QUASI-LIE ALGEBRAS AND A METHOD FOR DETERMINING NON-DEGENERACY OF THE QUASI-LIE PRODUCT

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The axioms of a quasi-Lie algebra are stated, followed by the definition of a non-degenerate quasi-Lie algebra which is given in the framework of Hopf algebra theory. Some quantum Lie algebras are shown to be examples of non-degenerate quasi-Lie algebras and a method of testing for this non-degeneracy is outlined. It is shown that non-degeneracy implies uniqueness of the symmetriser. For quantum Lie algebras this is surprising since classically several symmetrisers are known to exist.

1 Axioms for Quasi-Lie Algebras

An object L in a quasi tensor category equipped with functorial morphisms $\mu : L \otimes L \to L$ (quasi-Lie product) and $\gamma : L \otimes L \to L \otimes L$ (symmetriser) is called a quasi-Lie algebra if the following hold:

$$\mu\sigma = -q\mu \tag{1}$$
$$\ker\gamma \subseteq \ker\mu \tag{2}$$
$$\mu(\mu\otimes 1) = \mu(1\otimes\mu)(\gamma\otimes 1) \tag{3}$$

where q is some complex number and $\sigma : L \otimes L \to L \otimes L$ satisfies the braid relation. Condition(1) is a generalised antisymmetry property and condition(3) is a generalised Jacobi identity.

In what follows, we shall take the quasi tensor category to be the category of H-modules, where H is a quasi-triangular Hopf algebra.

2 Quasi-Lie Algebras from Quasi-Triangular Hopf Algebras

This section shows how the theory is developed in the framework of Hopf algebras, which lead to the examples of quantum Lie algebras[1,2,3].

Let (H, R) be a quasi-triangular Hopf algebra with universal R-matrix R. Let L be an irreducible H-module such that $L \otimes L$ is completely reducible and there exists a non-zero H-module homomorphism $\mu : L \otimes L \to L$ satisfying

$\mu\sigma = -q\mu$, where $\sigma = PR$ is the braid generator arising from the universal R-matrix and P, as usual, denotes the permutation operator on $L \otimes L$.

Now let $\{a_i^*\}$ be a basis for L^* dual to the basis $\{a_i\}$ for L so that

$$\xi = \sum_k a_k \otimes a_k^*$$

gives rise to the identity representation of H:

$$\Delta(x)\xi = \varepsilon(x)\xi,$$

$\forall x \in H$, where we take Δ to be the co-product of H and ε to be the co-unit. Under the previously mentioned assumptions on L, we have the following result.

LEMMA 1 $\mu(a \otimes L) = 0 \Rightarrow a = 0,\ \forall a \in L$.

proof Consider $\bar{\mu} : L \to L \otimes L^*$ defined by $\bar{\mu}(a) = (\mu \otimes 1)(a \otimes \xi)$. Then $\bar{\mu}$ is a module morphism with $\ker \bar{\mu} = \{a \in L | \mu(a \otimes L) = (0)\}$. Since L is irreducible and $\ker \bar{\mu}$ is a non-zero H-submodule of L, we must have that $\ker \bar{\mu} = (0)$, hence the result. □

Note: **(1)** This results indicates that any γ satisfying condition(3) will immediately satisfy condition(2).

2.1 *Splitting Morphisms and Projected Symmetrisers*

THEOREM 1 *L is a quasi-Lie algebra with (non-trivial) product μ if and only if there exists a (non-trivial) splitting morphism $\delta \in \hom(L, L \otimes L)$ satisfying*

$$\mu(1 \otimes \mu)(\delta \otimes 1) = \mu. \tag{4}$$

In such a case, L is a quasi-Lie algebra with symmetriser $\gamma_0 = \delta\mu$ such that $\ker \gamma_0 = \ker \mu$.

proof If we multiply equation(4) on the right by $\mu \otimes 1$ we obtain the Jacobi identity with symmetriser γ_0. Considering $\ker \mu \subseteq \ker \gamma_0$ and the result of LEMMA(1), we have a quasi-Lie algebra structure with $\ker \mu = \ker \gamma_0$. Conversely, there exists a splitting morphism $\hat{\delta} : L \to L \otimes L$ such that $\mu\hat{\delta} = I_L$ (where I_L is the identity on L) since $\ker \mu$ must split in $L \otimes L$. Hence, supposing a quasi-Lie algebra structure, we multiply the Jacobi identity

$$\mu(\mu \otimes 1) = \mu(1 \otimes \mu)(\gamma \otimes 1)$$

on the right by $\hat{\delta} \otimes 1$ to obtain equation(4) with $\delta = \gamma\hat{\delta} \in \hom(L, L \otimes L)$. Hence in this case L is a quasi-Lie algebra with symmetriser γ_0. However, $\gamma \neq \gamma_0$ in general. □

It is worth noting that by Schur's lemma, $\mu\delta = \alpha I_L$, where α is a complex constant. Thus γ_0 satisfies $\gamma_0^2 = \alpha\gamma_0$ and $\ker\gamma_0 = \ker\mu$. We call such a γ_0 a projected symmetriser.

2.2 Non-Degeneracy

Introduce the following subspace of $L \otimes L$:

$$V = \{\bar{a} \in L \otimes L | \mu(1 \otimes \mu)(\bar{a} \otimes L) = (0)\}.$$

Let $f : L \otimes L \to L \otimes L^*$ be defined by

$$\begin{aligned} f(a_i \otimes a_j) &= \sum_k \mu(1 \otimes \mu)(a_i \otimes a_j \otimes a_k) \otimes a_k^* \\ &= [\mu(1 \otimes \mu) \otimes 1](a_i \otimes a_j \otimes \xi). \end{aligned} \tag{5}$$

Then f is easily seen to give rise to a module morphism with $\ker f = V$.

