ELLIPTIC A_n SELBERG INTEGRALS

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Abstract. We use the elliptic interpolation kernel due to the second author to prove an A_n extension of the elliptic Selberg integral. More generally, we obtain elliptic analogues of the A_n Kadell, Hua–Kadell and Alba–Fateev–Litvinov–Tarnopolsky (or AFLT) integrals.

1. Introduction

In his famous 1944 paper [68], Atle Selberg evaluated the following multivariate extension of Euler's beta integral that now bears his name. For k a positive integer,

$$
\begin{aligned}
S_k(\alpha,\beta;\gamma) &:= \int\limits_{[0,1]^k} \prod_{i=1}^k x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |x_i-x_j|^{2\gamma}\, \mathrm{d}x_1\cdots \mathrm{d}x_k \\
&= \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(k+i-2)\gamma)\Gamma(1+\gamma)},
\end{aligned}
\tag{1.1}
$$

where $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$ and

$$\mathrm{Re}(\gamma) > -\min\{1/k, \mathrm{Re}(\alpha)/(k-1), \mathrm{Re}(\beta)/(k-1)\}.$$

The Selberg integral has come to be regarded as one of the most fundamental hypergeometric integrals, a reputation which is upheld by its appearance in numerous different areas of mathematics such as random matrix theory [6, 24, 25, 47], analytic number theory [4, 22, 27, 39, 40], enumerative combinatorics [38, 41, 42, 78], and conformal field theory [2, 20, 21, 49, 52, 67, 79, 80, 81]. For a review of the history and mathematics surrounding Selberg's integral the reader is referred to [26].

There are many important generalisations of the Selberg integral. One of the goals of this paper is to unify most of these by proving an elliptic analogue of the Selberg integral for the Lie algebra A_n, as well as elliptic analogues of the more general Kadell, Hua–Kadell and AFLT integrals for A_n. Before we describe the first of these generalisations, we remind the reader of the elliptic analogue of the ordinary (or A_1) Selberg integral and of the (non-elliptic) A_n Selberg integral.

Fix $p,q\in\mathbb{C}$ such that $|p|,|q|<1$, and let

$$\Gamma_{p,q}(z) := \prod_{i,j=0}^{\infty} \frac{1-p^{i+1}q^{j+1}/z}{1-p^iq^jz}$$

be the elliptic Gamma function [65]. This function, which has zeros at $p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1}$, poles at $p^{-\mathbb{N}_0}q^{-\mathbb{N}_0}$ and an essential singularity at the origin, is symmetric in p and q and satisfies the reflection formula

$$\Gamma_{p,q}(z)\Gamma_{p,q}(pq/z) = 1. \tag{1.2}$$

As is by now standard, in the following we adopt the multiplicative shorthand notation $\Gamma_{p,q}(z_1,\dots,z_n) := \Gamma_{p,q}(z_1)\cdots\Gamma_{p,q}(z_n)$ as well as the plus-minus notation

$$\begin{aligned}\Gamma_{p,q}(az^{\pm}) &:= \Gamma_{p,q}(az, az^{-1}),\\ \Gamma_{p,q}(az^{\pm}w^{\pm}) &:= \Gamma_{p,q}(azw, az^{-1}w, azw^{-1}, az^{-1}w^{-1}).\end{aligned}$$

Again assuming that $|q|<1$, let $(a;q)_\infty := \prod_{i\geqslant 0}(1-aq^i)$ be the infinite q-shifted factorial. Then the elliptic Selberg density is defined as [58, 70]

$$\Delta_{\mathrm{S}}^{(\mathrm{v})}(z_1,\dots,z_k;t_1,\dots,t_m;t;p,q) := \varkappa_k \prod_{1\leqslant i<j\leqslant k} \frac{\Gamma_{p,q}(tz_i^{\pm}z_j^{\pm})}{\Gamma_{p,q}(z_i^{\pm}z_j^{\pm})} \prod_{i=1}^{k} \frac{\Gamma_{p,q}(t)\prod_{r=1}^{m}\Gamma_{p,q}(t_r z_i^{\pm})}{\Gamma_{p,q}(z_i^{\pm 2})}, \tag{1.3}$$

where $z_1,\dots,z_k,t,t_1,\dots,t_m \in \mathbb{C}^*$ and

$$\varkappa_k := \frac{(p;p)_\infty^k (q;q)_\infty^k}{2^k k! (2\pi\mathrm{i})^k}. \tag{1.4}$$

The use of the superscript (v) is non-standard. Later we also need a companion density $\Delta_{\mathrm{S}}^{(\mathrm{e})}(\dots;\dots;c;p,q)$, and the superscripts (v) and (e) — v for vertex and e for edge of the A_n Dynkin diagram — have been added to avoid confusion. Assuming $0<|t|,|t_1|,\dots,|t_6|<1$ as well as the balancing condition $t^{2k-2}t_1\cdots t_6 = pq$, the elliptic Selberg integral corresponds to

$$\int_{\mathbb{T}^k} \Delta_{\mathrm{S}}^{(\mathrm{v})}(z_1,\dots,z_k;t_1,\dots,t_6;t;p,q)\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_k}{z_k} = \prod_{i=1}^{k}\biggl(\Gamma_{p,q}(t^i)\prod_{1\leqslant r<s\leqslant 6}\Gamma_{p,q}(t^{i-1}t_r t_s)\biggr), \tag{1.5}$$

where $\mathbb{T}^k$ denotes the complex k-torus. For $k=1$ the above integral is Spiridonov's elliptic beta integral [69]. For general k the integral evaluation (1.5) was conjectured by van Diejen and Spiridonov [17, 18] and proved by the second author [59]. Alternative proofs have since been given by Spiridonov [71] and by Ito and Noumi [34]. A rigorous proof that (1.5) simplifies to the Selberg integral (1.1) upon taking appropriate limits was presented in [58].

Elliptic beta and Selberg integrals are not just of interest from a special functions point of view, corresponding to the top-level results in the classical–basic–elliptic hierarchy of hypergeometric integrals. In 2009 Dolan and Osborn [19] made the important discovery that supersymmetric indices of supersymmetric 4-dimensional quantum field theories take the form of elliptic hypergeometric integrals. As a consequence, many conjectural Seiberg dualities for such quantum field theories imply transformation formulae for the corresponding indices, and hence for elliptic hypergeometric integrals. Since this discovery, elliptic hypergeometric integrals and their transformation properties play an important role in the study of dualities in quantum field theory, see e.g., [28, 29, 53, 62, 73, 74, 75, 76]. Another surprising application of elliptic hypergeometric integrals — not unrelated to the supersymmetric dualities, see the survey [30] — has been the construction of novel Yang–Baxter solvable models with continuous spin parameters [7, 8, 9, 72], generalising many famous exactly solvable discrete spin models such as the Ising and chiral Potts models. These connections between elliptic hypergeometric integrals and quantum field theory and integrable systems provide further motivation for generalising the integral evaluation (1.5) to A_n.

To succinctly describe the non-elliptic A_n Selberg integral, we define

$$\Delta(x) := \prod_{1\leqslant i<j\leqslant k} (x_i - x_j) \quad \text{and} \quad \Delta(x;y) := \prod_{i=1}^{k}\prod_{j=1}^{\ell}(x_i - y_j)$$

for $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_\ell)$. Let $0 =: k_0 \leqslant k_1 \leqslant \dots \leqslant k_n$ be nonnegative integers and, for $1 \leqslant r \leqslant n$, denote by $x^{(r)} = \big(x_1^{(r)}, \dots, x_{k_r}^{(r)}\big)$ a k_r-tuple of integration variables. Further let $\alpha_1, \dots, \alpha_n, \beta, \gamma \in \mathbb{C}$ satisfy

$$\operatorname{Re}(\beta) > 0, \qquad k_n|\operatorname{Re}(\gamma)| < 1, \qquad \operatorname{Re}\big(\beta + (k_n - 1)\gamma\big) > 0, \tag{1.6a}$$

$$\operatorname{Re}\big(\alpha_r + \dots + \alpha_s + (r - s + i - 1)\gamma\big) > 0 \quad \text{for } 1 \leqslant r \leqslant s \leqslant n \text{ and } 1 \leqslant i \leqslant k_r - k_{r-1}. \tag{1.6b}$$

Then the A_n Selberg integral refers to the integral evaluation

$$\begin{aligned}
\int\limits_{C_\gamma^{k_1,\dots,k_n}[0,1]} \prod_{r=1}^{n} \bigg(\big|\Delta\big(x^{(r)}\big)\big|^{2\gamma} \prod_{i=1}^{k_r} \big(x_i^{(r)}\big)^{\alpha_r - 1} \big(1 - x_i^{(r)}\big)^{\beta_r - 1} \bigg) \\
\times \prod_{r=1}^{n-1} \big|\Delta\big(x^{(r)}; x^{(r+1)}\big)\big|^{-\gamma} \,\mathrm{d}x^{(1)} \cdots \mathrm{d}x^{(n)} \\
= \prod_{r=1}^{n}\prod_{i=1}^{k_r} \frac{\Gamma(i\gamma)\Gamma(\beta_r + (i - k_{r+1} - 1)\gamma)}{\Gamma(\gamma)} \\
\times \prod_{1\leqslant r\leqslant s\leqslant n} \prod_{i=1}^{k_r - k_{r-1}} \frac{\Gamma(\alpha_r + \dots + \alpha_s + (r - s + i - 1)\gamma)}{\Gamma(\beta_s + \alpha_r + \dots + \alpha_s + (k_s - k_{s+1} + i + r - s - 2)\gamma)},
\end{aligned} \tag{1.7}$$

where $\beta_1 = \dots = \beta_{n-1} = 1$, $\beta_n := \beta$ and $k_{n+1} := 0$.

The origin of the restrictions $\beta_1 = \dots = \beta_{n-1} = 1$ and $k_1 \leqslant \ldots \leqslant k_n$ is representation theoretic. Let $\mathfrak{g} := \mathfrak{sl}_{n+1}$, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^*$ its dual. For $I := \{1, \dots, n\}$, let $\{\alpha_i\}_{i\in I} \in \mathfrak{h}^*$, $\{\omega_i\}_{i\in I} \in \mathfrak{h}^*$ and $\{\alpha_i^\vee\}_{i\in I} \in \mathfrak{h}$ be the set of simple roots, fundamental weights and simple coroots of $\mathfrak{g}$, so that $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{i,j}$. Finally, let $P_+ \subset \mathfrak{h}^*$ be the set of dominant integral weights, i.e., $\mu \in P_+$ if $\langle \mu, \alpha_i^\vee \rangle \in \mathbb{N}_0$ for all $i \in I$. Now fix $\mu = \sum_{i\in I}(\mu_i - 1)\omega_i \in P_+$ and $\nu = \sum_{i=I}(\nu_i - 1)\omega_i \in P_+$ such that $\nu_1 = \dots = \nu_{n-1} = 1$, and let V_μ and V_ν be two irreducible $\mathfrak{g}$-modules of highest weight μ and ν respectively. Then the following multiplicity-free tensor-product decomposition holds:

$$V_\mu \otimes V_\nu = \bigoplus_{\substack{0\leqslant k_1\leqslant\dots\leqslant k_n \\ \mu+\nu-\sum_{i\in I} k_i\alpha_i \in P_+}} V_{\mu+\nu-\sum_{i=1}^{n} k_i\alpha_i}.$$

The $\alpha_1, \dots, \alpha_n$ and β_n in (1.7) are essentially continuous analogues of $\mu_1, \dots, \mu_n$ and ν_n, respectively, and $\beta_i = \nu_i = 1$ for all $1 \leqslant i \leqslant n - 1$.

The domain of integration $C_\gamma^{k_1,\dots,k_n}[0,1]$ in (1.7) takes the form of a $(k_1+\dots+k_n)$-dimensional chain. Its precise form is not needed in this paper, and the interested reader is referred to [3, 79, 82, 83] for details. For $n = 1$ the integration chain is independent of γ and simplifies to the k-simplex

$$C_\gamma^k[0,1] = \big\{x \in \mathbb{R}^k : 0 < x_1 < \dots < x_k < 1\big\}.$$

Up to a factor of $k!$, the $n=1$ case of (1.7) is thus the original Selberg integral (1.1). For $n=2$ the evaluation (1.7) was first given by Tarasov and Varchenko [79], and for general n it is due to the third author [83]. There is also a finite field analogue of (1.7) due to Rimányi and Varchenko [63] which is not covered in our elliptic generalisation below.

To state the elliptic A_n Selberg integral we introduce some further notation. For $z=(z_1,\dots,z_k)\in(\mathbb{C}^*)^k$, $w=(w_1,\dots,w_\ell)\in(\mathbb{C}^*)^\ell$ and $c\in\mathbb{C}^*$, define

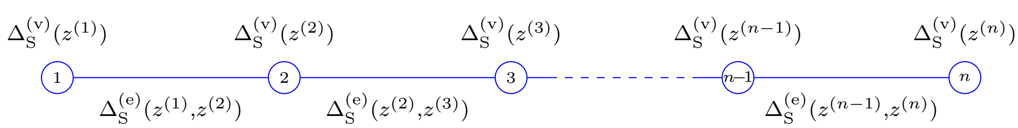
$$\Delta_{\mathrm{S}}^{(\mathrm{e})}(z;w;c;p,q):=\prod_{i=1}^{k}\prod_{j=1}^{\ell}\Gamma_{p,q}\big(cz_i^{\pm}w_j^{\pm}\big). \tag{1.8}$$

Whereas the elliptic Selberg density (1.3) should be viewed as the elliptic analogue of the integrand of the Selberg integral (1.1), the above function for $c=(pq/t)^{1/2}$ plays the role of $|\Delta(x;y)|^{-\gamma}$ in the elliptic analogue of (1.7). This same special case of (1.8) previously appeared in the study of elliptic integrable systems, see e.g., [5, 43, 66] and, as shown in [43, 66], satisfies a remarkable duality with respect to the 8-parameter van Diejen difference operator [16].

