,p')}$ into (5.26). The resulting formulas can be simplified by combining the $(\boldsymbol{m},\boldsymbol{n})$-system (5.25) with the t_{n+1}-dimensional $(\tilde{\boldsymbol{m}},\tilde{\boldsymbol{n}})$-system of $f(m_{N+k},\boldsymbol{u},\boldsymbol{v})$

$$
\tilde{\boldsymbol{m}} + \tilde{\boldsymbol{n}} = \frac{1}{2}\big(\mathcal{I}_B\tilde{\boldsymbol{m}} + \boldsymbol{u}^* + \boldsymbol{v}^* + m_{N+k}\boldsymbol{e}_1\big) \tag{5.28}
$$

as

$$
\hat{m}_j = \begin{cases} m_j & \text{for } 1 \leqslant j \leqslant N+k, \\ \tilde{m}_j & \text{for } N+k < j \leqslant t_{n+1} + N + k, \end{cases}
\tag{5.29}
$$

and a corresponding vector $\hat{\boldsymbol{n}}$ through the $(\hat{\boldsymbol{m}},\hat{\boldsymbol{n}})$-system (5.3), where

$$\hat{u}_j = \begin{cases} \delta_{j,\ell} & \text{for } 1 \leqslant j < N, \\ 0 & \text{for } N \leqslant j \leqslant N+k, \\ (\boldsymbol{u}^* + \boldsymbol{v}^*)_{j-N-k} & \text{for } N+k < j \leqslant N+k+t_{n+1}. \end{cases} \tag{5.30}$$

and $\mathcal{I}_{\hat{B}}$ is based on the continued fraction expansion (5.1) with $P = p'$ and $P' = p' + N(p + kp')$. Using $\hat{B} = 2I - \mathcal{I}_{\hat{B}}$ this yields

$$\hat{B} = \left(\begin{array}{c|c|c|c} T^{(N-1)} & & & \\ & 1 & & \\ \hline -1 & & & \\ & T^{(k)} & & \\ & & 1 & \\ \hline & -1 & 1 & 1 \\ \hline & & -1 & \\ & & & B \end{array}\right), \tag{5.31}$$

which is indeed also the matrix one obtains from the quadratic exponent by noticing that the top-left $(N+k) \times (N+k)$ entries of $\hat{B}$ correspond to $B^{(N+k)}$.

From (5.31) we see that $\hat{B}$ is based on the continued fraction expansion (5.1) with $P = p'$ and $P' = p' + N(p + kp')$ related to the continued fraction expansion of $M(p, p')$ by

$$\hat{\nu}_0 = N - 1, \quad \hat{\nu}_1 = k, \quad \hat{\nu}_2 = 1 \quad \text{and} \quad \hat{\nu}_i = \nu_{i-3} \quad \text{for } 3 \leqslant i \leqslant \hat{n}, \tag{5.32}$$

with $\hat{n} = n + 3$, which means that three additional zones have been added.

Upon further setting

$$\hat{A}_j = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant \hat{t}_2 + 1, \\ (\boldsymbol{A}_{\boldsymbol{u},\boldsymbol{v}})_{j-\hat{t}_2-1} & \text{for } \hat{t}_2 + 1 < j \leqslant \hat{t}_{\hat{n}+1}, \end{cases} \tag{5.33}$$

and

$$\hat{Q}_j = \begin{cases} Q_j & \text{for } 1 \leqslant j \leqslant \hat{t}_2 + 1, \\ (\boldsymbol{Q}_{\boldsymbol{u}+\boldsymbol{v}})_{j-\hat{t}_2-1} & \text{for } \hat{t}_2 + 1 < j \leqslant \hat{t}_{\hat{n}+1}, \end{cases} \tag{5.34}$$

we obtain

$$f(m_{N+k}, \boldsymbol{u}, \boldsymbol{v}) \xrightarrow{\text{BF}} q^{-\frac{\ell(N-\ell)+(Nk+1)\lambda^2}{4N}} \hat{g}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}}) \tag{5.35}$$

and similarly

$$\tilde{f}(2r_k + \lambda, \boldsymbol{u}, \boldsymbol{v}) \xrightarrow{\text{BF}} q^{-\frac{\ell(N-\ell)+(Nk+1)\lambda^2}{4N}}$$

