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$A-D-E$ POLYNOMIAL AND ROGERS–RAMANUJAN IDENTITIES

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We conjecture polynomial identities which imply Rogers–Ramanujan type identities for branching functions associated with the cosets $(\mathcal{G}^{(1)})_{\ell-1} \otimes (\mathcal{G}^{(1)})_1/(\mathcal{G}^{(1)})_\ell$, with $\mathcal{G} = A_{n-1}$ ($\ell \geq 2$), D_{n-1} ($\ell \geq 2$), $E_{6,7,8}$ ($\ell = 2$). In support of our conjectures we establish the correct behavior under level-rank duality for $\mathcal{G} = A_{n-1}$ and show that the $A-D-E$ Rogers–Ramanujan identities have the expected $q \to 1^-$ asymptotics in terms of dilogarithm identities. Possible generalizations to arbitrary cosets are also discussed briefly.

1. Introduction

Without doubt, the Rogers–Ramanujan identities

$$\sum_{m=0}^{\infty} \frac{q^{m(m+\sigma)}}{(q)_m} = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left\{ q^{j(10j+1-4\sigma)} - q^{(2j+1)(5j+3-2\sigma)} \right\}, \quad \sigma = 0, 1, \tag{1.1}$$

with $(q)_m = \prod_{k=1}^{m}(1-q^k)$ for $m > 0$ and $(q)_0 = 1$, are among the most beautiful and intriguing results of classical mathematics. Since their independent discovery by Rogers[1] and Ramanujan[2] many different methods of proof have been developed. A particularly fruitful approach was initiated by Schur.[3] The key idea is that identities of the Rogers–Ramanujan type are in fact limiting cases of polynomial identities. For example, the polynomial identities

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$$\sum_{m=0}^{\infty} q^{m(m+\sigma)} \begin{bmatrix} L-m-\sigma \\ m \end{bmatrix}_q$$
$$= \sum_{j=-\infty}^{\infty} \left\{ q^{j(10j+1-4\sigma)} \begin{bmatrix} L \\ \lfloor \frac{L}{2} \rfloor - 5j + \sigma \end{bmatrix}_q - q^{(2j+1)(5j+3-2\sigma)} \begin{bmatrix} L \\ \lfloor \frac{L-5}{2} \rfloor - 5j + \sigma \end{bmatrix}_q \right\}, \tag{1.2}$$

which hold for arbitrary $L \in \mathbb{Z}_{\geq 0}$, yield the Rogers–Ramanujan identities (1.1) in the limit $L \to \infty$ when q is restricted to $|q| < 1$. Here $\lfloor x \rfloor$ denotes the largest integer not exceeding x, and the Gaussian or q polynomial is defined by[4]

$$\begin{bmatrix} N \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q)_N}{(q)_m (q)_{N-m}}, & 0 \leq m \leq N, \\ 0 & \text{otherwise}. \end{cases} \tag{1.3}$$

By the use of the elementary formulae $\left[\begin{smallmatrix} N \\ m \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} N-1 \\ m \end{smallmatrix}\right]_q + q^{N-m} \left[\begin{smallmatrix} N-1 \\ m-1 \end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix} N \\ m \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} N-1 \\ m-1 \end{smallmatrix}\right]_q + q^{m} \left[\begin{smallmatrix} N-1 \\ m \end{smallmatrix}\right]_q$, it is easy to verify that both the left and the right hand side of Eq. (1.2) satisfy the recurrence $f_L = f_{L-1} + q^{L-1} f_{L-2}$. Given appropriate initial conditions, this recurrence has a unique solution, hence establishing the polynomial identity and its limiting form (1.1).

Identities of the Rogers–Ramanujan type occur in various branches of mathematics and physics. First, their connection with the theory of (affine) Lie algebras[5,6] and with partition theory[4] has led to many generalizations of (1.1). To illustrate these connections, it is for example easily established that for $\sigma = 1$ the right hand side of (1.1) can be rewritten in a more algebraic fashion as[a]

$$\frac{q^{c/24}}{\eta(q)} \sum_{\alpha \in Q} \sum_{w \in \bar{W}} \operatorname{sgn}(w) q^{\frac{1}{2} pp' \left| \alpha - \frac{p'\bar{\rho} - p w(\bar{\rho})}{pp'} \right|^2}, \qquad p = 2, \quad p' = 5, \tag{1.4}$$

with Q the root lattice, $\bar{W}$ the Weyl group and $\bar{\rho}$ the Weyl vector of the classical Lie algebra A_1, $\eta(q) = q^{1/24} (q)_\infty$ the Dedekind eta function and

$$c = 1 - \frac{6(p-p')^2}{pp'}. \tag{1.5}$$

Similarly, if $Q_{k,i}(n)$ denotes the number of partitions of n with each successive rank in the interval $[2-i, 2k-i-1]$, then by sieving methods the generating function of $Q_{k,i}$ can be seen to again yield the right hand side of (1.1) provided we choose $k = 2$ and $i = 3 - 2\sigma$.[4]

Second, Rogers–Ramanujan identities also appear in various areas of physics. Most notable is perhaps the fact that (1.4) can be identified as the normalized

[a] If a symbol x is used in the context of both classical and affine Lie algebras, we will throughout this paper write $\bar{x}$ to mean its classical (counter)part.

Rocha–Caridi form[7] for the identity character $\chi_{1,1}^{(p,p')}$ of the Virasoro algebra. Indeed, each pair of positive integers p, p', with p and p' coprime, labels a minimal conformal field theory[8] of *central charge* c given by (1.5). Another branch of physics where Rogers–Ramanujan type identities have occurred is in the theory of solvable lattice models.[9] Among other works, in Refs. 10 and 11 Andrews, Baxter and Forrester (ABF) encountered generalized Rogers–Ramanujan identities in their corner transfer matrix (CTM) calculation of one-point functions of an infinite series of restricted solid-on-solid (RSOS) models. In addition, in Refs. 12 and 13 Kedem *et al.* conjectured many identities motivated by a Bethe ansatz study of the row transfer matrix spectrum of the three-state Potts model.

Interestingly, though, it is in fact the combination of the approaches of Ref. 10 and Ref. 12 to solvable models that leads to polynomial identities of the type (1.2). In computing one-point functions of solvable RSOS models using CTM's along the lines of Ref. 10, one is naturally led to the computation of so-called one-dimensional configuration sums. These configuration sums take forms very similar to the right hand side of (1.2). On the other hand, in performing the Bethe ansatz, and more particularly thermodynamic Bethe ansatz (TBA) computations, one is led to expressions of a similar nature to the left hand side of (1.2).

Starting with the ABF models and pursuing the lines sketched above, Melzer conjectured[14] an infinite family of polynomial identities similar to those in (1.2). In the infinite limit these identities again lead to Rogers–Ramanujan type identities, but now for Virasoro characters $\chi_{r,s}^{(p,p+1)}$ of unitary minimal models, i.e. for characters with the Rocha–Caridi right hand side form [in the sense of (1.1)]

$$\chi_{r,s}^{(p,p+1)}(q) = \frac{q^{\Delta_{r,s}^{(p,p+1)}-c/24}}{(q)_\infty} \times \sum_{j=-\infty}^{\infty} \left\{q^{j(p(p+1)+(p+1)r-ps)} - q^{(pj+r)((p+1)j+s)}\right\}, \tag{1.6}$$

labeled by the conformal weights

$$\Delta_{r,s}^{(p,p+1)} = \frac{[(p+1)r-ps]^2-1}{4p(p+1)}, \qquad 1 \le r \le p-1, \quad 1 \le s \le p. \tag{1.7}$$

The corresponding left hand side forms of these characters were conjectured earlier in the work of Kedem *et al.*[13] and their *finitization* in Ref. 14 again provided a method of proof. For $p = 3$ and 4 the proof was carried out in Ref. 14 and Berkovich[15] subsequently generalized this to all $p \ge 3$ for $s = 1$.

