AN A_2 BAILEY LEMMA AND ROGERS–RAMANUJAN-TYPE IDENTITIES

GEORGE E. ANDREWS, ANNE SCHILLING, AND S. OLE WARNAAR

ABSTRACT. Using new q-functions recently introduced by Hatayama *et al.* and by (two of) the authors, we obtain an A_2 version of the classical Bailey lemma. We apply our result, which is distinct from the A_2 Bailey lemma of Milne and Lilly, to derive Rogers–Ramanujan-type identities for characters of the W_3 algebra.

1. INTRODUCTION

The celebrated Rogers–Ramanujan identities [35, 37, 39] are given by

$$(1.1) \qquad 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

and

$$(1.2) \qquad 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

for $|q| < 1$. The fame of these identities lies not only in their beauty and fascinating history [3, 17], but also in their relevance to the theory of partitions and many other branches of mathematics and physics. In particular, MacMahon [27] and Schur [39] independently noted that the left-hand side of (1.1) is the generating function for partitions into parts with difference at least two while the right-hand side generates partitions into parts congruent to ± 1 modulo 5. Similarly, the left-hand side of (1.2) is the generating function for partitions into parts with difference at least two and no parts equal to 1, while the right-hand side of (1.2) generates partitions into parts congruent to ± 2 modulo 5.

Over the years many generalizations of both the analytic and the combinatorial statement of the Rogers–Ramanujan identities have been found, see e.g., [2, 4, 7, 8, 9, 16]. All the cited analytic generalizations are accessible through the classical, or A_1 Bailey lemma and can thus be classified as "A_1 Rogers–Ramanujan-type identities". (We always mean identities of the "sum=product" form when referring to Rogers–Ramanujan-type identities.)

The proof of Bailey's lemma relies on a $_3\phi_2$ summation known as the q-Pfaff–Saalschütz formula. In [30, 31] Milne and Lilly used a $_6\phi_5$ sum for A_{n-1} basic hypergeometric functions to establish a higher-rank version of Bailey's lemma. Though

Milne and Lilly's result has been applied to yield many beautiful A_{n-1} generalizations of well-known basic hypergeometric function identities [31], it is remarkable that it has not led to A_{n-1} Rogers–Ramanujan-type identities. In [28, 29] Milne gives U(n) generalizations of the Rogers–Ramanujan identities related to partitions with differences between parts at least n, but in contrast to the $n = 2$ case Milne's identities are not of Rogers–Ramanujan type.

In this paper a first step is taken towards generalizing the Rogers–Ramanujan identities to A_{n-1}. After an introduction of the classical Bailey lemma (Sec. 2 and 3) we prove two summation formulas which give rise to an A_2 Bailey lemma and Bailey chain (Sec. 4). One of these summation formulas involves new q-functions recently introduced in [18, 21, 38]. In the second part of the paper applications of the A_2 Bailey lemma are presented. First we prove several Rogers–Ramanujan-type identities for characters of the W_3 algebra, which generalize the Rogers–Ramanujan identities (1.1) and (1.2) and their extensions due to Gordon and Andrews (Sec. 5.2–5.6). As a second application we give an A_2 generalization of Bressoud's identities for partitions with even moduli (Sec. 5.7). Finally our lemma is applied to yield summation formulas for Kostka polynomials (Sec. 5.8).

1.1. **Notation.** Throughout the paper the following notations will be used. The q-shifted factorial is defined for all integers n by

$$(a;q)_\infty = (a)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

and

$$(a;q)_n = (a)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

Note in particular that $1/(q)_n = 0$ when $n < 0$. We also use the condensed notation of Gaspar and Rahman [14], i.e.,

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1, a_2, \ldots, a_m)_n = (a_1)_n(a_2)_n \cdots (a_m)_n.$$

The Gaussian polynomial or q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \dfrac{(q)_n}{(q)_m(q)_{n-m}} & \text{for } 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases}$$

At times we find it convenient to display both lower entries of the q-binomial, writing $\begin{bmatrix} m+n \\ m,n \end{bmatrix}$. We often use the basic hypergeometric function notation

$$_{r+1}\phi_r\begin{bmatrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{bmatrix}; q, z \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1})_n}{(q, b_1, \ldots, b_r)_n} z^n.$$

Finally we set $\binom{n}{2} = n(n-1)/2$ for $n \in \mathbb{Z}$, and adopt the convention that $\sum_n$ stands for a sum over all integers n.

2. The A_1 Bailey lemma

One of the most elegant approaches towards proving q-series identities is provided by the Bailey lemma. In trying to catch the essence of Rogers' second proof of the Rogers–Ramanujan identities [36], Bailey [6] was led to the following definition.

Definition 2.1. *A pair of sequences* $\alpha = \{\alpha_L\}_{L \geq 0}$ *and* $\beta = \{\beta_L\}_{L \geq 0}$ *that satisfies*

$$(2.1) \qquad \beta_L = \sum_{r=0}^{L} \frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}}$$

forms a Bailey pair relative to a.

With this definition we have the following lemma, known as Bailey's lemma, but in its present form due to Andrews [4] and (in special cases) to Paule [34].

Lemma 2.2. *Let* (α, β) *form a Bailey pair relative to* a. *Then the pair* (α', β') *given by*

$$\alpha'_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L} \alpha_L$$

(2.2)

$$\beta'_L = \sum_{r=0}^{L} \frac{(\rho_1)_r(\rho_2)_r(aq/\rho_1\rho_2)^r(aq/\rho_1\rho_2)_{L-r}}{(aq/\rho_1)_L(aq/\rho_2)_L(q)_{L-r}} \beta_r$$

again forms a Bailey pair relative to a.

The strength of this lemma is that it can be iterated leading to the Bailey chain

$$(\alpha, \beta) \to (\alpha', \beta') \to (\alpha'', \beta'') \to \cdots .$$

An important special case of Bailey's lemma, and the one that will be generalized in Section 4, follows when ρ_1, ρ_2 tend to infinity. Using

$$\lim_{a \to \infty} a^{-n}(a)_n = (-1)^n q^{\binom{n}{2}},$$

equation (2.2) reduces to

$$(2.3) \qquad \alpha'_L = a^L q^{L^2} \alpha_L, \qquad \beta'_L = \sum_{r=0}^{L} \frac{a^r q^{r^2}}{(q)_{L-r}} \beta_r.$$

The simplest and perhaps most important application follows by taking the Bailey pair [4]

$$(2.4) \qquad \alpha_L = \frac{(1-aq^{2L})(a)_L(-1)^L q^{\binom{L}{2}}}{(1-a)(q)_L}, \qquad \beta_L = \delta_{L,0}$$

which is an immediate consequence of a simple ${}_4\phi_3$ summation [14, Eq. (2.3.4)]. Iterating this Bailey pair using (2.3) and setting $a = q^\ell$, $\ell = 0, 1$, yields the identities
(2.5)

$$\sum_r \frac{(-1)^r q^{((2k+1)r-1)r/2} a^{rk}}{(q)_{L-r}(q)_{L+r+\ell}} = \sum_{n_1,\ldots,n_{k-1}} \frac{a^{n_1+\cdots+n_{k-1}} q^{n_1^2+\cdots+n_{k-1}^2}}{(q)_{L-n_1}(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}.$$

Letting L tend to infinity and using Jacobi's triple product identity [14, Eq. (II.28)] gives ($|q| < 1$)

$$(2.6) \qquad \sum_{n_1,\ldots,n_{k-1}} \frac{q^{n_1^2+\cdots+n_{k-1}^2}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}} = \frac{(q^k, q^{k+1}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}$$

and

$$(2.7) \qquad \sum_{n_1,\ldots,n_{k-1}} \frac{q^{n_1^2+\cdots+n_{k-1}^2+n_1+\cdots+n_{k-1}}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}} = \frac{(q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}.$$

For $k = 2$ these are the Rogers–Ramanujan identities (1.1) and (1.2), whereas for $k \geq 3$ they are identities of Andrews [2], related to Gordon's partition theorem [16].

In the remainder of this paper we frequently use Bailey pairs such as those of equation (2.4), but unlike (2.4) they are generally very cumbersome to write down explicitly. We therefore adopt the practice of not writing down Bailey pairs explicitly, but to only present (polynomial) identities which imply Bailey pairs. The latter are generally much more compact. When $a = q^\ell$, the Bailey pair (2.4) can, for example, be caught in the identity

$$(2.8) \qquad \sum_r (-1)^r q^{r(r-1)/2} \begin{bmatrix} 2L + \ell \\ L - r \end{bmatrix} = \frac{q^{L(L+\ell)}(q)_\ell}{(q^{-\ell})_{-L}(q)_{-L}}, \qquad \ell = 0, 1, \dots$$

for $2L + \ell \geq 0$. When $L \geq 0$ the right-hand side simplifies to $(q)_\ell \delta_{L,0}$ and one can (after some work) recover (2.4). For general $2L + \ell \geq 0$ equation (2.8) follows by taking $x = 1$ in the q-binomial formula [14, Eq. (II.4)]

$$\sum_r (-1)^r x^r q^{r(r-1)/2} \begin{bmatrix} 2L + \ell \\ L - r \end{bmatrix} = (x)_L (q/x)_{L+\ell}.$$

In Section 5.2 we will consider an A_2 generalization of (2.8) that implies A_2 versions of (2.6) and (2.7).

3. Bypassing the Bailey lattice

Identities (2.6) and (2.7) are the $i = k$ and $i = 1$ instances of [2]

$$(3.1) \qquad \sum_{n_1,\dots,n_{k-1}} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-2} - n_{k-1}} (q)_{n_{k-1}}} = \frac{(q^i, q^{2k-i+1}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}$$

true for all $i = 1, \dots, k$ and $|q| < 1$. In [1], Agarwal, Andrews and Bressoud derive the above result using an extension of the Bailey chain known as the Bailey lattice (see also [10]). In Section 5 we wish to derive the A_2 counterpart of the identities (3.1), but we are faced with the problem that we do have an A_2 Bailey chain but not a lattice.

Here we describe a little trick which permits the derivation of (3.1) for all i from just the $a = 1$ Bailey pair of equation (2.4) and the Bailey chain, bypassing the need for a Bailey lattice. Later we apply the same procedure in the A_2 setting. As a first step we again take the Bailey pair of equation (2.4) with $a = 1$ and iterate once. This leads to the identity (2.5) for $a = 1$ and $k = 1$. Now the sum is split according to the parity of r giving the following bounded form of Euler's identity:

$$(3.2) \qquad \sum_r \left\{ q^{(6r+1)r} \begin{bmatrix} 2L \\ L - 2r \end{bmatrix} - q^{(2r+1)(3r+1)} \begin{bmatrix} 2L \\ L - 2r - 1 \end{bmatrix} \right\} = \frac{(q)_{2L}}{(q)_L}.$$

This can be rewritten as

$$(3.3) \qquad \sum_r \frac{q^{(6r+1)r}(1 - q^{4r+1})}{(q)_{L-2r}(q)_{L+2r+1}} = \frac{1}{(q)_L}$$

which is an equation that implies a Bailey pair relative to $a = q$. Iterating $k - i$ times leads to

$$\sum_r q^{2r(2r+1)(k-i)} \frac{q^{(6r+1)r}(1 - q^{4r+1})}{(q)_{L-2r}(q)_{L+2r+1}} = \sum_{n_1,\dots,n_{k-i}} \frac{q^{n_1^2 + \cdots + n_{k-i}^2 + n_1 + \cdots + n_{k-i}}}{(q)_{L-n_1} \cdots (q)_{n_{k-i-1} - n_{k-i}}(q)_{n_{k-i}}}.$$

Now rewrite this again in the form of equation (3.2), and combine the two terms in the summand of the left-hand side into one to arrive back at a form similar to (2.5). Explicitly,

$$\sum_r \frac{(-1)^r q^{((2k-2i+3)(r+1)-2)r/2}}{(q)_{L-r}(q)_{L+r}} = \sum_{n_1,\ldots,n_{k-i}} \frac{q^{n_1^2+\cdots+n_{k-i}^2+n_1+\cdots+n_{k-i}}}{(q)_{L-n_1}\cdots(q)_{n_{k-i-1}-n_{k-i}}(q)_{n_{k-i}}}$$

and we are back to an equation which implies a Bailey pair relative to $a = 1$. Iterating $i - 1$ times finally gives

$$\sum_r \frac{(-1)^r q^{((2k+1)(r+1)-2i)r/2}}{(q)_{L-r}(q)_{L+r}} = \sum_{n_1,\ldots,n_{k-1}} \frac{q^{n_1^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{L-n_1}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

so that, letting L tend to infinity and using the Jacobi triple product identity, we obtain (3.1).