We now combine the two elliptic Selberg densities to form the A_n elliptic Selberg density

$$\begin{aligned}
&\Delta_{\mathrm{S}}\big(z^{(1)},\dots,z^{(n)};t_1,\dots,t_{2n+4};c;t;p,q\big)\\
&\quad:=\prod_{r=1}^{n-1}\Big(\Delta_{\mathrm{S}}^{(\mathrm{v})}\big(z^{(r)};c^{r-n}t_{2r-1},c^{r-n}t_{2r},tc^{n-r}/t_{2r+1},tc^{n-r}/t_{2r+2};t;p,q\big)\\
&\qquad\qquad\times\Delta_{\mathrm{S}}^{(\mathrm{e})}\big(z^{(r)};z^{(r+1)};c;p,q\big)\Big)\\
&\qquad\times\Delta_{\mathrm{S}}^{(\mathrm{v})}\big(z^{(n)};t_{2n-1},t_{2n},t_{2n+1},t_{2n+2},t_{2n+3},t_{2n+4};t;p,q\big),
\end{aligned} \tag{1.9}$$

where $z^{(r)}=\big(z_1^{(r)},\dots,z_{k_r}^{(r)}\big)$. Suppressing the dependence on $c,t,t_1,\dots,t_{2n+4},p,q$, the individual densities making up the A_n density should be thought of as corresponding to the vertices and edges of the A_n Dynkin diagram as follows:



Finally, for $z=(z_1,\dots,z_k)$, we let $\frac{\mathrm{d}z}{z}:=\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_k}{z_k}$.

Theorem 1.1 (A_n elliptic Selberg integral)**.** *Let n be a positive integer and $k_1,\dots,k_n$ integers such that $0=:k_0\leqslant k_1\leqslant\cdots\leqslant k_n$. For $p,q,t\in\mathbb{C}^*$ such that $|p|,|q|,|t|,|pq/t|<1$, fix a branch of $c:=(pq/t)^{1/2}$, and let $t_1,\dots,t_{2n+4}\in\mathbb{C}^*$ such that the balancing condition*

$$t^{k_r-k_{r-1}+k_n-2}t_{2r-1}t_{2r}t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4}=pq \tag{1.10}$$

holds for all $1\leqslant r\leqslant n$. Then

$$\begin{aligned}
&\int\Delta_{\mathrm{S}}\big(z^{(1)},\dots,z^{(n)};t_1,\dots,t_{2n+4};c;t;p,q\big)\,\frac{\mathrm{d}z^{(1)}}{z^{(1)}}\cdots\frac{\mathrm{d}z^{(n)}}{z^{(n)}}\\
&\qquad=\prod_{r=1}^{n}\prod_{i=1}^{k_r-k_{r-1}}\Gamma_{p,q}(t^i,t^{i-1}c^{2r-2n}t_{2r-1}t_{2r})\prod_{2n+1\leqslant r<s\leqslant 2n+4}\prod_{i=1}^{k_n}\Gamma_{p,q}(t^{i-1}t_rt_s)
\end{aligned} \tag{1.11}$$

$$\times \prod_{1\leqslant r<s\leqslant n} \prod_{i=1}^{k_r-k_{r-1}} \Gamma_{p,q}(t^i t_{2r-1}/t_{2s-1}, t^i t_{2r}/t_{2s-1}, t^i t_{2r-1}/t_{2s}, t^i t_{2r}/t_{2s})$$
$$\times \prod_{r=1}^{n} \prod_{s=2n+1}^{2n+4} \prod_{i=1}^{k_r-k_{r-1}} \Gamma_{p,q}(t^{i-1} t_{2r-1} t_s, t^{i-1} t_{2r} t_s),$$

where $z^{(r)} = \big(z_1^{(r)}, \dots, z_{k_r}^{(r)}\big)$ *for all* $1 \leqslant r \leqslant n$.

The $(k_1 + \cdots + k_n)$-dimensional contour of integration of the A_n Selberg integral has the product structure

$$\underbrace{C_1 \times \cdots \times C_1}_{k_1\text{-times}} \times \underbrace{C_2 \times \cdots \times C_2}_{k_2\text{-times}} \times \cdots\cdots \times \underbrace{C_n \times \cdots \times C_n}_{k_n\text{-times}},$$

where C_r for each $1 \leqslant r \leqslant n$ is a positively oriented smooth Jordan curve around 0 such that $C_r = C_r^{-1}$. Moreover, for $1 \leqslant r \leqslant n-1$, the elements of the sets

$$c^{r-n} t_{2r+s-2} p^{\mathbb{N}_0} q^{\mathbb{N}_0}, \quad t c^{n-r} t_{2r+s}^{-1} p^{\mathbb{N}_0} q^{\mathbb{N}_0} \ (1 \leqslant s \leqslant 2), \quad t p^{\mathbb{N}_0} q^{\mathbb{N}_0} C_r, \quad c p^{\mathbb{N}_0} q^{\mathbb{N}_0} C_{r\pm 1} \tag{1.12a}$$

all lie in the interior of C_r, and the elements of

$$t_{s+2n-2} p^{\mathbb{N}_0} q^{\mathbb{N}_0} \ (1 \leqslant s \leqslant 6), \quad t p^{\mathbb{N}_0} q^{\mathbb{N}_0} C_n, \quad c p^{\mathbb{N}_0} q^{\mathbb{N}_0} C_{n-1} \tag{1.12b}$$

all lie in the interior of C_n, where $C_0 := 0$. These conditions on the C_r in particular imply that $c^2 C_r \in \operatorname{int}(C_r)$ for $2 \leqslant r \leqslant n$, explaining why $|c^2| = |pq/t| < 1$. For $n = 1$ this restriction can obviously be dropped. Furthermore, for $n = 1$ the balancing condition (1.10) simplifies to $t^{2k_1-2} t_1 t_2 \cdots t_6 = pq$. Taking $|t_1|, \dots, |t_6| < 1$ it then follows that (1.12b) is satisfied for $C = \mathbb{T}$, so that the integral reduces to (1.5). For $n \geqslant 2$ it is generally not possible to restrict the parameters such that $C_r = \mathbb{T}$ for all $1 \leqslant r \leqslant n$. For example, if $C_r = \mathbb{T}$ for all r, it follows from (1.12a) that $c^{r-n} t_{2r-1}, c^{r-n} t_{2r}, t c^{n-r} t_{2r+1}^{-1}, t c^{n-r} t_{2r+2}^{-1}$ all lie in the interior of $\mathbb{T}$. By (1.10) and $|t| < 1$ this would impose the condition that $k_{r+1} - 2k_r + k_{r-1} \geqslant -1$ for all $1 \leqslant r \leqslant n-1$.

All of the integral formulas listed thus far admit generalisations in which the integrand is multiplied by an appropriate symmetric function or BC_n-symmetric function. In the case of (1.1) the most general such integral was discovered by Alba, Fateev, Litvinov and Tarnopolsky (AFLT) [2] and contains a pair of Jack polynomials in the integrand. The AFLT integral includes the well-known Kadell integral [37] (which contains one Jack polynomial) and the Hua–Kadell integral [32, 36] (which contains two Jack polynomials but assumes $\beta = \gamma$) as special cases. In our previous paper [3] the AFLT integral was extended to the elliptic case, as well as to A_n. In Section 4 we unify both these results by proving an elliptic A_n AFLT integral. In this integral the Jack polynomials in the integrand of the non-elliptic A_n AFLT integral are replaced by a pair of elliptic interpolation functions [60]. Our approach to the elliptic A_n Selberg and AFLT integrals is based on a recursion for a generalisation of the elliptic interpolation functions, known as the elliptic interpolation kernel [61]. This differs from the approaches taken in [3], where the non-elliptic A_n AFLT integral is proved using Cauchy-type identities for Macdonald polynomials and the A_1 elliptic AFLT integral is proved using known integral identities for elliptic interpolation functions.

The remainder of the paper is organised as follows. In the next section we review some standard definitions and notation from the theory of elliptic beta integrals. Section 3 is devoted to several classes of elliptic special functions, including the elliptic interpolation functions and the elliptic interpolation kernel. The latter forms the basis of our approach to Theorem 1.1. In Section 4 we first discuss the original AFLT integral and its A_n analogue, and then state and prove an elliptic A_n AFLT integral. As a special case this yields Theorem 1.1.

2. Elliptic preliminaries

Throughout this paper we assume that $p,q\in\mathbb{C}^*$ such that $|p|,|q|<1$.

2.1. **Partitions.** A partition $\lambda=(\lambda_1,\lambda_2,\dots)$ is a weakly decreasing sequence of nonnegative integers λ_i such that only finitely many λ_i are nonzero. The nonzero λ_i are called the parts of λ, and the number of parts is the length of λ, denoted by $l(\lambda)$. Partitions are identified up to the number of trailing zeroes, so that, for example, $(3,1,1)=(3,1,1,0,\dots)$. We write $\mathscr{P}$ for the set of all partitions and $\mathscr{P}_n$ for the set of all partitions of length at most n. In particular, $\mathscr{P}_0=\{0\}$, with 0 the unique partition of 0. If the sum of the parts, denoted $|\lambda|$, is equal to some integer n, then λ is said to be a partition of n, which is also written $\lambda\vdash n$. If λ is a partition, we write $(i,j)\in\lambda$ to mean any pair of integers (i,j) such that $1\leqslant i\leqslant l(\lambda)$ and $1\leqslant j\leqslant\lambda_i$. If λ is a partition, its conjugate λ' is defined by $\lambda'_i:=|\{j\in\mathbb{N}:\lambda_j\geqslant i\}|$. For example $(7,4,2,1,1)'=(5,3,2,2,1,1,1)$. For a pair of partitions λ,μ we write $\mu\subseteq\lambda$ if $\mu_i\leqslant\lambda_i$ for all i. If λ,μ further satisfy $\lambda_1\geqslant\mu_1\geqslant\lambda_2\geqslant\mu_2\geqslant\cdots$ (i.e., $\mu\subseteq\lambda$ and $\lambda'_i-\mu'_i\in\{0,1\}$ for all $i\geqslant 1$), we write $\mu\prec\lambda$. (In this case the skew shape λ/μ is known as a horizontal strip.)

We refer to elements of $\mathscr{P}^2$ as bipartitions, and to distinguish partitions from bipartitions a bold font such as $\boldsymbol{\lambda}$ is used for the latter. In particular, $\mathbf{0}$ denotes the bipartition $(0,0)$. If $\boldsymbol{\lambda}=(\lambda^{(1)},\lambda^{(2)})$ and $\boldsymbol{\mu}=(\mu^{(1)},\mu^{(2)})$ are bipartitions then the notation $\boldsymbol{\mu}\subseteq\boldsymbol{\lambda}$ is shorthand for the termwise inclusions $\mu^{(1)}\subseteq\lambda^{(1)}$ and $\mu^{(2)}\subseteq\lambda^{(2)}$. The notation $\boldsymbol{\mu}\prec\boldsymbol{\lambda}$ is similarly defined. For $\boldsymbol{\lambda}\in\mathscr{P}_n^2$, the spectral vector $\langle\boldsymbol{\lambda}\rangle_{n;t;p,q}$ is given by

$$\langle\boldsymbol{\lambda}\rangle_{n;t;p,q}:=\Big(p^{\lambda_1^{(1)}}q^{\lambda_1^{(2)}}t^{n-1},p^{\lambda_2^{(1)}}q^{\lambda_2^{(2)}}t^{n-2},\dots,p^{\lambda_{n-1}^{(1)}}q^{\lambda_{n-1}^{(2)}}t,p^{\lambda_n^{(1)}}q^{\lambda_n^{(2)}}\Big),$$

so that

$$\big\langle\big(\lambda^{(1)},\lambda^{(2)}\big)\big\rangle_{n;t;p,q}=\big\langle\big(\lambda^{(2)},\lambda^{(1)}\big)\big\rangle_{n;t;q,p}.$$

2.2. **Elliptic preliminaries.** A key ingredient in the theory of elliptic hypergeometric functions is the modified theta function, defined as

$$\theta_p(z):=(z;p)_\infty(p/z;p)_\infty,$$

for $z\in\mathbb{C}^*$. This function is quasi periodic along annuli

$$\theta_p(pz)=-z^{-1}\theta_p(z), \tag{2.1}$$

satisfies the symmetry $\theta_p(z)=-z\,\theta_p(1/z)$, and features in the functional equation

$$\Gamma_{p,q}(pz)=\theta_q(z)\Gamma_{p,q}(z) \tag{2.2}$$

for the elliptic gamma function.

For n an integer, the elliptic shifted factorial is defined as

$$(z;q,p)_n:=\frac{\Gamma_{p,q}(q^nz)}{\Gamma_{p,q}(z)}, \tag{2.3}$$

where it is noted that for $n \geqslant 0$,

$$(z;q,p)_n = \prod_{i=1}^{n} \theta_p(zq^{i-1}).$$

The elliptic shifted factorial has three important generalisations to partitions, given by

$$\begin{aligned}
C^0_\lambda(z;q,t;p) &:= \prod_{(i,j)\in\lambda} \theta_p\bigl(zq^{j-1}t^{1-i}\bigr), \\
C^+_\lambda(z;q,t;p) &:= \prod_{(i,j)\in\lambda} \theta_p\bigl(zq^{\lambda_i+j-1}t^{2-\lambda'_j-i}\bigr), \\
C^-_\lambda(z;q,t;p) &:= \prod_{(i,j)\in\lambda} \theta_p\bigl(zq^{\lambda_i-j}t^{\lambda'_j-i}\bigr).
\end{aligned}$$

Note that $C^0_\lambda(z;q,t;p)$ is sometimes denoted $(z;q,t;p)_\lambda$ in the literature on elliptic hypergeometric series.

