Reversing time as an analytical tool: Isn’t that just Radon-Nikodym?

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AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

Menuetto al rovesci

Joseph Haydn’s Sonata No. 4 for Violin and Piano (piano part only) *Menuetto al rovescio*

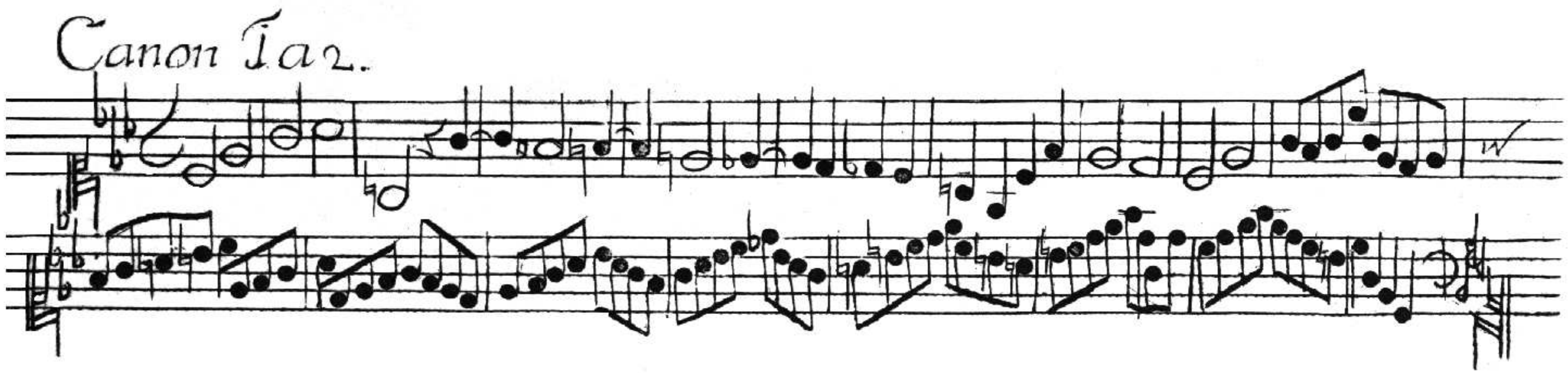
Motet *Diliges Dominum*

William Byrd's motet *Diliges Dominum*

Hammerklavier Sonata

Beethoven's Piano Sonata No. 29 in B flat, Op. 106 ("Hammerklavier"), Last movement (fugue) *Allegro risoluto*

Canon *cancrizans*



J.S. Bach's *Das Musikalische Opfer* (The Musical Offering), BWV 1079, Canon 1. a 2 *cancrizans*

Canon *cancrizans*

Setting

$(\Omega, \mathcal{F}, \mathbb{P})$ is our carrier triple.

$(X_t,\, t \in T)$ will denote a *stochastic process* with (ordered) *parameter set* T and *state space* $(E, \mathcal{E})$. (T would usually be "time": Z or Z_+, or, R or R_+.)

The "elementary picture" is: for each $t \in T$,

$$X_t : \Omega \to E \quad \text{and} \quad X_t^{-1} : \mathcal{E} \to \mathcal{F},$$

with X_t with $\mathcal{F}$-measurable.

We shall assume that $\mathcal{E}$ includes all point sets of E, that is, for all $x \in E$, $\{x\} \in \mathcal{E}$. At this stage, we make no further topological assumptions about the measurable space $(E, \mathcal{E})$.

The time reverse process

Definition. Let $(X_t, t \in T)$ and $(X_t^*, t \in T)$ be two stochastic processes with the same parameter set T and the same state space $(E, \mathcal{E})$. We say that X^* is a *time reverse* of X if, for any finite sequence $t_1 < t_2 < \cdots < t_n$ in T such that

$$t_n - t_{n-1} = t_2 - t_1,\ t_{n-2} - t_{n-1} = t_3 - t_2, \ldots,$$

and for any $A_1, A_2, \ldots, A_n \in \mathcal{E}$,

$$\mathbb{P}\left(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right) = \mathbb{P}\left(X_{t_1}^* \in A_n, \ldots, X_{t_n}^* \in A_1\right).$$