At the heart of the above sequence of steps was the rewriting of (3.2) into (3.3) and, later, the reverse of this. This rewriting comes down to

$$\frac{q^{-2r}}{(q)_{L-2r}(q)_{L+2r}} - \frac{q^{2r+1}}{(q)_{L-2r-1}(q)_{L+2r+1}} = \frac{q^{-2r} - q^{2r+1}}{(q)_{L-2r}(q)_{L+2r+1}}.$$

As we wish to generalize to higher rank it is essential to recognize the above equality as the $B_1 = -2r + 1$, $B_2 = 2r + 2$ case of the determinant evaluation

$$\det_{1 \le i,j \le 2} \left(\frac{q^{j(j-B_i)}}{(q)_{L+B_i-j}} \right) = \frac{q^{5-B_1-2B_2}(1 - q^{B_2-B_1})}{(q)_{L+B_1-1}(q)_{L+B_2-1}}.$$

4. AN A$_2$ BAILEY LEMMA

4.1. **Motivation.** In order to generalize Bailey's lemma we return to Definition 2.1 of the classical, or A$_1$ Bailey pair. Setting $a = q^\ell$ with $\ell = 0, 1, \ldots$, we can rewrite (2.1) as

$$(4.1) \qquad \beta_L = \frac{1}{(aq)_{2L}} \sum_{\substack{k_1 \ge k_2 \\ k_1+k_2=0}} \begin{bmatrix} 2L + \ell \\ L - k_1, L - k_2 + \ell \end{bmatrix} \alpha_k$$

where $k = (k_1, k_2)$ and where we have chosen to display both lower entries of the q-binomial coefficient. We now generalize the definition of the A$_1$ Bailey pair by replacing the q-binomial with its higher-rank analogues. Recall that the q-binomial is the q-deformation of the ordinary binomial coefficient, which can be defined through the expansion

$$(4.2) \qquad (x_1 + x_2)^L = \sum_{\lambda_1,\lambda_2} x_1^{\lambda_1} x_2^{\lambda_2} \binom{L}{\lambda_1, \lambda_2}.$$

The rth elementary symmetric function in n variables is defined as [26]

$$e_r(x_1, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Identifying the left-hand side of (4.2) as $(e_1(x_1, x_2))^L$ we consider the following A$_{n-1}$ generalization of the binomial coefficient

$$\prod_{a=1}^{n-1} \left(e_a(x_1, \ldots, x_n) \right)^{L_a} = \sum_\lambda x_1^{\lambda_1} \cdots x_n^{\lambda_n} \binom{L_1, \ldots, L_{n-1}}{\lambda_1, \ldots, \lambda_n}.$$

Since e_r is homogeneous of degree r one sees that $\binom{L_1,\dots,L_{n-1}}{\lambda_1,\dots,\lambda_n} = 0$ if $\sum_{i=1}^{n} \lambda_i \neq \sum_{j=1}^{n-1} jL_j$. It is straightforward to write down an explicit expression for the generalized binomial as

$$(4.3) \qquad \binom{L_1,\dots,L_{n-1}}{\lambda_1,\dots,\lambda_n} = \sum_r \frac{\prod_{a=1}^{n-1} L_a!}{\prod_{a=1}^{n-1} \prod_{1 \leq i_1 < \cdots < i_a \leq n} r_{i_1 i_2 \cdots i_a}},$$

where the summation over r denotes a sum over the $r_{i_1\cdots i_a}$ $(a = 1,\dots,n-1)$ such that

$$\sum_{a=1}^{n-1} \sum_{\substack{1 \leq i_1 < \cdots < i_a \leq n \\ i_b = p \, (1 \leq b \leq a)}} r_{i_1 i_2 \cdots i_a} = \lambda_p, \qquad p = 1,\dots,n,$$

$$\sum_{1 \leq i_1 < \cdots < i_a \leq n} r_{i_1 i_2 \cdots i_a} = L_a, \qquad a = 1,\dots,n-1.$$

In [18] and [38] a q-deformation of (4.3) was introduced, which we call (following the terminology of [38]) (completely antisymmetric) A_{n-1} supernomial. In particular, Hatayama *et al.* [18] (see also [21]) propose the following representation of the A_{n-1} supernomial. Let $\nu^{(n)}$ denote the conjugate of the partition $(1^{L_1} 2^{L_2} \cdots (n-1)^{L_{n-1}})$, i.e., $\nu_j^{(n)} = L_j + \cdots + L_{n-1}$. Then, for $\sum_i \lambda_i = |\nu^{(n)}|$,

$$(4.4) \qquad \begin{bmatrix} L_1,\dots,L_{n-1} \\ \lambda_1,\dots,\lambda_n \end{bmatrix} = \sum_\nu \prod_{a=1}^{n-1} \prod_{j=1}^{a} \begin{bmatrix} \nu_j^{(a+1)} - \nu_{j+1}^{(a+1)} \\ \nu_j^{(a)} - \nu_{j+1}^{(a+1)} \end{bmatrix},$$

where the sum over ν denotes a sum over sequences $\emptyset = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n)}$ of Young diagrams such that each skew diagram $\nu^{(a)} - \nu^{(a-1)}$ is a horizontal λ_a-strip (see for these notions [26]).

If we now restrict (4.4) to $n = 3$, and set $\nu^{(1)} = \lambda_1$, $\nu^{(2)} = (\lambda_1 + m, \lambda_2 - m)$ and $\nu^{(3)} = (L_1 + L_2, L_2)$ we get

$$(4.5) \qquad \begin{bmatrix} L_1, L_2 \\ \lambda_1, \lambda_2, \lambda_3 \end{bmatrix} = \sum_m \begin{bmatrix} \lambda_1 - \lambda_2 + 2m \\ m \end{bmatrix} \begin{bmatrix} L_1 \\ \lambda_1 - L_2 + m \end{bmatrix} \begin{bmatrix} L_2 \\ \lambda_2 - m \end{bmatrix}$$

for $\lambda_1 + \lambda_2 + \lambda_3 = L_1 + 2L_2$ and zero otherwise. In subsequent manipulations we also employ the following representation of the A_2 supernomial, which, unlike (4.5), is a manifest q-deformation of the $n = 3$ case of (4.3),

$$(4.6) \qquad \begin{bmatrix} L_1, L_2 \\ \lambda_1, \lambda_2, \lambda_3 \end{bmatrix} = \sum_r \frac{q^{r_1 r_{23}} (q)_{L_1} (q)_{L_2}}{(q)_{r_1} (q)_{r_2} (q)_{r_3} (q)_{r_{12}} (q)_{r_{13}} (q)_{r_{23}}},$$

where the summation over r denotes a sum over $r_1,\dots,r_{23}$ such that

$$r_1 + r_{12} + r_{13} = \lambda_1, \quad r_2 + r_{12} + r_{23} = \lambda_2, \quad r_3 + r_{13} + r_{23} = \lambda_3$$

and

$$r_1 + r_2 + r_3 = L_1, \quad r_{12} + r_{13} + r_{23} = L_2.$$

To pass from (4.6) to (4.5) one has to apply the q-Chu–Vandermonde summation [14, Eq. (II.7)]

$$(4.7) \qquad {}_2\phi_1 \begin{bmatrix} q^{-n}, b \\ c \end{bmatrix} ; q, cq^n/b \end{bmatrix} = \frac{(c/b)_n}{(c)_n}.$$

To conclude our discussion of the supernomials we point out a slightly different viewpoint, see e.g., [20, 21]. In this approach the ordinary q-binomial is recognized as

$$\begin{bmatrix} L \\ \lambda_1, \lambda_2 \end{bmatrix} = \sum_{\eta \vdash |\lambda|} K_{\eta\lambda} K_{\eta'\mu}(q),$$

where $\mu = (1^L)$ and $\lambda = (\lambda_1, \lambda_2)$ a composition such that $|\lambda| = L$. The $K_{\lambda\mu}$ and $K_{\lambda\mu}(q)$ are the Kostka number and Kostka polynomial, respectively [26]. Given this expression it becomes natural to define a generalized q-binomial as

$$(4.8) \qquad \begin{bmatrix} L_1, \ldots, L_{n-1} \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} = \sum_{\eta \vdash |\lambda|} K_{\eta\lambda} K_{\eta'\mu}(q),$$

where now $\mu = (1^{L_1} 2^{L_2} \cdots (n-1)^{L_{n-1}})$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ a composition such that $|\lambda| = |\mu|$. The generalized q-binomial defined this way coincides with (4.4). From its definition in (4.8) it is clear that the (completely antisymmetric) A_{n-1} supernomials are the connection coefficients between the elementary symmetric functions and the Hall–Littlewood polynomials in n variables [26],

$$e_\lambda(x_1, \ldots, x_n) = \sum_{\mu \vdash |\lambda|} \begin{bmatrix} L_1, \ldots, L_{n-1} \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} P_\mu(x_1, \ldots, x_n; q).$$

4.2. **An A_2 Bailey lemma.** We now come to the main result of this paper, a new Bailey lemma for the algebra A_2. Guided by Definition 2.1 of an A_1 Bailey pair and its rewriting as (4.1), we use the supernomial (4.6) to define the following A_2 Bailey pair.

Definition 4.1 (A_2 Bailey pair of type I). *Denote $\alpha_k = \alpha_{k_1, k_2, k_3}$ and $\beta_L = \beta_{L_1, L_2}$ and let $\alpha = \{\alpha_k\}_{\substack{k_1 \geq k_2 \geq k_3 \\ k_1+k_2+k_3=0}}$ and $\beta = \{\beta_L\}_{L_1, L_2 \geq 0}$ be a pair of sequences that satisfies*

$$\beta_L = \sum_{\substack{k_1 \geq k_2 \geq k_3 \\ k_1+k_2+k_3=0}} \alpha_k \sum_r \frac{q^{r_1 r_{23}}}{(q)_{r_1} (q)_{r_2} (aq)_{r_3} (q)_{r_{12}} (q)_{r_{13}} (q)_{r_{23}}},$$

where $\sum_r$ denotes a sum over $r_1, \ldots, r_{23}$ such that

$(4.9) \quad r_1 + r_{12} + r_{13} = L_2 - k_1, \quad r_2 + r_{12} + r_{23} = L_2 - k_2, \quad r_3 + r_{13} + r_{23} = L_2 - k_3$

and

$(4.10) \qquad r_1 + r_2 + r_3 = 2L_1 - L_2, \quad r_{12} + r_{13} + r_{23} = 2L_2 - L_1.$

Then (α, β) forms an A_2 Bailey pair of type I relative to a.

Note that for $a = q^\ell$, $\ell = 0, 1, \ldots$, the above definition can be recast in the form

$$\beta_L = \frac{1}{(aq)_{2L_1 - L_2} (q)_{2L_2 - L_1}} \sum_{\substack{k_1 \geq k_2 \geq k_3 \\ k_1+k_2+k_3=0}} \begin{bmatrix} 2L_1 - L_2 + \ell, 2L_2 - L_1 \\ L_2 - k_1, L_2 - k_2, L_2 - k_3 + \ell \end{bmatrix} \alpha_k,$$

which is indeed a generalization of (4.1) using the A_2 supernomial (4.6).

As a second definition we need

Definition 4.2 (A_2 Bailey pair of type II). *Denote* $\alpha_k = \alpha_{k_1,k_2,k_3}$ *and* $\beta_L = \beta_{L_1,L_2}$ *and let* $\alpha = \{\alpha_k\}_{\substack{k_1 \geq k_2 \geq k_3 \\ k_1+k_2+k_3=0}}$ *and* $\beta = \{\beta_L\}_{L_1,L_2 \geq 0}$ *be a pair of sequences that satisfies*

$$\beta_L = \sum_{\substack{k_1 \geq k_2 \geq k_3 \\ k_1+k_2+k_3=0}} \alpha_k \frac{(aq)_{L_1+L_2}}{(aq)_{L_1+k_1}(aq)_{L_1+k_2}(q)_{L_1+k_3}(q)_{L_2-k_1}(q)_{L_2-k_2}(aq)_{L_2-k_3}}.$$

Then (α,β) *forms an* A_2 *Bailey pair of type II relative to* a.