$$
\times \begin{cases} \hat{g}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}_1 - \boldsymbol{e}_{\hat{t}_2+1}, \hat{\boldsymbol{Q}}_1) \\ \quad +\hat{g}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_2+2} - \boldsymbol{e}_{\hat{t}_4-1} + \boldsymbol{e}_{\hat{t}_4}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) \\ \quad -\hat{g}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_2+2} - \boldsymbol{e}_{\hat{t}_4-1} + \boldsymbol{e}_{\hat{t}_4}, \hat{\boldsymbol{A}}_2 - 2\boldsymbol{e}_{\hat{t}_2+1}, \hat{\boldsymbol{Q}}_2) & \text{for } j=0, \\ q^{\frac{j}{2}}\big[\hat{g}(\hat{\boldsymbol{u}} + \boldsymbol{e}_{\hat{t}_2+2} - \boldsymbol{e}_{\hat{t}_4-j-1} + \boldsymbol{e}_{\hat{t}_4-j}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) \\ \quad -q^{\frac{1}{2}}\hat{g}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_4-j-1} + \boldsymbol{e}_{\hat{t}_4-j+1}, \hat{\boldsymbol{A}}_3 - \boldsymbol{e}_{\hat{t}_2+1}, \hat{\boldsymbol{Q}}_3)\big] & \text{for } 1 \leqslant j < \nu_0, \end{cases} \tag{5.36}
$$

where $\hat{\boldsymbol{u}}$ as in (5.30), $\hat{\boldsymbol{A}}_1, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{A}}_3$ as in (5.33) and $\hat{\boldsymbol{Q}}_1, \hat{\boldsymbol{Q}}_2, \hat{\boldsymbol{Q}}_3$ as in (5.34) with $\boldsymbol{u}$ replaced by $\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j+1}$, respectively.

Making the replacements (5.35) and (5.36) in (4.16)-(4.18), (4.23) and (4.25) gives the fermionic expression for the branching function $\hat{\chi}^{(p',p'+N(p+kp');N)}_{s,b+N(kb+r);\ell}(q)$.

5.2.2. The case $(Nk+1)P < P' < (Nk+N/2+1)P$ with $k \geqslant 1$

The fermionic representation of $\hat{\chi}^{(p',p'+N(kp'+p'-p);N)}_{s,b+N(kb+b-r);\ell}(q)$ with $s+b+N(kb+b-r)+\ell$ even, follows from substituting $\beta^{(k)}_L$ of equation (3.9) into the right-hand side of the higher-level Bailey lemma (2.18) using the explicit form of $F^{(p,p')}_{r,s}(2r_k+\lambda, b; 1/q)$. We first obtain

$$
\hat{\chi}^{(p',p'+N(kp'+p'-p);N)}_{s,b+N(kb+b-r);\ell}(q) = q^{-\frac{\ell(N-\ell)+(Nk+N+1)\lambda^2}{4N}} \sum_{\boldsymbol{m}\in\mathbb{Z}_+^{N+k},\ \boldsymbol{m}\equiv\boldsymbol{Q}\ (\mathrm{mod}\ 2)} q^{\frac{1}{4}\boldsymbol{m}B^{(N+k)}\boldsymbol{m}} \prod_{j=1}^{N-1} \begin{bmatrix} m_j+n_j \\ m_j \end{bmatrix} \frac{F^{(p,p')}_{r,s}(m_{N+k}, b; 1/q)}{(q)_{n_N}\cdots(q)_{n_{N+k-1}}(q)_{m_{N+k}}}, \tag{5.37}
$$

where

$$
Q_j = \begin{cases} (\delta_{\ell+1,j}+\delta_{\ell+3,j}+\ldots)+(kb+b-r+1)(\delta_{j,1}+\delta_{j,3}+\ldots) & \text{for } 1 \leqslant j < N, \\ 0 & \text{for } N \leqslant j < N+k, \\ \lambda & \text{for } j = N+k. \end{cases} \tag{5.38}
$$

The $(N+k)$-dimensional $(\boldsymbol{m},\boldsymbol{n})$-system is that of Eq. (5.25), but this time with $\mathcal{I}_{B^{(N+k)}}$ (and thus $B^{(N+k)}$) based on the continued fraction expansion with $\nu_0 = N-1$, $\nu_1 = k$ and $\nu_2 = 1$. We note that to get to this result the same variable change as in (5.24) has been carried out.