We say that X is *time reversible* if

$$\mathbb{P}\left(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right) = \mathbb{P}\left(X_{t_1} \in A_n, \ldots, X_{t_n} \in A_1\right).$$

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In particular, for all $A, B \in \mathcal{E}$ and $t, u \in T$,

$$\mathbb{P}\left(X_t \in A, X_u \in B\right) = \mathbb{P}\left(X_t^* \in B, X_u^* \in A\right).$$

On taking $B = E$, we see that $\mathbb{P}\left(X_t \in A\right) = \mathbb{P}\left(X_u^* \in A\right)$, which implies $\pi(A) := \mathbb{P}\left(X_t \in A\right) = \mathbb{P}\left(X_t^* \in A\right)$ (*the same for all t*).

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Conclusion. The above definition only makes sense if X^* and X are *stationary* with the same *stationary law* π.

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Example. Brownian motion has no time reverse.

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Conclusion. The above definition only makes sense if X^* and X are *stationary* with the same *stationary law* π.

Example. Brownian motion has no time reverse.

Exercise. Think of a diffusion that *does* have a time reverse.

Transition function

$$p_t(x, A) = \mathbb{P}(X_{s+t} \in A | X_s = x)$$

The time reverse of a Markov process

Definition. If $(E, \mathcal{E})$ is a measurable space, then a *transition function* $p = (p_t,\, t \geq 0)$ on $(E, \mathcal{E})$ is a family of mappings $p_t : E \times \mathcal{E} \to \mathrm{R}_+$ with the following properties:

(1) for all $A \in \mathcal{E}$, $p_t(\cdot, A)$ is an $\mathcal{E}$-measurable function,

(2) for all $x \in E$, $p_t(x, \cdot)$ is a subprobability measure on $(E, \mathcal{E})$ (that is, a measure on $(E, \mathcal{E})$ with $p_t(x, E) \leq 1$),

(3) the *Chapman-Kolmogorov equation* holds, that is, for all $x \in E$ and $A \in \mathcal{E}$, $p_{s+t}(x, A) = \int_E p_s(x, dy) p_t(y, A)$, $s, t \geq 0$, and

The transition function p is called *honest* if, for all $x \in E$ and $t \geq 0$, $p_t(x, \cdot)$ is a probability measure ($p_t(x, E) = 1$).

The time reverse of a Markov process

It is “usual” to have $p_0(x, A) = I_A(x)$ ($x \in E$, $A \in \mathcal{E}$), *but we certainly do not require* $\lim_{t \downarrow 0} p_t(x, A) = I_A(x)$.

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Interpretation. For an honest transition function p there always exists a (time-homogeneous) *Markov process* $(X_t, t \geq 0)$ with $p_t(x, A) = \mathbb{P}(X_{s+t} \in A | X_s = x)$ $(s, t \geq 0,\ A \in \mathcal{E})$.

(If p is dishonest, then we can append a coffin state ∂ making p honest over ($E^\partial, \mathcal{E}^\partial$), where $E^\partial = E \cup \partial$.)

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If X has *stationary law* π, that is, $\mathbb{P}(X_s \in A) = \pi(A)$, $s \geq 0$, then, by Total Probability,

$$\mathbb{P}(X_s \in A, X_{t+s} \in B) = \int_A \pi(dx) p_t(x, B) \quad (s, t \geq 0).$$

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Theorem 1. Let $(X_t, t \geq 0)$ and $(X_t^*, t \geq 0)$ be two Markov processes on the same state space $(E, \mathcal{E})$ with transition functions p and p^*, respectively. Then, X^* is the time reverse of X *if and only if*

(1) X and X^* are stationary with the same stationary law π.

(2) p^* is the reverse of p with respect to π.

In particular (**corollary!**), X is time reversible *if and only if* X is stationary with stationary law π and p is reversible with respect to π.

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So, what do I mean by "p^* is the reverse of p with respect to π" and "p is reversible with respect to π"?

The reverse transition function

Definition. Let p and p^* transition functions on the same measurable space $(E, \mathcal{E})$, and let m be a measure on $(E, \mathcal{E})$. Then, p^* is the *reverse of p with respect to m* if

$$\int_B m(dx) p_t(x, A) = \int_A m(dx) p^*_t(x, B) \quad (A, B \in \mathcal{E},\ t \geq 0).$$

If p is its own reverse with respect to m, that is,

$$\int_B m(dx) p_t(x, A) = \int_A m(dx) p_t(x, B) \quad (A, B \in \mathcal{E},\ t \geq 0),$$

then p is said to be *reversible with respect to m*.

