# Identifying Markov chains with a given invariant measure

by

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Let $Q = (q_{ij},\ i,j \in S)$ be a stable and conservative $q$-matrix of transition rates over a countable set $S$. Suppose that we are given a subinvariant for $Q$, that is, a collection of positive numbers $m = (m_i,\ i \in S)$ such that $\sum_{i\in S} m_i q_{ij} \le 0$, $j \in S$. Our problem is to identify a $Q$-process for which $m$ is invariant, that is, a standard transition function $P(\cdot) = (p_{ij}(\cdot),\ i,j \in S)$ that satisfies $p'_{ij}(0+) = q_{ij}$, $i,j \in S$, and

$$\sum_{i\in S} m_i p_{ij}(t) = m_j, \qquad j \in S,\ t > 0.$$

We begin by showing that if $m$ is invariant for $P$, then it is subinvariant for $Q$, and then *invariant* for $Q$ if and only if $P$ satisfies the backward differential equations. A simple corollary is that if $m$ is invariant for *minimal* $Q$-process, then it is invariant for $Q$.

The major result gives conditions for the existence of a $Q$-process $P$ for which the given measure $m$ (subinvariant for $Q$) is invariant for $P$; one such $Q$-process is specified through its resolvent. The invariance condition is shown to be necessary in the case where $Q$ is single-exit (there is a single escape route to infinity). We also give necessary and sufficient conditions for this process to be *honest*, that is $\sum_{j\in S} p_{ij}(t) = 1$ for all $i \in S$ and $t > 0$, as well as a simple sufficient condition for the existence of an honest $Q$-process for which the given measure $m$ is invariant for $P$. The case where $Q$ is symmetrically reversible with respect to $m$ is considered in some detail, leading to a complete solution of the existence and uniqueness problem for birth-death processes.