Risk Analysis at UQ

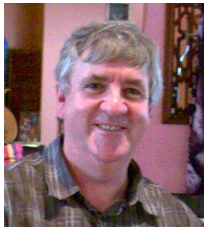
Phil Pollett

Department of Mathematics
The University of Queensland
http://www.maths.uq.edu.au/˜pkp

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

MASCOS Qld 2009

Research staff



Phil
Pollett (CI)

(Networks
and Risk)

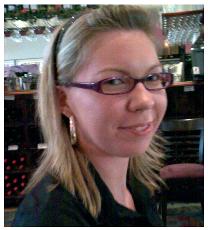
Ross
McVinish (RF)

(Networks
and Risk)

Iadine
Chadès (RF)

(Risk)

PhD students



Fionnuala
Buckley
April 2007

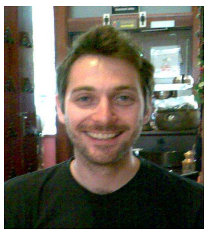
(Networks)

Dejan
Jovanović
March 2009

(Networks)

Andrew
Smith
July 2009

(Networks)



Daniel
Pagendam
March 2007

(Risk)

Nimmy
Thaliath
February 2009

(Risk)

Robert
Cope
July 2009

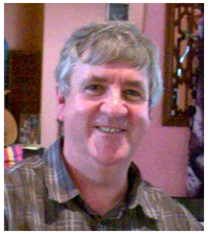
(Risk)

Honours/Masters students

Alex Ridley (Networks), Chung Kai Chan (Risk)

Expertise

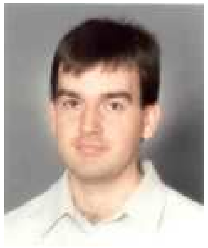
Phil Pollett



Mathematical modelling, stochastic process theory and applications: ecology, epidemiology, parasitology, telecommunications and chemical kinetics.

A current project: *Modelling population processes with random initial conditions*.

Ross McVinish



Lévy processes and stochastic processes displaying long memory, Bayesian nonparametrics, computation for Bayesian statistics and time series analysis.

A current project: *Statistical inference for partially observed population processes*.

Iadine Chadès

Markov decision processes. Mathematical modelling and decision making in ecology and conservation biology.

A current project: *Strategies for managing invasive species in space: deciding whether to eradicate, contain or control*.

PhD Projects

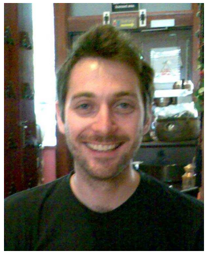
Robert Cope (July 2009 –)

Animal Movement Between Populations Deduced from Family Trees

The aim is to develop a new method for estimating animal movements using information contained in family trees. Movement estimates are essential to population models that assist natural resource managers to plan species recovery and to predict the effect of future challenges, such as human-mediated activities and climate change. We will evaluate ways of constructing family trees from genetic data and develop a statistic that describes animal movement between populations that is based on the families whose members were sampled in more than one population; empirical data has been sourced from a long-term mark-recapture study of dugongs in Moreton Bay, and new samples from two adjacent populations.

PhD Projects

Daniel Pagendam (March 2007 –)



Optimal Design for Statistical Inference in Stochastic Processes

Stochastic processes have been used to model a wide range of phenomena such as population dynamics, chemical reactions, epidemics and telecommunications traffic. However, the statistical methods for these processes have not received a great deal of attention. There are two key aspects of statistical inference that are being investigated: parameter estimation for stochastic processes, and optimal design of experiments that can be formulated as stochastic processes. Whilst the former has received attention by a number of authors, the latter is a largely unexplored, with great potential to improve the utility of stochastic processes as statistical models in an experimental context. Our approach is to use Gaussian diffusion approximations to obtain analytical approximations to Fisher's information matrix, which then leads to optimal sampling schemes for stochastic population models.

PhD Projects

Nimmy Thaliath (February 2009 –)

Minimum Risk Optimal Portfolio Allocation: a Game Theoretic Approach

We are concerned with allocating capital among a set of risky assets so as to obtain an optimal portfolio allocation. A game theoretic approach is proposed, based on the notion of Conditional Value at Risk. Since Conditional Value at Risk (CVaR) is a coherent risk measure, it can potentially reduce the likelihood of substantial losses.