In this paper we generalize Melzer's approach to finding polynomial identities from solvable lattice models. By considering the CTM as well as TBA calculations of various higher rank generalizations of the ABF models, we are led to conjectures for polynomial identities labeled by the Lie algebras of the $A-D-E$ type. In the infinite limit our $\mathcal{G} = A-D-E$ polynomial identities lead to Rogers–Ramanujan type expressions for the branching functions associated with the GKO coset pair[16,5]

$$\begin{array}{cccccc} & \mathcal{G}^{(1)} & \oplus & \mathcal{G}^{(1)} & \supset & \mathcal{G}^{(1)} \\ \text{level} & \ell-1 & & 1 & & \ell \end{array} . \tag{1.8}$$

The rest of this paper is organized as follows. In the next section we define polynomial expressions $F_q^{\mathcal{G}}(L)$ following from the TBA calculations of Bazhanov and Reshetikhin.[17] Since the polynomials are defined as restricted sums over the solutions of $\mathcal{G}$ type constraint equations, we adopt the terminology of Ref. 12 and call these polynomials *fermionic*. In Sec. 3, we define analogous polynomial expressions, originating from the CTM calculations of Refs. 18–20 for the same solvable lattice models as considered by Bazhanov and Reshetikhin. Again following the terminology of Ref. 12, we call these polynomials *bosonic*, and denote them by $B_q^{\mathcal{G}}(L)$. Our conjectures can then be formulated as

$$F_q^{\mathcal{G}}(L) = B_q^{\mathcal{G}}(L) . \tag{1.9}$$

In Sec. 4 we study the $L \to \infty$ behavior of our polynomial identities, thereby deriving $A-D-E$ type Rogers–Ramanujan identities. In the subsequent two sections we provide some indications for the correctness of our conjectures. In Sec. 5 we establish the expected level–rank duality for $\mathcal{G} = A_{n-1}$ and in Sec. 6 we show that in the $q \to 1^-$ limit the $A-D-E$ Rogers–Ramanujan identities lead to the correct dilogarithm identities. Finally, in Sec. 7, we summarize our results and discuss possible generalizations to non simply laced Lie algebras and to more general cosets than those listed under (1.8).

2. Fermionic $A-D-E$ Polynomials

Let us now turn to the definition of the fermionic $A-D-E$ polynomials as follow from the TBA calculations of Ref. 17.

We denote the Cartan matrix of the simply laced Lie algebra $\mathcal{G} = A, D, E$ by $C^{\mathcal{G}}$ and the corresponding incidence matrix by $\mathcal{I}^{\mathcal{G}} = 2\,\mathrm{Id} - C^{\mathcal{G}}$, choosing the labeling of the nodes of the $A-D-E$ Dynkin diagrams as shown in Fig. 1. We furthermore let $m_j^{(a)} \in \mathbb{Z}_{\geq 0}$ and $n_j^{(a)} \in \mathbb{Z}_{\geq 0}$ be the number of "particles" and "antiparticles" of type j and color a, respectively, satisfying the constraint system

$$m_j^{(a)} + n_j^{(a)} = \frac{1}{2}\left[L\,\delta_{a,p}\,\delta_{j,1} + \sum_{b=1}^{\mathrm{rank}\,\mathcal{G}} \mathcal{I}_{a,b}^{\mathcal{G}} n_j^{(b)} + \sum_{k=1}^{\ell-1} \mathcal{I}_{j,k}^{A_{\ell-1}} m_k^{(a)}\right] . \tag{2.1}$$

Here the particles are labeled from $j = 1,\ldots,\ell-1$ and the colors from $a = 1,\ldots,\mathrm{rank}\,\mathcal{G}$. The variable p in the first Kronecker delta is 1 except for $\mathcal{G} = E_{6,7}$ when we have $p = 6$. We note that the above equation is a parameter-independent version of the constraint equations for densities of strings and holes in the TBA calculations of Refs. 17 and 21, and given a set $\{m_j^{(a)}\}_{j=1,\ldots,\mathrm{rank}\,\mathcal{G}}^{a=1,\ldots,\ell-1}$ it determines a companion set $\{n_j^{(a)}\}_{j=1,\ldots,\mathrm{rank}\,\mathcal{G}}^{a=1,\ldots,\ell-1}$. Of course, only for special sets $\{m_j^{(a)}\}$ does it follow that $\{n_j^{(a)}\}$ again consists of only nonnegative integers.

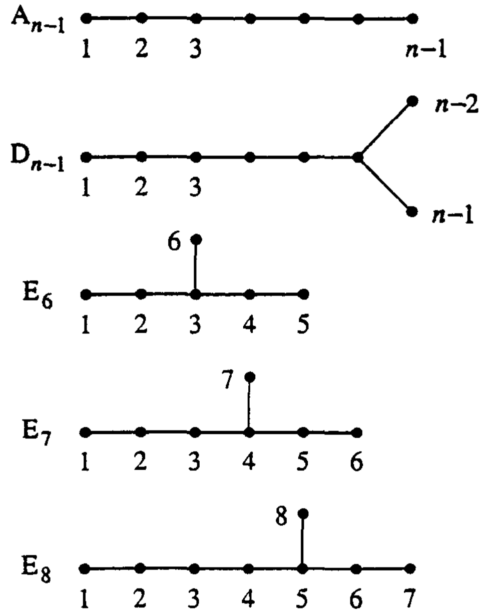

Fig. 1. The Dynkin diagrams of the simply laced Lie algebras with relevant labeling of the nodes.

We now define the fermionic polynomials $F_q^{\mathcal{G}}(L)$ as the following sum over the solutions to the constraint system (2.1):

$$F_q^{\mathcal{G}}(L) = \sum{}' q^{\frac{1}{2}\sum_{a,b=1}^{\operatorname{rank}\mathcal{G}}\sum_{j,k=1}^{\ell-1}(C^{\mathcal{G}})^{-1}_{a,b}C^{A_{\ell-1}}_{j,k}m_j^{(a)}m_k^{(b)}} \prod_{a=1}^{\operatorname{rank}\mathcal{G}}\prod_{j=1}^{\ell-1}\begin{bmatrix} m_j^{(a)}+n_j^{(a)} \\ m_j^{(a)} \end{bmatrix}_q . \tag{2.2}$$

Here the prime signifies the additional constraints

$$\sum_{a=1}^{\operatorname{rank}\mathcal{G}}(C^{\mathcal{G}})^{-1}_{r,a}m_{\ell-1}^{(a)} \in \mathbb{Z}\,, \tag{2.3}$$

or, equivalently,

$$\sum_{j=1}^{\ell-1}(C^{A_{\ell-1}})^{-1}_{\ell-1,j}n_j^{(r)} - L(C^{\mathcal{G}})^{-1}_{r,p}(C^{A_{\ell-1}})^{-1}_{\ell-1,1} \in \mathbb{Z}\,, \tag{2.4}$$

with $r = n-1$ for $\mathcal{G} = A_{n-1}$, $r = n-2$ and $n-1$ for $\mathcal{G} = D_{n-1}$, $r = 1$ or, equivalently, 5 for E_6 and $r = 1$ or, equivalently, 7 for E_7. Since all entries of $(C^{E_8})^{-1}$ are integers there is no additional constraint for $\mathcal{G} = E_8$.

We note that for $\mathcal{G} = A_1$ the fermionic polynomials coincide with a special case of those defined in Refs. 14 and 15.

3. Bosonic $A-D-E$ Polynomials

As mentioned in the introduction, the bosonic polynomials all arise naturally in corner transfer matrix calculations of order parameters of solvable lattice models of the RSOS type.[9,10] One of the key steps in such a calculation is the evaluation of the *one-dimensional configuration sums*

$$X_L(a,b,c) = \sum_{a_2,\dots,a_L} q^{\sum_{j=1}^{L} jH(a_j,a_{j+1},a_{j+2})}, \qquad a_1 = a\,, \quad a_{L+1} = b\,, \quad a_{L+2} = c\,. \tag{3.1}$$

The function H here is determined by the Boltzmann weights of the relevant solvable model, and each pair (a_j, a_{j+1}) is assumed to be admissible in the sense explained below.

Except for the case $\mathcal{G} = E_6$, the bosonic $A-D-E$ polynomials turn out to coincide with one particular configuration sum. For E_6 it can be defined as a linear combination of two such sums.