To define non-degeneracy, we say that μ is non-degenerate if

$$V = (0).$$

A quasi-Lie algebra is called non-degenerate if it is equipped with a non-degenerate quasi-Lie product.

We now show that if L is non-degenerate then it necessarily has a unique structure. Consider the Jacobi identity for L with symmetriser γ:

$$\mu(1 \otimes \mu)(\gamma \otimes 1) = \mu(\mu \otimes 1)$$

which can be written in the form

$$f(\gamma(a_i \otimes a_j)) = g(a_i \otimes a_j) \ \in L \otimes L^*$$

with f as given by (5) and $g : L \otimes L \to L \otimes L^*$ the morphism defined by

$$\begin{aligned} g(a_i \otimes a_j) &= \sum_k \mu(\mu \otimes 1)(a_i \otimes a_j \otimes a_k) \otimes a_k^* \\ &= [\mu(\mu \otimes 1) \otimes 1](a_i \otimes a_j \otimes \xi). \end{aligned} \tag{6}$$

Then, by non-degeneracy, f determines an isomorphism and so admits an inverse morphism $f^{-1} : L \otimes L^* \to L \otimes L$ satisfying $f^{-1}f = I_{L\otimes L}$. Thus γ is uniquely determined by

$$\gamma(a_i \otimes a_j) = f^{-1}(g(a_i \otimes a_j) \tag{7}$$

and is a morphism, since f^{-1} and g are both morphisms.

We have already seen (note (1)) that $\ker\gamma \subseteq \ker\mu$. Moreover,

$$\begin{aligned}\bar{v} \in \ker\mu \Rightarrow (0) &= \mu(1 \otimes \mu)(\gamma(\bar{v}) \otimes L)\\ &\Rightarrow \gamma(\bar{v}) \in V = (0),\end{aligned}$$

so $\ker\mu \subseteq \ker\gamma$ and hence $\ker\mu = \ker\gamma$. We have therefore proved the following theorem.

THEOREM 2 *If L is non-degenerate then it gives rise to a unique quasi-Lie algebra structure with unique symmetriser γ. Moreover $\ker\mu = \ker\gamma$.*

□

Since we have shown that the symmetriser is unique in this case, by THEOREM 1 we must have that the symmetriser is in fact the projected symmetriser.

2.3 Testing for Non-Degeneracy

In terms of structure constants C^i_{jk}, (5) is expressible

$$\begin{aligned}f(a_i \otimes a_j) &= \sum_k \mu(a_i \otimes \mu(a_j \otimes a_k)) \otimes a_k^*\\ &= C^l_{jk} C^r_{il} a_r \otimes a_k^*\\ &= A^{rk}_{ij} a_r \otimes a_k^*\end{aligned}$$

where A denotes the $(\dim L)^2 \times (\dim L)^2$ matrix

$$A^{rk}_{ij} = \sum_l C^r_{il} C^l_{jk},$$

with the rows of A determined by the indices rk and the columns determined by the indices ij. If A is non-singular then this is equivalent to the existence of the morphism f^{-1}. Hence L is non-degenerate if and only if A is non-singular. This gives a useful method of checking non-degeneracy from the structure constants. Also, we can express (6) in terms of structure constants by

$$\begin{aligned}g(a_i \otimes a_j) &= \sum_k \mu(\mu(a_i \otimes a_j) \otimes a_k) \otimes a_k^*\\ &= C^l_{ij} C^r_{lk} a_r \otimes a_k^*\\ &= B^{rk}_{ij} a_r \otimes a_k^*\end{aligned}$$

where B denotes the singular $(\dim L)^2 \times (\dim L)^2$ matrix

$$B_{ij}^{rk} = \sum_l C_{ij}^l C_{lk}^r,$$

with the rows of B determined by the indices rk and the columns determined by the indices ij. By writing the matrix elements of γ as $\gamma_{ij}^{i'j'}$, γ becomes the $(\dim L)^2 \times (\dim L)^2$ matrix with columns given by the indices ij and the rows given by $i'j'$.

So, provided A^{-1} exists, we have γ uniquely determined by the matrix equation (cf. equation (7))

$$\gamma = A^{-1}B.$$

3 Conclusions

This procedure has been tested for the following quantum Lie algebras from the classical series: A_{1q}, A_{2q}, B_{1q}, B_{2q}, C_{1q}, C_{2q}, D_{2q}. We find that for all of these quantum Lie algebras, the matrix A has a non-zero determinant except for D_{2q}. This is still consistent with our theory since D_{2q} is not irreducible. Therefore, for all the cases given above, with the exception of D_{2q}, the quantum Lie algebra is non-degenerate and hence we can obtain a unique symmetriser which is precisely the projected one. This seems surprising since in the classical case there are in general several symmetrisers. It is worth noting that the procedure relies on the action of μ having been calculated, which has already been done[1,2] for A_{lq}, B_{lq}, C_{lq} and D_{lq}.

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References

1. G. W. Delius, A. Hüffmann, M. D. Gould and Y. -Z. Zhang, Journal of Physics A: Mathematical and General **29**, 5611 (1996).
2. G. W. Delius, C. Gardner And M. D. Gould, Journal of Physics A: Mathematical and General **31**, 1995 (1998).
3. V. Lyubashenko and A. Sudberry, q-alg/9510004.