The reverse transition function

Some implications. Putting $B = E$ we get

$$\int_E m(dx) p_t(x, A) = \int_A m(dx) p_t^*(x, E) \quad (A \in \mathcal{E},\ t \geq 0).$$

Since p^* is a transition function, $p_t^*(x, \cdot)$ is a subprobability measure, we get

$$\int_E m(dx) p_t(x, A) \leq \int_A m(dx) = m(A) \quad (A \in \mathcal{E},\ t \geq 0).$$

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We say that m is a *subinvariant measure* for p. Moreover, m is an *invariant measure* for p, that is equality holds,

$$\int_E m(dx)p_t(x, A) = m(A) \quad (A \in \mathcal{E},\ t \geq 0),$$

if p^* is honest.

The reverse transition function

Conversely, if m is invariant for p, then

$$\int_A m(dx) = m(A) = \int_E m(dx) p_t(x, A) = \int_A m(dx) p_t^*(x, E)$$

for all $A \in \mathcal{E}$ and $t \geq 0$, that is,

$$\int_A m(dx)(1 - p_t^*(x, E)) \geq 0 \quad (A \in \mathcal{E},\ t \geq 0),$$

So, if m is, additionally, a σ-finite measure, we may apply Radon-Nikodym to show[a] that p^* is $m - a.e.$ *honest*, that is, for all $t > 0$, $p_t^*(x, E) = 1$ for $m-$*almost all* $x \in E$.

[a] I will write out the argument carefully later

Markov chains

This is trivial for MCs. For example, the discrete-time, discrete-state case . . . .

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Exercise. Let $P = (p(i,j),\, i,j \in E)$ be a transition matrix and let $m = (m(j),\, j \in E)$ be a collection of positive numbers. Define $P^* = (p^*(i,j),\, i,j, \in E)$ by $p^*(i,j) = m(j)p(j,i)/m(i)$ $(i,j \in E)$. Show that P^* is a transition matrix whenever m is invariant for P, that is,

$$\sum_{j \in E} m(j)p(j,i) = m(i) \qquad (i \in E).$$

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Exercise. Agree that the n-step transition matrices bear the same relationship: $m(i)p_n^*(i,j) = m(j)p_n(j,i)$ $(i,j \in E)$.

The reverse transition function

Question. Given a transition function p and a subinvariant measure m on $(E, \mathcal{E})$, can we always find a transition function p^* on $(E, \mathcal{E})$ that is the reverse of p with respect to m? That is,

$$\int_B m(dx) p_t(x, A) = \int_A m(dx) p_t^*(x, B) \quad (A, B \in \mathcal{E},\ t \geq 0).$$

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Exercise. (Hint!) Let m be a σ-finite measure on $(E, \mathcal{E})$. Show that, for every $B \in \mathcal{E}$ with $m(B) < \infty$, $\mu_B(\cdot) := \int_B m(dx)p_t(x, \cdot)$ is a finite measure on $(E, \mathcal{E})$.

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Exercise. (Bigger hint!) Let m be a σ-finite measure on $(E, \mathcal{E})$ that is subinvariant for p. Show that, for every $B \in \mathcal{E}$, μ_B is a absolutely continuous with respect to m.

The time reverse of a Markov process

Theorem 1. Let $(X_t, t \geq 0)$ and $(X_t^*, t \geq 0)$ be two Markov processes on the same state space $(E, \mathcal{E})$ with transition functions p and p^*, respectively. Then, X^* is the time reverse of X *if and only if*

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Proof. Suppose X^* is the time reverse of X.