We adopt the coalitional games concept, interpreting the different portfolios as different players. The Aumann Shapley Principle of game theory will then be used to compute allocations. If we consider risk assessment as a linear optimization problem, then the Shapely value can be computed more easily. Since CVaR allows for optimization shortcuts through linear programming, it can be used in this context. Both game theory and CVaR have been used independently in portfolio management, and we expect that, in combination, they will prove to be very effective.

Recent highlights

Let $n_t = (n_{1,t}, \ldots, n_{k,t})$, where $n_{i,t}$ is the number of individuals of type i in a population with k types and a total number of N individuals.

Recent highlights

Let $n_t = (n_{1,t}, \ldots, n_{k,t})$, where $n_{i,t}$ is the number of individuals of type i in a population with k types and a total number of N individuals.

Suppose that $(n_t,\ t \geq 0)$ is a continuous-time Markov chain taking values in a (finite) subset S of $\mathbb{Z}^k$.

Recent highlights

Let $n_t = (n_{1,t}, \ldots, n_{k,t})$, where $n_{i,t}$ is the number of individuals of type i in a population with k types and a total number of N individuals.

Suppose that $(n_t,\, t \geq 0)$ is a continuous-time Markov chain taking values in a (finite) subset S of $\mathbb{Z}^k$.

Suppose that the transition rates $Q = (q_{nm},\, n, m \in S)$ have the following property (*density dependence*): there is a subset E of $\mathbb{R}^k$ and a continuous function $f : \mathbb{Z}^k \times E \to \mathbb{R}$, such that

$$q_{n,n+l} = N f_l\left(\frac{n}{N}\right), \quad l \neq 0 \quad (l \in \mathbb{Z}^k).$$

Recent highlights

Let $X_{i,t}^{(N)} = n_{i,t}/N$ be the *proportion* of individuals of type i and call $(X_t^{(N)})$ the *density process*.

Recent highlights

Let $X_{i,t}^{(N)} = n_{i,t}/N$ be the *proportion* of individuals of type i and call $(X_t^{(N)})$ the *density process*.

Theorem Let $F(x) := \sum_{l \neq 0} l f_l(x)$ $(x \in E)$ and suppose that F is Lipschitz.

If $\lim_{N\to\infty} X_0^{(N)} = x_0$, then $(X_t^{(N)})$ converges (uniformly in probability over $[0,t]$) to (x_t), the unique (deterministic) trajectory satisfying

$$x_s' = F(x_s) \quad (x_s \in E,\ s \in [0,t]).$$

Recent highlights

Let $X_{i,t}^{(N)} = n_{i,t}/N$ be the *proportion* of individuals of type i and call $(X_t^{(N)})$ the *density process*.

Theorem Let $F(x) := \sum_{l \neq 0} l f_l(x)$ $(x \in E)$ and suppose that F is Lipschitz.

If $\lim_{N \to \infty} X_0^{(N)} = x_0$, then $(X_t^{(N)})$ converges (uniformly in probability over $[0,t]$) to (x_t), the unique (deterministic) trajectory satisfying

$$x_s' = F(x_s) \quad (x_s \in E, \; s \in [0,t]).$$

Recent highlights

Let $X_{i,t}^{(N)} = n_{i,t}/N$ be the *proportion* of individuals of type i and call $(X_t^{(N)})$ the *density process*.

Theorem Let $F(x) := \sum_{l \neq 0} l f_l(x)$ $(x \in E)$ and suppose that F is Lipschitz.

If $\lim_{N\to\infty} X_0^{(N)} = x_0$, then $(X_t^{(N)})$ converges (uniformly in probability over $[0,t]$) to (x_t), the unique (deterministic) trajectory satisfying

$$x_s' = F(x_s) \quad (x_s \in E,\ s \in [0,t]).$$

Recent highlights

Define $(Z_t^{(N)})$ (scaled fluctuations about the deterministic trajectory) by

$$Z_s^{(N)} = \sqrt{N}\left(X_s^{(N)} - x_s\right) \qquad (0 \le s \le t).$$

Recent highlights

Define $(Z_t^{(N)})$ (scaled fluctuations about the deterministic trajectory) by

$$Z_s^{(N)} = \sqrt{N}\left(X_s^{(N)} - x_s\right) \qquad (0 \leq s \leq t).$$

Idea: $(Z_t^{(N)})$ looks like a *Gaussian diffusion* for large N.