We now again define variables as in (5.29) and the corresponding $(\hat{\boldsymbol{m}},\hat{\boldsymbol{n}})$-system (5.3) with $\hat{\boldsymbol{u}}$ defined as in (5.30) and $\mathcal{I}_{\hat{B}}$ this time based on the continued fraction expansion (5.1) with $P=p'$ and $P' = p'+N(kp'+p'-p)$. This yields the matrix

$$\hat{B} = \left(\begin{array}{cc|cc|cc} T^{(N-1)} & & & & & \\ & & 1 & & & \\ \hline & -1 & & & & \\ & & T^{(k)} & & & \\ & & & & 1 & \\ \hline & & & -1 & 2 & -1 \\ & & & & -1 & \\ & & & & & B \end{array}\right) \tag{5.39}$$

which again agrees with the matrix one obtains from the quadratic exponent (up to arbitrary antisymmetric pieces).

The old and new continued fraction expansions are related by

$$\hat{\nu}_0 = N-1, \quad \hat{\nu}_1 = k, \quad \hat{\nu}_2 = \nu_0 + 1 \quad \text{and} \quad \hat{\nu}_i = \nu_{i-2} \ \text{ for } 3 \leqslant i \leqslant \hat{n} \tag{5.40}$$

with $\hat{n} = n+2$. This corresponds to the addition of two extra zones.

We further define $\hat{\boldsymbol{Q}}$ as in (5.34) with Q_j as in (5.38) and

$$\hat{A}_j = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant \hat{t}_2 + 1, \\ (\boldsymbol{u}^* + \boldsymbol{v}^* - \boldsymbol{A}_{u,v})_{j-\hat{t}_2-1} & \text{for } \hat{t}_2 + 1 < j \leqslant \hat{t}_{\hat{n}+1}. \end{cases} \tag{5.41}$$

Then

$$f(2r_k + \lambda, \boldsymbol{u}_b, \boldsymbol{u}_s; 1/q) \xrightarrow{\text{BF}} q^{-\frac{\ell(N-\ell)+(Nk+N+1)\lambda^2}{4N}} \hat{g}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{Q}}) \tag{5.42}$$

and

$$\begin{aligned} \tilde{f}(2r_k + \lambda, \boldsymbol{u}, \boldsymbol{v}; 1/q) &\xrightarrow{\text{BF}} q^{-\frac{\ell(N-\ell)+(Nk+N+1)\lambda^2}{4N}} \\ &\times \begin{cases} \hat{g}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{A}}_1 + \boldsymbol{e}_{\hat{t}_2+1}, \hat{\boldsymbol{Q}}_1) + \hat{g}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_2+2} - \boldsymbol{e}_{\hat{t}_3-1} + \boldsymbol{e}_{\hat{t}_3}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) \\ \quad -\hat{g}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_2+2} - \boldsymbol{e}_{\hat{t}_3-1} + \boldsymbol{e}_{\hat{t}_3}, \hat{\boldsymbol{A}}_2 + 2\boldsymbol{e}_{\hat{t}_2+1}, \hat{\boldsymbol{Q}}_2) & \text{for } j = 0, \\ q^{-\frac{j}{2}} \left[\hat{g}(\hat{\boldsymbol{u}} + \boldsymbol{e}_{\hat{t}_2+2} - \boldsymbol{e}_{\hat{t}_3-j-1} + \boldsymbol{e}_{\hat{t}_3-j}, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{Q}}_2) \right. \\ \quad \left. -q^{-\frac{1}{2}} \hat{g}(\hat{\boldsymbol{u}} - \boldsymbol{e}_{\hat{t}_3-j-1} + \boldsymbol{e}_{\hat{t}_3-j+1}, \hat{\boldsymbol{A}}_3 + \boldsymbol{e}_{\hat{t}_2+1}, \hat{\boldsymbol{Q}}_3) \right. & \text{for } 1 \leqslant j < \nu_0, \end{cases} \end{aligned} \tag{5.43}$$

where $\hat{\boldsymbol{u}}$ as in (5.30), $\hat{\boldsymbol{A}}_1, \hat{\boldsymbol{A}}_2, \hat{\boldsymbol{A}}_3$ as in (5.41) and $\hat{\boldsymbol{Q}}_1, \hat{\boldsymbol{Q}}_2, \hat{\boldsymbol{Q}}_3$ as in (5.34) with $\boldsymbol{u}$ replaced by $\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j}, \boldsymbol{u} - \boldsymbol{e}_{\nu_0-j-1} + \boldsymbol{e}_{\nu_0-j+1}$, respectively.