For all of the functions defined above, condensed notation such as

$$C^0_\lambda(z_1,\dots,z_k;q,t;p) := C^0_\lambda(z_1;q,t;p)\cdots C^0_\lambda(z_n;q,t;p)$$

will be employed. As further shorthand notation we define the following well-poised ratio of products of elliptic shifted factorials:

$$\Delta^0_\lambda(a|b_1,\dots,b_n;q,t;p) := \prod_{i=1}^{n} \frac{C^0_\lambda(b_i;q,t;p)}{C^0_\lambda(pqa/b_i;q,t;p)},$$

which satisfies the reflection equation

$$\Delta^0_\lambda(a|b_1,\dots,b_n;q,t;p) = \frac{1}{\Delta^0_\lambda(a|pqa/b_1,\dots,pqa/b_n;q,t;p)}. \tag{2.4}$$

To preserve p,q-symmetry in many of the elliptic functions and integrals considered in this paper, we require an extension of the above definitions to bipartitions, and for any function $f_\lambda(a_1,\dots,a_n;q,t;p)$ or $f_{\lambda/\mu}(a_1,\dots,a_n;q,t;p)$ we define

$$f_{\boldsymbol{\lambda}}(a_1,\dots,a_n;t;p,q) := f_{\lambda^{(1)}}(a_1,\dots,a_n;p,t;q)f_{\lambda^{(2)}}(a_1,\dots,a_n;q,t;p), \tag{2.5a}$$

$$f_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(a_1,\dots,a_n;t;p,q) := f_{\lambda^{(1)}/\mu^{(1)}}(a_1,\dots,a_n;p,t;q)f_{\lambda^{(2)}/\mu^{(2)}}(a_1,\dots,a_n;q,t;p). \tag{2.5b}$$

Interchanging p and q is thus the same as interchanging the two components of $\boldsymbol{\lambda}$ and, in the skew case, the two components of $\boldsymbol{\mu}$. By (1.2) and (2.3) followed by the use of the quasi periodicity (2.1), it may be shown that

$$\prod_{i=1}^{n} \frac{\Gamma_{p,q}\bigl(at^{1-i}p^{\lambda^{(1)}_i}q^{\lambda^{(2)}_i}, bt^{i-1}p^{-\lambda^{(1)}_i}q^{-\lambda^{(2)}_i}\bigr)}{\Gamma_{p,q}(at^{1-i},bt^{i-1})} = \Bigl(\frac{pq}{ab}\Bigr)^{\sum_{i=1}^{n}\lambda^{(1)}_i\lambda^{(2)}_i} \Delta^0_{\boldsymbol{\lambda}}(a/b|a;t;p,q), \tag{2.6}$$

for $\boldsymbol{\lambda} \in \mathscr{P}^2_n$.

2.3. **The Dixon and Selberg densities.** In addition to the elliptic Selberg density (1.3), we need the elliptic Dixon density

$$\Delta_{\mathrm{D}}(z_1,\dots,z_k;t_1,\dots,t_m;p,q):=\varkappa_k\prod_{1\leqslant i<j\leqslant k}\frac{1}{\Gamma_{p,q}(z_i^{\pm}z_j^{\pm})}\prod_{i=1}^{k}\frac{\prod_{r=1}^{m}\Gamma_{p,q}(t_rz_i^{\pm})}{\Gamma_{p,q}(z_i^{\pm 2})},$$

with $\varkappa_k$ given in (1.4). This is related to the Selberg density by

$$\begin{aligned}&\Delta_{\mathrm{S}}^{(\mathrm{v})}(z_1,\dots,z_k;t_1,\dots,t_m;t;p,q)\\&\quad=\Delta_{\mathrm{D}}(z_1,\dots,z_k;t_1,\dots,t_m;t;p,q)\,\Gamma_{p,q}^k(t)\prod_{1\leqslant i<j\leqslant k}\Gamma_{p,q}(tz_i^{\pm}z_j^{\pm}).\end{aligned}$$

Apart from possible balancing conditions, or restrictions to certain subsets of the complex plane, it will be assumed throughout this paper that parameters such as $t_1,\dots,t_m,p,q,t$ are in generic position.

We say that a function $f:(\mathbb{C}^*)^k\longrightarrow\mathbb{C}$ is BC_k-symmetric if $f(x_1,\dots,x_k)$ is invariant under the natural action of the hyperoctahedral group $\mathfrak{S}_k\ltimes(\mathbb{Z}/2\mathbb{Z})^k$. For f a BC_k-symmetric meromorphic function and $t,t_1,\dots,t_6\in\mathbb{C}^*$ such that $|t|<1$, we define the Selberg average of f as

$$\langle f\rangle^k_{t_1,\dots,t_6;t;p,q}:=\frac{1}{S_k(t_1,\dots,t_6;t;p,q)}\int f(z)\Delta_{\mathrm{S}}^{(\mathrm{v})}(z;t_1,\dots,t_6;t;p,q)\,\frac{\mathrm{d}z}{z},\tag{2.7}$$

where $S_k(t_1,\dots,t_6;t;p,q)$ denotes the A_1 elliptic Selberg integral (1.5) and where it is assumed that $t^{2k-2}t_1\cdots t_6=pq$. The contour of the integral on the right has the form C^k, where $C=C^{-1}$ is a positively oriented smooth Jordan curve around 0 such that

$$t_rp^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant r\leqslant 6),\quad tp^{\mathbb{N}_0}q^{\mathbb{N}_0}C,$$

as well as any sequence of poles of f tending to zero, excluding those cancelled by the univariate part of the Dixon density, all lie in the interior of C. If f is analytic on $(\mathbb{C}^*)^k$ and $|t_1|,\dots,|t_6|<1$, we may take $C=\mathbb{T}$.

3. Elliptic interpolation functions and the interpolation kernel

The purpose of this section is to introduce the BC_k-symmetric elliptic interpolation functions and the closely related interpolation kernel. The interpolation functions will play the role of Jack polynomials in our elliptic analogues of the A_n AFLT, Kadell and Hua–Kadell integrals. The interpolation kernel is a crucial ingredient in our proof of the various elliptic A_n Selberg integrals, allowing us to establish a recursion in the rank n.

3.1. **Elliptic interpolation functions.** Below we give a brief review of the elliptic interpolation functions. The reader may consult [15, 57, 59, 60, 61, 64] for more complete accounts.

For $\boldsymbol{\mu}\in\mathscr{P}_k^2$, $x=(x_1,\dots,x_k)\in(\mathbb{C}^*)^k$ and $a,b,t\in\mathbb{C}^*$, the BC_k-symmetric elliptic interpolation function is denoted by

$$R^*_{\boldsymbol{\mu}}(x;a,b;t;p,q),$$

and consists of a q-elliptic factor and p-elliptic factor:

$$R^*_{\boldsymbol{\mu}}(x;a,b;t;p,q)=R^*_{\mu^{(1)}}(x;a,b;p,t;q)R^*_{\mu^{(2)}}(x;a,b;q,t;p).$$

As usual in symmetric function theory, $R_0^*(x;a,b;q,t;p)=1$. We also adopt the convention that $R^*_{\boldsymbol{\mu}}(x;a,b;t;p,q)=0$ if $\boldsymbol{\mu}$ is a bipartition such that $\boldsymbol{\mu}\notin\mathscr{P}_k^2$, i.e., if the length of at least one of $\mu^{(1)},\mu^{(2)}$ exceeds k.

The fundamental property of the elliptic interpolation functions is the vanishing

$$R^*_{\boldsymbol{\mu}}(a\langle\boldsymbol{\lambda}\rangle_{k;t;p,q};a,b;t;p,q)=0$$

for all $\boldsymbol{\lambda}\in\mathscr{P}_k^2$ such that $\boldsymbol{\mu}\not\subseteq\boldsymbol{\lambda}$. The BC_k-symmetric interpolation function $R^*_\mu(x;a,b;q,t;p)$ generalises Okounkov's BC_k-symmetric interpolation Macdonald polynomial $P^*_\mu(x;q,t,s)$, which satisfies a similar vanishing property and contains the ordinary Macdonald polynomial $P_\mu(x;q,t)$ as its top-homogeneous degree component; see [54, 56] for details. The interpolation functions completely factorise under principal specialisation:

$$R^*_{\boldsymbol{\mu}}(v\langle\mathbf{0}\rangle_{k;t;p,q};a,b;t;p,q)=\Delta^0_{\boldsymbol{\mu}}(t^{k-1}a/b|t^{k-1}av,a/v;t;p,q).$$

If the parameters satisfy $t^kab=pq$ then the interpolation functions are said to be of Cauchy type and once again factorise:

$$R^*_{\boldsymbol{\mu}}(x;a,b;t;p,q)=\Delta^0_{\boldsymbol{\mu}}\big(t^{k-1}a/b\big|t^{k-1}ax_1^{\pm},\dots,t^{k-1}ax_k^{\pm};t;p,q\big). \tag{3.1}$$

The elliptic binomial coefficients

$$\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a,b];t;p,q}=\left\langle\begin{matrix}\lambda^{(1)}\\ \mu^{(1)}\end{matrix}\right\rangle_{[a,b];p,t;q}\left\langle\begin{matrix}\lambda^{(2)}\\ \mu^{(2)}\end{matrix}\right\rangle_{[a,b];q,t;p}$$

are defined as normalised connection coefficients between the elliptic BC_k interpolation functions:

$$\begin{aligned}R^*_{\boldsymbol{\lambda}}&(x;a,b;t;p,q)\\ &=\sum_{\boldsymbol{\mu}}\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[t^{k-1}a/b,a/a'];t;p,q}\frac{\Delta^0_{\boldsymbol{\lambda}}(t^{k-1}a/b|t^{k-1}aa';t;p,q)}{\Delta^0_{\boldsymbol{\mu}}(t^{k-1}a'/b|t^{k-1}aa';t;p,q)}\,R^*_{\boldsymbol{\mu}}(x;a',b;t;p,q).\end{aligned} \tag{3.2}$$

It may be shown that this definition is independent of the choice of k and that $\left\langle\begin{smallmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{smallmatrix}\right\rangle_{[a,b];t;p,q}$ vanishes unless $\boldsymbol{\mu}\subseteq\boldsymbol{\lambda}$. Moreover, for $b=t$ there is additional vanishing and

$$\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a,t];t;p,q}=0\quad\text{unless }\boldsymbol{\mu}\prec\boldsymbol{\lambda}. \tag{3.3}$$

For notational purposes it is convenient to extending the definition of the elliptic binomials to

$$\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a,b](v_1,\dots,v_k);t;p,q}:=\frac{\Delta^0_{\boldsymbol{\lambda}}(a|v_1,\dots,v_k;t;p,q)}{\Delta^0_{\boldsymbol{\mu}}(a/b|v_1,\dots,v_k;t;p,q)}\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a,b];t;p,q}. \tag{3.4}$$

By (2.4),

$$\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a,b](v_1,\dots,v_k,w,pqa/bw);t;p,q}=\frac{\Delta^0_{\boldsymbol{\lambda}}(a|w;t;p,q)}{\Delta^0_{\boldsymbol{\lambda}}(a|bw;t;p,q)}\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a,b](v_1,\dots,v_k);t;p,q}. \tag{3.5}$$

The reader is warned that the elliptic binomial coefficients for $\boldsymbol{\mu}=0$ or $\boldsymbol{\mu}=\boldsymbol{\lambda}$ do not simplify to 1:

$$\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \mathbf{0}\end{matrix}\right\rangle_{[a,b];t;p,q}=\Delta^0_{\boldsymbol{\lambda}}(a|b;t,p,q)\quad\text{and}\quad\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\lambda}\end{matrix}\right\rangle_{[a,b];t;p,q}=\frac{C^+_{\boldsymbol{\lambda}}(a;t;p,q)}{C^+_{\boldsymbol{\lambda}}(a/b;t;p,q)}. \tag{3.6}$$

For $b=1$ they trivialise to

$$\left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a,1](v_1,\dots,v_k);t;p,q} = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}, \tag{3.7}$$

as follows immediately from the definition (3.2). The elliptic binomial coefficients satisfy an analogue of the elliptic Jackson sum as follows [57, Theorem 4.1] (see also [15, Equation (3.7)]):

$$\sum_{\boldsymbol{\mu}} \Delta^0_{\boldsymbol{\mu}}(a/b|d,e;t;p,q) \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a,b];t;p,q} \left\langle \begin{matrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{matrix} \right\rangle_{[a/b,c/b];t;p,q} = \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\nu} \end{matrix} \right\rangle_{[a,c](bd,be);t;p,q}, \tag{3.8}$$

where $bcde=apq$.

Using (3.4), the connection coefficient formula may be written more succinctly as

$$R^*_{\boldsymbol{\lambda}}(x;a,b;t;p,q) = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[t^{k-1}a/b,a/a'](t^{k-1}aa');t;p,q} R^*_{\boldsymbol{\mu}}(x;a',b;t;p,q). \tag{3.9}$$

Choosing $a'=pq/t^kb$ and using (3.1) implies that

$$\begin{aligned} &R^*_{\boldsymbol{\lambda}}(x;a,b;t;p,q) \\ &\quad = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[t^{k-1}a/b,t^kab/pq](pqa/tb);t;p,q} \Delta^0_{\boldsymbol{\mu}}\big(pq/tb^2\big|pqx_1^{\pm}/tb,\dots,pqx_k^{\pm}/tb;t;p,q\big). \end{aligned} \tag{3.10}$$

The elliptic binomial coefficients may be used to define suitable skew analogues of the interpolation functions, and for arbitrary $\boldsymbol{\lambda},\boldsymbol{\nu}\in\mathscr{P}^2$ and k a nonnegative integer,

$$\begin{aligned} &R^*_{\boldsymbol{\lambda}/\boldsymbol{\nu}}([v_1,\dots,v_{2k}];a,b;t;p,q) \\ &\quad := \sum_{\boldsymbol{\mu}} \Delta^0_{\boldsymbol{\mu}}(pq/b^2|pq/bv_1,\dots,pq/bv_{2k};t;p,q) \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a/b,ab/pq];t;p,q} \left\langle \begin{matrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{matrix} \right\rangle_{[pq/b^2,pqV/ab];t;p,q}, \end{aligned} \tag{3.11}$$

where $a,b,t,v_1,\dots,v_{2k}\in\mathbb{C}^*$ and $V:=v_1\cdots v_{2k}$. Obviously, we have vanishing unless $\boldsymbol{\nu}\subseteq\boldsymbol{\lambda}$. It follows from the definition that the skew interpolation functions are $\mathfrak{S}_{2k}$-symmetric functions, rather than BC_k-symmetric. As explained in more detail in [3, 60], the use of the brackets around the $v_1,\dots,v_{2k}$ is a reflection of the close connection with plethystic notation, see for example, [3, Equation (6.7)]. Taking $\boldsymbol{\nu}=\mathbf{0}$ in (3.11) and using (3.6) it follows that

$$R^*_{\boldsymbol{\lambda}/\mathbf{0}}([v_1,\dots,v_{2k}];a,b;t;p,q)$$

is symmetric in $v_1,\dots,v_{2k},a/V$.