Recent highlights

Theorem Suppose that F is Lipschitz and has uniformly continuous first derivative on E, and that the $k \times k$ matrix $G(x)$, defined for $x \in E$ by $G_{ij}(x) = \sum_{l \neq 0} l_i l_j f_l(x)$, is uniformly continuous on E.

Let (x_t) be the unique deterministic trajectory starting at x_0 and suppose that $\lim_{N\to\infty} \sqrt{N}\left(X_0^{(N)} - x_0\right) = z$.

Then, $(Z_t^{(N)})$ converges weakly in $D[0,t]$ (the space of right-continuous, left-hand limits functions on $[0,t]$) to a Gaussian diffusion (Z_t) with initial value $Z_0 = z$ and with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = M_s z$, where $M_s = \exp(\int_0^s B_u \, du)$ and $B_s = \partial F(x_s)$, and

$$V_s := \mathrm{Cov}(Z_s) = M_s \left(\int_0^s M_u^{-1} G(x_u) (M_u^{-1})^\top \, du \right) M_s^\top .$$

Recent highlights

Theorem Suppose that F is Lipschitz and has uniformly continuous first derivative on E, and that the $k \times k$ matrix $G(x)$, defined for $x \in E$ by $G_{ij}(x) = \sum_{l \neq 0} l_i l_j f_l(x)$, is uniformly continuous on E.

Let (x_t) be the unique deterministic trajectory starting at x_0 and suppose that $\lim_{N \to \infty} \sqrt{N} \left(X_0^{(N)} - x_0 \right) = z$.

Then, $(Z_t^{(N)})$ converges weakly in $D[0,t]$ (the space of right-continuous, left-hand limits functions on $[0,t]$) to a Gaussian diffusion (Z_t) with initial value $Z_0 = z$ and with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = M_s z$, where $M_s = \exp(\int_0^s B_u \, du)$ and $B_s = \partial F(x_s)$, and

$$V_s := \mathrm{Cov}(Z_s) = M_s \left(\int_0^s M_u^{-1} G(x_u)(M_u^{-1})^\top \, du \right) M_s^\top .$$

Recent highlights

Theorem Suppose that F is Lipschitz and has uniformly continuous first derivative on E, and that the $k \times k$ matrix $G(x)$, defined for $x \in E$ by $G_{ij}(x) = \sum_{l \neq 0} l_i l_j f_l(x)$, is uniformly continuous on E.

Let (x_t) be the unique deterministic trajectory starting at x_0 and suppose that $\lim_{N \to \infty} \sqrt{N} \left(X_0^{(N)} - x_0 \right) = z$.

Then, $(Z_t^{(N)})$ converges weakly in $D[0,t]$ (the space of right-continuous, left-hand limits functions on $[0,t]$) to a Gaussian diffusion (Z_t) with initial value $Z_0 = z$ and with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = M_s z$, where $M_s = \exp(\int_0^s B_u \, du)$ and $B_s = \partial F(x_s)$, and

$$V_s := \mathrm{Cov}(Z_s) = M_s \left(\int_0^s M_u^{-1} G(x_u) (M_u^{-1})^\top \, du \right) M_s^\top .$$

Recent highlights

We conclude that, for N large, $X_t^{(N)}$ has an approximate Gaussian distribution with $\mathrm{Cov}(X_t^{(N)}) \simeq V_t/N$.