Making the replacements (5.42) and (5.43) in (4.16)-(4.18), (4.23) and (4.25) gives the fermionic form of $\hat{\chi}^{(p',p'+N(kp'+p'-p);N)}_{s,b+N(kb+b-r);\ell}(q)$.

To conclude Section 5.2, let us remark that for $N = 1$ the continued fraction expansion used in (5.1) is not the same as the one used in Refs. [41,42] for the case $P' > 2P$. Here we considered the continued fraction expansion of $P'/(P'-P)$ whereas in Refs. [41,42] that of P'/P was used. The difference between these two cases is that in the first one a zero zone of length zero (i.e. $\hat{\nu}_0 = 0$) is obtained whereas the second case starts with $\hat{\nu}'_0 = \hat{\nu}_1$. This of course changes the Takahashi decomposition, but the final results are the same (in particular the fractional-level incidence matrices for the $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}})$-system are the same).

6. Discussion

In the introduction we briefly discussed the relation of the massless RG flow of [1]-[7] to Bailey flow. Specifically we noted that the RG flow of [3] flows with a continuous parameter from $M(p,p+1)$ to $M(p-1,p)$ whereas, in contrast, Bailey flow is a discrete process which adds degrees of freedom and, in this special case, goes from $M(p-1,p)$ to $M(p,p+1)$. Moreover the parameter L which appears in the polynomial identities for $M(p,p')$ is, as emphasized by Melzer [57], an ultraviolet cutoff, which certainly is reminiscent of the RG flow phenomena. However, the precise (if any) connection between the two concepts is not yet understood and we will conclude this paper by discussing a few of the possibilities.

The special case of $M(p,p+1)$ might suggest that the Bailey construction could be generalized to include a free parameter so that it would be an exact inverse to RG flow. This is particularly the case with the flow between $M(3,4)$ and $M(4,5)$. Here, using the equivalence of $M(4,5)$ with the $N=1$ supersymmetric model $SM(3,5)$, the Bailey construction of Ref. [10], which utilizes the original lemma of Bailey (2.6) with $\rho_1 \to \infty$ provides a one parameter flow (in the variable ρ_2) from $SM(3,5) = M(4,5)$ to $M(3,4)$. The model $SM(3,5) = M(4,5)$ corresponds to $\rho_2 = q^{1/2}$ or q and $M(3,4)$ to $\rho_2 \to \infty$. The $q \to 1$ limit of the vacuum character has been studied in Ref. [58] and this provides a function which satisfies the property of the c function of Ref. [7] in interpolating between the two models.

Furthermore, in Ref. [10] the full two parameters in Bailey's lemma (2.6) are exploited to give flows from the $N=2$ supersymmetric models with central charge $c = 3(1-2/p)$ to $SM(p,p+2)$ to $M(p,p+1)$. The corresponding $q \to 1$ limit of the vacuum character is computed in Ref. [58] and this also interpolates between the central charges of the models. In this case no RG flow (or TBA) analysis with two continuous parameters has been carried out.

On the other hand it might be expected that the TBA equations of Ref. [3] which give the one parameter RG flow $M(p,p+1) \to M(p-1,p)$ should have an extension to a one parameter flow of characters. But it seems to us that if this is true it may require further identities that are not yet extant. The reason for this is that on the fermionic side the Bailey constructions add particles on one side of the genuine quasi particles (those which contribute $1/(q)_n$) and the tail of ghost (or pseudo) particles which correspond to the q-binomials lies on the other side of the genuine quasi particles. This leads to asymmetric fermionic representations of the characters whereas the TBA equations of Ref. [3] treat the two ends of the equations in a symmetric fashion. This problem needs to be explored.

We also note that the Bailey flows of this paper of $M(p,p')$ into the cosets (1.3) is not the inverse of the flow of Refs. [5,6] which is between different cosets of the form (1.3) and does not in general involve $M(p,p')$.