3.1. A_{n-1}

In this case we have to consider the family of $A_{n-1}^{(1)}$ RSOS models as introduced[22] and solved[18] by Jimbo, Miwa and Okado (JMO), generalizing the RSOS models of ABF[10] which are recovered for $n = 2$. Before we present JMO's result for the one-dimensional configuration sums X_L, we need some notation.

For the affine Lie algebra $A_{n-1}^{(1)}$ we let $\Lambda_0, \dots, \Lambda_{n-1}$ denote the fundamental weights, ρ the Weyl vector and δ the null root. We have the usual inner product on $\mathcal{H}^* = \mathbb{C}\Lambda_0 \oplus \cdots \oplus \mathbb{C}\Lambda_{n-1} \oplus \mathbb{C}\delta$ by

$$\langle \Lambda_\mu, \Lambda_\nu \rangle = \min(\mu,\nu) - \frac{\mu\nu}{n}\,, \qquad \langle \delta, \delta \rangle = 0\,, \qquad \langle \Lambda_\mu, \delta \rangle = 1\,. \tag{3.2}$$

For a general element $a \in \mathcal{H}^*$ its level is defined as $\mathrm{lev}(a) = \langle a, \delta \rangle$, and hence, according to (3.2), all fundamental weights have level 1. A weight a belongs to the set P_+^ℓ of *level ℓ dominant integral weights* if

$$a = \sum_{\mu=0}^{n-1} \mathbb{Z}_{\geq 0} \Lambda_\mu \quad \mathrm{lev}(a) = \ell\,, \tag{3.3}$$

i.e. $0 \leq \langle \rho, a \rangle \leq \ell$. We finally introduce the vectors e_j, $j \in J = \{1, \dots, n\}$:

$$e_j = \Lambda_j - \Lambda_{j-1}\,, \tag{3.4}$$

where we have set $\Lambda_n = \Lambda_0$. Clearly $\sum_{j \in J} e_j = 0$. An ordered pair of weights (a, b) both in P_+^ℓ is said to be *admissible* and denoted as $a \sim b$, if $b - a = e_j$ for some $j \in J$.

We now come to the definition of the one-dimensional configuration sums $X_L(a,b,c)$ of the $A_{n-1}^{(1)}$ RSOS models at level ℓ. They are given by (3.1), with

$a,\ b,\ c \in P_+^\ell$, where the sum is over all sequences $a \sim a_2 \sim \cdots \sim a_L \sim b \sim c$ of admissible level ℓ dominant integral weights. The function H in (3.1) takes the values

$$H(a, a+e_j, a+e_j+e_k) = H(j,k) = \begin{cases} 0, & 1 \le j < k \le n, \\ 1, & 1 \le k \le j \le n. \end{cases} \tag{3.5}$$

Using standard recurrence methods,[10] JMO found[18] the following expression for X_L:

$$X_L(a, b, b+e_j) = \sum_{w \in W} \det(w) x_L(b + \rho - w(a+\rho), j), \tag{3.6}$$

with W the affine Weyl group and, for $\lambda = \sum_{j \in J} \lambda_j e_j + z\delta \in \mathcal{H}^*$,

$$\begin{aligned} x_L(\lambda, j) &= q^{\sum_{k \in J} \lambda_k [H(k,j) + (\lambda_k - 1)/2]} \begin{bmatrix} L \\ \lambda \end{bmatrix}_q \\ &= q^{|\lambda - \Lambda_{j-1}|^2/2 + (L-j+1)(L-j+1+n)/(2n)} \begin{bmatrix} L \\ \lambda \end{bmatrix}_q . \end{aligned} \tag{3.7}$$

The q multinomial coefficient in the definition of x_L reads

$$\begin{bmatrix} L \\ \lambda \end{bmatrix}_q = \begin{cases} \dfrac{(q)_L}{\prod_{j \in J} (q)_{\lambda_j}}, & \sum_{j \in J} \lambda_j = L, \quad \lambda_j \ge 0, \\ 0 & \text{otherwise}. \end{cases} \tag{3.8}$$

Though Eq. (3.6) defines bosonic polynomials for any a, b and j, we only consider the special case $a = b = \ell\Lambda_0$ and $j = 1$ and define[b]

$$B_q^{A_{n-1}}(L) = q^{-L(L+n)/(2n)} X_L(\ell\Lambda_0, \ell\Lambda_0, \Lambda_1 + (\ell-1)\Lambda_0). \tag{3.9}$$

Our conjecture is that $F_q^{A_{n-1}}(L) = B_q^{A_{n-1}}(L)$ for all $L \in \mathbb{Z}_{\ge 0}$, $L = 0 \pmod{n}$. For $n = 2$ this was previously conjectured and partially proven in Ref. 14. A full proof for $n = 2$ can be found in Ref. 15. To test the conjecture we have generated both fermionic and bosonic polynomials $F_q^{A_{n-1}}(L)$ and $B_q^{A_{n-1}}(L)$ for all $n + \ell \le 8$ and for extensive ranges of L using Mathematica.[23]

3.2. D_{n-1}

For $\mathcal{G} = D_{n-1}$ we turn to the hierarchy of $D_{n-1}^{(1)}$ RSOS models, again introduced by JMO.[22] The result for the one-dimensional configuration sums[19] is almost completely analogous to that for $A_{n-1}^{(1)}$ as described in the previous subsection, and hence we only point out the relevant changes.

[b] To explicitly denote the fact that b is obtained via admissible sequences of L steps on P_+^ℓ, it may be better to employ the usual Young tableau notation assigning a tableau of signature $(f_1, \ldots, f_n)$ to each element of P_+^ℓ. In the convention of Ref. 18 we then would have $a = (0, \ldots, 0)$, $b = (L/n, \ldots, L/n)$ and $c = (L/n + 1, L/n, \ldots, L/n)$.

First of all, the inner product in the weight space changes to

$$\begin{aligned}\langle\Lambda_\mu,\Lambda_\nu\rangle &= \min(\mu,\nu)\,, & \langle\Lambda_\mu,\Lambda_m\rangle &= \frac{1}{2}\mu\,,\\ \langle\Lambda_m,\Lambda_m\rangle &= \frac{1}{4}(n-1)\,, & \langle\Lambda_m,\Lambda_{m'}\rangle &= \frac{1}{4}(n-3)\,,\\ \langle\Lambda_{0,1,n-2,n-1},\delta\rangle &= 1\,, & \langle\Lambda_{2,\dots,n-3},\delta\rangle &= 2\,,\end{aligned} \tag{3.10}$$

with $0 \le \mu \le n-3$, m, $m' = n-2$, $n-1$ and $m \neq m'$. We again introduce vectors e_j, $j \in J = \{\pm 1, \dots, \pm(n-1)\}$, defined by (3.4) for $j > 0$ and $e_{-j} = -e_j$. Two exceptions to (3.4) are $e_2 = \Lambda_2 - \Lambda_1 - \Lambda_0$ and $e_{n-2} = \Lambda_{n-2} - \Lambda_{n-3} + \Lambda_{n-1}$, and we note that $\langle e_j, e_k\rangle = jk\,\delta_{|j|,|k|}/|jk|$ for all j, $k \in J$. A pair (a,b), a, $b \in P_+^\ell$, is admissible if $b - a \in J$, where P_+^ℓ again denotes the level ℓ dominant integral weights. The values taken by the function H read[19]

$$\begin{aligned}&H(a, a+e_j, a+e_j+e_k)\\ &\quad = H(j,k) = \begin{cases} 0\,, & j<k \text{ and } jk\ 0 \text{ or } k<j \text{ and } jk<0\,,\\ 1 & \text{otherwise}\,,\end{cases}\end{aligned} \tag{3.11}$$

together with the exceptions $H(1,-1) = -1$ and $H\big(-(n-1), n-1\big) = 0$.

Again by application of recurrence methods it was shown in Ref. 19 that X_L can be expressed as the sum over the affine Weyl group as defined in (3.6), with x_L therein given by

$$x_L(\lambda, j) = \sum_{\eta_j - \eta_{-j} = \lambda_j} q^{-\eta_p\eta_{-p} + \sum_{k\in J}\eta_k[H(k,j)+(\eta_k-1)/2]} \begin{bmatrix} L \\ \eta \end{bmatrix}_q\,, \tag{3.12}$$

with $p = 1$ for $j < 0$ and $p = n-1$ for $j > 0$, and with the same definition of the multinomial coefficient as in (3.8).