Recent highlights

Corollary If x_{eq} satisfies $F(x_{\text{eq}}) = 0$, then, under the conditions of the previous theorem, the family $(Z_t^{(N)})$ defined by

$$Z_s^{(N)} = \sqrt{N}(X_s^{(N)} - x_{\text{eq}}) \qquad (0 \le s \le t),$$

converges weakly in $D[0,t]$ to an *OU process* (Z_t) with initial value $Z_0 = z$, local drift matrix $B = \partial F(x_{\text{eq}})$ and local covariance matrix $G(x_{\text{eq}})$. In particular, Z_s is normally distributed with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = e^{Bs}z$ and

$$V_s := \text{Cov}(Z_s) = \int_0^s e^{Bu} G(x_{\text{eq}}) e^{B^\top u}\, du\,.$$

Recent highlights

Corollary If $x_{\rm eq}$ satisfies $F(x_{\rm eq}) = 0$, then, under the conditions of the previous theorem, the family $(Z_t^{(N)})$ defined by

$$Z_s^{(N)} = \sqrt{N}(X_s^{(N)} - x_{\rm eq}) \qquad (0 \le s \le t),$$

converges weakly in $D[0,t]$ to an *OU process* (Z_t) with initial value $Z_0 = z$, local drift matrix $B = \partial F(x_{\rm eq})$ and local covariance matrix $G(x_{\rm eq})$. In particular, Z_s is normally distributed with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = e^{Bs}z$ and

$$V_s := \mathrm{Cov}(Z_s) = \int_0^s e^{Bu} G(x_{\rm eq}) e^{B^\top u}\, du\,.$$

Recent highlights

Corollary If x_{eq} satisfies $F(x_{\mathrm{eq}}) = 0$, then, under the conditions of the previous theorem, the family $(Z_t^{(N)})$ defined by

$$Z_s^{(N)} = \sqrt{N}(X_s^{(N)} - x_{\mathrm{eq}}) \qquad (0 \le s \le t),$$

converges weakly in $D[0,t]$ to an *OU process* (Z_t) with initial value $Z_0 = z$, local drift matrix $B = \partial F(x_{\mathrm{eq}})$ and local covariance matrix $G(x_{\mathrm{eq}})$. In particular, Z_s is normally distributed with mean and covariance given by $\mu_s := \mathbb{E}(Z_s) = e^{Bs}z$ and

$$V_s := \mathrm{Cov}(Z_s) = \int_0^s e^{Bu} G(x_{\mathrm{eq}}) e^{B^\top u}\, du\,.$$

Recent highlights

Note that

$$V_t = \int_0^t e^{Bu} G(x_{\mathrm{eq}}) e^{B^\top u}\, du = V_\infty - e^{Bt} V_\infty e^{B^\top t},$$

where V_∞, the stationary covariance matrix, satisfies

$$BV_\infty + V_\infty B^\top + G(x_{\mathrm{eq}}) = 0.$$

We conclude that, for N large, $X_t^{(N)}$ has an approximate Gaussian distribution with $\mathrm{Cov}(X_t^{(N)}) \simeq V_t/N$.

Recent highlights

Parameter estimation. Let $p_n(t) = \Pr(n_t = n)$ and $p_{nm}(t) = \Pr(n_{s+t} = m | n_s = n)$ (the state probabilities and transition probabilities of our Markov chain population model). The likelihood of observing a set of K observations $y_k = n_{t_k} (k = 1, \ldots, K)$ of the state of the Markov chain (n_t) at times $(0 \leq) \, t_1 < \cdots < t_K$ is

$$L(y|\theta) = p_{y_1}\left(\theta; t_1\right) \prod_{k=2}^{n} p_{y_{k-1}, y_k}\left(\theta; t_k - t_{k-1}\right),$$

which we can use to estimate a parameter (or vector of parameters) θ.

Recent highlights

Idea: We approximate the above likelihood using the likelihood of observing our Gaussian process (Z_t) at times $t_1, \ldots, t_K$.

Recent highlights

Idea: We approximate the above likelihood using the likelihood of observing our Gaussian process (Z_t) at times $t_1, \ldots, t_K$.

Ross, J.V., Taimre, T. and Pollett, P.K. (2006) On parameter estimation in population models, *Theoret. Pop. Biol.* 70, 498-510.

Ross, J.V., Pagendam, D.E. and Pollett, P.K. (2009) On parameter estimation in population models II: multi-dimensional processes and transient dynamics, *Theoret. Pop. Biol.* 75, 123-132.

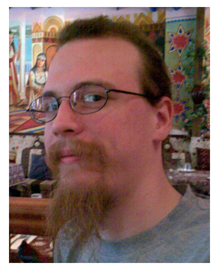
Recent highlights

Idea: We approximate the above likelihood using the likelihood of observing our Gaussian process (Z_t) at times $t_1, \ldots, t_K$.