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Proof. Suppose X^* is the time reverse of X. We have already seen that X and X^* are necessarily stationary with the same stationary law π, and that

$$\mathbb{P}(X_s \in A, X_{t+s} \in B) = \int_A \pi(dx) p_t(x, B) \quad (A, B \in \mathcal{E},\ s, t \geq 0).$$

The time reverse of a Markov process

Thus, for all $A, B \in \mathcal{E}$, $s, t \geq 0$,

$$\begin{aligned}
\int_A \pi(dx) p_t(x, B) &= \mathbb{P}\left(X_s \in A, X_{t+s} \in B\right) \\
&= \mathbb{P}\left(X_s^* \in B, X_{t+s}^* \in A\right) \\
&= \int_B \pi(dx) p_t^*(x, A).
\end{aligned}$$

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Thus, for all $A, B \in \mathcal{E}$, $s, t \geq 0$,

$$\begin{aligned}\int_A \pi(dx) p_t(x, B) &= \mathbb{P}\left(X_s \in A, X_{t+s} \in B\right) \\ &= \mathbb{P}\left(X_s^* \in B, X_{t+s}^* \in A\right) \\ &= \int_B \pi(dx) p_t^*(x, A).\end{aligned}$$

Conversely, if (1) and (2) hold, then, as we have already seen, π is an invariant measure for p (and for p^*):

$$\int_E \pi(dx) p_t(x, B) = \pi(B), \qquad \pi(A) = \int_E \pi(dx) p_t^*(x, A).$$

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Therefore, for any $t_1 < t_2 < \cdots < t_n$ in T such that $t_n - t_{n-1} = t_2 - t_1,\ t_{n-2} - t_{n-1} = t_3 - t_2, \ldots$, and for any $A_1, A_2, \ldots, A_n \in \mathcal{E}$,

$$\mathbb{P}\left(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right) =$$

$$\int_E \pi(dx) \int_{A_1} p_{t_1}(x, dx_1) \int_{A_2} p_{t_2-t_1}(x_1, dx_2) \ldots \int_{A_n} p_{t_n-t_{n-1}}(x_{n-1}, dx_n).$$

Since π is invariant for p, this becomes

$$\int_{A_1} \pi(dx_1) \int_{A_2} p_{t_2-t_1}(x_1, dx_2) \ldots \int_{A_n} p_{t_n-t_{n-1}}(x_{n-1}, dx_n)$$
$$= \int_{A_n} \pi(dx_n) \int_{A_{n-1}} p^*_{t_n-t_{n-1}}(x_n, dx_{n-1}) \ldots \int_{A_1} p^*_{t_2-t_1}(x_2, dx_1),$$

repeatedly applying $\int_A \pi(dx) p_t(x, B) = \int_B \pi(dx) p^*_t(x, A)$.

The time reverse of a Markov process

Now use $t_n - t_{n-1} = t_2 - t_1,\ t_{n-2} - t_{n-1} = t_3 - t_2, \ldots$, together with the fact that π is invariant for p^*, to complete the proof:

$$\begin{aligned}
&\mathbb{P}\left(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right) \\
&\quad = \int_{A_n} \pi(dx_n) \int_{A_{n-1}} p^*_{t_n - t_{n-1}}(x_n, dx_{n-1}) \ldots \int_{A_1} p^*_{t_2 - t_1}(x_2, dx_1) \\
&\quad = \int_{A_n} \pi(dx_1) \int_{A_{n-1}} p^*_{t_n - t_{n-1}}(x_1, dx_2) \ldots \int_{A_1} p^*_{t_2 - t_1}(x_{n-1}, dx_n) \\
&\quad = \int_{A_n} \pi(dx_1) \int_{A_{n-1}} p^*_{t_2 - t_1}(x_1, dx_2) \ldots \int_{A_1} p^*_{t_n - t_{n-1}}(x_{n-1}, dx_n) \\
&\qquad\qquad = \mathbb{P}\left(X^*_{t_1} \in A_n, \ldots, X^*_{t_n} \in A_1\right).
\end{aligned}$$

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Example. The *Ornstein-Uhlenbeck (OU) process*.

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$$dZ_t = -\beta Z_t \, dt + \sigma dB_t \quad (t \geq 0),$$

where $(B_t, t \geq 0)$ is standard Brownian motion on $(\mathrm{R}, \mathcal{B}(\mathrm{R}))$ and β and σ are positive constants.

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where $(B_t, t \geq 0)$ is standard Brownian motion on $(\mathrm{R}, \mathcal{B}(\mathrm{R}))$ and β and σ are positive constants.