As in the previous case, we only consider the bosonic polynomials obtained by specializing $a = b = \ell\Lambda_0$ and $j = 1$ in (3.6), and set

$$B_q^{D_{n-1}}(L) = q^{-L/2} X_L\big(\ell\Lambda_0, \ell\Lambda_0, \Lambda_1 + (\ell-1)\Lambda_0\big)\,. \tag{3.13}$$

We then conjecture that $F_q^{D_{n-1}}(L) = B_q^{D_{n-1}}(L)$ for all even $L \in \mathbb{Z}_{\ge 0}$. This conjecture has again been tested extensively for all $n + \ell \le 8$ using Mathematica.[23]

3.3. $E_{6,7,8}$

We now come to the exceptional simply laced Lie algebras: E_6, E_7 and E_8. Unfortunately, in this case we do not have an algebraic formulation of the bosonic polynomials, and as an immediate consequence we only have conjectures for $\ell = 2$.

In all three cases to follow we use the following definition of the q multinomial:

$$\begin{bmatrix} N \\ m_1, m_2 \end{bmatrix}_q = \begin{cases} \dfrac{(q)_N}{(q)_{m_1}(q)_{m_2}(q)_{N-m_1-m_2}}\,, & 0 \le m_1 + m_2 \le N,\ m_1,\ m_2 \ge 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.14}$$

3.3.1. E_6

Define the polynomials

$$\begin{aligned} B_q^{E_6}(L) = \sum_{j,k=-\infty}^{\infty} \Bigg\{ & q^{42j^2+j+k(k+14j)} \begin{bmatrix} L \\ k, k+14j \end{bmatrix}_q \\ & - q^{42j^2+13j+1+k(k+14j+2)} \begin{bmatrix} L \\ k, k+14j+2 \end{bmatrix}_q \Bigg\} \\ + q^5 \sum_{j,k=-\infty}^{\infty} \Bigg\{ & q^{42j^2+29j+k(k+14j+5)} \begin{bmatrix} L \\ k, k+14j+5 \end{bmatrix}_q \\ & - q^{42j^2+41j+5+k(k+14j+7)} \begin{bmatrix} L \\ k, k+14j+7 \end{bmatrix}_q \Bigg\}\,. \end{aligned} \tag{3.15}$$

It is to be noted that 5 is the integer value of the weight $\Delta_{5,1}^{(6,7)}$ and hence that the above form corresponds to a finitization of the *extended* identity character (with respect to the Virasoro algebra). The bosonic expression (3.15) can be identified with the normalized sum of the two configuration sums [$X_L(1,1,2)$ and $X_L(1,6,5)$ in the notation of Ref. 20] of the dilute A_6 lattice model.[24]

Our conjecture is that $F_q^{E_6}(L) = B_q^{E_6}(L)$, for all $L \in \mathbb{Z}_{\ge 0}$. We note that for $\ell = 2$ the restriction (2.3) is implied by the constraint system (2.1) and as a result we can drop the prime in the sum (2.2). We have checked the correctness of the E_6 polynomial identity for all $L \le 33$ by direct expansion.

3.3.2. E_7

Define the polynomials

$$\begin{aligned} B_q^{E_7}(L) = \sum_{j,k=-\infty}^{\infty} \Bigg\{ & q^{20j^2+j+k(k+10j)} \begin{bmatrix} L \\ k, k+10j \end{bmatrix}_q \\ & - q^{20j^2+9j+1+k(k+10j+2)} \begin{bmatrix} L \\ k, k+10j+2 \end{bmatrix}_q \Bigg\}\,. \end{aligned} \tag{3.16}$$

This bosonic expression is related to one of the configuration sums [$X_L(1,1,2)$ in the notation of Ref. 20] of the dilute A_4 model.[24]

Our conjecture is that $F_q^{E_7}(L) = B_q^{E_7}(L)$, for all $L \in \mathbb{Z}_{\geq 0}$. Tests have again confirmed the polynomial identity for all $L \leq 37$.

3.3.3. E_8

Define the polynomials

$$B_q^{E_8}(L) = \sum_{j,k=-\infty}^{\infty} \left\{ q^{12j^2+j+k(k+8j)} \begin{bmatrix} L \\ k, k+8j \end{bmatrix}_q \right.$$
$$\left. - q^{12j^2+7j+1+k(k+8j+2)} \begin{bmatrix} L \\ k, k+8j+2 \end{bmatrix}_q \right\}. \tag{3.17}$$

Comparison with the one-dimensional configuration sums for the dilute A_3 model in regime 2 as computed in Ref. 20 shows that $B_q^{E_8}(L) = X_L(1,1,2)$.

The assertion is now that $F_q^{E_8}(L) = B_q^{E_8}(L)$, for all $L \in \mathbb{Z}_{\geq 0}$. The E_8 polynomial identity has in fact been proven,[25] and is directly related to the E_8 structure of the critical Ising model in a magnetic field.[26,27]

4. *A–D–E* Rogers–Ramanujan identities

In this section we present the Rogers–Ramanujan type identities that follow if one takes the $L \to \infty$ limit in the various polynomial identities listed in the previous section. To put the results in the context of coset conformal field theories based on the GKO construction of Eq. (1.8), we first give a brief reminder on theta functions and *branching rules*; see e.g. Refs. 5, 6, 16, 18, 28, 29.

4.1. *Branching functions*

For μ, $z \in \bar{\mathcal{H}}^*$ the classical $\mathcal{G}$ theta function of characteristic μ and degree m is defined as[c]

$$\Theta_{\mu,m}(z,\tau) = \sum_{\alpha \in Q + \frac{\mu}{m}} e^{m\pi i\tau|\alpha|^2 - 2\pi i m\langle\alpha,z\rangle}, \tag{4.1}$$

with Q the root lattice of $\mathcal{G}$. Then, according to the Weyl–Kac formula,[6] the character of highest weight representation $a \in P_+^\ell$ of a $\mathcal{G}^{(1)}$ Kac–Moody algebra is given by

$$\chi_{a,\ell}(z,\tau) = \frac{\sum_{w\in\bar{W}} \operatorname{sgn}(w)\Theta_{w(\bar{a}+\bar{\rho}),g+\ell}(z,\tau)}{\sum_{w\in\bar{W}} \operatorname{sgn}(w)\Theta_{w(\bar{\rho}),g}(z,\tau)}, \tag{4.2}$$

with $\bar{W}$ the Weyl group and g the dual Coxeter number of $\mathcal{G}$.

We now come to the definition of the *branching functions* $b_{a,b}^{(t)}(\tau)$ for dominant integral weights $a \in P_+^{\ell-1}$, $b \in P_+^\ell$ and $t \in P_+^1$, $\bar{t} = \bar{b} - \bar{a} \pmod{Q}$, via the decomposition or branching rule

[c]As in the rest of the paper, $\mathcal{G}$ denotes a simply laced Lie algebra.

$$\chi_{a,\ell-1}(z,\tau)\chi_{t,1}(z,\tau) = \sum_{b\in P_+^\ell} b_{a,b}^{(t)}(\tau)\chi_{b,\ell}(z,\tau)\,. \tag{4.3}$$