Ross, J.V., Taimre, T. and Pollett, P.K. (2006) On parameter estimation in population models, *Theoret. Pop. Biol.* 70, 498-510.

Ross, J.V., Pagendam, D.E. and Pollett, P.K. (2009) On parameter estimation in population models II: multi-dimensional processes and transient dynamics, *Theoret. Pop. Biol.* 75, 123-132.

Joshua Ross 

Thomas Taimre 

Recent highlights

This is made possible because the cross-covariance can be evaluated:

$$\begin{aligned} V_{t,t+s} &:= \mathrm{Cov}(Z_t, Z_{t+s}) \\ &= M_t \int_0^t M_u^{-1}\, G(x_u)\, (M_u^{-1})^\top du\, M_{t+s}^\top \\ &= \mathrm{Cov}(Z_t)(M_t^\top)^{-1} M_{t+s}^\top \\ &= V_t\, (M_{t+s} M_t^{-1})^\top \\ &= V_t \exp(\textstyle\int_t^{t+s} \partial F(x_u)^\top du). \end{aligned}$$

Recent highlights

Random initial conditions.

Pollett, P.K., Dooley, A.H. and Ross, J.V. (2010) Modelling population processes with random initial conditions, *Math. Biosci.* (to appear).

Recent highlights

Random initial conditions.

Pollett, P.K., Dooley, A.H. and Ross, J.V. (2010) Modelling population processes with random initial conditions, *Math. Biosci.* (to appear).

(I have internodally collaborated with Tony Dooley!)

Recent highlights

Ingredients . . .

Recent highlights

Ingredients ...

Population model $(n_t,\, t \geq 0)$ $(n_t \in \mathbb{Z}^k)$, $X_t^{(N)} = n_t/N$.

Recent highlights

Ingredients . . .

Population model $(n_t,\, t \geq 0)$ $(n_t \in \mathbb{Z}^k)$, $X_t^{(N)} = n_t/N$.

$$q_{n,n+l} = N f_l\left(\tfrac{n}{N}\right) \qquad (l \neq 0,\ l \in \mathbb{Z}^k)$$

Recent highlights

Ingredients . . .

Population model $(n_t,\, t \geq 0)$ $(n_t \in \mathbb{Z}^k)$, $X_t^{(N)} = n_t/N$.

$$q_{n,n+l} = N f_l\left(\tfrac{n}{N}\right) \qquad (l \neq 0,\; l \in \mathbb{Z}^k)$$

$$F(x) := \textstyle\sum_{l \neq 0} l f_l\,(x) \qquad (x \in E)$$

Recent highlights

Ingredients ...

Population model $(n_t,\, t \geq 0)$ $(n_t \in \mathbb{Z}^k)$, $X_t^{(N)} = n_t/N$.

$$q_{n,n+l} = N f_l\left(\tfrac{n}{N}\right) \qquad (l \neq 0,\ l \in \mathbb{Z}^k)$$

$$F(x) := \textstyle\sum_{l \neq 0} l\, f_l\,(x) \qquad (x \in E)$$

$$(X_t^{(N)}) \simeq (x_t) \qquad (\text{ large } N)$$

Recent highlights

Ingredients . . .

Population model $(n_t,\, t \geq 0)$ $(n_t \in \mathbb{Z}^k)$, $X_t^{(N)} = n_t/N$.

$$q_{n,n+l} = N f_l\left(\tfrac{n}{N}\right) \qquad (l \neq 0,\ l \in \mathbb{Z}^k)$$

$$F(x) := \textstyle\sum_{l \neq 0} l f_l\,(x) \qquad (x \in E)$$

$$(X_t^{(N)}) \simeq (x_t) \qquad (\text{ large } N)$$

$$x_t' = F(x_t) \qquad (x_t \in E,\ s \in [0,t]) \qquad (\text{initial value } x_0)$$

Recent highlights

Ingredients ...