Its transition function p is absolutely continuous with respect to Lebesgue measure[a] in that $p_t(x, A) = \int_A p_t(x, y) \, dy$, where $p_t(x, y)$ (the *transition density*) is the Gaussian density with mean $xe^{-\beta t}$ and variance $\sigma^2(1 - e^{-2\beta t})/(2\beta)$.

[a] True for all diffusions!

The time reverse of a Markov process

Exercise. Show that π given by $\pi(A) = \int_A \phi(y)\,dy$, where ϕ is the Gaussian density with mean 0 and variance $\sigma^2/(2\beta)$, is an invariant probability measure for p.

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Exercise. Show that π given by $\pi(A) = \int_A \phi(y)\,dy$, where ϕ is the Gaussian density with mean 0 and variance $\sigma^2/(2\beta)$, is an invariant probability measure for p.

You will need to verify that

$$\int_{\mathbb{R}} \pi(dx) p_t(x, A) = \pi(A),$$

or, equivalently, for the transition density,

$$\int_{-\infty}^{\infty} \phi(x) p_t(x, y)\,dx = \phi(y) \quad (t \geq 0,\; y \in \mathbb{R}).$$

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It is considerably easier to verify that

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We conclude that

$$\int_A \pi(dx)p_t(x,B) = \int_B \pi(dy)p_t(y,A).$$

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Hence, the transition function p is reversible with respect to π, and (as before, putting $A = E$ $(= \mathrm{R})$), π is invariant for p.

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Hence, the transition function p is reversible with respect to π, and (as before, putting $A = E\ (= \mathrm{R})$), π is invariant for p.

Also, by Theorem 1, our process Z (the stationary OU process) is *time reversible*.

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But, its transition density

$$p_t(x, y) = (2\pi t)^{-n/2} \exp\left(-|y - x|/2t\right) \quad (x, y \in \mathrm{R}^n)$$

satisfies $p_t(x, y) = p_t(y, x)$, and so its transition function p is *reversible with respect to Lebesgue measure* (and hence invariant for p).

The time reverse of a Markov process

Example. *Brownian motion* (this time on $(\mathrm{R}^n, \mathcal{B}(\mathrm{R}^n))$). We have already agreed that there is no time reverse.

But, its transition density

$$p_t(x, y) = (2\pi t)^{-n/2} \exp\left(-|y - x|/2t\right) \quad (x, y \in \mathrm{R}^n)$$

satisfies $p_t(x, y) = p_t(y, x)$, and so its transition function p is *reversible with respect to Lebesgue measure* (and hence invariant for p).

For the complete story on *reversible diffusions* (not necessarily time-reversible!), see John Kent's 1978 paper*.

*Kent, J. (1978) Time-reversible diffusions, Adv. Appl. Probab. 10, 819–835.

The reverse transition function

The reverse transition function as an analytical tool.
Suppose that we are given a transition function p and measure m on $(E, \mathcal{E})$. If we can determine the reverse transition function, that is, a transition function p^* on $(E, \mathcal{E})$ satisfying $\int_B m(dx) p_t(x, A) = \int_A m(dx) p_t^*(x, B)$, $A, B \in \mathcal{E}$, $t \geq 0$, then, as already remarked, m will be subinvariant for p and invariant for p if p^* is honest (and only if p^* is $m - a.e.$ honest).

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Of course, in most cases, we would be given more fundamental information, such as the *diffusion coefficients* (diffusions), or the *transition rates* (chains). We might hope to be able to establish the existence, and then the honesty of p^* *without actually exhibiting* p^* *explicitly*.

The reverse transition function

Recall the hints:

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Exercise. (Hint!) Let m be a σ-finite measure on $(E, \mathcal{E})$. Show that, for every $B \in \mathcal{E}$ with $m(B) < \infty$, $\mu_B(\cdot) := \int_B m(dx) p_t(x, \cdot)$ is a finite measure on $(E, \mathcal{E})$.

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Solution. Clearly $\mu_B(\cdot) \geq 0$, $\mu_B(\emptyset) = 0$ and $\mu_B(\cdot)$ is countably additive by Fubini (since m is σ-finite). Also, since $p_t(x, E) \leq 1$, we have $\mu_B(E) \leq \int_B m(dx) = m(B) < \infty$.