The branching functions can be expressed in terms of the $\mathcal{G}$ theta functions as

$$b_{a,b}^{(t)}(\tau) = \frac{1}{[\eta(q)]^{\operatorname{rank}\mathcal{G}}} \sum_{w\in\bar{W}} \operatorname{sgn}(w)\Theta_{-(p+1)(\bar{a}+\bar{\rho})+pw(\bar{b}+\bar{\rho}),p(p+1)}(0,\tau)\,, \tag{4.4}$$

with $p = \ell + g - 1$. From the definition (4.1) of the theta functions this can be rewritten in a form generalizing Eq. (1.6) for the unitary minimal Virasoro characters[29]

$$b_{r,s}^{(t)}(q) = \frac{1}{[\eta(q)]^{\operatorname{rank}\mathcal{G}}} \sum_{\alpha\in Q}\sum_{w\in\bar{W}} \operatorname{sgn}(w) q^{\frac{1}{2}p(p+1)\left|\alpha - \frac{(p+1)\bar{r}-pw(\bar{s})}{p(p+1)}\right|^2}\,, \tag{4.5}$$

with $q = \exp(2\pi i\tau)$, $r = a+\rho$ and $s = b+\rho$. The lowest order term in this expression occurs for $w = \text{id}$, $\alpha = 0$, and thus

$$b_{r,s}^{(t)}(q) = q^{\Delta_{r,s}^{(p,p+1)}-c/24}\sum_{n=0}^{\infty} a_n q^n\,, \qquad a_n \in \mathbb{Z}_{\geq 0}\,, \tag{4.6}$$

generalizing (1.5) and (1.7) to

$$c = \operatorname{rank}\mathcal{G}\left[1 - \frac{g(g+1)}{p(p+1)}\right]\,, \tag{4.7}$$

$$\Delta_{r,s}^{(p,p+1)} = \frac{[(p+1)r - ps]^2 - \frac{g\dim\mathcal{G}}{12}}{2p(p+1)}\,. \tag{4.8}$$

4.2. *Rogers–Ramanujan type identities*

We now consider the $L \to \infty$ limit of the polynomial identities as conjectured in Secs. 2 and 3.

Eliminating $n_j^{(a)}$ from the fermionic polynomials (2.2) using (2.1), and then letting $L \to \infty$ by $\lim_{L\to\infty} \left[{N \atop m}\right]_q = 1/(q)_m$, yields

$$\lim_{L\to\infty} F_q^{\mathcal{G}}(L) \equiv F_q^{\mathcal{G}} = \sum{}' q^{\frac{1}{2}\sum_{a,b=1}^{\operatorname{rank}\mathcal{G}}\sum_{j,k=1}^{\ell-1}(C^{\mathcal{G}})_{a,b}^{-1}C_{j,k}^{A_{\ell-1}}m_j^{(a)}m_k^{(b)}} \times \prod_{a=1}^{\operatorname{rank}\mathcal{G}}\frac{1}{(q)_{m_1^{(a)}}}\prod_{j=2}^{\ell-1}\left[{m_j^{(a)} + P_j^{(a)} \atop m_j^{(a)}}\right]_q\,. \tag{4.9}$$

Here we have introduced the symbol $P_j^{(a)}$ to mean

$$P_j^{(a)} = \begin{cases} \infty\,, & j = 1\,,\\ -\displaystyle\sum_{b=1}^{\operatorname{rank}\mathcal{G}}\sum_{k=1}^{\ell-1}(C^{\mathcal{G}})_{a,b}^{-1}C_{j,k}^{A_{\ell-1}}m_k^{(b)}\,, & j = 2,\ldots,\ell-1\,,\end{cases} \tag{4.10}$$

with $P_1^{(a)}$ included for later convenience. Some caution has to be taken in interpreting the primed sum in (4.9). No longer is the sum over all solutions to the constraint system (2.1), but simply over all $m_j^{(a)} \in \mathbb{Z}_{\geq 0}$. The prime still denotes the condition (2.3), where it is to be noted that we can no longer drop the prime for $\mathcal{G} = E_6$.

The $L \to \infty$ limit of the bosonic polynomials $B_q^{A_{n-1}}(L)$ defined in (3.9) was considered in Ref. 18. The result reads

$$\lim_{L\to\infty} B_q^{A_{n-1}}(L) \equiv B_q^{A_{n-1}} = q^{c/24} b_{r,s}^{(t)}(q) \begin{cases} r = (\ell-1)\Lambda_0 + \rho\,, \\ s = \ell\Lambda_0 + \rho\,, \\ t = \Lambda_0\,, \end{cases} \tag{4.11}$$

where the branching function is that of the previous subsection based on $\mathcal{G} = A_{n-1}$.

In taking the infinite limit of the D_{n-1} polynomials (3.13) we get exactly the same result as in (4.11) with A_{n-1} replaced by D_{n-1}[19] and interpreting the branching function as that of $\mathcal{G} = D_{n-1}$.

For the exceptional cases we only have $\ell = 2$ and a simplification occurs as the exceptional branching functions at level 2 reproduce the ordinary Virasoro characters $\chi_{r,s}^{(p,p+1)}$ in the bosonic representation of Eq. (1.6). To illustrate this consider, for example, $\mathcal{G} = E_8$. Then $P_+^1 = \{\Lambda_0\}$ and $P_+^2 = \{2\Lambda_0, \Lambda_1, \Lambda_7\}$ and we compute from (4.7) $c = 1/2$ and from (4.8)

$$\begin{aligned} \Delta^{(31,32)}_{\Lambda_0+\rho,2\Lambda_0+\rho} &= 0 = \Delta^{(3,4)}_{1,1}\,, \\ \Delta^{(31,32)}_{\Lambda_0+\rho,\Lambda_1+\rho} &= \frac{1}{16} = \Delta^{(3,4)}_{1,2}\,, \\ \Delta^{(31,32)}_{\Lambda_0+\rho,\Lambda_7+\rho} &= \frac{1}{2} = \Delta^{(3,4)}_{2,1}\,. \end{aligned} \tag{4.12}$$

Here the weights on the left hand side are to be understood as those of (4.8) based on $\mathcal{G} = E_8$ and the weights on the right hand side as those of Eq. (1.7) or, equivalently, as those of (4.8) with $\mathcal{G} = A_1$. Hence, writing the three infinite forms of the bosonic polynomials in terms of Virasoro characters yields[20,24]

$$\lim_{L\to\infty} B_q^{E_n}(L) \equiv B_q^{E_n} = \begin{cases} q^{c/24}\big(\chi_{1,1}^{(6,7)} + \chi_{5,1}^{(6,7)}\big)\,, & n = 6\,, \\ q^{c/24}\chi_{1,1}^{(4,5)}\,, & n = 7\,, \\ q^{c/24}\chi_{1,1}^{(3,4)}\,, & n = 8\,. \end{cases} \tag{4.13}$$

With the definitions (4.9)–(4.13) the $\mathcal{G}$ Rogers–Ramanujan type identities can be written as

$$F_q^{\mathcal{G}} = B_q^{\mathcal{G}}\,. \tag{4.14}$$

For $\ell = 2$ and arbitrary $\mathcal{G}$ this has been conjectured by Kedem *et al.* in Ref. 12. For $\mathcal{G} = A_1$ and arbitrary ℓ this was again conjectured by Kedem *et al.*, this time in Ref. 13. Also, the general form of (4.14) can be inferred from Refs. 12 and 13, but details such as the precise form of the restrictions (2.3) on the sum in (4.9) were not conjectured in full generality.

5. Level-Rank Duality

An important consideration in the confirmation of the conjectured polynomial identities is their behavior under the transformation $q \to 1/q$. In particular, for the $\mathcal{G} = A_{n-1}$ case we show that the identity $F_q^{\mathcal{G}}(L) = B_q^{\mathcal{G}}(L)$ displays the correct invariance under the simultaneous transformations

$$\ell \leftrightarrow n\,, \qquad q \leftrightarrow \frac{1}{q}\,, \tag{5.1}$$

provided we choose $L(C^{A_{n-1}})^{-1}_{n-1,1}(C^{A_{\ell-1}})^{-1}_{\ell-1,1} = L/n\ell \in \mathbb{Z}$.