With these definitions our A_2 version of Bailey's lemma reads

Theorem 4.3. *Let* (α,β) *form an* A_2 *Bailey pair of type T=I,II relative to* a. *Then the pair* (α',β') *given by*

$$\alpha'_k = a^{k_1+k_2} q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)} \alpha_k$$

(4.11)
$$\beta'_L = f_L^{(T)} \sum_{r_1=0}^{L_1} \sum_{r_2=0}^{L_2} \frac{a^{r_1} q^{r_1^2 - r_1 r_2 + r_2^2}}{(q)_{L_1-r_1}(q)_{L_2-r_2}} \beta_r$$

forms an A_2 *Bailey pair of type II relative to* a. *Here* $f_L^{(I)} = (aq)_{L_1+L_2}^{-1}$ *and* $f_L^{(II)} = 1$.

Iteration of Theorem 4.3 gives the A_2 Bailey chains

$$(\alpha,\beta)_I \to (\alpha',\beta')_{II} \to (\alpha'',\beta'')_{II} \to \cdots$$

and

$$(\alpha,\beta)_{II} \to (\alpha',\beta')_{II} \to (\alpha'',\beta'')_{II} \to \cdots.$$

Proof of Theorem 4.3. The transformation of a Bailey pair of type I to a Bailey pair of type II follows from the summation formula

(4.12)
$$\sum_{L_1=0}^{M_1} \sum_{L_2=0}^{M_2} \frac{a^{L_1} q^{L_1^2 - L_1 L_2 + L_2^2}}{(q)_{M_1-L_1}(q)_{M_2-L_2}} \sum_r \frac{q^{r_1 r_{23}}}{(q)_{r_1}(q)_{r_2}(aq)_{r_3}(q)_{r_{12}}(q)_{r_{13}}(q)_{r_{23}}}$$
$$= \frac{a^{k_1+k_2} q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)}(aq)_{M_1+M_2}^2}{(aq)_{M_1+k_1}(aq)_{M_1+k_2}(q)_{M_1+k_3}(q)_{M_2-k_1}(q)_{M_2-k_2}(aq)_{M_2-k_3}},$$

where $k_1 + k_2 + k_3 = 0$ and with the same restrictions on the sum over r as given by equations (4.9) and (4.10). We begin by rewriting the sum over r as a sum over just r_{12} and r_{13} using these restrictions and by then making the shifts $r_{12} \to r_{12} + L_2$ and $r_{13} \to r_{13} + L_2$. After replacing r_{12} by m and r_{13} by n the left-hand side of (4.12) thus becomes

$$\sum_{m,n} q^{(k_1+m+n)(m+n)} \sum_{L_1=0}^{M_1} \frac{a^{L_1} q^{L_1(L_1+k_1+m+n)}}{(q)_{M_1-L_1}(q)_{L_1-k_2+n}(aq)_{L_1+m-k_3}(q)_{-L_1-m-n}}$$
$$\times \sum_{L_2=0}^{M_2} \frac{q^{L_2(L_2+m+n)}}{(q)_{M_2-L_2}(q)_{-L_2-k_1-m-n}(q)_{L_2+m}(q)_{L_2+n}}.$$

The sums over L_1 and L_2 have completely decoupled and can both be carried out using the q-Chu–Vandermonde sum (4.7). Collecting terms we are lucky again as the remaining double sum has once more decoupled, leaving us to sum

$$(aq)_{M_1+k_1}(q)_{M_2-k_1} \sum_m \frac{q^{m(m+k_1+k_2)}}{(aq)_{M_1+m-k_3}(q)_{M_2+m}(q)_{-k_2-m}(q)_{-k_1-m}}$$

$$\times \sum_n \frac{a^{k_2-n}q^{n(n+k_1+k_3)}}{(q)_{M_1+n-k_2}(q)_{M_2+n}(aq)_{-k_3-n}(q)_{-k_1-n}}.$$

Now the other instance of q-Chu–Vandermonde is needed [14, Eq. (II.6)]

$$(4.13) \qquad {}_2\phi_1\left[\begin{matrix} q^{-n},b \\ c \end{matrix}; q,q\right] = \frac{b^n(c/b)_n}{(c)_n}$$

allowing both summations to be performed. After a few simplifications this yields the right-hand side of (4.12).

The transformation of a type II to a type II Bailey pair follows from

$$(4.14) \quad \sum_{L_1=0}^{M_1}\sum_{L_2=0}^{M_2} \frac{a^{L_1}q^{L_1^2-L_1L_2+L_2^2}}{(q)_{M_1-L_1}(q)_{M_2-L_2}}$$

$$\times \frac{(aq)_{L_1+L_2}}{(aq)_{L_1+k_1}(aq)_{L_1+k_2}(q)_{L_1+k_3}(q)_{L_2-k_1}(q)_{L_2-k_2}(aq)_{L_2-k_3}}$$

$$= \frac{a^{k_1+k_2}q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)}(aq)_{M_1+M_2}}{(aq)_{M_1+k_1}(aq)_{M_1+k_2}(q)_{M_1+k_3}(q)_{M_2-k_1}(q)_{M_2-k_2}(aq)_{M_2-k_3}}$$

for $k_1+k_2+k_3=0$. Without loss of generality we can assume $k_1 \geq k_2 \geq k_3$ so that $k_1 \geq 0$ and $k_3 \leq 0$. Shifting $L_2 \to L_2+k_1$ the left-hand side becomes

$$\sum_{L_1} \frac{a^{L_1}q^{L_1^2-L_1k_1+k_1^2}}{(q)_{M_1-L_1}(aq)_{L_1+k_1}(aq)_{L_1+k_2}(q)_{L_1+k_3}}$$

$$\times \sum_{L_2} \frac{q^{L_2(L_2-L_1+2k_1)}(aq)_{L_1+L_2+k_1}}{(q)_{M_2-L_2-k_1}(q)_{L_2}(q)_{L_2+k_1-k_2}(aq)_{L_2+k_1-k_3}}.$$

Next we require the transformation

$$\lim_{c\to\infty} {}_3\phi_2\left[\begin{matrix} q^{-n},b,c \\ d,e \end{matrix}; q, \frac{deq^n}{bc}\right] = \frac{1}{(e)_n} {}_2\phi_1\left[\begin{matrix} q^{-n},d/b \\ d \end{matrix}; q, eq^n\right]$$

which is a special case of the q-Kummer–Thomae–Whipple transformation for ${}_3\phi_2$ series [14, Eq. (III.9)] (see also (5.11)). Applying this to the sum over L_2 and making some simplifications leads to

$$\frac{q^{k_1^2}}{(q)_{M_2-k_2}} \sum_j \frac{q^{j(j+k_1-k_2-k_3)}}{(q)_j(q)_{M_2-k_1-j}(aq)_{k_1-k_3+j}} \sum_{L_1} \frac{a^{L_1}q^{L_1(L_1-k_1-j)}}{(q)_{M_1-L_1}(aq)_{L_1+k_2}(q)_{L_1+k_3-j}}.$$

Now shift $L_1 \to L_1-k_3+j$ and perform the sum over L_1 using (4.7) with $b \to \infty$. This gives

$$\frac{a^{-k_3}q^{k_1^2-k_2k_3}}{(q)_{M_2-k_2}(aq)_{M_1+k_2}} \sum_j \frac{a^jq^{j(j+k_1-k_3)}}{(q)_j(q)_{M_1+k_3-j}(q)_{M_2-k_1-j}(aq)_{k_1-k_3+j}}.$$

The final sum can again be carried out using (4.7) yielding the desired right-hand side of (4.14). □

5. Applications

In this section several applications of our A_2 Bailey lemma will be given. Many of the resulting Rogers–Ramanujan-type identities can be recognized as identities for characters of the W_3 algebra, and to facilitate discussions we give a brief introduction to the characters of the W_n algebras in Section 5.1. In Section 5.2 a supernomial identity is proven which serves as a seed to the A_2 Bailey lemma. Then, in Sections 5.3–5.7 A_2 Rogers–Ramanujan-type identities are derived. Finally, some identities for Kostka polynomials are proven in Section 5.8.

5.1. Characters of the W_n algebra.
The W_n algebra, introduced by Zamolodchikov [40] and Fateev and Lykyanov [12], is an A_{n-1} generalization of the well-known Virasoro algebra. The minimal series of W_n algebras is labelled by two integers p and p' such that $\gcd(p, p') = 1$ and $n \leq p < p'$, with central charge

$$c = (n-1)\Big(1 - \frac{n(n+1)(p'-p)^2}{pp'}\Big).$$

In the remainder we denote this series by $M(p, p')_n$. Let $\Lambda_0, \ldots, \Lambda_{n-1}$ be the fundamental weights and $P_+(n, \ell)$ the set of level-ℓ dominant integral weights of $A_{n-1}^{(1)}$. Then the $M(p, p')_n$ character of the highest-weight representation labelled by the pair $\xi \in P_+(n, p-n)$ and $a \in P_+(n, p'-n)$ is given by [32]

$$\chi_{\xi,a}^{(p,p')}(q) = \frac{1}{\eta(q)^{n-1}} \sum_{\alpha \in Q} \sum_{w \in \mathcal{W}_n} \epsilon(w) q^{\frac{1}{2}pp'|\alpha - (\xi+\rho)/p + w(a+\rho)/p'|^2}.$$

Here Q is the root lattice and $\mathcal{W}_n$ the Weyl group of A_{n-1}, $\epsilon(w) = (-1)^{\ell(w)}$ with $\ell(w)$ the length of w, ρ is the Weyl vector and $\eta(q)$ is the Dedekind η-function. We remark that the characters of $M(p, p')_n$ can be identified with the branching coefficients of the coset pair $(A_{n-1}^{(1)} \oplus A_{n-1}^{(1)}, A_{n-1}^{(1)})$ at levels $p/(p'-p) - n$, 1 and $p'/(p'-p) - n$, respectively [19, 33].

Of particular interest to us here are the characters of $M(n, k)_n$. In this case $\xi = 0$ and the characters are labelled by a single level-$(k-n)$ dominant integral weight which we write as $j = \sum_{a=0}^{n-1}(j_a - 1)\Lambda_a$, with $j_0, \ldots, j_{n-1} \geq 1$ and $j_0 + \cdots + j_{n-1} = k$. Invoking the A_{n-1} Macdonald identity [11, 25]

$$(5.1) \qquad \sum_{k_1 + \cdots + k_n = 0} \prod_{i=1}^{n} x_i^{nk_i} q^{\frac{n}{2}k_i^2 + ik_i} \prod_{1 \leq i < j \leq n} (1 - x_i x_j^{-1} q^{k_i - k_j})$$

$$= (q)_\infty^{n-1} \prod_{1 \leq i < j \leq n} (x_i x_j^{-1}, q x_j x_i^{-1})_\infty$$

the sum over Q and $\mathcal{W}_n$ (i.e., the affine Weyl group) can be carried out to yield the following product representation of the (normalized) $M(n, k)_n$ characters

$$(5.2) \quad \chi_j^{(n,k)}(q) = \chi_{(j_0, \ldots, j_{n-1})}^{(n,k)}(q) = \frac{(q^k; q^k)_\infty^{n-1}}{(q; q)_\infty^{n-1}} \prod_{a=1}^{n-1} \prod_{b=0}^{n-1} (q^{j_b + j_{b+1} + \cdots + j_{a+b-1}}; q^k)_\infty$$

with the convention that $j_i = j_{i'}$ if $i \equiv i' \pmod{n}$.

5.2. **A supernomial identity.** As a first application of the A_2 Bailey lemma we wish to repeat the working of Section 3 to obtain A_2 analogues of equation (3.1). Our starting point is the following supernomial analogue of Identity (2.8), which is the first instance of a hierachy of supernomial identities conjectured in [38].

Proposition 5.1. *For L_1, L_2, ℓ integers such that $2L_1 - L_2 + \ell \geq 0$, $2L_2 - L_1 \geq 0$ and $\ell \geq 0$,*

(5.3)
$$\sum_{k_1+k_2+k_3=0} \sum_{\sigma \in S_3} \epsilon(\sigma) q^{\frac{1}{2}\sum_{s=1}^{3}(3k_s-2\sigma_s)k_s} \begin{bmatrix} 2L_1-L_2+\ell, 2L_2-L_1 \\ L_2-3k_1+\sigma_1-1+\ell, L_2-3k_2+\sigma_2-2, L_2-3k_3+\sigma_3-3 \end{bmatrix}$$
$$= \frac{q^{L_1^2-L_1L_2+L_2^2+\ell L_1}(q)_\ell}{(q^{-\ell})_{-L_1}(q)_{-L_2}},$$

where S_3 is the permutation group on $1, 2, 3$ and $\epsilon(\sigma)$ is the sign of the permutation σ.