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Exercise. (Bigger hint!) Let m be a σ-finite measure on $(E, \mathcal{E})$ that is subinvariant for p. Show that, for every $B \in \mathcal{E}$, μ_B is a absolutely continuous with respect to m.

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Solution. We are told that $\int_E m(dx) p_t(x, A) \leq m(A)$ $(A \in \mathcal{E})$. So, $\mu_B(A) \leq m(A)$, and hence $m(N) = 0 \Rightarrow \mu_B(N) = 0$.

The reverse transition function

We're in business! Radon-Nikodym provides us with an $\mathcal{E}$-measurable (m-integrable) non-negative function f define uniquely $m - a.e.$ by $\mu_B(A) = \int_A f(x)m(dx)$ $(A \in \mathcal{E})$.

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Writing this out carefully: if m is a σ-finite measure that is subinvariant for p, then, for all $t \geq 0$, and for every $B \in \mathcal{E}$ with $m(B) < \infty$, there is an $\mathcal{E}$-measurable $f_t(\cdot, B)$ that satisfies

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Why?

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First, m being totally finite is *too restrictive*.

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Second, we need to show

(2) that $f_t(x, \cdot)$ is a subprobability measure on $(E, \mathcal{E})$, and

(3) that f satisfies the Chapman-Kolmogorov equation $f_{s+t}(x, A) = \int_E f_s(x, dy) f_t(y, A)$,

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And third, we need a refinement of the usual definition of a transition function (because our reverse transition function can at best be known $m - a.e.$ uniquely), which was $\ldots$

The reverse transition function

Definition. If $(E, \mathcal{E})$ is a measurable space, then a transition function $p = (p_t,\ t \geq 0)$ on $(E, \mathcal{E})$ is a family of mappings $p_t : E \times \mathcal{E} \to \mathrm{R}_+$ with the following properties:

(1) for all $A \in \mathcal{E}$, $p_t(\cdot, A)$ is an $\mathcal{E}$-measurable function,

(2) for all $x \in E$, $p_t(x, \cdot)$ is a subprobability measure on $(E, \mathcal{E})$ (that is, a measure on $(E, \mathcal{E})$ with $p_t(x, E) \leq 1$),

(3) the Chapman-Kolmogorov equation holds, that is, for all $x \in E$ and $A \in \mathcal{E}$, $p_{s+t}(x, A) = \int_E p_s(x, dy) p_t(y, A)$, $s, t \geq 0$ (unmarked sums shall be over E), and

The transition function p is called honest if, for all $x \in E$ and $t \geq 0$, $p_t(x, \cdot)$ is a probability measure ($p_t(x, E) = 1$).

The reverse transition function

Definition. If $(E, \mathcal{E})$ be a measurable space and let m be a σ-finite measure on $(E, \mathcal{E})$. Then, an $m - a.e.$ *transition function* $p = (p_t,\ t \geq 0)$ on $(E, \mathcal{E})$ is a family of mappings $p_t : E \times \mathcal{E} \to \mathrm{R}_+$ with the usual properties (1)–(3), *but (2) and (3) are required to hold for* $m-$*almost all* $x \in E$.

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Next we will show that our candidate reverse transition function f satisfies *(2)* and *(3)* assuming that out subinvariant measure m is a *finite measure*–thus avoiding technicalities.

The reverse transition function

Recall that f is determined $m - a.e.$ uniquely: for all $t \geq 0$, and for every $B \in \mathcal{E}$, there is an $\mathcal{E}$-measurable $f_t(\cdot, B)$ that satisfies

$$\int_B m(dx) p_t(x, A) = \int_A m(dx) f_t(x, B) \quad (A \in \mathcal{E}).$$

Remember we are assuming that $m(B) < \infty$ for all $B \in \mathcal{E}$.

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First we prove that for m-almost all x, $f_t(x, \cdot)$ is a subprobability measure on $(E, \mathcal{E})$, that is, a measure on $(E, \mathcal{E})$ with $f_t(x, E) \leq 1$.