5.1. *The transformation* $q \to 1/q$

Let us first consider how the general fermionic sum (2.2) transforms under inversion of q. To do so we rewrite (2.1) in a form expressing the number of antiparticles in terms of the number of particles, and vice versa:

$$n_j^{(a)} = L\,\delta_{j,1}(C^{\mathcal{G}})^{-1}_{a,p} - \sum_{b=1}^{\operatorname{rank}\mathcal{G}}\sum_{k=1}^{\ell-1}(C^{\mathcal{G}})^{-1}_{a,b}C^{A_{\ell-1}}_{j,k}m_k^{(b)}\,, \tag{5.2}$$

$$m_j^{(a)} = L\,\delta_{a,p}(C^{A_{\ell-1}})^{-1}_{j,1} - \sum_{b=1}^{\operatorname{rank}\mathcal{G}}\sum_{k=1}^{\ell-1}C^{\mathcal{G}}_{a,b}(C^{A_{\ell-1}})^{-1}_{j,k}n_k^{(b)}\,. \tag{5.3}$$

Now, upon using the inversion

$$\begin{bmatrix} N \\ m \end{bmatrix}_{1/q} = q^{m(m-N)}\begin{bmatrix} N \\ m \end{bmatrix}_q\,, \tag{5.4}$$

we find that the exponent of q in the expression for $F^{\mathcal{G}}_{1/q}(L)$ reads

$$-\frac{1}{2}\sum_{a,b=1}^{\operatorname{rank}\mathcal{G}}\sum_{j,k=1}^{\ell-1}(C^{\mathcal{G}})^{-1}_{a,b}C^{A_{\ell-1}}_{j,k}m_j^{(a)}m_k^{(b)} - \sum_{a=1}^{\operatorname{rank}\mathcal{G}}\sum_{j=1}^{\ell-1}m_j^{(a)}n_j^{(a)}\,. \tag{5.5}$$

If we substitute (5.2) to eliminate $n_j^{(a)}$, this becomes

$$\begin{aligned} -\frac{L^2}{2}(C^{\mathcal{G}})^{-1}_{p,p}(C^{A_{\ell-1}})^{-1}_{1,1} &+ \frac{1}{2}\sum_{a,b=1}^{\operatorname{rank}\mathcal{G}}\sum_{j,k=1}^{\ell-1}(C^{\mathcal{G}})^{-1}_{a,b}C^{A_{\ell-1}}_{j,k} \\ &\times\left[m_j^{(a)} - L\,\delta_{a,p}(C^{A_{\ell-1}})^{-1}_{j,1}\right]\left[m_k^{(b)} - L\,\delta_{b,p}(C^{A_{\ell-1}})^{-1}_{k,1}\right]. \end{aligned} \tag{5.6}$$

Then, using (5.3), we arrive at

$$-\frac{L^2}{2}(C^{\mathcal{G}})^{-1}_{p,p}(C^{A_{\ell-1}})^{-1}_{1,1} + \frac{1}{2}\sum_{a,b=1}^{\operatorname{rank}\mathcal{G}}\sum_{j,k=1}^{\ell-1}C^{\mathcal{G}}_{a,b}(C^{A_{\ell-1}})^{-1}_{j,k}n_j^{(a)}n_k^{(b)}\,. \tag{5.7}$$

We now carry out a transformation of variables. Replacing $m_j^{(a)} \to n_a^{(j)}$ and $n_j^{(a)} \to m_a^{(j)}$ followed by $j \leftrightarrow a$ and $k \leftrightarrow b$ yields

$$F^{\mathcal{G}}_{1/q}(L) = q^{-\frac{L^2}{2}(C^{\mathcal{G}})^{-1}_{p,p}(C^{A_{\ell-1}})^{-1}_{1,1}} \sum{}' q^{\frac{1}{2}\sum_{a,b=1}^{\ell-1}\sum_{j,k=1}^{\mathrm{rank}\,\mathcal{G}}(C^{A_{\ell-1}})^{-1}_{a,b}C^{\mathcal{G}}_{j,k}m_j^{(a)}m_k^{(b)}}$$
$$\times \prod_{a=1}^{\ell-1}\prod_{j=1}^{\mathrm{rank}\,\mathcal{G}} \begin{bmatrix} m_j^{(a)} + n_j^{(a)} \\ m_j^{(a)} \end{bmatrix}_q . \tag{5.8}$$

Of course, now the sum is over all solutions of

$$m_j^{(a)} + n_j^{(a)} = \frac{1}{2}\left[L\,\delta_{a,1}\delta_{j,p} + \sum_{b=1}^{\ell-1}\mathcal{I}^{A_{\ell-1}}_{a,b}n_j^{(b)} + \sum_{k=1}^{\mathrm{rank}\,\mathcal{G}}\mathcal{I}^{\mathcal{G}}_{j,k}m_k^{(a)}\right], \tag{5.9}$$

where the prime denotes the additional condition

$$\sum_{a=1}^{\ell-1}(C^{A_{\ell-1}})^{-1}_{\ell-1,a}m_r^{(a)} - L(C^{A_{\ell-1}})^{-1}_{\ell-1,1}(C^{\mathcal{G}})^{-1}_{r,p} \in \mathbb{Z}, \tag{5.10}$$

or, equivalently,

$$\sum_{j=1}^{\mathrm{rank}\,\mathcal{G}} (C^{\mathcal{G}})^{-1}_{r,j}n_j^{(\ell-1)} \in \mathbb{Z}. \tag{5.11}$$

In the following we denote the fermionic sum (5.8) without the irrelevant factor in front of the summation symbol by $G^{\mathcal{G}}_q(L)$. We note the apparent similarity of $F^{\mathcal{G}}_q(L)$ and $G^{\mathcal{G}}_q(L)$ under interchange of $\mathcal{G}$ and $A_{\ell-1}$ in either of the two expressions.

5.2. *The case $\mathcal{G} = A_{n-1}$: level–rank duality*

We now proceed to consider the inversion properties of the polynomial expressions in the case $\mathcal{G} = A_{n-1}$ only.

To explicitly exhibit the dependence of the fermionic polynomials on A_{n-1} as well as on $A_{\ell-1}$, we write $F_q^{A_{n-1}}(L) = F_q^{(n,\ell)}(L)$. Then, since $p = 1$ and $r = n-1 = \mathrm{rank}\,A_{n-1}$, we clearly have $F_q^{(n,\ell)}(L) = G_q^{(\ell,n)}(L)$, given that we choose L such that $L = 0 \pmod{n\ell}$. This duality of the fermionic polynomials under the transformation (5.1) is an example of so-called *level-rank* duality.[30] Of course, for our conjecture $F_q^{(n,\ell)}(L) = B_q^{(n,\ell)}(L)$ [$= B_q^{A_{n-1}}(L)$] to be correct, the bosonic polynomials $B_q^{A_{n-1}}(L)$ in (3.9) must also remain invariant under the transformation (5.1). As pointed out in Ref. 18, this is indeed the case. Apart from the overall factor

$$q^{-\frac{1}{2}L^2(C^{\mathcal{G}})^{-1}_{1,1}(C^{A_{\ell-1}})^{-1}_{1,1}} = q^{-\frac{L^2(n-1)(\ell-1)}{2n\ell}}, \tag{5.12}$$

we again have level–rank duality provided $L = 0 \pmod{n\ell}$.

It is in fact precisely this level–rank duality for $\mathcal{G} = A_{n-1}$ that led us to the conjecture (2.2). That is to say, from the already known and proven[14,15] polynomial identity for A_1 at level $\ell-1$ we can immediately infer a polynomial identity for $A_{\ell-1}$

at level 2. To then write down polynomial identities for general rank, level and $\mathcal{G}$ is a matter of straightforward generalization.

6. Dilogarithm Identities

It was first shown by Richmond and Szekeres[31] that in studying Rogers–Ramanujan identities in the limit $q \to 1$, one obtains identities involving dilogarithms. We here show that all $A-D-E$ identities of the previous section indeed yield the expected dilogarithm identities. In particular, we will show that for $q \to 1^-$ the Rogers–Ramanujan identities imply the identities

$$s^{\mathcal{G}}(\ell-1) + s^{\mathcal{G}}(1) - s^{\mathcal{G}}(\ell) = c\,, \tag{6.1}$$

corresponding to the coset conformal field theories (1.8) with central charges c as given by (4.7). Here $s^{\mathcal{G}}$ is defined as a sum over the Rogers dilogarithm function[32]

$$L(z) = \frac{1}{2}\int_0^z \left[\frac{\log(1-\zeta)}{\zeta} + \frac{\log\zeta}{1-\zeta}\right] d\zeta = -\int_0^z \frac{\log(1-\zeta)}{\zeta}\,d\zeta + \frac{1}{2}\log z\log(1-z) \tag{6.2}$$

as follows:

$$s^{\mathcal{G}}(\ell) = \frac{6}{\pi^2}\sum_{a=1}^{\text{rank}\,\mathcal{G}}\sum_{j=1}^{\ell} L(\xi_j^{(a)})\,. \tag{6.3}$$

The numbers $\xi_\ell^{(a)}$ in the above sum are the solutions to the TBA equations[17,33–35]

$$\sum_{b=1}^{\text{rank}\,\mathcal{G}} C_{a,b}^{\mathcal{G}} \log\left(1-\xi_j^{(b)}\right) = \sum_{k=1}^{\ell-1} C_{j,k}^{A_{\ell-1}} \log\left(\xi_k^{(a)}\right), \tag{6.4}$$

where by definition $\xi_\ell^{(a)} = 1$.