The definition of the skew interpolation functions combined with the elliptic Jackson summation (3.8) implies the branching rule

$$\begin{aligned} &R^*_{\boldsymbol{\lambda}/\boldsymbol{\nu}}([v_1,\dots,v_{2k},w_1,w_2];a,b;t;p,q) \\ &\quad = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a/b,w_1w_2](a/w_1,a/w_2);t;p,q} R^*_{\boldsymbol{\mu}/\boldsymbol{\nu}}([v_1,\dots,v_{2k}];a/w_1w_2,b;t;p,q). \end{aligned} \tag{3.12}$$

Taking $w_1w_2=1$ and using (3.7), this shows that

$$R^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_1,\dots,v_{2k}];a,b;t;p,q)|_{v_{2k-1}v_{2k}=1} = R^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_1,\dots,v_{2k-2}];a,b;t;p,q). \tag{3.13}$$

By symmetry this extends to any pair of variables whose product is 1. Similarly, from (3.8) with $c=1$ (so that $\Delta^0_{\boldsymbol{\mu}}(a/b|d,e;t,p,q)=1$) and (3.7), it follows that

$$R^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([\,];a,b;t;p,q)=\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}. \tag{3.14}$$

By (3.12) with $k=0$ this generalises to

$$R^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_1,v_2];a,b;t;p,q)=\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a/b,v_1v_2](a/v_1,a/v_2);t;p,q}.$$

Let $v_{2i-1}v_{2i}=t$ for all $1\leqslant i\leqslant k$. Then, by (3.3), (3.12), (3.14) and induction on k, it follows that $R^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_1,\dots,v_{2k}];a,b;t;p,q)$ vanishes unless there exist $\boldsymbol{\kappa}^{(1)},\dots,\boldsymbol{\kappa}^{(k)}\in\mathscr{P}^2$ such that $\boldsymbol{\mu}\prec\boldsymbol{\kappa}^{(1)}\prec\cdots\prec\boldsymbol{\kappa}^{(k)}\prec\boldsymbol{\lambda}$. In particular, for $\boldsymbol{\mu}=\mathbf{0}$ we have vanishing if $\boldsymbol{\lambda}\not\in\mathscr{P}^2_k$.

A further consequence of (3.7) is that for $ab=pq$

$$R^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_1,\dots,v_{2k}];a,b;t;p,q)=\Delta^0_{\boldsymbol{\lambda}}(a/b|a/v_1,\dots,a/v_{2k};t;p,q)\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a/b,V];t;p,q},$$

so that in particular for $ab=pq$,

$$R^*_{\boldsymbol{\lambda}/\mathbf{0}}([v_1,\dots,v_{2k}];a,b;t;p,q)=\Delta^0_{\boldsymbol{\lambda}}(a/b|a/v_1,\dots,a/v_{2k},V;t;p,q). \tag{3.15}$$

Specialising $(v_1,\dots,v_{2k};\boldsymbol{\nu})$ to $(x_1,x_1^{-1},\dots,x_k,x_k^{-1};\mathbf{0})$ in (3.11), using (3.6), and then comparing the resulting equation with (3.10) yields the nonvanishing case (i.e., $\boldsymbol{\lambda}\in\mathscr{P}^2_k$) of

$$\begin{aligned}&R^*_{\boldsymbol{\lambda}/\mathbf{0}}\big([t^{1/2}x_1^{\pm},\dots,t^{1/2}x_k^{\pm}];t^{k-1/2}a,t^{1/2}b;t;p,q\big)\\&\quad=\Delta^0_{\boldsymbol{\lambda}}(t^{k-1}a/b|t^k;t;p,q)R^*_{\boldsymbol{\lambda}}(x_1,\dots,x_k;a,b;t;p,q).\end{aligned}$$

The above identity shows that, up to simple factor, the non-skew elliptic interpolation functions are a special instances of the skew interpolation functions.

For our purposes it will be convenient to define the hybrid interpolation function

$$\begin{aligned}&R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;v_1,\dots,v_{2\ell};a,b;t;p,q)\\&\quad:=\frac{R^*_{\boldsymbol{\mu}/\mathbf{0}}\big([t^{1/2}x_1^{\pm},\dots,t^{1/2}x_k^{\pm},t^{1/2}v_1,\dots,t^{1/2}v_{2\ell}];t^{k-1/2}a,t^{1/2}b;t;p,q\big)}{\Delta^0_{\boldsymbol{\mu}}(t^{k-1}a/b|t^{k+\ell}v_1\cdots v_{2\ell};t;p,q)}\end{aligned} \tag{3.16}$$

for arbitrary $\boldsymbol{\mu}\in\mathscr{P}^2$, so that

$$R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;-;a,b;t;p,q)=R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;a,b;t;p,q).$$

By (3.13),

$$R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;v_1,\dots,v_{2\ell};a,b;t;p,q)|_{v_{2\ell-1}v_{2\ell}=1/t}=R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;v_1,\dots,v_{2\ell-2};a,b;t;p,q),$$

and, from the definition,

$$\begin{aligned}&R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;v_1,\dots,v_{2\ell};a,b;t;p,q)|_{(v_{2\ell-1},v_{2\ell})=(x_{k+1},x_{k+1}^{-1})}\\&\quad=R^*_{\boldsymbol{\mu}}(x_1,\dots,x_{k+1};v_1,\dots,v_{2\ell-2};a/t,b;t;p,q).\end{aligned}$$

Also, from (3.15) it follows that for $t^kab=pq$ the following generalisation of (3.1) holds:

$$\begin{aligned}&R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;v_1,\dots,v_{2\ell};a,b;t;p,q)\\&\quad=\Delta^0_{\boldsymbol{\mu}}\big(t^{k-1}a/b\big|t^{k-1}ax_1^{\pm},\dots,t^{k-1}ax_k^{\pm},t^{k-1}a/v_1,\dots,t^{k-1}a/v_{2\ell};t,p,q\big).\end{aligned} \tag{3.17}$$

Finally, by (3.12),

$$\begin{aligned}(3.18)\qquad & R^*_{\boldsymbol{\lambda}}(x_1,\dots,x_k;v_1,v_2;at,b;t;p,q)\\ &\quad=\sum_{\boldsymbol{\mu}}\Big\langle{\boldsymbol{\lambda}\atop\boldsymbol{\mu}}\Big\rangle_{[t^ka/b,tv_1v_2](t^ka/v_1,t^ka/v_2,pqa/tbv_1v_2);t;p,q}R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;a/v_1v_2,b;t;p,q).\end{aligned}$$

Recall our convention that parameters are assumed to be in generic position. Then both $R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;a,b;t;p,q)$ and $R^*_{\boldsymbol{\mu}}(x_1,\dots,x_k;v_1,v_2;a,b;t;p,q)$ have sequences of poles in the complex x_i-plane converging to zero at

$$(3.19a)\qquad b^{-1}t^{1-j}q^{\mathbb{N}_0+1}p^{\ell},\quad bt^{j-1}q^{\mathbb{N}_0}p^{-\ell},$$

for $1\leqslant j\leqslant l(\mu^{(1)})$, $1\leqslant\ell\leqslant\mu^{(1)}_i$, and at

$$(3.19b)\qquad b^{-1}t^{1-j}p^{\mathbb{N}_0+1}q^{\ell},\quad bt^{j-1}p^{\mathbb{N}_0}q^{-\ell},$$

for $1\leqslant j\leqslant l(\mu^{(2)})$, $1\leqslant\ell\leqslant\mu^{(2)}_i$. By symmetry, it has diverging sequences of poles in the complex x_i-plane at the reciprocals of the above points.

3.2. **The elliptic interpolation kernel.** We now turn our attention to the elliptic interpolation kernel, which was introduced by the second author in [61]. The interpolation kernel generalises the elliptic interpolation functions and has many remarkable properties, making it a powerful tool for proving results for elliptic hypergeometric functions. For more details the interested reader should consult [45, 61], and for applications of the elliptic interpolation kernel to elliptic hypergeometric integrals and dualities, see e.g., [10, 11, 14, 33, 55, 61].

All the integrals described in this section are of the form $\int f(z)\frac{\mathrm{d}z}{z}$, where $z:=(z_1,\dots,z_k)$, $\frac{\mathrm{d}z}{z}:=\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_k}{z_k}$ and $f(z)$ is BC_k-symmetric. Moreover, the contour of integration is assumed to always have the product structure $C^k=C\times C\times\cdots\times C$, where $C=C^{-1}$ is a positively oriented smooth Jordan curve around 0 such that a given set of points I_C lies in the interior of C. For each of the integrals below we will explicitly describe this set.

For $x,y\in(\mathbb{C}^*)^k$ and $c,t\in\mathbb{C}^*$, the elliptic interpolation kernel $\mathcal{K}_c(x;y;t;p,q)$ may be defined recursively by fixing one of the initial conditions

$$\mathcal{K}_c(-;-;t;p,q)=1\quad\text{or}\quad\mathcal{K}_c(x_1;y_1;t;p,q)=\frac{\Gamma_{p,q}(cx_1^{\pm}y_1^{\pm})}{\Gamma_{p,q}(t,c^2)},$$

and imposing the branching rule

$$\begin{aligned}(3.20)\quad \mathcal{K}_c(x_1,\dots,x_{k+1};y_1,\dots,y_{k+1};t;p,q)&=\frac{\prod_{i=1}^{k+1}\Gamma_{p,q}(cx_i^{\pm}y_{k+1}^{\pm})}{\Gamma^{k+1}_{p,q}(t)\Gamma_{p,q}(c^2)\prod_{1\leqslant i<j\leqslant k+1}\Gamma_{p,q}(tx_i^{\pm}x_j^{\pm})}\\ &\times\int\mathcal{K}_{ct^{-1/2}}(z;y_1,\dots,y_k;t;p,q)\Delta_{\mathrm{D}}\big(z;t^{1/2}x_1^{\pm},\dots,t^{1/2}x_{k+1}^{\pm},pqy_{k+1}^{\pm}/ct^{1/2};p,q\big)\,\frac{\mathrm{d}z}{z},\end{aligned}$$

where $z:=(z_1,\dots,z_k)$ and I_C is the union of the sets

$$\begin{gathered}t^{-1}p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1}C,\quad t^{1/2}x_i^{\pm}p^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant i\leqslant k+1),\\ ct^{-1/2}y_i^{\pm}p^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant i\leqslant k),\quad c^{-1}t^{-1/2}y_{k+1}^{\pm}p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1}.\end{gathered}$$

The condition that $t^{-1}p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1}C$ lies in the interior of C (which can be dropped if $k=1$) requires that $|pq/t|<1$. However, by the symmetry [58, Proposition 3.5]

$$\mathcal{K}_c(x;y;pq/t;p,q)=\Gamma_{p,q}^{2k}(t)\mathcal{K}_c(x;y;t;p,q)\prod_{1\leqslant i<j\leqslant k}\Gamma_{p,q}\big(tx_i^{\pm}x_j^{\pm},ty_i^{\pm}y_j^{\pm}\big),$$

the interpolation kernel may be meromorphically extended to $t\in\mathbb{C}^*$. Additional symmetries of the interpolation kernel, beyond the BC_k-symmetry in both x and y, are

$$\mathcal{K}_c(x;y;t;p,q)=\mathcal{K}_c(y;x;t;p,q)=\mathcal{K}_c(x;y;t;q,p)=\mathcal{K}_{-c}(-x;y;t;p,q).$$

Replacing $(c,x)\mapsto(-c,-x)$ in (3.20) and using $\mathcal{K}_c(x;y;t;p,q)=\mathcal{K}_{-c}(-x;y;t;p,q)$, it follows that the branching rule, and hence the interpolation kernel, is independent of the choice of branch of $t^{1/2}$. It should also be remarked that the symmetry in y is not at all evident from the definition and is a consequence of the same symmetry for the formal interpolation kernel of [61, Section 2].

By specialising one of x,y to $a\langle\boldsymbol{\lambda}\rangle_{k;t;p,q}/c$ for $\boldsymbol{\lambda}\in\mathscr{P}_k^2$ the interpolation kernel reduces to an elliptic interpolation function:

$$\mathcal{K}_c(x;a\langle\boldsymbol{\lambda}\rangle_{k;t;p,q}/c;t;p,q)=R^*_{\boldsymbol{\lambda}}(x;a,b;t;p,q)\prod_{i=1}^k\frac{(pq/ab)^{2\lambda_i^{(1)}\lambda_i^{(2)}}\Gamma_{p,q}(ax_i^{\pm},bx_i^{\pm})}{\Gamma_{p,q}(t^i,t^{i-1}ab)},\tag{3.21}$$

where b on the right is fixed by $c^2=t^{k-1}ab$.[1] The kernel also factors if $c=(pq/t)^{1/2}$ [61, Proposition 2.10]:

$$\mathcal{K}_{(pq/t)^{1/2}}(x;y;t;p,q)=\prod_{i,j=1}^k\Gamma_{p,q}\big((pq/t)^{1/2}x_i^{\pm}y_j^{\pm}\big)=\Delta_{\mathrm{S}}^{(\mathrm{e})}\big(x;y;(pq/t)^{1/2};p,q\big),\tag{3.22}$$

where we recall the definition of $\Delta_{\mathrm{S}}^{(\mathrm{e})}\big(x;y;c;p,q\big)$ given in (1.8) (which does not necessarily assume that the alphabets x and y have the same cardinality).