The reverse transition function

Claim. $f_t(x, E) \leq 1 \ldots\ldots$

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Take $B = E$ in the definition of f:

$$\int_E m(dx) p_t(x, A) = \int_A m(dx) f_t(x, E) \quad (A \in \mathcal{E},\, t \geq 0).$$

So, for all $A \in \mathcal{E}$,

$$\int_A m(dx) f_t(x, E) = \int_E m(dx) p_t(x, A) \leq m(A) = \int_A m(dx),$$

and hence

$$\int_A m(dx) \left(1 - f_t(x, E)\right) = \int_A m(dx) - \int_A m(dx) f_t(x, E) \geq 0.$$

The reverse transition function

It follows that $\int_{(\cdot)} m(dx)\,(1 - f_t(x, E))$ is a totally finite positive measure on $(E, \mathcal{E})$.

It is absolutely continuous with respect to m, because if N is an m-null set in $\mathcal{E}$, then

$$(0 \leq)\;\; \int_N m(dx)\,(1 - f_t(x, E)) \leq \int_N m(dx) = m(N) = 0,$$

and hence it has the $m - a.e.$ uniquely determined Radon-Nikodym derivative $1 - f_t(x, E)$.

Since the latter is $m - a.e.$ unique, it is therefore $m - a.e.$ positive, that is, $f_t(x, E) \leq 1$, for $m-$almost all $x \in E$.

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Claim. $f_t(x, \cdot)$ is a measure for m-almost all x . . . . . .

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Claim. $f_t(x,\cdot)$ is a measure for m-almost all x ......

First,

$$0 = \int_{\varnothing} m(dx) p_t(x,A) = \int_A m(dx) f_t(x,\varnothing) \quad (A \in \mathcal{E}).$$

and hence, by the Radon-Nikodym Theorem, $f_t(x,\varnothing) = 0$ for m-almost all x.

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We already have $f_t(\cdot, B) \geq 0$, $m - a.e.$, so we only need to check σ-additivity.

The reverse transition function

Let $B_1, B_2, \ldots$ be a sequence of disjoint sets in $\mathcal{E}$. Using the definition of f and Fubini, we have, for all $A \in \mathcal{E}$,

$$\begin{aligned}
\int_A m(dx) f_t\left(x, \cup_j B_j\right) &= \int_{(\cup_j B_j)} m(dx) p_t(x, A) \\
&= \sum_j \int_{B_j} m(dx) p_t(x, A) \\
&= \sum_j \int_A m(dx) f_t(x, B_j) \\
&= \int_A m(dx) \sum_j f_t(x, B_j) \ (< \infty),
\end{aligned}$$

That is, $\int_A m(dx) \left(f_t\left(x, \cup_j B_j\right) - \sum_j f_t(x, B_j) \right)$, for all $A \in \mathcal{E}$, and hence $f_t\left(\cdot, \cup_j B_j\right) = \sum_j f_t(\cdot, B_j)$, $m - a.e.$ (again by the Radon-Nikodym Theorem).

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Since $\int_B m(dx) p_t(x, A) = \int_A m(dx) f_t(x, B)$ for all $A, B \in \mathcal{E}$, we have that

$$\begin{aligned} \int_A m(dx) f_{s+t}(x, B) &= \int_B m(dx) p_{s+t}(x, A) \\ &= \int_B m(dx) \int_E p_t(x, dy) p_s(y, A) \\ &= \int_A m(dx) \int_E f_s(x, dy) f_t(y, B). \end{aligned}$$

The Radon-Nikodym Theorem then tells us that

$$f_{s+t}(\,\cdot\,, B) = \int_E f_s(\,\cdot\,, dy) f_t(y, B), \quad m - a.e.$$

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We have proved the following simple result.

The reverse transition function

Proposition 1. Let p be a transition function on a measurable space $(E, \mathcal{E})$ and suppose that m is a totally finite measure on $(E, \mathcal{E})$ that is subinvariant for p. Then there exists an $m - a.e.$ transition function p^* which is the $m - a.e.$ unique reverse of p with respect to m.

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I am happy to report the following pleasing result.

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Theorem 2. Let $(E, \mathcal{O}, \mathcal{E})$ be an inner-regular, measurable topological space with a countable basis. Let p be a transition function on $(E, \mathcal{E})$ and suppose that m is a σ-finite measure on $(E, \mathcal{E})$ that is subinvariant for p. Then, there exists an $m - a.e.$ transition function p^* which is the $m - a.e.$ unique reverse of p with respect to m.

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What is $(E, \mathcal{O}, \mathcal{E})$ and why?

An application

But first, an application . . .