Before proceeding to derive (6.1) from the Rogers–Ramanujan identities (4.14), let us note that the above dilogarithm identities were first conjectured in Ref. 17 in the computation of central charges of $A-D-E$ TBA systems. For the occurrence of these same identities in related work on TBA, see e.g. Refs. 33–35. A proof of (6.1) for $\mathcal{G} = A_{n-1}$ has been given by Kirillov in Ref. 36.

To establish (6.1), we follow the working of Nahm *et al.*[37] (see also Refs. 31, 38, 39) in evaluating $F_q^{\mathcal{G}}$, $q \to 1^-$, using steepest descent. Writing

$$F_q^{\mathcal{G}} = \sum_{\{m_j^{(a)}\}} f_q^{\mathcal{G}}(\{m_j^{(a)}\}) = \sum_{M=0}^{\infty} a_M\, q^M \tag{6.5}$$

we have

$$a_{M-1} = \frac{1}{2\pi \mathrm{i}}\sum_{\{m_j^{(a)}\}} \oint q^{-M} f_q^{\mathcal{G}}(\{m_j^{(a)}\})dq\,. \tag{6.6}$$

Treating $m_j^{(a)}$ as continuous variables, we now approximate the integration kernel by

$$\log\left(q^{-M} f_q^{\mathcal{G}}(\{m_j^{(a)}\})\right)$$
$$\approx \left[\frac{1}{2}\sum_{a,b=1}^{\mathrm{rank}\,\mathcal{G}}\sum_{j,k=1}^{\ell-1}(C^{\mathcal{G}})^{-1}_{a,b}C^{A_{\ell-1}}_{j,k}m_j^{(a)}m_k^{(b)} - M\right]\log q$$
$$+\sum_{a=1}^{\mathrm{rank}\,\mathcal{G}}\sum_{j=1}^{\ell-1}\left(\int_0^{P_j^{(a)}+m_j^{(a)}} - \int_0^{P_j^{(a)}} - \int_0^{m_j^{(a)}}\right)\log\left(1-q^t\right)dt\,. \tag{6.7}$$

Differentiating with respect to $m_j^{(a)}$ to find the saddle point results in the TBA equations (6.4), with $\xi_j^{(a)}$ defined by

$$\xi_j^{(a)} = \frac{q^{m_j^{(a)}}\left(1-q^{P_j^{(a)}}\right)}{1-q^{P_j^{(a)}+m_j^{(a)}}}\,. \tag{6.8}$$

Using the definition (6.2) of the dilogarithm function L, the simple relation

$$L(z) + L(1-z) = L(1) = \frac{\pi^2}{6} \tag{6.9}$$

and the pentagonal relation[32]

$$L(1-x) + L(1-y) - L(1-xy) = L\left(\frac{x(1-y)}{1-xy}\right) - L\left(\frac{1-y}{1-xy}\right) \tag{6.10}$$

yields

$$\log\left(q^{-M} f_q^{\mathcal{G}}(\{m_j^{(a)}\})\right) \approx -M\log q + \frac{1}{\log q}\sum_{a=1}^{\mathrm{rank}\,\mathcal{G}}\sum_{j=1}^{\ell-1}\left[L(\xi_j^{(a)}) - L(\eta_j^{(a)})\right]$$
$$+\frac{1}{2}\sum_{a=1}^{\mathrm{rank}\,\mathcal{G}}\sum_{j=1}^{\ell-1}\left\{\log q\sum_{b=1}^{\mathrm{rank}\,\mathcal{G}}\sum_{k=1}^{\ell-1}(C^{\mathcal{G}})^{-1}_{a,b}C^{A_{\ell-1}}_{j,k}m_j^{(a)}m_k^{(b)}\right.$$
$$+\left(P_j^{(a)}+m_j^{(a)}\right)\log\left(1-q^{P_j^{(a)}+m_j^{(a)}}\right)$$
$$\left. - P_j^{(a)}\log\left(1-q^{P_j^{(a)}}\right) - m_j^{(a)}\log\left(1-q^{m_j^{(a)}}\right)\right\}. \tag{6.11}$$

The variables $\eta_j^{(a)}$ in this expression are given by

$$\eta_j^{(a)} = \frac{1-q^{P_j^{(a)}}}{1-q^{P_j^{(a)}+m_j^{(a)}}} \tag{6.12}$$

and, by (4.10) and (6.8), they satisfy

$$\begin{gathered}\eta_1^{(a)} = 1\,,\\ \sum_{b=1}^{\operatorname{rank}\mathcal{G}} C^{\mathcal{G}}_{a,b} \log\left(1-\eta^{(a)}_{j+1}\right) = \sum_{k=1}^{\ell-2} C^{A_{\ell-2}}_{j,k} \log\left(\eta^{(b)}_{k+1}\right), \quad j = 1,\ldots,\ell-2\,.\end{gathered} \tag{6.13}$$

Clearly, $\eta_j^{(a)}$, $j = 2,\ldots,\ell-1$, satisfy the same TBA equations as $\xi_j^{(a)}$, $j = 1,\ldots,\ell-1$, but with ℓ replaced by $\ell-1$.

The following elementary manipulations serve to show that the last two lines in (6.11) cancel as a consequence of the TBA equations (6.4):

$$\begin{aligned}
&\log q \sum_{a,b=1}^{\operatorname{rank}\mathcal{G}} \sum_{j,k=1}^{\ell-1} (C^{\mathcal{G}})^{-1}_{a,b} C^{A_{\ell-1}}_{j,k} m_j^{(a)} m_k^{(b)}\\
&= \sum_{a,b=1}^{\operatorname{rank}\mathcal{G}} \sum_{j,k=1}^{\ell-1} (C^{\mathcal{G}})^{-1}_{a,b} C^{A_{\ell-1}}_{j,k} m_j^{(a)} \log\left(\frac{\xi_k^{(b)}}{\eta_k^{(b)}}\right)\\
&= \sum_{a=1}^{\operatorname{rank}\mathcal{G}} \sum_{j=1}^{\ell-1} m_j^{(a)} \log\left(1-\xi_j^{(a)}\right) - \sum_{a,b=1}^{\operatorname{rank}\mathcal{G}} \sum_{j,k=1}^{\ell-1} (C^{\mathcal{G}})^{-1}_{a,b} C^{A_{\ell-1}}_{j,k} m_j^{(a)} \log\left(\eta_k^{(b)}\right)\\
&= \sum_{a=1}^{\operatorname{rank}\mathcal{G}} \sum_{j=1}^{\ell-1} \Bigg\{ P_j^{(a)} \log\left(1-q^{P_j^{(a)}}\right)\\
&\quad + m_j^{(a)} \log\left(1-q^{m_j^{(a)}}\right) - \left(P_j^{(a)} + m_j^{(a)}\right) \log\left(1-q^{P_j^{(a)}+m_j^{(a)}}\right)\\
&\quad - \left(P_j^{(a)} + \sum_{b=1}^{\operatorname{rank}\mathcal{G}} \sum_{k=1}^{\ell-1} (C^{\mathcal{G}})^{-1}_{a,b} C^{A_{\ell-1}}_{j,k} m_k^{(b)}\right) \log\left(\eta_j^{(a)}\right) \Bigg\}\\
&= \sum_{a=1}^{\operatorname{rank}\mathcal{G}} \sum_{j=1}^{\ell-1} \Big\{ P_j^{(a)} \log\left(1-q^{P_j^{(a)}}\right)\\
&\quad + m_j^{(a)} \log\left(1-q^{m_j^{(a)}}\right) - \left(P_j^{(a)} + m_j^{(a)}\right) \log\left(1-q^{P_j^{(a)}+m_j^{(a)}}\right) \Big\}\,.
\end{aligned} \tag{6.14}$$