Note that for $L_1, L_2 \geq 0$ the right-hand side simplifies to $(q)_\ell \delta_{L_1,0} \delta_{L_2,0}$. Proposition 5.1 is a corollary of the following stronger result.

Lemma 5.2. *For L, a, ℓ integers such that $0 \leq 2a \leq 3L + \ell$ and $\ell \geq 0$,*

(5.4)
$$\sideset{}{'}\sum_{\substack{k_1+k_2+k_3=0 \\ \sigma \in S_3}} \epsilon(\sigma) q^{\frac{1}{2}\sum_{s=1}^{3}(3k_s-2\sigma_s)k_s} \begin{bmatrix} 3L-2a+\ell, a \\ L-3k_1+\sigma_1-1+\ell, L-3k_2+\sigma_2-2, L-3k_3+\sigma_3-3 \end{bmatrix}$$
$$= (-1)^t q^{\binom{t+1}{2}+(L-a)(L-a+t)} \begin{bmatrix} 3L-a+\ell \\ 2L-a+t \end{bmatrix},$$

where the sum on the left-hand side is such that $t = 3k_1 + 1 - \sigma_1$ is fixed.

Replacing $L \to L_2$ and $a \to 2L_2 - L_1$, summing over t using the q-binomial theorem $_1\phi_0(q^{-n}; -; q, z) = (zq^{-n})_n$ [14, Eq. (II.4)] and respecting the ranges of L_1, L_2 and ℓ yields (5.3). Hence we are left to prove (5.4).

For the proof of Lemma 5.2 we need the following three functions introduced by Andrews *et al.* [5],

$$\gamma(N, M) = \sum_i \left\{ q^{6i^2+2i} \begin{bmatrix} N+M \\ N-3i \end{bmatrix} - q^{6i^2-2i} \begin{bmatrix} N+M \\ N-3i+1 \end{bmatrix} \right\}$$

(5.5)
$$\delta(N, M) = \sum_i \left\{ q^{6i^2-i} \begin{bmatrix} N+M \\ N-3i \end{bmatrix} - q^{6i^2-5i+1} \begin{bmatrix} N+M \\ N-3i+1 \end{bmatrix} \right\}$$

$$\varepsilon(N, M) = \sum_i \left\{ q^{6i^2-4i} \begin{bmatrix} N+M \\ N-3i \end{bmatrix} - q^{6i^2-8i+2} \begin{bmatrix} N+M \\ N-3i+1 \end{bmatrix} \right\}.$$

Observe that γ and ε obey the symmetries

(5.6)
$$\gamma(N, M) = -\gamma(M-1, N+1)$$
$$\varepsilon(N, M) = -\varepsilon(M+2, N-2).$$

All three functions have another sum representation stated in the following lemma.

Lemma 5.3. *The functions γ, δ and ε as defined in* (5.5) *satisfy*

$$\gamma(N, M) = \sum_{h=0}^{\infty} (-1)^{h+1} q^{(2h+1)(N+1)+\binom{h+1}{2}} \frac{(q^{M-N-h-1})_{2h+1}}{(q)_{2h+1}}$$

(5.7) $$\delta(N, M) = \sum_{h=0}^{\infty} (-1)^h q^{2hN+\binom{h+1}{2}} \frac{(q^{M-N-h+1})_{2h}}{(q)_{2h}}$$

$$\varepsilon(N, M) = \sum_{h=0}^{\infty} (-1)^h q^{(2h+1)N+\binom{h}{2}-1} \frac{(q^{M-N-h+2})_{2h+1}}{(q)_{2h+1}}.$$

Proof. Using the q-binomial recurrences

(5.8) $$\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{bmatrix} m+n-1 \\ m \end{bmatrix} + q^n \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix} = q^m \begin{bmatrix} m+n-1 \\ m \end{bmatrix} + \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}$$

one can show that γ, δ and ε of equation (5.5) as well as the expressions in (5.7) obey the recurrences

$$\gamma(N, M) = \gamma(N-1, M) + q^N \delta(N, M-1)$$
$$\delta(N, M) = \delta(N-1, M) + q^N \varepsilon(N, M-1) = \delta(N, M-1) + q^M \gamma(N-1, M)$$
$$\varepsilon(N, M) = \varepsilon(N, M-1) + q^M \delta(N-1, M).$$

Together with the initial conditions $\delta(N, N) = \delta(N+1, N) = 1$ and $\gamma(N, N+1) = \varepsilon(N, N-2) = 0$ this specifies the three functions uniquely. For the representations in (5.7) these initial conditions can be easily checked. For γ and ε in (5.5) they follow from the symmetries (5.6), and for δ of equation (5.5) these are Schur's bounded analogues [39] of Euler's identity. $\square$

Also necessary for the proof of Lemma 5.2 is the next result.

Lemma 5.4. *For integers M, b, a, h such that $M \geq 0$, $0 \leq b \leq M$, $0 \leq a \leq M$ and $0 \leq 2h \leq a$, the following identity holds:*
(5.9)
$$\sum_{k,m} (-1)^{k+m} q^{\binom{m}{2}-mh+\binom{k+1}{2}+k(b-2h)} \begin{bmatrix} M-k-m \\ b-m \end{bmatrix} \begin{bmatrix} a \\ m \end{bmatrix} \begin{bmatrix} a-m \\ k \end{bmatrix} \frac{(q^{h-k-m+1})_{a-2h}}{(q)_{a-2h}} = \begin{bmatrix} M \\ b \end{bmatrix} \delta_{h,0}.$$

Proof. First we establish a symmetry of equation (5.9) which will be used later. More precisely, we consider the sum on the left-hand side of the above identity divided by $\begin{bmatrix} M \\ b \end{bmatrix}$ which we denote, suppressing the M and h dependence, by $g(a, b)$. (In the following we always assume that M, b, a, h satisfy the ranges as specified by the lemma.)

We start by transforming the sum over k as follows. When $h-k-m \geq 0$ we apply Sears' $_3\phi_2$ transformation [14, Eq. (III.11)]

(5.10) $$_3\phi_2 \begin{bmatrix} q^{-n}, b, c \\ d, e \end{bmatrix}; q, q \end{bmatrix} = \frac{(de/bc)_n}{(e)_n} \left(\frac{bc}{d}\right)^n {}_3\phi_2 \begin{bmatrix} q^{-n}, d/b, d/c \\ d, de/bc \end{bmatrix}; q, q \end{bmatrix},$$

with $n = h-m$ and $d = q^{-(M-m)}$. When $h-k-m < 0$ we replace $k \to a-m-k$ and apply Sears' transformation [14, Eq. (III.9)]

(5.11) $$_3\phi_2 \begin{bmatrix} q^{-n}, b, c \\ d, e \end{bmatrix}; q, \frac{deq^n}{bc} \end{bmatrix} = \frac{(de/bc)_n}{(e)_n} {}_3\phi_2 \begin{bmatrix} q^{-n}, d/b, d/c \\ d, de/bc \end{bmatrix}; q, eq^n \end{bmatrix},$$

with $n = h - 1$ and $d = q^{M+m-a-b+1}$, followed by $k \to b - m - k$. As a result of all this one finds that

$$g(a,b) = \sum_{k,m}(-1)^{k+m}q^{\binom{m}{2}-mh+\binom{k+1}{2}+k(a-2h)}\begin{bmatrix}M-k-m\\b-k-m\end{bmatrix}\begin{bmatrix}a\\m\end{bmatrix}\begin{bmatrix}M-a\\k\end{bmatrix}/\begin{bmatrix}M\\b\end{bmatrix}$$

$$\times\left\{\frac{(q^{b-2h+1})_{h-k-m}}{(q)_{h-k-m}} + (-1)^b q^{\frac{1}{2}(b-2h)(b-2k-2m+1)}\frac{(q^{b-2h+1})_{k+m-b+h-1}}{(q)_{k+m-b+h-1}}\right\}.$$

Using

$$(q^{-n})_k = (-1)^k q^{\binom{k}{2}-nk}\frac{(q)_n}{(q)_{n-k}}, \qquad n \geq 0,$$

it can be observed that the two terms in the summand cancel when $b - 2h + 1 \leq 0$. Hence we can assume $b - 2h \geq 0$ and use on the first term of the summand

$$\frac{(q^{n+1})_k}{(q)_k} = \frac{(q^{k+1})_n}{(q)_n}, \qquad k, n \geq 0$$

and on the second term

$$\frac{(q^{n+1})_k}{(q)_k} = (-1)^n q^{\binom{n+1}{2}+nk}\frac{(q^{-k-n})_n}{(q)_n}, \qquad k, n \geq 0.$$

This leads to the identity

$$g(a,b) =$$

$$\sum_{k,m}(-1)^{k+m}q^{\binom{m}{2}-mh+\binom{k+1}{2}+k(a-2h)}\begin{bmatrix}M-k-m\\b-k-m\end{bmatrix}\begin{bmatrix}a\\m\end{bmatrix}\begin{bmatrix}M-a\\k\end{bmatrix}\frac{(q^{h-k-m+1})_{b-2h}}{(q)_{b-2h}}/\begin{bmatrix}M\\b\end{bmatrix}.$$

(The two contributing terms to this equation correspond to $k + m < h + 1$ and $m + k \geq h + 1$, respectively). Reshuffling the four q-binomials this can be recognized as $g(b,a)$. Note that in establishing this symmetry of g we have also proven (5.9) for $b - 2h < 0$.

We now come to the actual proof of Lemma 5.4. First assume that $h = 0$. The third q-binomial implies that the summand vanishes unless $a - m - k \geq 0$. But in this range $(q^{1-k-m})_a$ is nonzero for $m = k = 0$ only and the result is immediate.

The case when h is strictly positive is not so straight-forward and will be proven by induction on M and b. Let us denote the left-hand side of (5.9) by $f(M,b,a,h)$. By (5.8) f satisfies the recurrence

$$f(M,b,a,h) = f(M-1,b,a,h) + q^{M-b}f(M-1,b-1,a,h),$$

for $0 \leq b \leq M$ and $0 \leq a < M$, so that (5.9) will be proven if we can show its validity in the three cases $b = 0$, $b = M$ and $a = M$. By the symmetry between a and b it is sufficient to treat the cases $b = 0$ and $b = M$ only.

The case $b = 0$ is the simplest. All we need to show is that

$$\sum_k(-1)^k q^{\binom{k+1}{2}-2kh}\begin{bmatrix}a\\k\end{bmatrix}\frac{(q^{h-k+1})_{a-2h}}{(q)_{a-2h}} = 0,$$

for $0 < 2h \leq a$. To achieve this we split the summand according to whether $h - k \geq 0$ or $h - k + 1 \leq 0$. In the latter case we shift $k \to k - h + a + 1$. Now both terms admit the sum over k to be performed using the q-Chu–Vandermonde summation (4.13), yielding $(-1)^h q^{-\frac{h}{2}(3h-1)}\begin{bmatrix}2h-1\\h\end{bmatrix}$ and its negative, and we are done.