The key property of the kernel from which our A_n integrals follow is [61, Theorem 2.16].

Theorem 3.1. *Let k,ℓ be nonnegative integers such that $k\leqslant\ell$, and $b,c,d,t\in\mathbb{C}^*$, $x:=(x_1,\dots,x_\ell)\in(\mathbb{C}^*)^\ell$, $y:=(y_1,\dots,y_k)\in(\mathbb{C}^*)^k$ such that $|t|,|pq/t|<1$. Then*

$$\begin{aligned}\int &\mathcal{K}_c(x;z,b,bt,\dots,bt^{\ell-k-1};t;p,q)\mathcal{K}_d(z;y;t;p,q)\Delta_{\mathrm{S}}^{(\mathrm{v})}(z;t^{\ell-k}b,pq/bc^2d^2;t;p,q)\,\frac{\mathrm{d}z}{z}\\ &=\mathcal{K}_{cd}(x;y_1,\dots,y_k,bd,bdt,\dots,bdt^{\ell-k-1};t;p,q)\\ &\quad\times\prod_{i=1}^{\ell-k}\frac{\Gamma_{p,q}(t^{1-i}c^2d^2)}{\Gamma_{p,q}(t^{1-i}c^2)}\prod_{i=1}^{\ell}\frac{\Gamma_{p,q}(bcx_i^{\pm})}{\Gamma_{p,q}(bcd^2x_i^{\pm})}\prod_{i=1}^k\frac{\Gamma_{p,q}(t^{\ell-k}bdy_i^{\pm})}{\Gamma_{p,q}(bc^2dy_i^{\pm})},\end{aligned}$$

where $z:=(z_1,\dots,z_k)$ and I_C is the union of the sets

$$tp^{\mathbb{N}_0}q^{\mathbb{N}_0}C,\quad t^{-1}p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1}C,\quad t^{\ell-k}bp^{\mathbb{N}_0}q^{\mathbb{N}_0},\quad (bc^2d^2)^{-1}p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1},$$
$$cx_i^{\pm}p^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant i\leqslant\ell),\quad dy_i^{\pm}p^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant i\leqslant k).$$

Specialising $c=(pq/t)^{1/2}$ we can use (3.22) and (1.2) to obtain the following corollary.

[1] Note that this expression is independent of the choice of branch for c.

Corollary 3.2. *Let k,ℓ be nonnegative integers such that $k\leqslant \ell$, and $b,d,t\in\mathbb{C}^*$, $x:=(x_1,\dots,x_\ell)\in(\mathbb{C}^*)^\ell$, $y:=(y_1,\dots,y_k)\in(\mathbb{C}^*)^k$ such that $|t|,|pq/t|<1$. Fix $c:=(pq/t)^{1/2}$. Then*

$$\begin{aligned}\int \mathcal{K}_d(z;y;t;p,q)\Delta_{\mathrm{S}}^{(\mathrm{v})}(z;t^{\ell-k}b,t/bd^2;t;p,q)\Delta_{\mathrm{S}}^{(\mathrm{e})}\big(z;x;c;t;p,q\big)\,\frac{\mathrm{d}z}{z}\\ =\mathcal{K}_{cd}\big(x;y_1,\dots,y_k,bd,bdt,\dots,bdt^{\ell-k-1};t;p,q\big)\\ \times\prod_{i=1}^{\ell-k}\frac{\Gamma_{p,q}(t^i)}{\Gamma_{p,q}(t^i/d^2)}\cdot\frac{\prod_{i=1}^k\Gamma_{p,q}\big(t^{\ell-k}bdy_i^{\pm},ty_i^{\pm}/bd\big)}{\prod_{i=1}^{\ell}\Gamma_{p,q}(bcd^2x_i^{\pm},ct^{k-\ell+1}x_i^{\pm}/b)},\end{aligned}$$

where $z:=(z_1,\dots,z_k)$ and I_C is as in Theorem 3.1 with c specialised accordingly.

If we further fix $\ell=k$, specialise $y=a\langle\boldsymbol{\mu}\rangle_{k;t;p,q}/d$ and make the substitution

$$(a,b,d^2)\mapsto(t_1,t_3,t^{k-1}t_1t_2),$$

we obtain the elliptic beta integral

$$\begin{aligned}\int R^*_{\boldsymbol{\mu}}(z;t_1,t_2;t;p,q)\Delta_{\mathrm{S}}^{(\mathrm{v})}\big(z;t_1,t_2,t_3,t_4,cx_1^{\pm},\dots,cx_k^{\pm};t,p,q\big)\,\frac{\mathrm{d}z}{z}\\ =R^*_{\boldsymbol{\mu}}(x;ct_1,ct_2;t;p,q)\Delta^0_{\boldsymbol{\mu}}\big(t^{k-1}t_1/t_2|t^{k-1}t_1t_3,t^{k-1}t_1t_4\big)\\ \times\prod_{i=1}^k\bigg(\prod_{1\leqslant r<s\leqslant 4}\Gamma_{p,q}(t^{i-1}t_rt_s)\prod_{r=1}^4\Gamma_{p,q}(ct_rx_i^{\pm})\bigg),\end{aligned}\tag{3.23}$$

where $z:=(z_1,\dots,z_k)$, $t,t_1,t_2,t_3,t_4,x_1,\dots,x_k\in\mathbb{C}^*$ such that $|t|<1$ and $t^{k-2}t_1t_2t_3t_4=1$. As before, $c:=(pq/t)^{1/2}$, and I_C is the union of the sets

$$t_rp^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant r\leqslant 4),\quad tp^{\mathbb{N}_0}q^{\mathbb{N}_0}C,\quad cx_r^{\pm}p^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant r\leqslant k),$$

and the sets (3.19a) and (3.19b) with $b\mapsto t_2$. For $\boldsymbol{\mu}=\mathbf{0}$ this is [12, Theorem 3.1, $m=0$] due to van der Bult, see also [60, 73].

A final result for the interpolation kernel that is needed is [61, Corollary 3.25].

Proposition 3.3. *Let $\boldsymbol{\mu}\in\mathscr{P}_k^2$, $x=(x_1,\dots,x_k)\in(\mathbb{C}^*)^k$ and $c,t,t_1,t_2,t_3,t_4,t_5\in\mathbb{C}^*$ such that $|t|,|pq/t|<1$ and*

$$c^2t^{k-1}t_2t_3t_4t_5=pq.$$

Then

$$\begin{aligned}\int \mathcal{K}_c(x;z;t;p,q)R^*_{\boldsymbol{\mu}}(z;t_1,t_2;t;p,q)\Delta_{\mathrm{S}}^{(\mathrm{v})}(z;t_2,t_3,t_4,t_5;t,p,q)\,\frac{\mathrm{d}z}{z}\\ =\prod_{i=1}^k\bigg(\prod_{2\leqslant r<s\leqslant 5}\Gamma(t^{i-1}t_rt_s)\prod_{r=2}^5\Gamma\big(ct_rx_i^{\pm}\big)\bigg)\\ \times\sum_{\boldsymbol{\nu}}\bigg\langle{\boldsymbol{\mu}\atop\boldsymbol{\nu}}\bigg\rangle_{[t^{k-1}t_1/t_2,c^2](t^{k-1}t_1t_3,t^{k-1}t_1t_4,t^{k-1}t_1t_5);t;p,q}R^*_{\boldsymbol{\nu}}(x;t_1/c,ct_2;t,p,q),\end{aligned}\tag{3.24}$$

where $z:=(z_1,\dots,z_k)$ and I_C is the union of

$$tp^{\mathbb{N}_0}q^{\mathbb{N}_0}C,\quad t^{-1}p^{\mathbb{N}_0+1}q^{\mathbb{N}_0+1}C,\quad t_rp^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (2\leqslant r\leqslant 5),\quad cx_i^{\pm}p^{\mathbb{N}_0}q^{\mathbb{N}_0}\ (1\leqslant i\leqslant k)$$

and the sets (3.19a) *and* (3.19b) *with $b\mapsto t_2$.*

We will use this to prove the following key result.

Theorem 3.4. *Let* $\boldsymbol{\mu} \in \mathscr{P}^2$, $x = (x_1, \dots, x_k) \in (\mathbb{C}^*)^k$ *and* $c, t, t_1, t_2, t_3, t_4, v_1, v_2 \in \mathbb{C}^*$ *such that*

$$t_4 = tv_1 \quad \text{and} \quad c^2 t^{k-1} t_1 t_2 t_3 t_4 = pq.$$

Then

$$\begin{aligned}
&\int \mathcal{K}_c(x; z; t; p, q) R^*_{\boldsymbol{\mu}}(z; v_1, v_2; tt_1v_1v_2, t_2; t; p, q) \Delta_{\mathrm{S}}^{(\mathrm{v})}\big(z; t_1, t_2, t_3, t_4; t; p, q\big) \,\frac{\mathrm{d}z}{z} \\
&= \prod_{i=1}^k \bigg(\prod_{1 \leqslant r < s \leqslant 4} \Gamma(t^{i-1} t_r t_s) \prod_{r=1}^4 \Gamma\big(ct_r x_i^{\pm}\big) \bigg) \\
&\quad \times \frac{\Delta^0_{\boldsymbol{\mu}}(t^k t_1 v_1 v_2 / t_2 | t^k t_1 v_1; t; p, q)}{\Delta^0_{\boldsymbol{\mu}}(t^k t_1 v_1 v_2 / t_2 | c^2 t^k t_1 v_1; t; p, q)}\, R^*_{\boldsymbol{\mu}}(x; cv_1, v_2/c; ctt_1v_1v_2, ct_2; t; p, q),
\end{aligned}$$

where $z := (z_1, \dots, z_k)$ *and* I_C *is as in Proposition 3.3 with* $t_5 \mapsto t_1$.

Proof of Theorem 3.4. In the following we write $\Gamma(x)$ instead of $\Gamma_{p,q}(x)$, $\mathcal{K}_c(x; z)$ instead of $\mathcal{K}_c(x; z; t; p, q)$, and so on.

Setting $t_5 = t_1$ in (3.24), which implies the balancing condition $c^2 t^{k-1} t_1 t_2 t_3 t_4 = pq$, gives

$$\begin{aligned}
&\int \mathcal{K}_c(x; z) R^*_{\boldsymbol{\mu}}(z; t_1, t_2) \Delta_{\mathrm{S}}^{(\mathrm{v})}(z; t_1, t_2, t_3, t_4) \,\frac{\mathrm{d}z}{z} \\
&= \prod_{i=1}^k \bigg(\prod_{1 \leqslant r < s \leqslant 4} \Gamma(t^{i-1} t_r t_s) \prod_{r=1}^4 \Gamma\big(ct_r x_i^{\pm}\big) \bigg) \\
&\quad \times \Delta^0_{\boldsymbol{\mu}}(t^{k-1} t_1/t_2 | t^{k-1} t_1 t_3, t^{k-1} t_1 t_4) \sum_{\boldsymbol{\nu}} \left\langle \begin{matrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{matrix} \right\rangle_{[t^{k-1} t_1/t_2, c^2](t^{k-1} t_1^2)} R^*_{\boldsymbol{\nu}}(x; t_1/c, ct_2) \\
&= \prod_{i=1}^k \bigg(\prod_{1 \leqslant r < s \leqslant 4} \Gamma(t^{i-1} t_r t_s) \prod_{r=1}^4 \Gamma\big(ct_r x_i^{\pm}\big) \bigg) \\
&\quad \times \Delta^0_{\boldsymbol{\mu}}(t^{k-1} t_1/t_2 | t^{k-1} t_1 t_3, t^{k-1} t_1 t_4) R^*_{\boldsymbol{\mu}}(x; ct_1, ct_2),
\end{aligned}$$

where the second equality follows from the connection coefficient formula (3.9). Multiplying both sides by

$$\left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[t^k t_1 v_1 v_2/t_2, tv_1 v_2](t^k t_1 v_1, t^k t_1 v_2, pqt_1/tt_2)},$$

where $\boldsymbol{\lambda} \in \mathscr{P}^2$, and then summing over $\boldsymbol{\mu}$, yields

$$\begin{aligned}
&\int \mathcal{K}_c(x; z) R^*_{\boldsymbol{\lambda}}(z; v_1, v_2; tt_1v_1v_2, t_2) \Delta_{\mathrm{S}}^{(\mathrm{v})}(z; t_1, t_2, t_3, t_4) \,\frac{\mathrm{d}z}{z} \\
&= \prod_{i=1}^k \bigg(\prod_{1 \leqslant r < s \leqslant 4} \Gamma(t^{i-1} t_r t_s) \prod_{r=1}^4 \Gamma\big(ct_r x_i^{\pm}\big) \bigg) \Delta^0_{\boldsymbol{\lambda}}(t^k t_1 v_1 v_2/t_2 | t^k t_1 t_3 v_1 v_2, t^k t_1 t_4 v_1 v_2) \\
&\quad \times \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[t^k t_1 v_1 v_2/t_2, tv_1 v_2](t^k t_1 v_1, t^k t_1 v_2, pqt_1/tt_2, pq/t_2 t_3, pq/t_2 t_4)} R^*_{\boldsymbol{\mu}}(x; ct_1, ct_2).
\end{aligned}$$