As a result of this we have

$$\begin{aligned}
\log\left(q^{-M} f_q^{\mathcal{G}}(\{m_j^{(a)}\})\right) &\approx -M\log q + \frac{1}{\log q} \sum_{a=1}^{\operatorname{rank}\mathcal{G}} \sum_{j=1}^{\ell-1} \left[L(\xi_j^{(a)}) - L(\eta_j^{(a)})\right]\\
&= -M\log q - \frac{\pi^2}{6\log q}\left[s^{\mathcal{G}}(\ell-1) + s^{\mathcal{G}}(1) - s^{\mathcal{G}}(\ell)\right].
\end{aligned} \tag{6.15}$$

Finally we have to fix the value of q at the saddle point. From $\frac{d}{dq}\log f_q = 0$ this is found to be

$$(\log q)^2 = \frac{\pi^2}{6M}\left[s^{\mathcal{G}}(\ell-1) + s^{\mathcal{G}}(1) - s^{\mathcal{G}}(\ell)\right]. \tag{6.16}$$

Returning to the expression (6.6) hence yields the following result for the asymptotics of a_M:

$$a_M \sim \exp\left\{2\pi\sqrt{\left[s^{\mathcal{G}}(\ell-1) + s^{\mathcal{G}}(1) - s^{\mathcal{G}}(\ell)\right]M/6}\right\}. \tag{6.17}$$

To actually obtain the identity (6.1) we have to show that in addition to (6.17) we also have $a_M \sim \exp(2\pi\sqrt{cM/6})$ with c given by (4.7). However, from the automorphic properties of the $\mathcal{G}$ theta functions this can indeed be established.[6,40,18,29] That is to say, if we carry out the modular transformation $\tau \to -1/\tau$ which relates the $q \to 1$ to the $q \to 0$ limit of the branching functions (4.5) and then use for $(q)_\infty^{-1} = \sum a_M q^M$ the fact that $\log a_M \sim 2\pi\sqrt{cM/6}$, the central charge follows as given in (4.7). In fact, it is precisely the prefactor $q^{\Delta_{r,s}^{(p,p+1)}-c/24}$ in the expansion (4.6) of the branching functions that ensures proper modular invariance of the branching rule (4.3).

Before ending this section let us make some concluding remarks. We have presented the dilogarithm identities (6.1) as special sums over the solution to the TBA equations (6.4) avoiding the problem of explicitly solving these equations. In Ref. 36 an elegant algebraic formulation of the solution for $\mathcal{G} = A_{n-1}$ and D_{n-1} has, however, been given, based on the work of Ref. 41 on the representations of Yangians. For $\mathcal{G} = A_{n-1}$ the solution is especially simple,

$$\xi_j^{(a)} = \frac{\sin\left(\frac{a\pi}{n+\ell}\right)\sin\left(\frac{(n-a)\pi}{n+\ell}\right)}{\sin\left(\frac{(j+a)\pi}{n+\ell}\right)\sin\left(\frac{(n+j-a)\pi}{n+\ell}\right)}, \tag{6.18}$$

and can in fact be easily checked by direct substitution in (6.4). For the exceptional cases no explicit form of the solutions is known, but some conjectures towards a solution have been made in Ref. 21.

Another remark to be made is that the identities (6.1) are in fact consequences of the stronger identities[17,36]

$$s^{\mathcal{G}}(\ell) \equiv \frac{6}{\pi^2}\sum_{a=1}^{\operatorname{rank}\mathcal{G}}\sum_{j=1}^{\ell} L(\xi_j^{(a)}) = \frac{\ell \dim \mathcal{G}}{\ell+g}, \tag{6.19}$$

where we recall that $p = \ell + g - 1$ in (4.7).

7. Summary and Discussion

In this paper we have presented conjectures for polynomial identities of the $A-D-E$ type following the "solvable lattice model" approach of Melzer[14] (see also Ref. 15). All polynomials identities can be viewed as finitizations of Rogers–Ramanujan type identities for branching functions associated with the GKO pair $\mathcal{G}^{(1)} \oplus \mathcal{G}^{(1)} \supset \mathcal{G}^{(1)}$ at levels $\ell - 1$, 1 and ℓ, respectively. Apart from extensive computer tests we have corroborated our conjectures by studying the behavior under level–rank duality for $\mathcal{G} = A_{n-1}$ and by establishing the expected dilogarithm identities following from the $A{-}D{-}E$ Rogers–Ramanujan identities in the asymptotic limit $q \to 1^-$.

Given that we have no proofs of our conjectures, a lot of course remains to be done. First of all it is to be noted that we have only presented fermionic polynomials associated with one particular choice of one-dimensional configuration sum, or, equivalently, we have only given fermionic finitizations of one particular branching function associated with the above-mentioned coset pair. This is in contrast to Melzer's original work, where fermionic polynomials for all Virasoro characters $\chi^{(p,p+1)}_{r,s}$ were conjectured. In fact, the method of proof for the A_1 polynomial identities[42] relies heavily on the completeness of a set of fermionic polynomials in order to apply recurrence methods similar to those used for obtaining the bosonic polynomials. However, despite some efforts to find fermionic counterparts to all configuration sums $X_L(a,b,c)$, we did not succeed in general. So, for example for $\mathcal{G} = A_{n-1}$, we only managed to find fermionic forms corresponding to configuration sums of the form $X_L(\ell\Lambda_0, b, b+e_j)$ with $b = p\Lambda_\mu + (\ell - p)\Lambda_\nu$, $0 \le p \le \ell$. Even so, assuming that we would have been able to establish completeness, it still seems that the recurrence method developed in Ref. 15 is far from ideal to tackle the general A_{n-1} case. In this respect, exploiting higher rank partition theory might be a more promising way to go. (For recent progress on proofs of polynomial identities for finitized Virasoro characters using solely partition theoretic arguments, see Ref. 43.)

Setting aside the problem of proof, it is quite clear that many of the results of this paper admit further generalization. Instead of defining the fermionic polynomials (2.2) based on the constraint system (2.1), we could have started with the more general equation

$$m_j^{(a)} + n_j^{(a)} = \frac{1}{2}\left[L\,\delta_{a,p}\delta_{j,s} + \sum_{b=1}^{\mathrm{rank}\,\mathcal{G}} \mathcal{I}^{\mathcal{G}}_{a,b} n_j^{(b)} + \sum_{k=1}^{\ell-1} \mathcal{I}^{A_{\ell-1}}_{j,k} m_k^{(a)}\right] \tag{7.1}$$

as considered in parameter-dependent form in Ref. 17. The parameter s here provides the generalization to the cosets

$$\begin{array}{lccccc} & \mathcal{G}^{(1)} & \oplus & \mathcal{G}^{(1)} & \supset & \mathcal{G}^{(1)} \\ \text{level} & \ell - s & & s & & \ell \end{array}, \tag{7.2}$$

and by letting p taking any of the values $1, \ldots, \mathrm{rank}\,\mathcal{G}$, instead of fixing it as in Sec. 2, we can obtain fermionic finitizations to several and not just 1 branching function associated with (7.2). The problem is then of course to also define appropriate bosonic polynomials to match the fermionic expressions. For the case $\mathcal{G} = A_1$, $s > 1$, these should be provided by CTM calculations of Ref. 44 for the fused ABF models.

In Ref. 21 an even further generalization to (2.1) was proposed, extending (7.1) to the case of nonsimply laced Lie algebras. Clearly, in defining the appropriately generalized form of the fermionic expression (2.2), this should for $s = 1$ correspond to the one-dimensional configuration sums for the $B_n^{(1)}$ and $C_n^{(1)}$ RSOS models of Ref. 19.

We hope to address some of the above-mentioned problems and generalizations in future publications.

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Note Added in Proof

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