When $b = M$ we are to show that

$$\sum_m (-1)^m q^{\binom{m}{2}-mh} \begin{bmatrix} a \\ m \end{bmatrix} \frac{(q^{h-m+1})_{a-2h}}{(q)_{a-2h}} = 0,$$

for $0 < 2h \le a$. Again we treat $h - m \ge 0$ and $h - m + 1 \le 0$ separately and shift $m \to m - h + a + 1$ in the latter case. In hypergeometric notation we then need to show the identity

$$(-1)^h q^{\binom{h+1}{2}} \begin{bmatrix} a - h \\ h \end{bmatrix} {}_2\phi_1 \begin{bmatrix} q^{-h}, q^{-a} \\ q^{-a+h} \end{bmatrix}; q, q^h \end{bmatrix} = \begin{bmatrix} a \\ h - 1 \end{bmatrix} {}_2\phi_1 \begin{bmatrix} q^{-h+1}, q^{a-2h+1} \\ q^{a-h+2} \end{bmatrix}; q, q^h \end{bmatrix}.$$

To make further progress we apply Jackson's transformation [14, (III.7)]

$$(5.12) \qquad {}_2\phi_1 \begin{bmatrix} q^{-n}, b \\ c \end{bmatrix}; q, z \end{bmatrix} = \frac{(c/b)_n}{(c)_n} {}_3\phi_2 \begin{bmatrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c, 0 \end{bmatrix}; q, q \end{bmatrix}.$$

Dropping common factors this yields

$${}_3\phi_2 \begin{bmatrix} q^{-h}, q^{-h}, q^{-a} \\ q^{-2h+1}, 0 \end{bmatrix}; q, q \end{bmatrix} = q^{h(h-a-1)} {}_3\phi_2 \begin{bmatrix} q^{-h}, q^{-h+1}, q^{a-2h+1} \\ q^{-2h+1}, 0 \end{bmatrix}; q, q \end{bmatrix},$$

which can be recognized as a specialization of Sears' ${}_3\phi_2$ transformation (5.10). $\quad\square$

We finally come to the proof of Lemma 5.2.

Proof of Lemma 5.2. Since in equation (5.4) σ_1 only takes the values $1, 2$ or 3, the condition $t = 3k_1 + 1 - \sigma_1$ fixes both k_1 and σ_1. Therefore the restrictions $k_1 + k_2 + k_3 = 0$ and $\sigma_1 + \sigma_2 + \sigma_3 = 6$ reduce the primed sum in (5.4) to a sum over only two independent summation variables. These we choose to be $\tau = \sigma_2 - 2$ and $i = k_2$. Further setting $k_1 = \eta$ and using the representation (4.5) of the supernomial with $m \to L - 3i - m + \tau$, the left-hand side of (5.4) reads

$$(5.13) \quad \sum_{i,m,\tau} \epsilon(\tau) q^{3(\eta^2+i^2+i\eta)+(4-\tau-2\sigma_1)\eta+(2-2\tau-\sigma_1)i}$$

$$\times \begin{bmatrix} 3L-2a+\ell \\ 2L-a+\ell-3i-m-t+\tau \end{bmatrix} \begin{bmatrix} 2L+\ell-3i-2m-t+\tau \\ L-3i-m+\tau \end{bmatrix} \begin{bmatrix} a \\ m \end{bmatrix}.$$

Here $\epsilon(\tau)$ denotes the sign of the permutation $(\sigma_1, \tau + 2, 4 - \sigma_1 - \tau) \in S_3$. Now apply

$$\begin{bmatrix} A \\ B \end{bmatrix} = \sum_{k=0}^n (-1)^k q^{\binom{k+1}{2}+Bk} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} A + n - k \\ B + n \end{bmatrix},$$

(which is a specialization of the q-Chu–Vandermonde summation (4.13)) with $n = a - m$ to the first q-binomial in (5.13), to get

$$(5.14)$$
$$\sum_{m,k} (-1)^k q^{\binom{k+1}{2}+(L-a+\ell-t)k+(L-m)(L+m-a+t)+\eta(t-\sigma_1+3)} \begin{bmatrix} 3L-a+\ell-m-k \\ L+\ell-m-t \end{bmatrix} \begin{bmatrix} a-m \\ k \end{bmatrix} \begin{bmatrix} a \\ m \end{bmatrix}$$

$$\times \left\{ \sum_{i,\tau} \epsilon(\tau) q^{6i^2+i(2t+6m-3a-1-4\tau)-\tau(2m-a+t-\eta-\tau)} \begin{bmatrix} 2L-a-k+t \\ L-3i-m+\tau \end{bmatrix} \right\}_{1/q}.$$

The expression within curly brackets can be expressed in terms of the functions in (5.5) depending on the parity of a. When $a = 2\alpha$ one finds

$$q^{-\frac{1}{2}\{3(\alpha-\eta-m)^2+(2\sigma_1-1)(\alpha-\eta-m)+(\sigma_1-1)(\sigma_1-2)\}}$$

$$\times\Big\{\chi(\alpha+m+t \text{ even})\, \delta(L+\frac{1}{2}(t-3\alpha+m), L+\frac{1}{2}(t-\alpha-m)-k)$$

$$-\chi(\alpha+m+t \text{ odd})\, \delta(L+\frac{1}{2}(t-\alpha-m+1)-k, L+\frac{1}{2}(t-3\alpha+m-1))\Big\}$$

and when $a = 2\alpha + 1$

$$q^{-\frac{1}{2}\{3(\alpha-\eta-m)^2+(2\sigma_1+2)(\alpha-\eta-m)+(\sigma_1-1)^2\}}$$

$$\times\Big\{\chi(\alpha+m+t \text{ even})\, \varepsilon(L+\frac{1}{2}(t-3\alpha+m), L+\frac{1}{2}(t-\alpha-m-2)-k)$$

$$-\chi(\alpha+m+t \text{ odd})\, q^{-\frac{1}{2}}\gamma(L+\frac{1}{2}(t-\alpha-m-1)-k, L+\frac{1}{2}(t-3\alpha+m-1))\Big\}$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

We now insert the representations for γ, δ and ε provided by Lemma 5.3. After the shift $h \to \alpha - h$, expression (5.14) becomes for both parities of a

$$(-1)^t q^{\binom{t+1}{2}+(L-a)(L-a+t)} \sum_{0\leq h\leq a/2} \sum_{k,m}(-1)^{h+k+m}q^{\binom{h+1}{2}+h(h+2L+t-2a)}$$

$$\times q^{\binom{m}{2}-mh+\binom{k+1}{2}+k(L+\ell-t-2h)}\begin{bmatrix}3L-a+\ell-m-k\\L+\ell-t-m\end{bmatrix}\begin{bmatrix}a-m\\k\end{bmatrix}\begin{bmatrix}a\\m\end{bmatrix}\frac{(q^{h-m-k+1})_{a-2h}}{(q)_{a-2h}}.$$

Applying Lemma 5.4 to this expression with $M = 3L - a + \ell$ and $b = L + \ell - t$ (so that indeed $0 \leq 2a \leq 3L + \ell$) yields the right-hand side of (5.4). $\square$

5.3. **Character identities for $M(3, 3k+1)_3$.** We now use the supernomial identity (5.3) to derive q-series identities for $M(3, 3k+1)_3$ characters. Since we will be dealing with quite lengthy expressions some shorthand notation is needed. For $v \in \mathbb{Z}^2$, set $|v| = v_1 + v_2$, $(a)_v = (a)_{v_1}(a)_{v_2}$ and $vAv = \sum_{i,j=1}^2 v_i A_{i,j} v_j$, with A a two-by-two matrix. Thus $v_1^2 - v_1 v_2 + v_2^2 = \frac{1}{2}vCv$, with C the Cartan matrix of A_2. With this notation (4.11) can, for example, be rewritten as

$$\beta'_L = f_L^{(T)} \sum_{r\in\mathbb{Z}_+^2} \frac{a^{r_1}q^{\frac{1}{2}rCr}}{(q)_{L-r}}\, \beta_r.$$

We now proceed as in Section 3. First we take (5.3) with $\ell = 0$ and iterate the corresponding A_2 Bailey pair of type I (relative to $a = 1$) once. This gives a Bailey pair of type II in the form of a doubly bounded version of the A_2 Euler identity, (5.15)

$$\sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)} \sum_{\sigma\in S_3} \epsilon(\sigma)\prod_{s=1}^3 q^{\frac{1}{2}(3k_s-\sigma_s+s)^2-\sigma_s k_s}\begin{bmatrix}L_1+L_2\\L_1+3k_s-\sigma_s+s\end{bmatrix} = \begin{bmatrix}L_1+L_2\\L_1\end{bmatrix}.$$

According to our recipe we should rewrite this as a sum over determinants such that these determinants can be evaluated explicitly, whilst retaining an expression of A_2 Bailey type. This can be achieved by making the variable changes $k_i \to -k_{\sigma_i}$

yielding the following determinantal form for the left-hand side of (5.15)

$$(5.16) \qquad \sum_{k_1+k_2+k_3=0} q^{6(k_1^2+k_2^2+k_3^2)+4(k_1+2k_2+3k_3)} \det_{1\le s,t\le 3}\left(q^{t(t-s-3k_s)}\begin{bmatrix}L_1+L_2\\L_1-3k_s+t-s\end{bmatrix}\right).$$

Determinants of the above type have been frequently encountered in the theory of plane partitions, and Krattenthaler [23, page 189] has shown that
(5.17)

$$\det_{1\le s,t\le n}\left(q^{t(t-B_s)}\begin{bmatrix}L_1+L_2\\L_1-B_s+t\end{bmatrix}\right) = \prod_{1\le s<t\le n}(1-q^{B_t-B_s})\prod_{s=1}^{n}\frac{q^{s(s-B_s)}(q)_{L_1+L_2+s-1}}{(q)_{L_1-B_s+n}(q)_{L_2+B_s-1}}.$$

Hence the determinants in (5.16) can be evaluated leading to

$$\sum_{k_1+k_2+k_3=0} q^{6(k_1^2+k_2^2+k_3^2)+k_1+2k_2+3k_3}\prod_{1\le s<t\le 3}(1-q^{3k_t-3k_s+t-s})\prod_{s=1}^{3}\begin{bmatrix}L_1+L_2+2\\L_1-3k_s-s+3\end{bmatrix}$$

$$= \frac{(q)_{L_1+L_2+2}^2}{(q)_{L_1+L_2+1}(q)_{L_1}(q)_{L_2}}.$$

We observe a slight difference with the case treated in Section 3. There we started with a Bailey pair relative to $a = 1$ and rewrote it as a Bailey pair relative to $a = q$. Here the analogous rewriting using determinants has transformed a Bailey pair relative to $a = 1$ into a new Bailey pair relative to 1. Iterating (the Bailey pair of type II implied by) the previous equation $k - i$ times leads to

$$\sum_{k_1+k_2+k_3=0} q^{\sum_{s=1}^{3} k_s\left(6k_s+s+\frac{3}{2}(k-i)(3k_s+2s)\right)}$$

$$\times \prod_{1\le s<t\le 3}(1-q^{3k_t-3k_s+t-s})\prod_{s=1}^{3}\begin{bmatrix}L_1+L_2+2\\L_1-3k_s-s+3\end{bmatrix}$$

$$= \sum_{r^{(j)}\in\mathbb{Z}^2}\frac{q^{\sum_{j=1}^{k-i}(\frac{1}{2}r^{(j)}Cr^{(j)}+|r^{(j)}|)}(q)_{L_1+L_2+2}^2}{(q)_{L-r^{(1)}}\cdots(q)_{r^{(k-i-1)}-r^{(k-i)}}(q)_{r^{(k-i)}}(q)_{|r^{(k-i)}|+1}}.$$

Now we again apply (5.17) for $n = 3$ so that

$$\sum_{k_1+k_2+k_3=0} q^{\frac{1}{2}(3k-3i+4)\sum_{s=1}^{3}(3k_s+2s)k_s}\det_{1\le s,t\le 3}\left(q^{t(t-s-3k_s)}\begin{bmatrix}L_1+L_2\\L_1-3k_s+t-s\end{bmatrix}\right)$$

$$= \sum_{r^{(j)}\in\mathbb{Z}^2}\frac{q^{\sum_{j=1}^{k-i}(\frac{1}{2}r^{(j)}Cr^{(j)}+|r^{(j)}|)}(q)_{L_1+L_2}(q)_{L_1+L_2+1}}{(q)_{L-r^{(1)}}\cdots(q)_{r^{(k-i-1)}-r^{(k-i)}}(q)_{r^{(k-i)}}(q)_{|r^{(k-i)}|+1}}.$$

As a final step we iterate this equation $i - 1$ times arriving at

$$(5.18) \qquad \sum_{k_1+k_2+k_3=0} q^{\frac{1}{2}(3k+1)\sum_{s=1}^{3}(3k_s+2s)k_s}\det_{1\le s,t\le 3}\left(q^{it(t-s-3k_s)}\begin{bmatrix}L_1+L_2\\L_1-3k_s+t-s\end{bmatrix}\right)$$

$$= \sum_{r^{(j)}\in\mathbb{Z}^2}\frac{q^{\frac{1}{2}\sum_{j=1}^{k-1}r^{(j)}Cr^{(j)}+\sum_{j=i}^{k-1}|r^{(j)}|}(1-q^{|r^{(i-1)}|+1})(q)_{L_1+L_2}^2}{(q)_{L-r^{(1)}}\cdots(q)_{r^{(k-2)}-r^{(k-1)}}(q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}.$$

for all $k \ge 1$ and $i = 1,\ldots,k$. When $i = 1$ one should identify $(1-q^{|r^{(0)}|+1})$ with $(1-q^{L_1+L_2+1})$.