Population model $(n_t,\, t \geq 0)$ $(n_t \in \mathbb{Z}^k)$, $X_t^{(N)} = n_t/N$.

$$q_{n,n+l} = N f_l\left(\tfrac{n}{N}\right) \qquad (l \neq 0,\ l \in \mathbb{Z}^k)$$

$$F(x) := \textstyle\sum_{l\neq 0} l\, f_l\,(x) \qquad (x \in E)$$

$$(X_t^{(N)}) \simeq (x_t) \qquad (\text{ large } N)$$

$$x_t' = F(x_t) \qquad (x_t \in E,\ \ s \in [0,t]) \qquad (\text{initial value } x_0)$$

$$\mathrm{Cov}(X_t^{(N)}) \simeq V_t/N$$

$$V_t = M_t \left(\textstyle\int_0^t M_u^{-1} G(x_u) (M_u^{-1})^\top \, du \right) M_t^\top$$

$$M_t = \exp(\textstyle\int_0^t B_u \, du) \quad \text{and} \quad B_t = \partial F(x_t)$$

Recent highlights

Write $x_t = x_t(x_0)$, and,

$$V_t(x_0) = M_t \left(\int_0^t M_u^{-1} G(x_u(x_0))(M_u^{-1})^\top \, du \right) M_t^\top ,$$

$$M_t = \exp(\textstyle\int_0^t B_u \, du) \quad \text{and} \quad B_t = \partial F(x_t(x_0)).$$

Recent highlights

Write $x_t = x_t(x_0)$, and,

$$V_t(x_0) = M_t \left(\int_0^t M_u^{-1} G(x_u(x_0))(M_u^{-1})^\top \, du \right) M_t^\top,$$

$$M_t = \exp(\textstyle\int_0^t B_u \, du) \quad \text{and} \quad B_t = \partial F(x_t(x_0)).$$

Now think of x_0 as a random variable X_0.

Recent highlights

Write $x_t = x_t(x_0)$, and,

$$V_t(x_0) = M_t \left(\int_0^t M_u^{-1} G(x_u(x_0))(M_u^{-1})^\top \, du \right) M_t^\top,$$

$$M_t = \exp(\int_0^t B_u \, du) \quad \text{and} \quad B_t = \partial F(x_t(x_0)).$$

Now think of x_0 as a random variable X_0.

Use conditional expectation (drop superscript (N)):

$$\mathbb{E}\, X_t = \mathbb{E}\, \mathbb{E}(X_t | X_0) \simeq \mathbb{E}\, x_t(X_0).$$

$$\begin{aligned} \mathrm{Cov}(X_t) &= \mathrm{Cov}(\mathbb{E}(X_t|X_0)) + \mathbb{E}\, \mathrm{Cov}(X_t|X_0) \\ &\simeq \mathrm{Cov}(x_t(X_0)) + \tfrac{1}{N}\, \mathbb{E} V_t(X_0). \end{aligned}$$

Recent highlights

Suppose X_0 has pdf f_0.

Recent highlights

Suppose X_0 has pdf f_0.

In determining the action of the map $x_0 \mapsto x_t(x_0)$ (for simplicity, assumed to be injective) on f_0, we obtain a pdf f_t that summarises the effect of random initial conditions in our population assuming deterministic dynamics: for any $t > 0$,

$$f_t(y) = |J_t(y)| f_0 \left(x_t^{-1}(y) \right) \qquad (y \in \mathcal{R}_t),$$

where $J_t(y)$ is the Jacobian of $x_t^{-1}(y)$ and $\mathcal{R}_t = x_t(E)$ is the image of E under x_t.

Recent highlights

In the one-dimensional case $(k = 1)$ we can exhibit f_t explicitly.

Recent highlights

In the one-dimensional case $(k = 1)$ we can exhibit f_t explicitly.

Let $L(u)$ be the primitive $L(u) = \int^u dw/F(w)$. Suppose L is injective (it is sufficient that F be everywhere positive or everywhere negative).

Recent highlights

In the one-dimensional case $(k = 1)$ we can exhibit f_t explicitly.

Let $L(u)$ be the primitive $L(u) = \int^u dw/F(w)$. Suppose L is injective (it is sufficient that F be everywhere positive or everywhere negative).

Then,

$$f_t(y) = \left| \frac{F(L^{-1}(L(y)-t))}{F(y)} \right| f_0(L^{-1}(L(y) - t)).$$