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Corollary. Let $(E, \mathcal{O}, \mathcal{E})$ be an inner-regular, measurable topological space with a countable basis. Let p be a transition function on $(E, \mathcal{E})$ and suppose that m is a σ-finite measure on $(E, \mathcal{E})$ that is subinvariant for p. Let p^* be the $m - a.e.$ unique reverse transition function with respect to m. Then, m is invariant for p *if and only if* p^* is $m - a.e.$ honest.

The reverse transition function

Why $\mathcal{O}$?

The reverse transition function

Why $\mathcal{O}$? We need to take limits.

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Let m be a σ-finite measure that is subinvariant for p and let $\mathcal{E}_0$ be the subset of $\mathcal{E}$ that consists of all sets B with $m(B) < \infty$. Then, for every $B \in \mathcal{E}_0$, Radon-Nikodym gave us an $\mathcal{E}$-measurable $f_t(\cdot, B)$ such that

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We showed that f satisfied the properties of an $m - a.e.$ transition function.

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We extend this from $(E, \mathcal{E}_0, m)$ to $(E, \mathcal{E}, m)$ by approximating $(E, \mathcal{E}, m)$ by *finite* measure spaces $(E, \mathcal{E}_n, \mu)$, $n = 1, 2, \ldots$.

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The idea. Statements (like the one immediately above) concerning a given σ-finite measure which hold over $\mathcal{E}_0$ are extended to $\mathcal{E}$, for they are show to hold over each of the σ-algebras $\{\mathcal{E}_n\}_{n=1}^{\infty}$ defined by $\mathcal{E}_n = \{A \cap E_n : A \in \mathcal{E}\}$.

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For example, our elementary argument (above) gives us an $f^{(n)}$ for each n with the right properties. We then set

$$p_t^*(x, A) = \sum_{n=1}^{\infty} f_t^{(n)}(x, A \cap E_n) \quad (x \in E,\ A \in \mathcal{E},\ t \geq 0)$$

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If *every* finite measure on $(E, \mathcal{E})$ is inner regular, we say that the *space* is inner regular.

A simple stress release model

Stress accumulates in small amounts of expected size ϵ, and at rate λ, and all the stress accumulated so far is released completely (a seismic event–say a mine collapse) at points of a Poisson process with rate σ.

Let $E = \mathrm{R}_+$ and $\mathcal{E} = \mathcal{B}(\mathrm{R}_+)$. Define “rates”

$$q(x, A) = \sigma I_A(0) + \int_A g(x, y)\, dy \quad (A \in \mathcal{E},\ x \in E),$$

where

$$g(x, y) = \frac{\lambda}{\epsilon^2} \exp\left(-(y - x)/\epsilon\right) I_{(x,\infty)}(y).$$

A simple stress release model

This is a version of a stochastic slip-predictable model for earthquake occurrences*. The state of the process represents the accumulated stress on a fault. The process waits a time which is exponentially distributed with mean $1/q(x) = 1/(\sigma + \lambda/\epsilon)$ and then either jumps to 0, which is identified as a seismic event (stress release), with probability $\alpha = \sigma/(\sigma + \lambda/\epsilon)$ or otherwise jumps up a distance which is exponentially distributed with mean ϵ. For small ϵ, the later is a pure-jump analogue of a continuous constant stress increase.

*Kiremidjian, A.S. and Anagnos, T. (1984) Stochastic slip-predictable model for earthquake occurrences, Bull. Seism. Soc. Amer. 74, 739–755.

A simple stress release model

Define $m = (m(x),\, x \in E)$ by $m(\{0\}) = \alpha$ and, for $x > 0$, $m(dx) = \epsilon^{-1}\alpha(1-\alpha)\exp(-\alpha x/\epsilon)dx$, so that $m((0,\infty)) = 1-\alpha$, $m(E) = m([0,\infty)) = 1$, and,

$$m((0,x)) = (1-\alpha)(1-e^{-\alpha x/\epsilon}) \qquad (x > 0)$$

$$m([0,x)) = 1-(1-\alpha)e^{-\alpha x/\epsilon} \qquad (x \geq 0).$$

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Is m an invariant measure?

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Is m an invariant measure?

This shall remain one of life's mysteries, at least until I have the opportunity to speak again on this topic.