Many more identities can be derived using (5.3) for general ℓ. However, for $\ell \geq 1$ one cannot evaluate any determinants thus yielding a one-parameter family of identities for each choice of $\ell(\geq 1)$ only. A further complication arises from the fact that the right-hand side of (5.3) is only a simple $(q)_\ell \delta_{L_1,0} \delta_{L_2,0}$ for $\ell = 0, 1$. As a consequence we find

$$(5.19) \quad \sum_{k_1+k_2+k_3=0} q^{\frac{1}{2}(3k+1)\sum_{s=1}^3(3k_s+2s)k_s}$$

$$\times \det_{1 \leq s,t \leq 3} \left(q^{k(t+\ell\chi(t>1))(t-s-3k_s)} \begin{bmatrix} L_1+L_2+\ell \\ L_1-3k_s+t-s+\ell\chi(t>1) \end{bmatrix} \right)$$

$$= \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^k (1+\delta_{j,k})(\frac{1}{2}r^{(j)}Cr^{(j)}+\ell r_1^{(j)})}(q)_{L_1+L_2+\ell}^2 (q)_\ell}{(q)_{L-r^{(1)}} \cdots (q)_{r^{(k-1)}-r^{(k)}}(q)_{|r^{(k-1)}|+\ell}(q)_{Cr^{(k)}+\ell e_1}(q^{-\ell})_{-r_1^{(k)}}(q)_{-r_2^{(k)}}},$$

where for arbitrary ℓ we cannot perform the sum over $r^{(k)}$. In this equation $\chi(\text{true}) = 1$, $\chi(\text{false}) = 0$, $Cr^{(k)} = (2r_1^{(k)} - r_2^{(k)}, 2r_2^{(k)} - r_1^{(k)})$ and $e_1 = (1,0)$. The sum over $r^{(k)} \in \mathbb{Z}^2$ is understood to be a sum such that $Cr^{(k)} + \ell e_1 \in \mathbb{Z}_+^2$. (Of course $1/(q)_{Cr^{(k)}+\ell e_1}$ is zero outside this range, but we want to stay clear from $r_1^{(k)} + \ell < 0$.)

As mentioned earlier, for $\ell = 0, 1$ the sum over $r^{(k)}$ trivializes since the kernel is only non-zero for $r^{(k)} = 0$. This simplifies the right-hand side of (5.19) to

$$(5.20) \quad \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^{k-1}(\frac{1}{2}r^{(j)}Cr^{(j)}+\ell r_1^{(j)})}(q)_{L_1+L_2+\ell}^2}{(q)_{L-r^{(1)}}(q)_{r^{(1)}-r^{(2)}} \cdots (q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+\ell}}, \qquad \ell = 0, 1.$$

For $\ell = 2$ we get contributions to the sum for $r^{(k)} = 0$ and $r^{(k)} = -e_1$. Combining these two terms and making the shift $r^{(j)} \to r^{(j)} - e_1$ ($j = 1, \ldots, k-1$) one finds that the right-hand side of (5.19) with $\ell = 2$ simplifies to

$$(5.21) \quad -q^{1-k} \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^{k-1}(\frac{1}{2}r^{(j)}Cr^{(j)}+r_2^{(j)})+r_1^{(k-1)}}(q)_{L_1+L_2+2}^2}{(q)_{L+e_1-r^{(1)}}(q)_{r^{(1)}-r^{(2)}} \cdots (q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}.$$

Note that for $k = 2$ equations (5.20) with $\ell = 1$ and (5.21) are equal up to an overall factor and a shift in L_1.

To transform the above polynomial identities into identities of the Rogers–Ramanujan type we let L_1, L_2 tend to infinity and apply the Macdonald identity (5.1) with $n = 3$. We can thus claim the following q-series identities.

Theorem 5.5. *Let $|q| < 1$ and $k \geq 2$. Then*

$$(5.22) \quad \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\frac{1}{2}\sum_{j=1}^{k-1} r^{(j)}Cr^{(j)}+\sum_{j=i}^{k-1}|r^{(j)}|}(1-q^{|r^{(i-1)}|+1})}{(q)_{r^{(1)}-r^{(2)}} \cdots (q)_{r^{(k-2)}-r^{(k-1)}}(q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}$$

$$= \frac{(q^i, q^i, q^{2i}, q^{3k-2i+1}, q^{3k-i+1}, q^{3k-i+1}, q^{3k+1}, q^{3k+1}; q^{3k+1})_\infty}{(q)_\infty^3}$$

for $i = 1, \ldots, k$, and

$$(5.23) \quad \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^{k-1}(\frac{1}{2}r^{(j)}Cr^{(j)}+r_1^{(j)})+\sigma r_2^{(k-1)}}}{(q)_{r^{(1)}-r^{(2)}} \cdots (q)_{r^{(k-2)}-r^{(k-1)}}(q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}$$

$$= \frac{(q, q^{k-\sigma}, q^{k+1-\sigma}, q^{2k+\sigma}, q^{2k+1+\sigma}, q^{3k}, q^{3k+1}, q^{3k+1}; q^{3k+1})_\infty}{(q)_\infty^3}$$

for $\sigma = 0, 1$.

Recalling the remark made after (5.18) we have $(1-q^{|r^{(0)}|+1}) = 1$ in the $i = 1$ case of (5.22). Note that despite the factor $(1 - q^{|r^{(i-1)}|+1})$ the left-hand side of (5.22) is a series with positive coefficients since the summand is zero for $r^{(k-1)} \notin \mathbb{Z}_+^2$ and

$$(1 - q^{|r^{(i-1)}|+1})/(q)_{|r^{(k-1)}|+1} = \begin{bmatrix} |r^{(i-1)}|+1 \\ 1 \end{bmatrix}/(q^2)_{|r^{(k-1)}|}.$$

Comparing the right-hand sides of the above two formulas with the character expression (5.2) of the W_n algebra, one can identify (5.22) and (5.23) as identities for the $M(3, 3k+1)_3$ characters $\chi_{(i,i,3k+1-2i)}^{(3,3k+1)}(q)$ and $\chi_{(1,k-\sigma,2k+\sigma)}^{(3,3k+1)}(q)$, respectively. However, in doing so we are confronted with the unpleasant fact that we have to multiply both sides of (5.22) and (5.23) by a factor $(q)_\infty$. But then the left-hand sides are no longer series with manifestly positive (integer) coefficients. It thus is desirable to find a summation formula that allows the left-hand sides of (5.22) and (5.23) to be rewritten in a form that has an explicit factor $(q)_\infty^{-1}$ such that the remaining expressions are manifestly positive. Such a summation formula is not known to us generally.

5.4. **A_2 analogues of the Rogers–Ramanujan identities.** In the following the $k = 2$ instance of Theorem 5.5 is treated in some further detail. Not only because this case warrants special attention as the A_2 generalization of the Rogers–Ramanujan identities, but also because one can actually eliminate the spurious $(q)_\infty$. All that is required is

$$(5.24) \qquad \lim_{b \to \infty} {}_2\phi_1\begin{bmatrix} q^{-n}, b \\ c \end{bmatrix}; q, zq^n/b\end{bmatrix} = \frac{1}{(c)_n} {}_2\phi_1\begin{bmatrix} q^{-n}, z/c \\ 0 \end{bmatrix}; q, cq^n\end{bmatrix}$$

which is a special case of Jackson's transformation (5.12). For example, taking the right-hand side of (5.18) with $i = k = 2$ and applying the above transformation to the sum over r_2, we find the rewriting

$$\sum_{r_1, r_2} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q)_{L_1-r_1}(q)_{L_2-r_2}(q)_{r_1}(q)_{r_2}(q)_{r_1+r_2}}$$

$$= \sum_{r_1, r_2} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q)_{L_1-r_1}(q)_{L_2-r_2}(q)_{L_2+r_1}(q)_{r_1}} \begin{bmatrix} 2r_1 \\ r_2 \end{bmatrix}.$$

When L_1, L_2 tend to infinity the right-hand side has an extra factor $1/(q)_\infty$ as desired. The two other $M(3, 7)_3$ identities ((5.18) with $i = 1$ and $k = 2$ and (5.19) with $\ell = 1$ and $k = 2$) can be treated similarly. As a result we have the following A_2 versions of the Rogers–Ramanujan identities

Theorem 5.6 (A_2 Rogers–Ramanujan identities). *For $|q| < 1$,*

$$\sum_{r_1,r_2 \geq 0} \frac{q^{r_1^2 - r_1 r_2 + r_2^2}}{(q)_{r_1}} \begin{bmatrix} 2r_1 \\ r_2 \end{bmatrix} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{7n-1})^2 (1 - q^{7n-3})(1 - q^{7n-4})(1 - q^{7n-6})^2},$$

$$\sum_{r_1,r_2 \geq 0} \frac{q^{r_1^2 - r_1 r_2 + r_2^2 + r_1 + r_2}}{(q)_{r_1}} \begin{bmatrix} 2r_1 \\ r_2 \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1 - q^{7n-2})(1 - q^{7n-3})^2 (1 - q^{7n-4})^2 (1 - q^{7n-5})}$$

and

$$\sum_{r_1,r_2 \geq 0} \frac{q^{r_1^2 - r_1 r_2 + r_2^2 + r_1}}{(q)_{r_1}} \begin{bmatrix} 2r_1 + 1 \\ r_2 \end{bmatrix} = \sum_{r_1,r_2 \geq 0} \frac{q^{r_1^2 - r_1 r_2 + r_2^2 + r_2}}{(q)_{r_1}} \begin{bmatrix} 2r_1 \\ r_2 \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1 - q^{7n-1})(1 - q^{7n-2})(1 - q^{7n-3})(1 - q^{7n-4})(1 - q^{7n-5})(1 - q^{7n-6})}.$$

We have not been able to find an identity for the fourth $M(3,7)_3$ character, corresponding to (5.2) with $n = 3, k = 7$ and $j = (1,3,3)$ (or a permutation thereof).

5.5. **Character identities for $M(3, 3k - 1)_3$.** Further families of identities follow by making the replacement $q \to 1/q$ in (5.16) and multiplying the resulting equation by $q^{3L_1 L_2}$. Then proceeding exactly as in Section 5.3, one readily finds that

$$(5.25) \quad \sum_{k_1 + k_2 + k_3 = 0} q^{\frac{1}{2}(3k-1) \sum_{s=1}^{3}(3k_s + 2s)k_s} \det_{1 \leq s,t \leq 3} \left(q^{it(t - s - 3k_s)} \begin{bmatrix} L_1 + L_2 \\ L_1 + 3k_s + s - t \end{bmatrix} \right)$$

$$= \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\frac{1}{2} \sum_{j=1}^{k-1} r^{(j)} C r^{(j)} + \sum_{j=i}^{k-1} |r^{(j)}| + 2r_1^{(k-1)} r_2^{(k-1)}} (1 - q^{|r^{(i-1)}|+1})(q)_{L_1 + L_2}^2}{(q)_{L - r^{(1)}} \cdots (q)_{r^{(k-2)} - r^{(k-1)}} (q)_{r^{(k-1)}} (q)_{|r^{(k-1)}|+1}}.$$

Again, for $i = 1$ we must take $(1 - q^{|r^{(0)}|+1}) = (1 - q^{L_1 + L_2 + 1})$. If we use the $q \to 1/q$ variant of (5.19) with $k = 1$ and iterate we find the polynomial identities

$$(5.26) \quad \sum_{k_1 + k_2 + k_3 = 0} q^{\frac{1}{2}(3k-1) \sum_{s=1}^{3}(3k_s + 2s)k_s}$$

$$\times \det_{1 \leq s,t \leq 3} \left(q^{k(t + \ell \chi(t > 1))(t - s - 3k_s)} \begin{bmatrix} L_1 + L_2 + \ell \\ L_1 - 3k_s + t - s + \ell \chi(t > 1) \end{bmatrix} \right)$$

$$= \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{(-1)^\ell q^{\sum_{j=1}^{k}(\frac{1}{2} r^{(j)} C r^{(j)} + \ell r_1^{(j)})} (q)_{r_1^{(k)} + \ell} (1)_{r_2^{(k)}} (q)_{L_1 + L_2 + \ell}^2}{(q)_{L - r^{(1)}} \cdots (q)_{r^{(k-1)} - r^{(k)}} (q)_{|r^{(k-1)}| + \ell} (q)_{Cr^{(k)} + \ell e_1}}$$

$$\times q^{3r_1^{(k-1)} r_2^{(k-1)} + \ell(r_2^{(k-1)} - |r^{(k)}|) - (r_1^{(k-1)} + r_2^{(k)})(r_2^{(k-1)} + r_1^{(k)}) - \binom{\ell+1}{2}}.$$