Here the sum over $\boldsymbol{\mu}$ on the left has been carried out by (3.18) with $(a,b)\mapsto(t_1v_1v_2,t_2)$. We now also assume that $t_4=tv_1$. Then the sum on the right may be simplified by (3.5) with $w\mapsto t^kt_1v_1$ to

$$\frac{\Delta^0_{\boldsymbol{\lambda}}(t^kt_1v_1v_2/t_2|t^kt_1v_1)}{\Delta^0_{\boldsymbol{\lambda}}(t^kt_1v_1v_2/t_2|t^{k+1}t_1v_1^2v_2)}\sum_{\boldsymbol{\mu}}\left\langle\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[t^kt_1v_1v_2/t_2,tv_1v_2](t^kc^2t_1v_1,t^kt_1v_2,pqt_1/tt_2)}R^*_{\boldsymbol{\mu}}(x;ct_1,ct_2)$$

$$=\frac{\Delta^0_{\boldsymbol{\lambda}}(t^kt_1v_1v_2/t_2|t^kt_1v_1)}{\Delta^0_{\boldsymbol{\lambda}}(t^kt_1v_1v_2/t_2|t^{k+1}t_1v_1^2v_2)}R^*_{\boldsymbol{\lambda}}(x;tcv_1,v_2/c;ctt_1v_1v_2,ct_2),$$

where the second equality follows from another application of (3.18), now with

$$(a,b,v_1,v_2)\mapsto(ct_1v_1v_2,ct_2,cv_1,v_2/c).$$

As a result,

$$\begin{aligned}&\int\mathcal{K}_c(x;z)R^*_{\boldsymbol{\lambda}}(z;v_1,v_2;tt_1v_1v_2,t_2)\Delta^{(\mathrm{v})}_{\mathrm{S}}(z;t_1,t_2,t_3,t_4)\,\frac{\mathrm{d}z}{z}\\&\quad=\prod_{i=1}^k\biggl(\Gamma\bigl(ctv_1x_i^{\pm}\bigr)\prod_{1\leqslant r<s\leqslant 3}\Gamma(t^{i-1}t_rt_s)\prod_{r=1}^3\Gamma\bigl(t^it_rv_1,ct_rx_i^{\pm}\bigr)\biggr)\\&\qquad\times\Delta^0_{\boldsymbol{\lambda}}(t^kt_1v_1v_2/t_2|t^kt_1v_1,t^kt_1t_3v_1v_2)R^*_{\boldsymbol{\lambda}}(x;cv_1,v_2/c;ctt_1v_1v_2,ct_2).\end{aligned}$$

Replacing $\boldsymbol{\lambda}$ by $\boldsymbol{\mu}$ and applying the reflection equation (2.4) completes the proof. ☐

4. Proof and generalisations of Theorem 1.1

The goal of this section is to prove the A_n elliptic Selberg integral of Theorem 1.1. As mentioned in the introduction, we will in fact prove an AFLT-type generalisation of the theorem in which the integrand is multiplied by an appropriate product of BC_n-symmetric functions.

Throughout this section we suppress dependence on p,q,t.

4.1. **An A_n elliptic AFLT integral.** Before stating our main theorem we discuss the original AFLT integral of Alba, Fateev, Litvinov and Tarnopolsky [2] and some of its special cases due to Kadell [37] and Hua and Kadell [32, 36]. For convenience these results will be expressed in terms of Selberg-type averages, and for $f\in\mathbb{C}[x_1,\dots,x_k]^{\mathfrak{S}_k}=:\Lambda_k$, we define

$$\bigl\langle f\bigr\rangle^k_{\alpha,\beta;\gamma}:=\frac{1}{S_k(\alpha,\beta;\gamma)}\int\limits_{[0,1]^k}f(x_1,\dots,x_k)\prod_{i=1}^k x_i^{\alpha-1}(1-x_i)^{\beta-1}\prod_{1\leqslant i<j\leqslant k}|x_i-x_j|^{2\gamma}\,\mathrm{d}x_1\cdots\mathrm{d}x_k,$$

where $S_k(\alpha,\beta;\gamma)$ is the Selberg integral (1.1).

For $\gamma\in\mathbb{C}^*$, let $P^{(1/\gamma)}_\lambda(x_1,\dots,x_k)$ be the Jack polynomial indexed by the partition λ, see [46, 77]. Also define the normalised Jack polynomial

$$\tilde{P}^{(1/\gamma)}_\lambda(x_1,\dots,x_k):=\frac{P^{(1/\gamma)}_\lambda(x_1,\dots,x_k)}{P^{(1/\gamma)}_\lambda(1,\dots,1)}.$$

Then Kadell's generalised Selberg integral is [37]

$$\bigl\langle\tilde{P}^{(1/\gamma)}_\lambda\bigr\rangle^k_{\alpha,\beta;\gamma}=\prod_{i\geqslant 1}\frac{(\alpha+(k-i)\gamma)_{\lambda_i}}{(\alpha+\beta+(2k-i-1)\gamma)_{\lambda_i}},\tag{4.1}$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ is the ordinary shifted factorial. In the case $\beta=\gamma$, Kadell further generalised this to a product of two Jack polynomials as [36]

$$\big\langle \tilde{P}_\lambda^{(1/\gamma)}\,\tilde{P}_\mu^{(1/\gamma)}\big\rangle_{\alpha,\gamma;\gamma}^k = \prod_{i,j=1}^k \frac{(\alpha+(2k-i-j)\gamma)_{\lambda_i+\mu_j}}{(\alpha+(2k-i-j+1)\gamma)_{\lambda_i+\mu_j}}. \tag{4.2}$$

Since in the Schur case, $\gamma=1$, this integral was previously discovered by Hua [32], this last result is commonly referred to as the Hua–Kadell integral.

To describe the AFLT integral, which unifies (4.1) and (4.2), we need some basic plethystic notation, see e.g., [3, 31, 44]. Let Λ be the ring of symmetric functions in infinitely (but countably) many variables over $\mathbb{C}$. Then the power sum symmetric functions are defined as $p_0 := 1$ and

$$p_r = x_1^r + x_2^r + \cdots,$$

for $r \geqslant 1$. Since $\Lambda = \mathbb{C}[p_1,p_2,\dots]$, any $f\in\Lambda$ admits an expansion of the form $f=\sum_\lambda c_\lambda p_\lambda$, where $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. Then for any $\xi\in\mathbb{C}$ and any alphabet x (infinite or finite), the expression $f[x+\xi]$ is defined as

$$f[x+\xi] := \sum_\lambda c_\lambda \prod_{i=1}^{l(\lambda)} \big(p_{\lambda_i}(x)+\xi\big). \tag{4.3}$$

Clearly, if $x=(x_1,\dots,x_k)$ then $f[x+\xi]\in\Lambda_k$. Moreover, $f[x]=f(x)$ and (4.3) unambiguously defines

$$f[k] = f(\underbrace{1,\dots,1}_{k \text{ times}}).$$

Indeed, setting $x = -$ (the empty alphabet) and $\xi = k$ for $k\in\mathbb{N}_0$ gives the same result as setting $x=(1,\dots,1)$ (k ones) and $\xi=0$.

For $x=(x_1,\dots,x_k)$, let $\tilde{P}_\lambda[x+\xi] = P_\lambda[x+\xi]/P_\lambda[k+\xi]$. Then the AFLT integral [2, Appendix A] may be stated as

$$\begin{aligned}&\big\langle \tilde{P}_\lambda^{(1/\gamma)}[x]\tilde{P}_\mu^{(1/\gamma)}[x+\beta/\gamma-1]\big\rangle_{\alpha,\beta;\gamma}^k \\ &\quad= \prod_{i=1}^k \frac{(\alpha+(k-i)\gamma)_{\lambda_i}}{(\alpha+\beta+(2k-m-i-1)\gamma)_{\lambda_i}} \prod_{i=1}^k\prod_{j=1}^m \frac{(\alpha+\beta+(2k-i-j-1)\gamma)_{\lambda_i+\mu_j}}{(\alpha+\beta+(2k-i-j)\gamma)_{\lambda_i+\mu_j}},\end{aligned} \tag{4.4}$$

where $\lambda\in\mathscr{P}_k$, $\mu\in\mathscr{P}$ and m is any integer such that $m\geqslant l(\mu)$. The Kadell and Hua–Kadell integrals correspond to $\mu=0$ and $\beta=\gamma$ respectively. As shown by Alba et al. [2], the AFLT integral is important in conformal field theory, particularly in the verification of the AGT conjecture for SU(2), see [1]. For further work on Selberg-type integrals and the AGT conjecture the reader is referred to [13, 23, 35, 48, 49, 50, 51, 85, 86].

In our previous paper [3] we gave generalisations of the AFLT integral to the elliptic level and to (non-elliptic) A_n. Our next theorem unifies these results by providing an elliptic A_n AFLT integral. In the following we assume all the conditions of Theorem 1.1 to hold, including the fixing of a branch of $(pq/t)^{1/2}$. For brevity we also suppress the dependence on p,q and t in most of our functions, such as $\Delta_{\mathrm{S}}(\dots;t;p,q)$, $R^*_{\boldsymbol{\lambda}}(\dots;t;p,q)$ and $\Gamma_{p,q}(z)$.

For $f:(\mathbb{C}^*)^{k_1}\times\cdots\times(\mathbb{C}^*)^{k_n}\longrightarrow\mathbb{C}$ a function which is BC_{k_r}-symmetric in the rth set of variables, we define the elliptic A_n Selberg average as

$$\langle f\rangle^{k_1,\dots,k_n}_{t_1,\dots,t_{2n+4}}:=\frac{1}{S^{\mathrm{A}_n}_{k_1,\dots,k_n}(t_1,\dots,t_{2n+4})}\\ \times\int\limits_C f\big(z^{(1)},\dots,z^{(n)}\big)\Delta_{\mathrm{S}}\big(z^{(1)},\dots,z^{(n)};t_1,\dots,t_{2n+4};(pq/t)^{1/2}\big)\frac{\mathrm{d}z^{(1)}}{z^{(1)}}\cdots\frac{\mathrm{d}z^{(n)}}{z^{(n)}},$$

where $S^{\mathrm{A}_n}_{k_1,\dots,k_n}(t_1,\dots,t_{2n+4})$ denotes the elliptic A_n Selberg integral (1.11). In addition to the conditions (1.12a) and (1.12b), the contour $C=C_1^{k_1}\times\cdots\times C_n^{k_n}$ (where as before C_r is a positively oriented smooth Jordan curve around 0 such that $C_r=C_r^{-1}$) should be such that any sequence of poles of f in $z_i^{(r)}$ tending to zero lies in the interior of C_r, excluding those which are cancelled by the univariate part of $\Delta_{\mathrm{S}}(z^{(r)};t_1,\dots,t_{2n+4})$.

Theorem 4.1 (Elliptic A_n AFLT integral)**.** *Assume the conditions of Theorem 1.1 and let $\tau_n:=t_{2n+1}t_{2n+2}t_{2n+3}/t^2$. Then*

$$\Big\langle R^*_{\boldsymbol{\lambda}}\big(z^{(1)};c^{1-n}t_1,c^{1-n}t_2\big)R^*_{\boldsymbol{\mu}}\big(z^{(n)};t_{2n+2}/t,t_{2n+3}/t;t\tau_n,t_{2n+4}\big)\Big\rangle^{k_1,\dots,k_n}_{t_1,\dots,t_{2n+4}}\\ =\prod_{r=3}^{2n}\Delta^0_{\boldsymbol{\lambda}}(t^{k_1-1}t_1/t_2\,|\,t^{k_1}t_1/t_r)\prod_{r=2n+1}^{2n+4}\Delta^0_{\boldsymbol{\lambda}}(t^{k_1-1}t_1/t_2\,|\,t^{k_1-1}t_1t_r)\\ \times\prod_{r=2n+2}^{2n+3}\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n-1}t_{2n+1}t_r)\prod_{r=2}^{n}\frac{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n}t_{2r-1}\tau_n)}{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n+k_r-k_{r-1}}t_{2r-1}\tau_n)}\\ \times\frac{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n}t_1\tau_n\langle\boldsymbol{\lambda}\rangle_{k_1;t;p,q})}{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n+1}t_1\tau_n\langle\boldsymbol{\lambda}\rangle_{k_1;t;p,q})},\tag{4.5}$$

where $\boldsymbol{\lambda}\in\mathscr{P}^2_{k_1}$ and $\boldsymbol{\mu}\in\mathscr{P}^2$.