It thus seems that Bailey flow and RG flow give somewhat different relations between models of CFT. However, what is lacking at present in the method of Bailey flow is an abstract understanding of why Bailey's construction is related to CFT at all. At present we are only able to make the identification of the Bailey flow characters with the CFT characters by comparing the results of two separate computations. Since all examples of Bailey flow have been identified with CFT models this cannot be an accidental coincidence and it is most desirable to prove that there is a connection between the Bailey flow and CFT which allows us to identify the CFT model without the need of doing a separate Feigin–Fuchs computation of the CFT characters. Such a theorem would allow the Bailey flow to be a complete constructive procedure which could serve as an alternative route to the construction of CFT.

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Appendix A. $A_1^{(1)}$ bosonic branching functions

The bosonic form of the branching functions for the coset model $(P,P')_N$ with $N \geqslant 2$ been calculated in Refs. [32–36] for integer levels N and N' (unitary models). The cosets (1.3) with fractional N' have been considered in Ref. [38]. Ahn et al. [39] determined the branching functions for fractional levels and specializing their results to integer N, we find (for all $N \geqslant 1$)

$$\chi_{r,s;\ell}^{(P,P';N)}(q) = q^{-\frac{c}{24}+\Delta_{r,s}+\frac{\ell(N-\ell)}{2N(N+2)}}\hat{\chi}_{r,s;\ell}^{(P,P';N)}(q) \tag{A.1}$$

with the normalized branching function

$$\hat{\chi}_{r,s;\ell}^{(P,P';N)}(q) = q^{-\frac{\ell(N-\ell)}{2N(N+2)}} \times \sum_{0\leqslant m\leqslant N/2} c_{2m}^{\ell}(q)\left(\sum_{j\in\mathbb{Z},\ m_{r-s}(j)\equiv\pm m\ (\mathrm{mod}\ N)} q^{\frac{j}{N}(jPP'+P'r-Ps)} - \sum_{j\in\mathbb{Z},\ m_{r+s}(j)\equiv\pm m\ (\mathrm{mod}\ N)} q^{\frac{1}{N}(jP'+s)(jP+r)}\right). \tag{A.2}$$

Here $1\leqslant r\leqslant P-1$, $1\leqslant s\leqslant P'-1$, $0\leqslant \ell\leqslant N$, and $r-s$ and ℓ are either both even or both odd. The first sum in (A.2) runs over integer m when ℓ is even and half an odd integer m when ℓ is odd, generalizing the Neveu–Schwarz and Ramond sector of the supersymmetric case corresponding to $N=2$. The restriction on $m_a(j) := (a/2+P'j)$

in the sum over j indicates that we only sum over those values of j for which $m_a(j) \equiv \pm m \pmod N$.

The central charge and conformal dimensions in (A.1) are given by

$$c = \frac{3N}{N+2}\left(1 - \frac{2(N+2)}{N^2}\frac{(P'-P)^2}{P'P}\right) = \frac{3NN'(N'+N+4)}{(N+N'+2)(N+2)(N'+2)} \tag{A.3}$$

$$= 1 - \frac{6N}{(N'+2)(N'+N+2)} + \frac{2(N-1)}{N+2} \tag{A.4}$$

and

$$\Delta_{r,s} = \frac{(P'r-Ps)^2-(P'-P)^2}{4NP'P} = \frac{[(N+N'+2)r-(N'+2)s]^2-N^2}{4N(N+N'+2)(N'+2)}. \tag{A.5}$$

Expression (A.4) for the central charge reflects that in the Feigin and Fuchs construction one deals with a Z_N-parafermion field with central charge equal to the third term in (A.4) and a bosonic field with a background charge with central charge given by the first two terms in (A.4).

The function c^ℓ_m in (A.2) is the level-N $A^{(1)}_1$ string function [59-62] which for our purposes in Section 3.3 we need in the form given by Lepowsky and Primc [62]

$$c^\ell_m(q) = \frac{q^{\frac{\ell(N-\ell)}{2N(N+2)}}}{(q)_\infty} \sum_{\boldsymbol{n}\in\mathbb{Z}^{N-1}_+,\ \frac{m+\ell}{2N}+(C^{-1}\boldsymbol{n})_1\in\mathbb{Z}} \frac{q^{\boldsymbol{n}C^{-1}(\boldsymbol{n}-\boldsymbol{e}_\ell)}}{(q)_{n_1}\cdots(q)_{n_{N-1}}}. \tag{A.6}$$

We also need the symmetry properties

$$c^\ell_m = c^\ell_{-m} = c^\ell_{m+2N} = c^{N-\ell}_{N-m}. \tag{A.7}$$

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