For small ℓ the sum over $r^{(k)}$ can be performed so that the right-hand side of (5.26) can be replaced by
(5.27)

$$(-q)^{-\ell} \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^{k-1}(\frac{1}{2} r^{(j)} C r^{(j)} + \ell r_1^{(j)}) + 2r_1^{(k-1)} r_2^{(k-1)} + \ell r_2^{(k-1)}} (q)_{L_1 + L_2 + \ell}^2}{(q)_{L - r^{(1)}} (q)_{r^{(1)} - r^{(2)}} \cdots (q)_{r^{(k-1)}} (q)_{|r^{(k-1)}| + \ell}}, \qquad \ell = 0, 1,$$

and

$$q^{-2-k} \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^{k-1}(\frac{1}{2}r^{(j)}Cr^{(j)}+r_2^{(j)})+2r_1^{(k-1)}r_2^{(k-1)}}(q)_{L_1+L_2+2}^2}{(q)_{L+e_1-r^{(1)}}(q)_{r^{(1)}-r^{(2)}}\cdots(q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}, \qquad \ell = 2.$$

Letting L_1, L_2 tend to infinity and once again using the A_2 Macdonald identity we obtain further A_2 Rogers–Ramanujan-type identities.

Theorem 5.7. *Let $|q| < 1$ and $k \geq 2$. Then*

$$(5.28) \quad \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\frac{1}{2}\sum_{j=1}^{k-1} r^{(j)}Cr^{(j)}+\sum_{j=i}^{k-1}|r^{(j)}|+2r_1^{(k-1)}r_2^{(k-1)}}(1-q^{|r^{(i-1)}|+1})}{(q)_{r^{(1)}-r^{(2)}}\cdots(q)_{r^{(k-2)}-r^{(k-1)}}(q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}$$
$$= \frac{(q^i, q^i, q^{2i}, q^{3k-2i-1}, q^{3k-i-1}, q^{3k-i-1}, q^{3k-1}, q^{3k-1}; q^{3k-1})_\infty}{(q)_\infty^3}$$

for $i = 1, \ldots, k$, and

$$(5.29) \quad \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\sum_{j=1}^{k-1}(\frac{1}{2}r^{(j)}Cr^{(j)}+r_1^{(j)})+2r_1^{(k-1)}r_2^{(k-1)}+\sigma r_2^{(k-1)}}}{(q)_{r^{(1)}-r^{(2)}}\cdots(q)_{r^{(k-2)}-r^{(k-1)}}(q)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}}$$
$$= \frac{(q, q^{k-\sigma}, q^{k+1-\sigma}, q^{2k-2+\sigma}, q^{2k-1+\sigma}, q^{3k-2}, q^{3k-1}, q^{3k-1}; q^{3k-1})_\infty}{(q)_\infty^3}$$

for $\sigma = 0, 1$.

In equations (5.25) and (5.27)–(5.29) the sum over $r^{(k-1)}$ can be simplified to a one-dimensional sum by the q-Chu–Vandermonde summation (4.7) and transformation (5.24). This simplification will be carried out explicitly in the next section where we deal with the case $k = 2$. Comparison with (5.2) shows that (5.28) and (5.29) are identities for the $M(3, 3k-1)_3$ characters $\chi_{(i,i,3k-2i-1)}^{(3,3k-1)}(q)$ and $\chi_{(1,k-\sigma,2k-2+\sigma)}^{(3,3k-1)}(q)$, respectively.

5.6. The Rogers–Ramanujan identities.

Again the case $k = 2$, corresponding to $M(3,5)_3$, deserves special attention. First we note that the W_2 and W_3 algebras $M(2,5)_2$ and $M(3,5)_3$ are equivalent under level-rank duality [24]. Since the $M(2,5)_2$ identities are nothing but the Rogers–Ramanujan identities (1.1) and (1.2), so should be the $M(3,5)_3$ identities. To transform the $k = 2$ identities of the previous section into the Rogers–Ramanujan identities we simplify the two-dimensional sum over $r^{(1)}$. This can be achieved by observing that for $\sigma = 0, 1$,
$$(5.30)$$
$$\sum_{r_1, r_2} \frac{q^{r_1^2+r_1 r_2+r_2^2+\sigma(r_1+r_2)}}{(q)_{A-r_1}(q)_{B-r_2}(q)_{r_1}(q)_{r_2}(q)_{r_1+r_2+\sigma}} = \sum_r \frac{q^{r(r+\sigma)}}{(q)_{A+B+\sigma}(q)_{A-r}(q)_{B-r}(q)_r}$$

by first summing over r_2 using the q-Chu–Vandermonde summation (4.7) with $b \to \infty$ and then applying transformation (5.24). Now, since $\frac{1}{2}rCr + 2r_1 r_2 =$

$r_1^2 + r_1 r_2 + r_2^2$, one can use (5.30) to rewrite (5.25) and (5.27) for $k = 2$ as

(5.31)
$$\sum_{k_1+k_2+k_3=0} q^{\frac{15}{2}(k_1^2+k_2^2+k_3^2)+5(k_1+2k_2+3k_3)} \det_{1\leq s,t\leq 3}\left(q^{(2-\sigma)t(t-s-3k_s)}\begin{bmatrix} L_1+L_2 \\ L_1-3k_s+t-s \end{bmatrix}\right)$$
$$= \sum_{r\geq 0} \frac{q^{r(r+\sigma)}(q)_{L_1+L_2}}{(q)_{L_1-r}(q)_{L_2-r}(q)_r}$$

for $\sigma = 0, 1$ and

(5.32)
$$-\sum_{k_1+k_2+k_3=0} q^{\frac{15}{2}(k_1^2+k_2^2+k_3^2)+5(k_1+2k_2+3k_3)+1}$$
$$\times \det_{1\leq s,t\leq 3}\left(q^{2(t+\chi(t>1))(t-s-3k_s)}\begin{bmatrix} L_1+L_2+1 \\ L_1-3k_s+t-s+\chi(t>1) \end{bmatrix}\right)$$
$$= \sum_{r\geq 0} \frac{q^{r(r+1)}(q)_{L_1+L_2+1}}{(q)_{L_1-r}(q)_{L_2-r}(q)_r}.$$

In the large L_1, L_2 limit this reproduces the Rogers–Ramanujan identities as it should. It is intriguing to observe that the level-rank duality, which predicts the equality of characters (which are infinite q-series), pertains at the polynomial level. That is, the doubly bounded versions of the Rogers–Ramanujan identities (5.31) and (5.32) admit an A$_1$ representation even at finite L_1 and L_2. Explicitly, one can replace (5.31) by

(5.33)
$$\sum_j (-1)^j q^{\frac{1}{2}j(5j+2\sigma+1)} \begin{bmatrix} L_1+L_2 \\ L_1-j \end{bmatrix}\begin{bmatrix} L_1+L_2 \\ L_2-j \end{bmatrix} = \sum_{r\geq 0} \frac{q^{r(r+\sigma)}(q)_{L_1+L_2}}{(q)_{L_1-r}(q)_{L_2-r}(q)_r}$$

and (5.32) by

(5.34)
$$\sum_j (-1)^j q^{\frac{1}{2}j(5j+3)} \begin{bmatrix} L_1+L_2+1 \\ L_1-j \end{bmatrix}\begin{bmatrix} L_1+L_2+1 \\ L_2-j \end{bmatrix} = \sum_{r\geq 0} \frac{q^{r(r+1)}(q)_{L_1+L_2+1}}{(q)_{L_1-r}(q)_{L_2-r}(q)_r}.$$

To see that this is indeed correct, take Watson's $_8\phi_7$ transformation formula [14, Eq. (III.18)]

$$_8\phi_7\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, \frac{a^2 q^{n+2}}{bcde}\right]$$
$$= \frac{(aq, aq/de)_n}{(aq/d, aq/e)_n} \, _4\phi_3\left[\begin{matrix} q^{-n}, d, e, aq/bc \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q\right],$$

replace $n \to L_1$ and $e \to q^{-L_2}$ and let $b, c, d \to \infty$. Specializing a to 1 one finds (5.33) for $\sigma = 0$, whereas specializing a to q yields (5.34). To obtain (5.33) for

$\sigma = 1$ we observe that

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}j(5j+3)} \begin{bmatrix} L_1+L_2 \\ L_1-j \end{bmatrix} \begin{bmatrix} L_1+L_2 \\ L_2-j \end{bmatrix}$$

$$= \sum_{j=0}^{\infty} (-1)^j q^{\frac{1}{2}j(5j+3)} \left(\begin{bmatrix} L_1+L_2 \\ L_1-j \end{bmatrix} \begin{bmatrix} L_1+L_2 \\ L_2-j \end{bmatrix} - q^{2j+1} \begin{bmatrix} L_1+L_2 \\ L_1-j-1 \end{bmatrix} \begin{bmatrix} L_1+L_2 \\ L_2-j-1 \end{bmatrix} \right)$$

$$= \frac{(q)_{L_1+L_2}}{(q)_{L_1+L_2+1}} \sum_{j=0}^{\infty} (-1)^j q^{\frac{1}{2}j(5j+3)} (1-q^{2j+1}) \begin{bmatrix} L_1+L_2+1 \\ L_1-j \end{bmatrix} \begin{bmatrix} L_1+L_2+1 \\ L_2-j \end{bmatrix}$$

which corresponds to (5.34). The equality of the last two lines follows by application of the determinant identity (5.17) with $n = 2$, $B_1 = 1 - j$ and $B_2 = 2 + j$.

5.7. Rogers–Ramanujan-type identities for "$M(3, 3k)_3$". As a further application we derive a family of identities generalizing Bressoud's series [8]

$$(5.35) \qquad \sum_{n_1,\dots,n_{k-1}} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q^2; q^2)_{n_{k-1}}} = \frac{(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty}$$

true for all $k \geq 2$, $1 \leq i \leq k$ and $|q| < 1$. Comparing the right-hand side with (5.2) we are led to label (5.35) as identities for the anomalous series $M(2, 2k)_2$. To generalize this to $M(3, 3k)_3$, we do not rely on the supernomial identity (5.3) as initial condition, but on the following polynomial identity of Gessel and Krattenthaler [15, Eq. (6.18)]

$$\sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}\sum_{s=1}^{3}(3k_s+2s)k_s} \det_{1\leq s,t \leq 3} \left(q^{t(t-s-3k_s)} \begin{bmatrix} L_1+L_2 \\ L_1+3k_s+s-t \end{bmatrix} \right) = \begin{bmatrix} L_1+L_2 \\ L_1 \end{bmatrix}_{q^3}.$$

Clearly this gives rise to a Bailey pair of type II relative to $a = 1$. Along the now clear-trodden path we thus derive for $1 \leq i \leq k$,

$$\sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}k\sum_{s=1}^{3}(3k_s+2s)k_s} \det_{1\leq s,t \leq 3} \left(q^{it(t-s-3k_s)} \begin{bmatrix} L_1+L_2 \\ L_1+3k_s+s-t \end{bmatrix} \right)$$

$$= \sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\frac{1}{2}\sum_{j=1}^{k-1} r^{(j)} C r^{(j)} + \sum_{j=i}^{k-1} |r^{(j)}|}(1-q^{|r^{(i-1)}|+1})(q^3; q^3)_{|r^{(k-1)}|}(q)_{L_1+L_2}^2}{(q)_{L-r^{(1)}} \cdots (q)_{r^{(k-2)}-r^{(k-1)}}(q^3; q^3)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}(q)_{|r^{(k-1)}|}}.$$

When L_1, L_2 approach infinity this yields the following theorem.