For $n=1$ the theorem reduces to the A_1 elliptic AFLT integral [3, Theorem 1.4]. In that paper we applied the symmetry-breaking trick introduced in [58] to obtain the following AFLT integral for Macdonald polynomials [3, Corollary 1.5]:

$$\frac{1}{k!(2\pi\mathrm{i})^k}\int\limits_{\mathbb{T}^k}P_\lambda(z;q,t)P_\mu\bigg(\bigg[z+\frac{t/c-b}{1-t}\bigg];q,t\bigg)\\ \times\prod_{i=1}^k\frac{(a/z_i,qz_i/a;q)_\infty}{(b/z_i,cz_i;q)_\infty}\prod_{1\leqslant i<j\leqslant k}\frac{(z_i/z_j,z_j/z_i;q)_\infty}{(tz_i/z_j,tz_j/z_i;q)_\infty}\,\frac{\mathrm{d}z}{z}\\ =b^{|\lambda|}(t/c)^{|\mu|}P_\lambda\bigg(\bigg[\frac{1-t^k}{1-t}\bigg];q,t\bigg)P_\mu\bigg(\bigg[\frac{1-bct^{k-1}}{1-t}\bigg];q,t\bigg)\\ \times\prod_{i=1}^k\frac{(t,act^{k-l(\mu)-i}q^{\lambda_i},at^{1-i}/b,qt^{i-1}b/a;q)_\infty}{(q,t^i,bct^{i-1},at^{1-i}q^{\lambda_i}/b;q)_\infty}\prod_{i=1}^k\prod_{j=1}^{l(\mu)}\frac{(act^{k-i-j+1}q^{\lambda_i+\mu_j};q)_\infty}{(act^{k-i-j}q^{\lambda_i+\mu_j};q)_\infty}.$$

Here $\lambda \in \mathscr{P}_k$, $\mu \in \mathscr{P}$ and $a, b, c \in \mathbb{C}^*$ such that $|b|, |c| < 1$.[2] Thus far, we have not been able to replicate this procedure for the full A_n elliptic AFLT integral, nor for the A_n elliptic Selberg integral of Theorem 1.1. However, one can show that under the natural generalisation of the limiting procedure of our previous paper the evaluation of either the elliptic A_n Selberg integral or elliptic AFLT integral reduce to q-analogues of their ordinary counterparts. To be more specific, assume that $0 < p, q < 1$ and scale the parameters $t_1, \dots, t_{2n+4}$ by

$$(t_{2r-1}, t_{2r}) \mapsto (t_{2r-1}, p^{1/2} t_{2r}),$$

for $1 \leqslant r \leqslant n$ and

$$\big(t_{2n+1}, t_{2n+2}, t_{2n+3}, t_{2n+4}\big) \mapsto \big(t_{2n+1}, p^{-1/4} t_{2n+2}, p^{1/4} t_{2n+3}, p^{1/2} t_{2n+4}\big).$$

Then the $p \to 0$ limit of the right-hand side of (1.11) exists and may be expressed as a product of q-shifted factorials. Now let $\alpha_1, \dots, \alpha_n, \beta, \gamma$ be as in the A_n Selberg integral (1.7). By setting $t = q^\gamma$, $t_{2n+2} t_{2n+3} = q^\beta$ and $t_{2r-1} t_{2n+1} = q^{\alpha_r + \cdots + \alpha_n + (r-n)\gamma}$ for $1 \leqslant r \leqslant n$, so that by the balancing conditions (1.10) we have $t_{2r} t_{2n+4} = q^{1-\beta-\alpha_r-\cdots-\alpha_n-(k_r-k_{r-1}+k_n+r-n-2)\gamma}$, one obtains a q-analogue of the A_n Selberg integral evaluation, up to factor induced by the q-reflection formula for the q-gamma function. Taking the $q \to 1$ limit of this expression then produces the A_n Selberg integral evaluation up to a scalar. The same procedure works for (4.5), but one additionally needs the limit of the elliptic interpolation functions [3, Equations (6.7)].

Setting $\boldsymbol{\mu} = 0$ in Theorem 4.1 leads to the following generalisation of the Kadell integral.

Corollary 4.2 (Elliptic A_n Kadell integral)**.** *With the same conditions as Theorem 1.1 and for $\boldsymbol{\lambda} \in \mathscr{P}_{k_1}^2$,*

$$\begin{aligned}
&\Big\langle R^*_{\boldsymbol{\lambda}}\big(z^{(1)}; c^{1-n} t_1, c^{1-n} t_2\big) \Big\rangle^{k_1,\dots,k_n}_{t_1,\dots,t_{2n+4}} \\
&\quad = \prod_{r=3}^{2n} \Delta^0_{\boldsymbol{\lambda}}(t^{k_1-1} t_1/t_2 | t^{k_1} t_1/t_r) \prod_{r=2n+1}^{2n+4} \Delta^0_{\boldsymbol{\lambda}}(t^{k_1-1} t_1/t_2 | t^{k_1-1} t_1 t_r).
\end{aligned}$$

Similarly, imposing the constraint $t_{2n+2} t_{2n+3} = t$ and using (3.13) results in a generalisation of the Hua–Kadell integral.

Corollary 4.3 (Elliptic A_n Hua–Kadell integral)**.** *Assume the same conditions as in Theorem 1.1 with the additional constraint $t_{2n+2} t_{2n+3} = t$ Then, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{k_1}^2$,*

$$\begin{aligned}
&\Big\langle R^*_{\boldsymbol{\lambda}}\big(z^{(1)}; c^{1-n} t_1, c^{1-n} t_2\big) R^*_{\boldsymbol{\mu}}\big(z^{(n)}; t_{2n+1}, t_{2n+4}\big) \Big\rangle^{k_1,\dots,k_n}_{t_1,\dots,t_{2n+4}} \\
&\quad = \prod_{r=3}^{2n} \Delta^0_{\boldsymbol{\lambda}}(t^{k_1-1} t_1/t_2 | t^{k_1} t_1/t_r) \prod_{r=2n+1}^{2n+4} \Delta^0_{\boldsymbol{\lambda}}(t^{k_1-1} t_1/t_2 | t^{k_1-1} t_1 t_r) \\
&\qquad \times \prod_{r=2n+2}^{2n+3} \Delta^0_{\boldsymbol{\mu}}(t^{k_n-1} t_{n+1}/t_{2n+4} | t^{k_n-1} t_{2n+1} t_r) \\
&\qquad \times \prod_{r=2}^{n} \frac{\Delta^0_{\boldsymbol{\mu}}(t^{k_n-1} t_{2n+1}/t_{2n+4} | t^{k_n-1} t_{2r-1} t_{2n+1})}{\Delta^0_{\boldsymbol{\mu}}(t^{k_n-1} t_{2n+1}/t_{2n+4} | t^{k_n+k_r-k_{r-1}-1} t_{2r-1} t_{2n+1})}
\end{aligned}$$

[2] In [3, Corollary 1.5] this was inadvertently stated with $c = 1$, which would require a small indentation of the contour $\mathbb{T}$ at 1.

$$\times \frac{\Delta^0_{\boldsymbol{\mu}}(t^{k_n-1}t_{2n+1}/t_{2n+4}|t^{k_n-1}t_1t_{2n+1}\langle\boldsymbol{\lambda}\rangle_{k_1;t;p,q})}{\Delta^0_{\boldsymbol{\mu}}(t^{k_n-1}t_{2n+1}/t_{2n+4}|t^{k_n}t_1t_{2n+1}\langle\boldsymbol{\lambda}\rangle_{k_1;t;p,q})}.$$

4.2. **Proof of Theorems 1.1 and 4.1.** Let $0 \leqslant k_1 \leqslant k_2 \leqslant \dots \leqslant k_n$, $c := (pq/t)^{1/2}$ (with a branch of c fixed) and let $t_1, \dots, t_{2n+4}$ satisfy the balancing conditions (1.10), i.e.,

$$t^{k_1+k_n-2}t_1t_2t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4} = pq$$

and

$$t^{k_r-k_{r-1}+k_n-2}t_{2r-1}t_{2r}t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4} = pq$$

for $2 \leqslant r \leqslant n$. The reason for restating these conditions as per the above, separating out the $r=1$ case, is that in what follows we will introduce an integer k_0 which, unlike in Theorem 1.1, will not be 0.

The task is to evaluate the integral

$$\begin{aligned}(4.6)\qquad & S^{k_1,\dots,k_n}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t_1,\dots,t_{2n+4}) \\ &:= \int \Big(R^*_{\boldsymbol{\lambda}}\big(z^{(1)};c^{1-n}t_1,c^{1-n}t_2\big)R^*_{\boldsymbol{\mu}}\big(z^{(n)};t_{2n+2}/t,t_{2n+3}/t;t\tau_n,t_{2n+4}\big) \\ &\qquad \times \Delta_{\mathrm{S}}\big(z^{(1)},\dots,z^{(n)};t_1,\dots,t_{2n+4};c\big)\Big)\frac{\mathrm{d}z^{(1)}}{z^{(1)}}\cdots\frac{\mathrm{d}z^{(n)}}{z^{(n)}},\end{aligned}$$

where $\tau_n := t_{2n+1}t_{2n+2}t_{2n+3}/t^2$. To this end we consider the more general problem of evaluating

$$\begin{aligned}& S^{k_0,k_1,\dots,k_n}_{\boldsymbol{\mu}}(x;t_1,\dots,t_{2n+4}) \\ &:= \int \Bigg(\mathcal{K}_d\big(z^{(1)};x\big)R^*_{\boldsymbol{\mu}}\big(z^{(n)};t_{2n+2}/t,t_{2n+3}/t;t\tau_n,t_{2n+4}\big) \\ &\qquad \times \frac{\Delta_{\mathrm{S}}\big(z^{(1)},\dots,z^{(n)};t_1,\dots,t_{2n+4};c\big)}{\prod_{i=1}^{k_1}\prod_{r=1}^{2}\Gamma\big(c^{1-n}t_r(z^{(1)}_i)^{\pm}\big)}\Bigg)\frac{\mathrm{d}z^{(1)}}{z^{(1)}}\cdots\frac{\mathrm{d}z^{(n)}}{z^{(n)}}.\end{aligned}$$

Here $k_0,k_1,\dots,k_n$ are integers such that $0 \leqslant k_1 \leqslant \dots \leqslant k_n$, $x := (x_1,\dots,x_{k_1})$,

$$d^2 := c^{2-2n}t^{k_1-k_0-1}t_1t_2 \tag{4.7}$$

and the $t_1,\dots,t_{2n+4}$ satisfy the modified balancing conditions

$$t^{k_r-k_{r-1}+k_n-2}t_{2r-1}t_{2r}t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4} = pq \tag{4.8}$$

for all $1 \leqslant r \leqslant n$. By (3.21) and (1.2),

$$\begin{aligned}(4.9)\qquad S^{k_1,\dots,k_n}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t_1,\dots,t_{2n+4}) &= \prod_{i=1}^{k_1}(c^{2n-2}pq/t_1t_2)^{-2\lambda^{(1)}_i\lambda^{(2)}_i}\Gamma(t^i,c^{2-2n}t^{i-1}t_1t_2) \\ &\quad \times S^{0,k_1,\dots,k_n}_{\boldsymbol{\mu}}\big(c^{1-n}t_1\langle\boldsymbol{\lambda}\rangle_{k_1}/d;t_1,\dots,t_{2n+4}\big),\end{aligned}$$

where d on the right is given by (4.7) with $k_0 = 0$.

Proposition 4.4. *With the parameters satisfying the conditions* (4.7) *and* (4.8)*,*

$$S^{k_0,k_1,\dots,k_n}_{\boldsymbol{\mu}}(x;t_1,\dots,t_{2n+4}) \tag{4.10}$$

$$
\begin{aligned}
&= \prod_{i=1}^{k_1} \biggl(\Delta^0_{\boldsymbol{\mu}}\bigl(t^{k_n}\tau_n/t_{2n+4}\big|t^{k_n}c^{n-1}d\tau_n x_i^{\pm}\bigr) \\
&\qquad\quad \times \prod_{r=3}^{2n} \Gamma\bigl(c^{n-1}dtx_i^{\pm}/t_r\bigr) \prod_{r=2n+1}^{2n+4} \Gamma\bigl(c^{n-1}dt_r x_i^{\pm}\bigr)\biggr) \\
&\quad \times \prod_{r=2}^{n} \prod_{i=1}^{k_r-k_{r-1}} \Gamma(t^i, t^{i-1}c^{2r-2n}t_{2r-1}t_{2r}) \prod_{2n+1\leqslant r<s\leqslant 2n+4} \prod_{i=1}^{k_n} \Gamma(t^{i-1}t_r t_s) \\
&\quad \times \prod_{2\leqslant r<s\leqslant n} \prod_{i=1}^{k_r-k_{r-1}} \Gamma(t^i t_{2r-1}/t_{2s-1}, t^i t_{2r}/t_{2s-1}, t^i t_{2r-1}/t_{2s}, t^i t_{2r}/t_{2s}) \\
&\quad \times \prod_{r=2}^{n} \prod_{s=2n+1}^{2n+4} \prod_{i=1}^{k_r-k_{r-1}} \Gamma(t^{i-1}t_{2r-1}t_s, t^{i-1}t_{2r}t_s) \\
&\quad \times \prod_{r=2n+2}^{2n+3} \Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n-1}t_{2n+1}t_r) \\
&\quad \times \prod_{r=2}^{n} \frac{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n}t_{2r-1}\tau_n)}{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n+k_r-k_{r-1}}t_{2r-1}\tau_n)}.
\end{aligned}
$$

It is readily checked using (4.9) that by this implies Theorems 1.1 and 4.1. In particular, from (4.7) and the $r=1$ case of (4.8), $pq = t^{k_n-1}c^{2n-2}d^2\tau_n t_{2n+4}$. Combined with (2.4) this yields

$$
\begin{aligned}
&\prod_{i=1}^{k_1} \Delta^0_{\boldsymbol{\mu}}\bigl(t^{k_n}\tau_n/t_{2n+4}\big|t^{k_n}c^{n-1}d\tau_n x_i^{\pm}\bigr)\big|_{x_i\mapsto c^{1-n}t_1(\langle\boldsymbol{\lambda}\rangle_{k_1})_i/d} \\
&\qquad = \frac{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n}t_1\tau_n\langle\boldsymbol{\lambda}\rangle_{k_1})}{\Delta^0_{\boldsymbol{\mu}}(t^{k_n}\tau_n/t_{2n+4}|t^{k_n+1}t_1\tau_n\langle\boldsymbol{\lambda}\rangle_{k_1})}.
\end{aligned}
$$

Furthermore, by the same specialisation of the x_i, (4.7), (4.8) and (2.6) with

$$
n\mapsto k_1,\ a\mapsto t^{k_1}t_1/t_r,\ b\mapsto c^{2n-2}d^2t^{2-k_1}/t_1t_r = t^{1-k_0}t_2/t_r,
$$

and

$$
n\mapsto k_1,\ a\mapsto t^{k_1-1}t_1t_r,\ b\mapsto c^{2n-2}d^2t^{1-k_1}t_r/t_1 = t^{-k_0}t_2t_r,
$$

respectively, we get

$$
\begin{aligned}
\prod_{i=1}^{k_1}\prod_{r=3}^{2n} \Gamma\bigl(c^{n-1}dtx_i^{\pm}/t_r\bigr) &\mapsto (c^{2n-2}t^{k_1-k_n})^{2\sum_{i=1}^{k_1}\lambda_i^{(1)}\lambda_i^{(2)}} \\
&\qquad \times \prod_{r=3}^{2n} \Gamma(t^i t_1/t_r, t^{i-k_0}t_2/t_r)\Delta^0_{\boldsymbol{\lambda}}(t^{k_0+k_1-1}t_1/t_2|t^{k_1}t_1/t_r)
\end{aligned}
$$

and

$$\prod_{i=1}^{k_1}\prod_{r=2n+1}^{2n+4}\Gamma\big(c^{n-1}dt_r x_i^{\pm}\big)\mapsto\Big(\frac{pqt^{k_0-k_1+k_n}}{t_1t_2}\Big)^{2\sum_{i=1}^{k_1}\lambda_i^{(1)}\lambda_i^{(2)}}$$
$$\times\prod_{r=2n+1}^{2n+4}\Gamma\big(t^{i-1}t_1t_r,t^{i-k_0-1}t_2t_r\big)\Delta^0_{\boldsymbol{\lambda}}\big(t^{k_0+k_1-1}t_1/t_2|t^{k_1-1}t_1t_r\big).$$

Combining these three results, setting $k_0=0$ and using (4.9) implies Theorems 1.1 and 4.1.

Proof of Proposition 4.4. Recalling the definition of the A_n Selberg density (1.9), and assuming that $n\geqslant 2$, we have

$$\begin{aligned}
&S^{k_0,k_1,\dots,k_n}_{\boldsymbol{\mu}}(x;t_1,\dots,t_{2n+4})\\
&\quad=\int\Big(\mathcal{K}_d\big(z^{(1)};x\big)R^*_{\boldsymbol{\mu}}\big(z^{(n)};t_{2n+2}/t,t_{2n+3}/t;t\tau_n,t_{2n+4}\big)\\
&\qquad\times\Delta^{(\mathrm{v})}_{\mathrm{S}}\big(z^{(1)};c^{n-1}t/t_3,c^{n-1}t/t_4\big)\Delta^{(\mathrm{e})}_{\mathrm{S}}\big(z^{(1)};z^{(2)};c\big)\\
&\qquad\times\Delta_{\mathrm{S}}\big(z^{(2)},\dots,z^{(n)};t_3,\dots,t_{2n+4};c\big)\Big)\frac{\mathrm{d}z^{(1)}}{z^{(1)}}\cdots\frac{\mathrm{d}z^{(n)}}{z^{(n)}}.
\end{aligned}$$

By Corollary 3.2 with d as given in (4.7) and

$$(k,\ell,b,x,y,z)\mapsto\big(k_1,k_2,c^{n-1}t^{k_1-k_2+1}/t_3,z^{(2)},x,z^{(1)}\big),$$

we can carry out the integration over $z^{(1)}$. In particular we note that the above substitutions imply that

$$t/bd^2\mapsto c^{n-1}t^{k_0-2k_1+k_2+1}t_3/t_1t_2=c^{n-1}t/t_4,$$

where the last equality follows from by taking the ratio of the balancing conditions (4.8) for $r=1$ and $r=2$. From these same balancing conditions it also follows that

$$(cd)^2=c^{4-2n}t^{k_2-k_1-1}t_3t_4.$$

As a result,

$$\begin{aligned}
&S^{k_0,k_1,\dots,k_n}_{\boldsymbol{\mu}}(x;t_1,\dots,t_{2n+4})\\
&\quad=\prod_{i=1}^{k_2-k_1}\Gamma(t^i,t^{i-1}c^{4-2n}t_3t_4)\prod_{i=1}^{k_1}\Gamma\big(c^{n-1}dtx_i^{\pm}/t_3,c^{n-1}dtx_i^{\pm}/t_4\big)\\
&\qquad\times S^{k_1,k_2,\dots,k_n}_{\boldsymbol{\mu}}(x';t_3,\dots,t_{2n+4}),
\end{aligned}$$

where

$$x':=\big(x_1,\dots,x_{k_1},c^{n-1}dt^{k_1-k_2+1}/t_3,c^{n-1}dt^{k_1-k_2+2}/t_3,\dots,c^{n-1}d/t_3\big).$$

A straightforward but somewhat tedious calculations shows that the right-hand side of (4.10) satisfies the same recursion. The proof is thus reduced to checking validity of the claim for $n=1$. This is

$$S^{k_0,k_1}_{\boldsymbol{\mu}}(x;t_1,\dots,t_6)=\int\mathcal{K}_d\big(z;x\big)R^*_{\boldsymbol{\mu}}\big(z;t_4/t,t_5/t;t\tau_1,t_6\big)\Delta^{(\mathrm{v})}_{\mathrm{S}}\big(z;t_3,t_4,t_5,t_6\big)\frac{\mathrm{d}z}{z},$$

where $z=(z_1,\dots,z_{k_1})$, $d^2:=t^{k_1-k_0-1}t_1t_2$ and $t^{2k_1-k_0-2}t_1\cdots t_6=pq$. But this is nothing but Theorem 3.4 with

$$(n,c,t_1,t_2,t_3,v_1,v_2)\mapsto(k_1,d,t_3,t_6,t_5,t_4/t,t_5/t).$$

Hence

$$\begin{aligned}S_{\boldsymbol{\mu}}^{k_0,k_1}(x;t_1,\dots,t_6)&=\prod_{i=1}^{k_1}\biggl(\prod_{3\leqslant r<s\leqslant 6}\Gamma(t^{i-1}t_rt_s)\prod_{r=3}^{6}\Gamma\bigl(dt_rx_i^{\pm}\bigr)\biggr)\\&\quad\times\frac{\Delta_{\boldsymbol{\mu}}^0(t^{k_1-2}t_3t_4t_5/t_6|t^{k_1-1}t_3t_4)}{\Delta_{\boldsymbol{\mu}}^0(t^{k_1-2}t_3t_4t_5/t_6|d^2t^{k_1-1}t_3t_4)}\,R_{\boldsymbol{\mu}}^*(x;dt_4/t,t_5/td;dt_3t_4t_5/t,dt_6).\end{aligned}$$

Since

$$t^{k_1}(dt_3t_4t_5/t)(dt_6)=t^{2k_1-k_0-2}t_1t_2t_3t_4t_5t_6=pq,$$

the interpolation function on the right is of Cauchy type and factors by (3.17). Therefore,

$$\begin{aligned}S_{\boldsymbol{\mu}}^{k_0,k_1}(x;t_1,\dots,t_6)&=\prod_{i=1}^{k_1}\biggl(\prod_{3\leqslant r<s\leqslant 6}\Gamma(t^{i-1}t_rt_s)\prod_{r=3}^{6}\Gamma\bigl(dt_rx_i^{\pm}\bigr)\biggr)\\&\quad\times\prod_{r=4}^{5}\Delta_{\boldsymbol{\mu}}^0(t^{k_1}\tau_1/t_6|t^{k_1-1}t_3t_r)\prod_{i=1}^{k_1}\Delta_{\boldsymbol{\mu}}^0\bigl(t^{k_1}\tau_1/t_6|t^{k_1}d\tau_1x_i^{\pm}\bigr).\end{aligned}$$

This is exactly the right-hand side of (4.10) for $n=1$. □

To conclude this section we remark that for $k_1=k_2=\cdots=k_n=k$ the evaluation of (4.6) that does not require the heavy machinery of the elliptic interpolation kernel. As per the above proof, for $n\geqslant 2$,

$$\begin{aligned}&S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k_1,\dots,k_n}(t_1,\dots,t_{2n+4})\\&\quad=\int\Bigl(R_{\boldsymbol{\lambda}}^*\bigl(z^{(1)};c^{1-n}t_1,c^{1-n}t_2\bigr)R_{\boldsymbol{\mu}}^*\bigl(z^{(n)};t_{2n+2}/t,t_{2n+3}/t;t\tau_n,t_{2n+4}\bigr)\\&\qquad\times\Delta_{\mathrm{S}}^{(\mathrm{v})}\bigl(z^{(1)};c^{1-n}t_1,t^{1-n}t_2,c^{n-1}t/t_3,c^{n-1}t/t_4\bigr)\Delta_{\mathrm{S}}^{(\mathrm{e})}\bigl(z^{(1)};z^{(2)};c\bigr)\\&\qquad\times\Delta_{\mathrm{S}}\bigl(z^{(2)},\dots,z^{(n)};t_3,\dots,t_{2n+4};c\bigr)\Bigr)\,\frac{\mathrm{d}z^{(1)}}{z^{(1)}}\cdots\frac{\mathrm{d}z^{(n)}}{z^{(n)}}.\end{aligned}$$

If $k_1=k_2=k$ then the integral over $z^{(1)}$ is exactly the elliptic beta integral (3.23) with

$$(t_1,t_2,t_3,t_4)\mapsto(c^{1-n}t_1,c^{1-n}t_2,c^{n-1}t/t_3,c^{n-1}t/t_4),$$

$\boldsymbol{\mu}\mapsto\boldsymbol{\lambda}$ and $x_i\mapsto z_i^{(2)}$ for $1\leqslant i\leqslant k$. In particular, by (1.10), $t^kt_1t_2/t_3t_4=1$, as required. Therefore,

$$\begin{aligned}S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k,k,k_3,\dots,k_n}(t_1,\dots,t_{2n+4})&=\prod_{i=1}^{k}\biggl(\frac{\Gamma(t^{i-1}c^{2-2n}t_1t_2)}{\Gamma(t^{i-1}c^{4-2n}t_1t_2)}\prod_{r=1}^{2}\prod_{s=3}^{4}\Gamma(t^it_r/t_s)\biggr)\\&\quad\times\Delta_{\boldsymbol{\lambda}}^0(t^{k-1}t_1/t_2\bigl|t^kt_1/t_3,t^kt_1/t_4)S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k,k_3,\dots,k_n}(t_1,t_2,t_5,\dots,t_{2n+4}),\end{aligned}$$

where $t_3t_4=t^kt_1t_2$. For $k=1$ and $\boldsymbol{\lambda}=\mathbf{0}$ this is the elliptic analogue of the recursion at the bottom of page 299 of [83]. Iterating the recursion yields

$$\begin{aligned}S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\overbrace{k,\dots,k}^{m\text{ times}},k_{m+1},\dots,k_n}(t_1,\dots,t_{2n+4})&=\prod_{i=1}^{k}\bigg(\frac{\Gamma(t^{i-1}c^{2-2n}t_1t_2)}{\Gamma(t^{i-1}c^{2m-2n}t_1t_2)}\prod_{r=1}^{2}\prod_{s=3}^{2m}\Gamma(t^it_r/t_s)\bigg)\\&\quad\times\prod_{r=3}^{2m}\Delta_{\boldsymbol{\lambda}}^{0}(t^{k-1}t_1/t_2|t^kt_1/t_r)\\&\quad\times S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k,k_{m+1},\dots,k_n}(t_1,t_2,t_{2m+1},\dots,t_{2n+4};t;p,q),\end{aligned}$$

where $1\leqslant m\leqslant n$ and $t_{2m-1}t_{2m}=\dots=t_3t_4=t^kt_1t_2$. In particular, for $m=n$,

$$\begin{aligned}S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k,\dots,k}(t_1,\dots,t_{2n+4})&=\prod_{i=1}^{k}\bigg(\frac{\Gamma(t^{i-1}c^{2-2n}t_1t_2)}{\Gamma(t^{i-1}t_1t_2)}\prod_{r=1}^{2}\prod_{s=3}^{2n}\Gamma(t^it_r/t_s)\bigg)\\&\quad\times\prod_{r=3}^{2n}\Delta_{\boldsymbol{\lambda}}^{0}(t^{k-1}t_1/t_2|t^kt_1/t_r)\\&\quad\times S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k}(t_1,t_2,t_{2n+1},\dots,t_{2n+4};t;p,q)\end{aligned}$$

This final integral is the elliptic AFLT integral of [3, Theorem 1.4], evaluated in [3] without the use of the interpolation kernel. Hence

$$\begin{aligned}S_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{k,\dots,k}(t_1,\dots,t_{2n+4})&=\prod_{i=1}^{k}\bigg(\Gamma(t^i,t^{i-1}c^{2-2n}t_1t_2)\prod_{r=1}^{2}\prod_{s=3}^{2n}\Gamma(t^it_r/t_s)\\&\qquad\times\prod_{r=1}^{2}\prod_{s=2n+1}^{2n+4}\Gamma(t^{i-1}t_rt_s)\prod_{2n+1\leqslant r<s\leqslant 2n+4}\Gamma(t^{i-1}t_rt_s)\bigg)\\&\quad\times\prod_{r=3}^{2n}\Delta_{\boldsymbol{\lambda}}^{0}(t^{k-1}t_1/t_2|t^kt_1/t_r)\prod_{r=2n+1}^{2n+4}\Delta_{\boldsymbol{\lambda}}^{0}(t^{k-1}t_1/t_2|t^{k-1}t_1t_r)\\&\quad\times\prod_{r=2n+2}^{2n+3}\Delta_{\boldsymbol{\mu}}^{0}(t^{k_n}\tau_n/t_{2n+4}|t^{k-1}t_{2n+1}t_r)\\&\quad\times\frac{\Delta_{\boldsymbol{\mu}}^{0}(t^k\tau_n/t_{2n+4}|t^kt_1\tau_n\langle\boldsymbol{\lambda}\rangle_k)}{\Delta_{\boldsymbol{\mu}}^{0}(t^k\tau_n/t_{2n+4}|t^{k+1}t_1\tau_n\langle\boldsymbol{\lambda}\rangle_{k;t})},\end{aligned}$$

where

$$\begin{aligned}t^{2k-2}t_1t_2t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4}&=t^{k-2}t_3t_4t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4}\\&=\dots=t^{k-2}t_{2n-1}t_{2n}t_{2n+1}t_{2n+2}t_{2n+3}t_{2n+4}=pq.\end{aligned}$$

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