Theorem 5.8. *For $|q| < 1$, $k \geq 2$ and $i = 1, \dots, k$,*

$$\sum_{r^{(j)} \in \mathbb{Z}^2} \frac{q^{\frac{1}{2}\sum_{j=1}^{k-1} r^{(j)} C r^{(j)} + \sum_{j=i}^{k-1} |r^{(j)}|}(1-q^{|r^{(i-1)}|+1})(q^3; q^3)_{|r^{(k-1)}|}}{(q)_{r^{(1)}-r^{(2)}} \cdots (q)_{r^{(k-2)}-r^{(k-1)}}(q^3; q^3)_{r^{(k-1)}}(q)_{|r^{(k-1)}|+1}(q)_{|r^{(k-1)}|}}$$

$$= \frac{(q^i, q^i, q^{2i}, q^{3k-2i}, q^{3k-i}, q^{3k-i}, q^{3k}, q^{3k}; q^{3k})_\infty}{(q)_\infty^3}.$$

Comparing with equation (5.2) this corresponds to identities for the "characters" $\chi_{(i,i,3k-2i)}^{(3,3k)}(q)$.

5.8. **Identities for Kostka polynomials.** A different type of application of the A_2 Bailey lemma arises from the representation (4.8) of the supernomials in terms of the Kostka polynomials. Using the summation [13, page 76]

$$\sum_{\sigma \in S_n} \epsilon(\sigma) K_{\eta(\lambda_1+\sigma_1-1,\dots,\lambda_n+\sigma_n-n)} = \delta_{\eta\lambda}$$

one can invert (4.8), resulting in [38]

$$K_{\lambda'\mu}(q) = \sum_{\sigma \in S_n} \epsilon(\sigma) \begin{bmatrix} L_1,\dots,L_n \\ \lambda_1+\sigma_1-1,\dots,\lambda_n+\sigma_n-n \end{bmatrix},$$

where $\mu = (1^{L_1} 2^{L_2} \cdots (n-1)^{L_{n-1}})$ and λ is a partition such that $|\lambda| = |\mu|$. For $n = 3$ this supernomial identity can be used as input to the A_2 Bailey lemma yielding the following identity for the Kostka polynomials for $k \geq 1$ and $i = 1, \dots, k$,

$$(5.36) \quad (q)_{L_1+L_2}^2 \sum_{r^{(j)} \in \mathbb{Z}_+^2} \frac{q^{\frac{1}{2}\sum_{j=1}^k r^{(j)} C r^{(j)} + \sum_{j=i}^{k-1} |r^{(j)}|}(1 - q^{|r^{(i-1)}|+1}) K_{\lambda'\mu}(q)}{(q)_{L-r^{(1)}} \cdots (q)_{r^{(k-1)}-r^{(k)}} (q)_{|r^{(k-1)}|+1} (q)_{Cr^{(k)}}}$$

$$= q^{\frac{k}{2}(k_1^2+k_2^2+k_3^2)-k(k_1+2k_2+3k_3)} \det_{1 \leq s,t \leq 3}\left(q^{it(t-s+k_s)} \begin{bmatrix} L_1+L_2 \\ L_1-k_s+s-t \end{bmatrix} \right)$$

where $\mu = (1^{(Cr^{(k)})_1} 2^{(Cr^{(k)})_2}) = (1^{2r_1^{(k)}-r_2^{(k)}} 2^{2r_2^{(k)}-r_1^{(k)}})$ and $\lambda = (|\mu|^3) + (k_1, k_2, k_3)$ with $k_1 \geq k_2 \geq k_3$, $k_1 + k_2 + k_3 = 0$. Also recall that $|r^{(0)}| = L_1 + L_2$. Taking the limit $L_1, L_2 \to \infty$ of (5.36) and evaluating the determinant leads to our final theorem.

Theorem 5.9. *For $|q| < 1$, $k \geq 1$ and $i = 1, \dots, k$,*

$$\sum_{r^{(j)} \in \mathbb{Z}_+^2} \frac{q^{\frac{1}{2}\sum_{j=1}^k r^{(j)} C r^{(j)} + \sum_{j=i}^{k-1} |r^{(j)}|}(1 - q^{|r^{(i-1)}|+1}) K_{\lambda'\mu}(q)}{(q)_{r^{(1)}-r^{(2)}} \cdots (q)_{r^{(k-1)}-r^{(k)}} (q)_{|r^{(k-1)}|+1} (q)_{Cr^{(k)}}}$$

$$= \frac{1}{(q)_\infty^3} q^{\frac{k}{2}(k_1^2+k_2^2+k_3^2)-(k-i)(k_1+2k_2+3k_3)} \prod_{1 \leq s < t \leq 3} (1 - q^{i(k_s-k_t-s+t)}).$$

The above results can either be viewed as some formal identities for the Kostka polynomials, or, by for example taking the representation for the Kostka polynomials due to Kirillov and Reshetikhin [22] (see also [26, page 245]), as explicit q-series identities.

For $k = 1$ the identity (5.36) can be written as

$$\frac{(q)_{L_1+L_2+2}^2}{(q)_{L_1+L_2+1}} \sum_{C^{-1}r \in \mathbb{Z}_+^2} \frac{q^{\frac{1}{2}rC^{-1}r} K_{\eta\mu}(q)}{(q)_{L-C^{-1}r}(q)_r}$$

$$= q^{\frac{1}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2)} \prod_{1 \leq s < t \leq 3} (1 - q^{\lambda_s-\lambda_t-s+t}) \prod_{s=1}^3 \begin{bmatrix} L_1+L_2+2 \\ L_1-\lambda_s+s-1 \end{bmatrix},$$

where $\mu = (1^{r_1} 2^{r_2})$ and $\eta = (1^{\lambda_1-\lambda_2} 2^{\lambda_2-\lambda_3} 3^{\lambda_3+|\mu|/3})$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3$, $|\lambda| = 0$. This is a bounded analogue of the $l = 1$, $n = 3$ case of [18, Eq. (4.39)]. Indeed, letting L_1, L_2 tend to infinity and recognizing

$$\frac{q^{\frac{1}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2)}}{(q)_\infty^2} \prod_{1 \leq s < t \leq 3} (1 - q^{\lambda_s-\lambda_t-s+t})$$

as the branching function $b_\lambda(q)$ of the level-1 basic representation $V(\Lambda_0)$ of $A_{n-1}^{(1)}$ we find

$$b_\lambda(q) = \sum_{C^{-1}r \in \mathbb{Z}_+^2} \frac{q^{\frac{1}{2}rC^{-1}r} K_{\eta\mu}(q)}{(q)_r}$$

of [18].

References

[1] A. K. Agarwal, G. E. Andrews and D. M. Bressoud, *The Bailey lattice*, J. Ind. Math. Soc. **51** (1987), 57–73. 3

[2] G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Prod. Nat. Acad. Sci. USA **71** (1974), 4082–4085. 1, 2, 3

[3] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, (Addison-Wesley, Reading, Massachusetts, 1976). 1

[4] G. E. Andrews, *Multiple series Rogers–Ramanujan type identities*, Pacific J. Math. **114** (1984), 267–283. 1, 2, 2

[5] G. E. Andrews, R. J. Baxter, D. M. Bressoud, W. H. Burge, P. J. Forrester and G. Viennot, *Partitions with prescribed hook differences*, Europ. J. Combinatorics **8** (1987), 341–350. 5.2

[6] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1949), 1–10. 2

[7] D. M. Bressoud, *A generalization of the Rogers–Ramanujan identities for all moduli*, J. Combin. Theory Ser. A **27** (1979), 64–68. 1

[8] D. M. Bressoud, *An analytic generalization of the Rogers–Ramanujan identities with interpretation*, Quart. J. Maths. Oxford (2) **31** (1980), 385–399. 1, 5.7

[9] D. M. Bressoud, *Analytic and combinatorial generalizations of the Rogers–Ramanujan identities*, Memoirs Amer. Math. Soc. **24** (1980), 1–54. 1

[10] D. M. Bressoud, *The Bailey lattice: An introduction*, in *Ramanujan Revisited*, pp. 57–67, G. E. Andrews *et al.* eds., (Academic Press, New York, 1988). 3

[11] F. Dyson, *Missed opportunities*, Bull. Amer. Math. Soc. **78** (1972), 140–156. 5.1

[12] V. A. Fateev and S. L. Lykyanov, *The models of two-dimensional conformal quantum field theory with Z_n symmetry*, Int. J. Mod. Phys. A **3** (1988), 507–520. 5.1

[13] W. Fulton, *Young tableaux: with applications to representation theory and geometry*, London Math. Soc. student texts **35**, Cambridge University Press (1997). 5.8

[14] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35, (Cambridge University Press, Cambridge, 1990). 1.1, 2, 2, 2, 4.1, 4.2, 4.2, 5.2, 5.2, 5.2, 5.2, 5.6

[15] I. M. Gessel and C. Krattenthaler, *Cylindric partitions*, Trans. Amer. Math. Soc. **349** (1997), 429–479. 5.7

[16] B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399. 1, 2

[17] G. H. Hardy, *Ramanujan*, (Cambridge University Press, London and New York, 1940). 1

[18] G. Hatayama, A. N. Kirillov, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Character formulae of $\widehat{sl}_n$-modules and inhomogeneous paths*, Nuclear Phys. B **536** (1999), 575–616. 1, 4.1, 5.8

[19] M. Jimbo, T. Miwa and M. Okado, *Local state probabilities of solvable lattice models: An $A_{n-1}^{(1)}$ family*, Nucl. Phys. B **300** [**FS22**] (1988), 74–108. 5.1

[20] A. N. Kirillov, *Dilogarithm identities*, Prog. Theor. Phys. Suppl. **118** (1995), 61–142. 4.1

[21] A. N. Kirillov, *New combinatorial formula for Hall–Littlewood polynomials*, in *q-series from a contemporary perspective*, pp. 283–333, Contemp. Math., 254, (Amer. Math. Soc., Providence, RI, 2000). 1, 4.1, 4.1

[22] A. N. Kirillov and N. Yu. Reshetikhin, *The Bethe Ansatz and the combinatorics of Young tableaux*, J. Soviet Math. **41** (1988), 925–955. 5.8

[23] C. Krattenthaler, *Generating functions for plane partitions of a given shape*, Manuscripta Math. **69** (1990), 173–202. 5.3

[24] A. Kuniba, T. Nakanishi and J. Suzuki, *Ferro– and antiferro-magnetizations in RSOS models*, Nucl. Phys. B **356** (1991), 750–774. 5.6

[25] I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Inv. Math. **15** (1972), 91–143. 5.1

[26] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, (Clarendon Press, Oxford, 1995). 4.1, 4.1, 4.1, 4.1, 5.8

[27] P. A. MacMahon, *Combinatory Analysis*, Vol. 2, (Cambridge University Press, London and New York, 1916). 1

[28] S. C. Milne *Classical partition functions and the $U(n+1)$ Rogers–Selberg identity*, Discrete Math. **99** (1992), 199-246. 1

[29] S. C. Milne *A q-analog of a Whipple's transformation for hypergeometric series in $U(n)$*, Adv. in Math. **108** (1994), 1–76. 1

[30] S. C. Milne and G. M. Lilly, *The A_ℓ and C_ℓ Bailey transform and lemma*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 258–263. 1

[31] S. C. Milne and G. M. Lilly, *Consequences of the A_ℓ and C_ℓ Bailey transform and Bailey lemma*, Discrete Math. **139** (1995), 319–346. 1

[32] S. Mizoguchi, *The structure of representations for the W_3 minimal model*, Int. J. Mod. Phys. A **6** (1991), 133-162. 5.1

[33] T. Nakanishi, *Non-unitary minimal models and RSOS models*, Nucl. Phys. B **334** (1990), 745–766. 5.1

[34] P. Paule, *On identities of the Rogers–Ramanujan type*, J. Math. Anal. Appl. **107** (1985), 225–284. 2

[35] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343. 1

[36] L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (2) **16** (1917), 315–336. 2

[37] L. J. Rogers and S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Phil. Soc. **19** (1919), 211–216. 1

[38] A. Schilling and S. O. Warnaar, *Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials*, Comm. Math. Phys. **202** (1999), 359–401. 1, 4.1, 5.2, 5.8

[39] I. J. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917), 302–321. 1, 1, 5.2

[40] A. B. Zamolodchikov, *Infinite additional symmetries in two-dimensional conformal quantum field theory*, Teo. Mat. Fiz. **65** (1985), 347–359. 5.1

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands