Birth-Death Processes and Orthogonal Polynomials

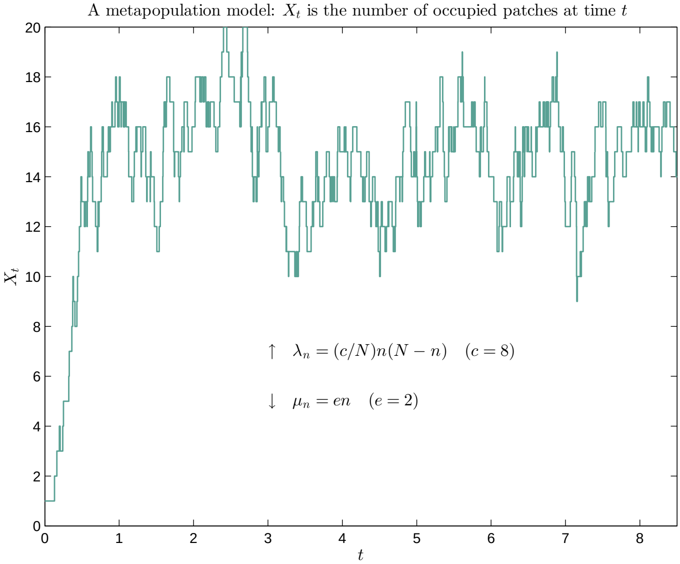
Phil. Pollett

The University of Queensland

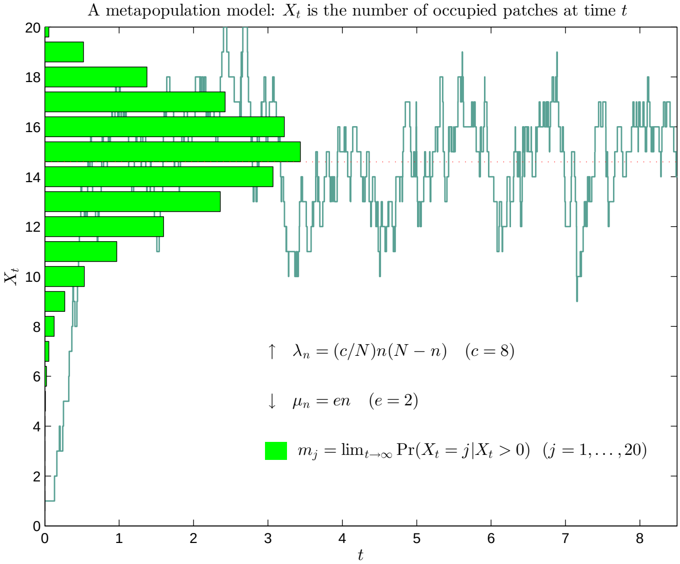
India Institute of Technology Bombay

26 October 2015

Example: a metapopulation model (illustrating quasi stationarity)



Quasi-stationary distribution



$$p_{ij}(t) := \Pr(X_{s+t} = j | X_s = i)$$

$$= \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x)$$

Birth-death processes

A *birth-death* process is a continuous-time Markov chain $(X_t,\ t \geq 0)$ taking values in $S \cup \{-1\}$, where $S \subseteq \{0, 1, \ldots\}$, with

$$\Pr(X_{t+h} = n+1 | X_t = n) = \lambda_n h + \circ(h)$$

$$\Pr(X_{t+h} = n-1 | X_t = n) = \mu_n h + \circ(h)$$

$$\Pr(X_{t+h} = n | X_t = n) = 1 - (\lambda_n + \mu_n) h + \circ(h)$$

(as $h \to 0$). Other transitions happen with probability $\circ(h)$.

Birth-death processes

A *birth-death* process is a continuous-time Markov chain $(X_t,\ t \geq 0)$ taking values in $S \cup \{-1\}$, where $S \subseteq \{0, 1, \ldots\}$, with

$$\Pr(X_{t+h} = n+1 | X_t = n) = \lambda_n h + \circ(h)$$

$$\Pr(X_{t+h} = n-1 | X_t = n) = \mu_n h + \circ(h)$$

$$\Pr(X_{t+h} = n | X_t = n) = 1 - (\lambda_n + \mu_n) h + \circ(h)$$

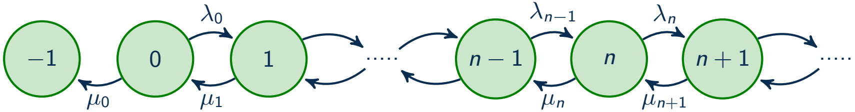
(as $h \to 0$). Other transitions happen with probability $\circ(h)$.

The *birth rates* $(\lambda_n,\ n \geq 0)$ and the *death rates* $(\mu_n,\ n \geq 0)$ are all strictly positive except perhaps μ_0, which could be 0. State -1 is a "extinction state", which can be reached if $\mu_0 > 0$.

Birth-death processes

The *birth rates* $(\lambda_n,\ n \geq 0)$ and the *death rates* $(\mu_n,\ n \geq 0)$ are all strictly positive, except perhaps μ_0, which could be 0. State -1 is a "extinction state", which can be reached if $\mu_0 > 0$.

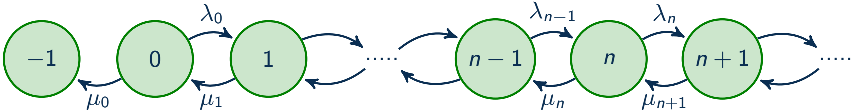
$\mu_0 > 0$



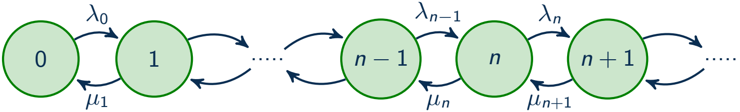
Birth-death processes

The *birth rates* ($\lambda_n,\ n \geq 0$) and the *death rates* ($\mu_n,\ n \geq 0$) are all strictly positive, except perhaps μ_0, which could be 0. State -1 is a "extinction state", which can be reached if $\mu_0 > 0$.

$\mu_0 > 0$



$\mu_0 = 0$



Explosive birth-death processes

Suppose that $\lambda_n = 2^{2n}$, $\mu_n = 2^{2n-1}$ $(n \geq 1)$, and $\mu_0 = 0$, with $S = \{0, 1, \dots\}$.

Explosive birth-death processes

Suppose that $\lambda_n = 2^{2n}$, $\mu_n = 2^{2n-1}$ $(n \geq 1)$, and $\mu_0 = 0$, with $S = \{0, 1, \ldots\}$.

An "equilibrium distribution" exists: $\pi_n = (\frac{1}{2})^{n+1}$. But ...

Explosive birth-death processes

Suppose that $\lambda_n = 2^{2n}$, $\mu_n = 2^{2n-1}$ $(n \geq 1)$, and $\mu_0 = 0$, with $S = \{0, 1, \dots\}$.

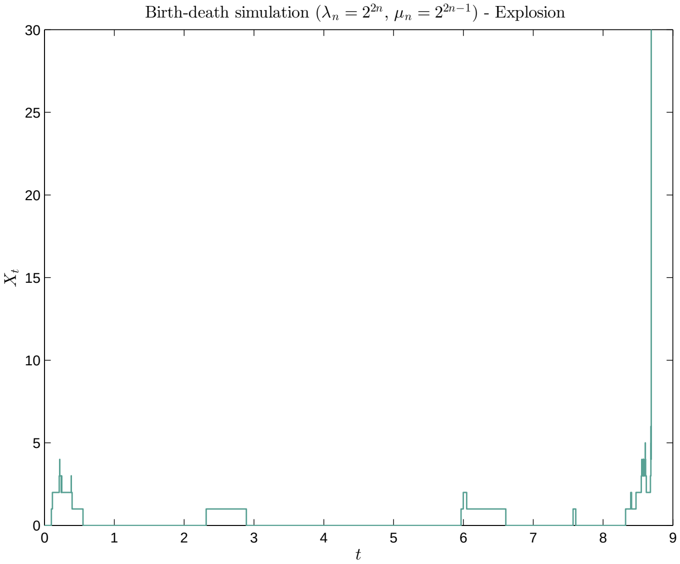
An "equilibrium distribution" exists: $\pi_n = (\frac{1}{2})^{n+1}$. But ...

When a jump occurs it is a birth with probability

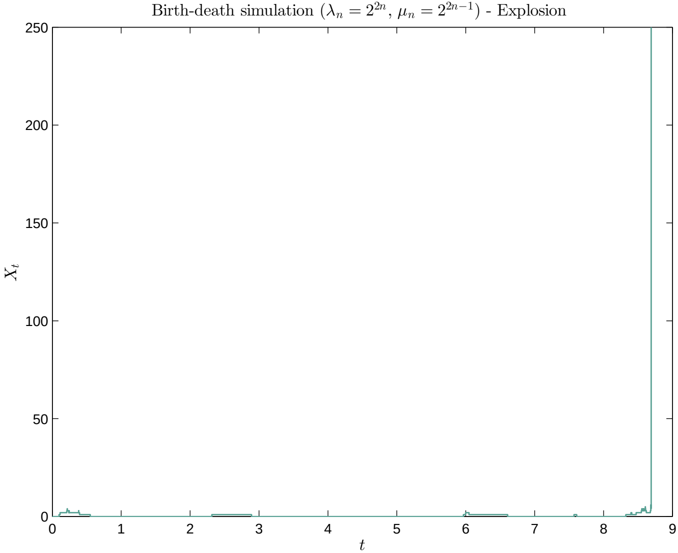
$$p_n = \frac{2^{2n}}{2^{2n} + 2^{2n-1}} = \frac{2}{3}.$$

Thus births are twice as likely as deaths, and so the process will have positive drift.

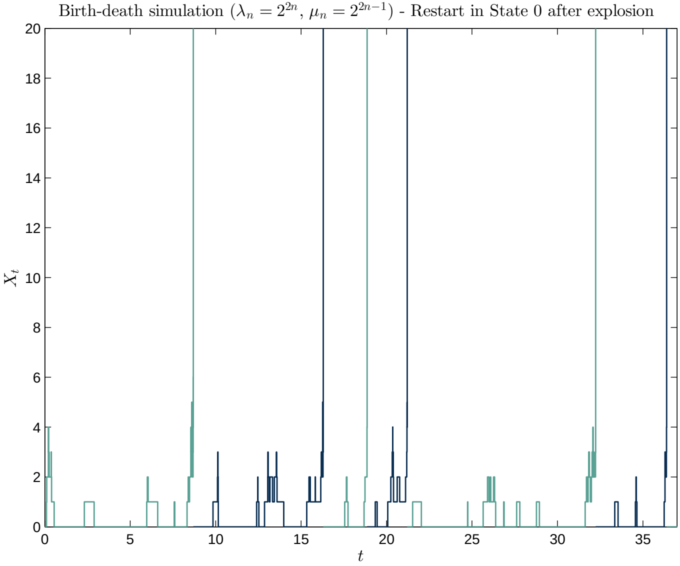
Explosive birth-death processes



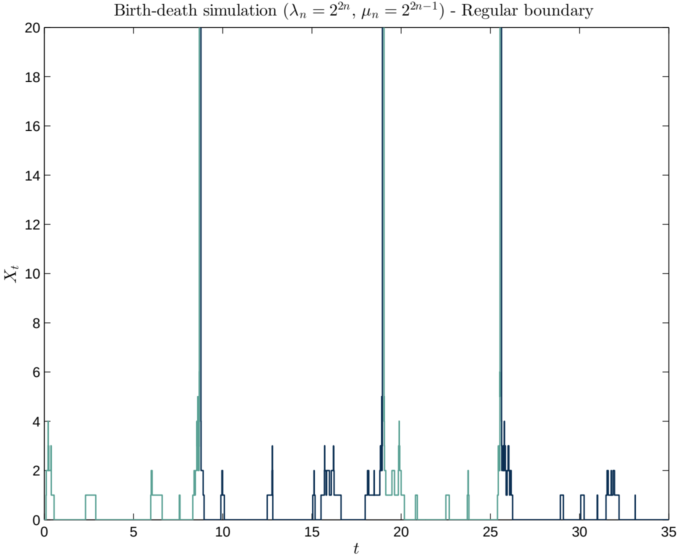
Explosive birth-death processes



Explosive birth-death processes



Explosive birth-death processes



The Kolmogorov differential equations

The conditions we have imposed ensure that the *transition probabilities* $p_{ij}(t) = \Pr(X_{s+t} = j | X_s = i)$ $(i, j \in S,\ s, t \geq 0)$ do not depend on s.

The Kolmogorov differential equations

The conditions we have imposed ensure that the *transition probabilities* $p_{ij}(t) = \Pr(X_{s+t} = j | X_s = i)$ $(i, j \in S,\ s, t \geq 0)$ do not depend on s.

For any such *time-homogeneous* continuous-time Markov chain with (conservative) transition rate matrix $Q = (q_{ij})$, the *transition function* $P(t) = (p_{ij}(t))$ satisfies the *backward equations*

$$P'(t) = QP(t) \qquad (BE)$$

but not necessarily the *forward equations*

$$P'(t) = P(t)Q \qquad (FE)$$

(the derivative is taken elementwise). Note that $Q = P'(0+)$.

The Kolmogorov differential equations

The conditions we have imposed ensure that the *transition probabilities* $p_{ij}(t) = \Pr(X_{s+t} = j | X_s = i)$ $(i, j \in S,\ s, t \geq 0)$ do not depend on s.

For any such *time-homogeneous* continuous-time Markov chain with (conservative) transition rate matrix $Q = (q_{ij})$, the *transition function* $P(t) = (p_{ij}(t))$ satisfies the *backward equations*

$$P'(t) = QP(t) \qquad (BE)$$

but not necessarily the *forward equations*

$$P'(t) = P(t)Q \qquad (FE)$$

(the derivative is taken elementwise). Note that $Q = P'(0+)$.

There is however a *minimal* solution $F(t) = (f_{ij}(t))$ to (BE) and this *satisfies* (FE).

The Kolmogorov differential equations

The conditions we have imposed ensure that the *transition probabilities* $p_{ij}(t) = \Pr(X_{s+t} = j | X_s = i)$ $(i, j \in S,\ s, t \geq 0)$ do not depend on s.

For any such *time-homogeneous* continuous-time Markov chain with (conservative) transition rate matrix $Q = (q_{ij})$, the *transition function* $P(t) = (p_{ij}(t))$ satisfies the *backward equations*

$$P'(t) = QP(t) \qquad (BE)$$

but not necessarily the *forward equations*

$$P'(t) = P(t)Q \qquad (FE)$$

(the derivative is taken elementwise). Note that $Q = P'(0+)$.

There is however a *minimal* solution $F(t) = (f_{ij}(t))$ to (BE) and this *satisfies* (FE).

Non-explosivity corresponds to F being the *unique* solution to (BE). Otherwise F governs the process *up to the time of the (first) explosion*.

The Kolmogorov differential equations

For birth-death processes the full range of behaviour is possible.

Here the transition rate matrix restricted to $S = \{0, 1, \dots\}$ has the form

$$Q = \begin{pmatrix} -(\lambda_0+\mu_0) & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Kolmogorov differential equations

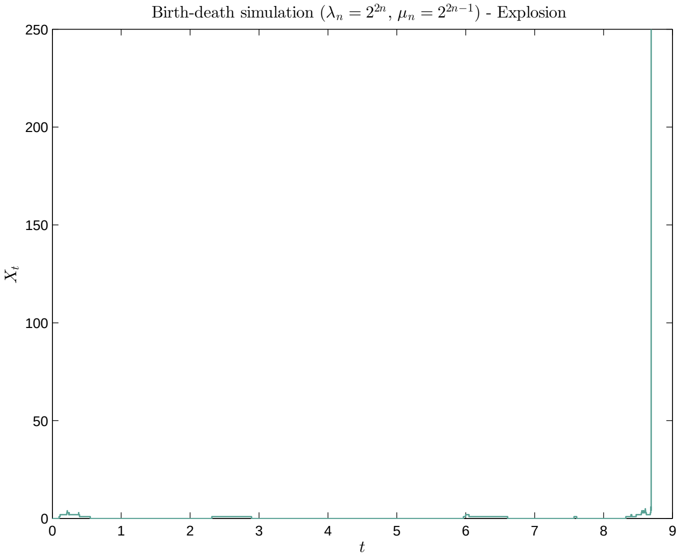
For birth-death processes the full range of behaviour is possible.

Here the transition rate matrix restricted to $S = \{0, 1, \dots\}$ has the form

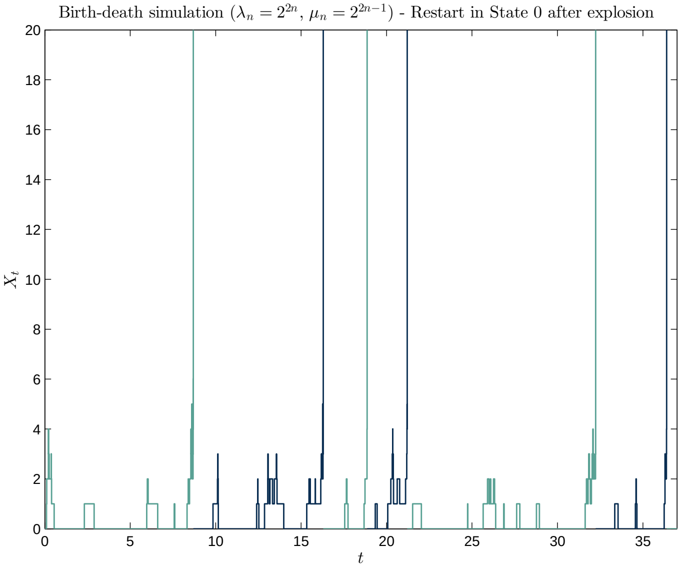
$$Q = \begin{pmatrix} -(\lambda_0+\mu_0) & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Returning to the example where $\lambda_n = 2^{2n}$, $\mu_n = 2^{2n-1}$ $(n \geq 1)$, and $\mu_0 = 0$, we have ...

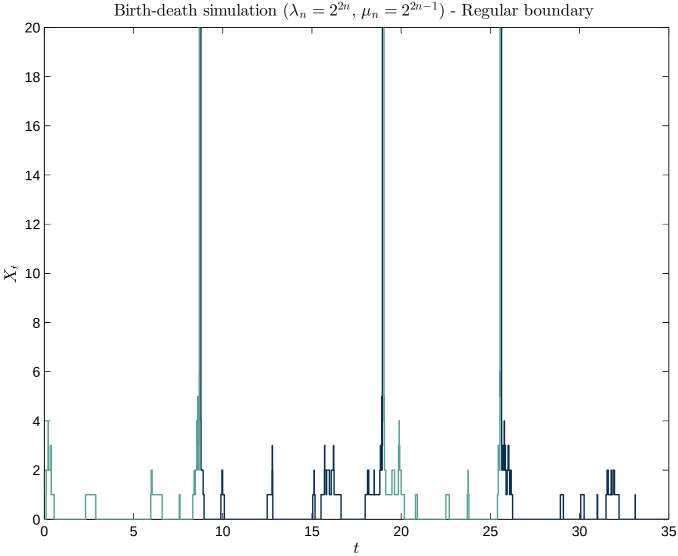
The process governed by F (the "minimal process")



A process where P satisfies (BE) *but not* (FE)



A process where P satisfies *both* (BE) and (FE)



The birth-death polynomials

Define a sequence $(\mathcal{Q}_n,\ n \in S)$ of polynomials by

$$\begin{aligned} \mathcal{Q}_0(x) &= 1 \\ -x\mathcal{Q}_0(x) &= -(\lambda_0 + \mu_0)\mathcal{Q}_0(x) + \lambda_0\mathcal{Q}_1(x) \\ -x\mathcal{Q}_n(x) &= \mu_n\mathcal{Q}_{n-1}(x) - (\lambda_n + \mu_n)\mathcal{Q}_n(x) + \lambda_n\mathcal{Q}_{n+1}(x), \end{aligned}$$

and a sequence of strictly positive numbers $(\pi_n,\ n \in S)$ by $\pi_0 = 1$ and, for $n \geq 1$,

$$\pi_n = \frac{\lambda_0\lambda_1 \dots \lambda_{n-1}}{\mu_1\mu_2 \dots \mu_n}.$$

A explicit expression for $p_{ij}(t)$

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \; t \geq 0).$$

[1] Karlin, S. and McGregor, J.L. (1957) The differential equations of birth-and-death processes, and the Stieltjes Moment Problem. *Trans. Amer. Math. Soc.* 85, 489–546.

A explicit expression for $p_{ij}(t)$

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

Since $p_{ij}(0) = \delta_{ij}$, it is clear that $(\mathcal{Q}_n)$ are orthogonal with orthogonalizing measure ψ:

$$\int_0^\infty \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x) = \frac{\delta_{ij}}{\pi_j} \qquad (i, j \geq 0).$$

A explicit expression for $p_{ij}(t)$

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

Proof. We use mathematical induction: on i using (BE) with $j = 0$ and then on j using (FE). But, showing that there is a probability measure ψ with support $[0, \infty)$ whose Laplace transform is $p_{00}(t)$, that is

$$p_{00}(t) = \int_0^\infty e^{-tx}\, d\psi(x) \ \left(= \pi_0 \int_0^\infty e^{-tx} \mathcal{Q}_0(x) \mathcal{Q}_0(x)\, d\psi(x) \right),$$

is not completely straightforward. More on this later.

A explicit expression for $p_{ij}(t)$ - Why is it useful?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \ t \geq 0).$$

A explicit expression for $p_{ij}(t)$ - Why is it useful?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \ t \geq 0).$$

A explicit expression for $p_{ij}(t)$ - Why is it useful?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

A explicit expression for $p_{ij}(t)$ - Why is it useful?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \pi_j \mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

A explicit expression for $p_{ij}(t)$ - Why is it useful?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \pi_j \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \; t \geq 0).$$

A explicit expression for $p_{ij}(t)$ - Why is it useful?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

This formula, together with the myriad of properties of $(\mathcal{Q}_n)$ and ψ, are used to develop theory peculiar to birth-death processes.

Some properties of $(\mathcal{Q}_n)$ and ψ

Of particular interest and importance is the "interlacing" property of the zeros $x_{n,i}$ $(i = 1, \cdots, n)$ of $\mathcal{Q}_n$: they are strictly positive, simple, and they satisfy

$$0 < x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \quad (i = 1, \cdots, n, \; n \geq 1),$$

from which it follows that the limits $\xi_i = \lim_{n\to\infty} x_{n,i}$ $(i \geq 1)$ exist and satisfy $0 \leq \xi_i \leq \xi_{i+1} < \infty$.

Some properties of $(\mathcal{Q}_n)$ and ψ

Of particular interest and importance is the "interlacing" property of the zeros $x_{n,i}$ $(i = 1, \cdots, n)$ of $\mathcal{Q}_n$: they are strictly positive, simple, and they satisfy

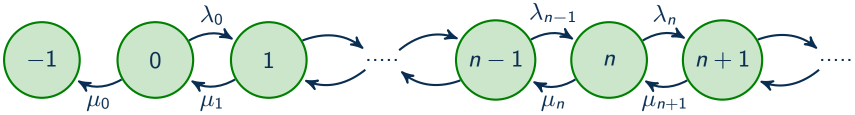
$$0 < x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \quad (i = 1, \cdots, n, \; n \geq 1),$$

from which it follows that the limits $\xi_i = \lim_{n\to\infty} x_{n,i}$ $(i \geq 1)$ exist and satisfy $0 \leq \xi_i \leq \xi_{i+1} < \infty$.

Interestingly, $\xi_1 := \inf \mathrm{supp}(\psi)$ and $\xi_2 := \inf\{\mathrm{supp}(\psi) \cap (\xi_1, \infty)\}$, quantities that are particularly important in the theory of *quasi-stationary distributions*.

The time to extinction

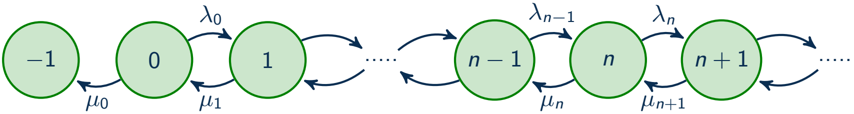
Consider the case $\mu_0 > 0$:



[2]Van Doorn, E.A. and Pollett, P.K. (2013) Quasi-stationary distributions for discrete-state models. Invited paper. *European J. Operat. Res.* 230, 1–14.

The time to extinction

Consider the case $\mu_0 > 0$:

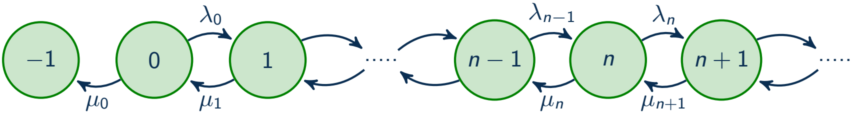


Suppose that $\sum_{n=0}^{\infty}(\lambda_n \pi_n)^{-1} = \infty$, which ensures that the extinction state -1 is reached with probability 1 (and necessarily the process is non-explosive).

[2]Van Doorn, E.A. and Pollett, P.K. (2013) Quasi-stationary distributions for discrete-state models. Invited paper. *European J. Operat. Res.* 230, 1–14.

The time to extinction

Consider the case $\mu_0 > 0$:



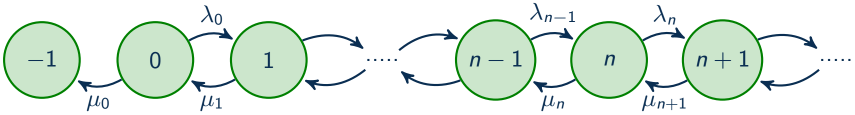
Suppose that $\sum_{n=0}^{\infty}(\lambda_n\pi_n)^{-1} = \infty$, which ensures that the extinction state -1 is reached with probability 1 (and necessarily the process is non-explosive).

Let $T = \inf\{t \geq 0 : X_t = -1\}$ be the time to extinction.

[2]Van Doorn, E.A. and Pollett, P.K. (2013) Quasi-stationary distributions for discrete-state models. Invited paper. *European J. Operat. Res.* 230, 1–14.

The time to extinction

Consider the case $\mu_0 > 0$:



Suppose that $\sum_{n=0}^{\infty}(\lambda_n\pi_n)^{-1} = \infty$, which ensures that the extinction state -1 is reached with probability 1 (and necessarily the process is non-explosive).

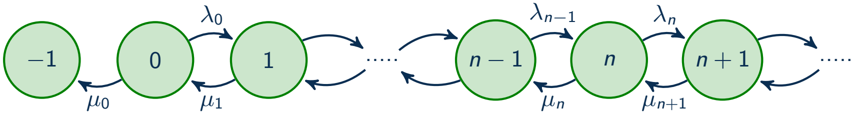
Let $T = \inf\{t \geq 0 : X_t = -1\}$ be the time to extinction.

Clearly $\Pr(T > t | X_0 = i) \to 0$ as $t \to \infty$, but how fast?

[2]Van Doorn, E.A. and Pollett, P.K. (2013) Quasi-stationary distributions for discrete-state models. Invited paper. *European J. Operat. Res.* 230, 1–14.

The time to extinction

Consider the case $\mu_0 > 0$:



Suppose that $\sum_{n=0}^{\infty}(\lambda_n\pi_n)^{-1} = \infty$, which ensures that the extinction state -1 is reached with probability 1 (and necessarily the process is non-explosive).

Let $T = \inf\{t \geq 0 : X_t = -1\}$ be the time to extinction.

Clearly $\Pr(T > t | X_0 = i) \to 0$ as $t \to \infty$, but how fast?

Claim. $\inf\left\{a \geq 0 : \int_0^\infty e^{at}\Pr(T > t|X_0 = i)\,dt = \infty\right\} = \xi_1$.

[2]Van Doorn, E.A. and Pollett, P.K. (2013) Quasi-stationary distributions for discrete-state models. Invited paper. *European J. Operat. Res.* 230, 1–14.

Quasi-stationary distributions

A distribution $\boldsymbol{u} = (u_n,\ n \geq 0)$ is called a *limiting conditional distribution* (or sometimes *quasi-stationary distribution*) if $u_{ij}(t) := \Pr(X_t = j | T > t, X_0 = i) \to u_j$ as $t \to \infty$.

[3] van Doorn, E.A. (1991) Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.* 23, 683–700.

[4] Kijima, M., Nair, M.G., Pollett, P.K. and van Doorn, E.A. (1997) Limiting conditional distributions for birth-death processes. *Adv. Appl. Probab.* 29, 185–204.

Quasi-stationary distributions

A distribution $\boldsymbol{u} = (u_n,\ n \geq 0)$ is called a *limiting conditional distribution* (or sometimes *quasi-stationary distribution*) if $u_{ij}(t) := \Pr(X_t = j | T > t, X_0 = i) \to u_j$ as $t \to \infty$.

Theorem

If $\xi_1 > 0$ then $u_{ij}(t) \to u_j := \mu_0^{-1}\xi_1\pi_j\mathcal{Q}_j(\xi_1)$. If $\xi_1 = 0$ then $u_j(t) \to 0$.

[3]van Doorn, E.A. (1991) Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.* 23, 683–700.

[4]Kijima, M., Nair, M.G., Pollett, P.K. and van Doorn, E.A. (1997) Limiting conditional distributions for birth-death processes. *Adv. Appl. Probab.* 29, 185–204.

Quasi-stationary distributions

A distribution $\boldsymbol{u} = (u_n,\ n \geq 0)$ is called a *limiting conditional distribution* (or sometimes *quasi-stationary distribution*) if $u_{ij}(t) := \Pr(X_t = j | T > t, X_0 = i) \to u_j$ as $t \to \infty$.

Theorem

If $\xi_1 > 0$ then $u_{ij}(t) \to u_j := \mu_0^{-1} \xi_1 \pi_j \mathcal{Q}_j(\xi_1)$. If $\xi_1 = 0$ then $u_j(t) \to 0$.

Again, how fast?

[3] van Doorn, E.A. (1991) Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.* 23, 683–700.

[4] Kijima, M., Nair, M.G., Pollett, P.K. and van Doorn, E.A. (1997) Limiting conditional distributions for birth-death processes. *Adv. Appl. Probab.* 29, 185–204.

Quasi-stationary distributions

A distribution $\boldsymbol{u} = (u_n,\ n \geq 0)$ is called a *limiting conditional distribution* (or sometimes *quasi-stationary distribution*) if $u_{ij}(t) := \Pr(X_t = j | T > t, X_0 = i) \to u_j$ as $t \to \infty$.

Theorem

If $\xi_1 > 0$ *then* $u_{ij}(t) \to u_j := \mu_0^{-1}\xi_1\pi_j\mathcal{Q}_j(\xi_1)$. *If* $\xi_1 = 0$ *then* $u_j(t) \to 0$.

Again, how fast?

Claim. $\inf\left\{a \geq 0 : \int_0^\infty e^{at}|u_{ij}(t) - u_j|\,dt = \infty\right\} = \xi_2 - \xi_1$ (same for all $i, j \in S$).

[3] van Doorn, E.A. (1991) Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.* 23, 683–700.

[4] Kijima, M., Nair, M.G., Pollett, P.K. and van Doorn, E.A. (1997) Limiting conditional distributions for birth-death processes. *Adv. Appl. Probab.* 29, 185–204.

Recall . . .

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x)\mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

Proof. We use mathematical induction: on i using (BE) with $j = 0$ and then on j using (FE). But, showing that there is a probability measure ψ with support $[0, \infty)$ whose Laplace transform is $p_{00}(t)$, that is

$$p_{00}(t) = \int_0^\infty e^{-tx}\, d\psi(x) \ \left(= \pi_0 \int_0^\infty e^{-tx} \mathcal{Q}_0(x)\mathcal{Q}_0(x)\, d\psi(x) \right),$$

is not completely straightforward. More on this later.

Why does this work?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \; t \geq 0).$$

Why does this work?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \ t \geq 0).$$

Why is there a ψ whose Laplace transform is $p_{00}(t)$: $p_{00}(t) = \int_0^\infty e^{-tx} \, d\psi(x)$?

Why does this work?

Theorem (Karlin and McGregor (1957))

Let $P(t) = (p_{ij}(t))$ be any transition function that satisfies both the backward and the forward equations (for example the minimal one). Then, there is a probability measure ψ with support $[0, \infty)$ such that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x)\, d\psi(x) \qquad (i, j \geq 0,\ t \geq 0).$$

Why is there a ψ whose Laplace transform is $p_{00}(t)$: $p_{00}(t) = \int_0^\infty e^{-tx}\, d\psi(x)$?

Answer. Weak symmetry: $\pi_i q_{ij} = \pi_j q_{ji}$ $(\pi_i \lambda_i = \pi_{i+1} \mu_{i+1})$

Finite state Markov chains - some linear algebra

Let $(X_t,\ t \geq 0)$ be a continuous-time Markov chain taking values in $S = \{0, 1, \dots, N\}$ with (conservative) transition rate matrix Q. So, there is collection $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ of strictly positive numbers such that $\boldsymbol{\pi} Q = 0$, that is

$$\sum_{i \in S} \pi_i q_{ij} = \pi_j \sum_{i \in S} q_{ji} \qquad (j \in S).$$

Finite state Markov chains - some linear algebra

Let $(X_t,\ t \geq 0)$ be a continuous-time Markov chain taking values in $S = \{0, 1, \ldots, N\}$ with (conservative) transition rate matrix Q. So, there is collection $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ of strictly positive numbers such that $\boldsymbol{\pi} Q = 0$, that is

$$\sum_{i \in S} \pi_i q_{ij} = \pi_j \sum_{i \in S} q_{ji} \qquad (j \in S).$$

Suppose that Q is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i q_{ij} = \pi_j q_{ji}$.

Finite state Markov chains - some linear algebra

Let $(X_t,\ t \geq 0)$ be a continuous-time Markov chain taking values in $S = \{0, 1, \ldots, N\}$ with (conservative) transition rate matrix Q. So, there is collection $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ of strictly positive numbers such that $\boldsymbol{\pi} Q = 0$, that is

$$\sum_{i \in S} \pi_i q_{ij} = \pi_j \sum_{i \in S} q_{ji} \qquad (j \in S).$$

Suppose that Q is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i q_{ij} = \pi_j q_{ji}$.

Let A be the symmetric matrix with entries $a_{ij} = \sqrt{\pi_i} q_{ij} / \sqrt{\pi_j}$. It is orthogonally similar to a diagonal matrix $D = \mathrm{diag}\{d_0, d_1, \ldots, d_N\}$: $A = MDM^\top$ ..., et cetera, ...

Finite state Markov chains - some linear algebra

Let $(X_t,\ t \geq 0)$ be a continuous-time Markov chain taking values in $S = \{0, 1, \ldots, N\}$ with (conservative) transition rate matrix Q. So, there is collection $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ of strictly positive numbers such that $\boldsymbol{\pi} Q = 0$, that is

$$\sum_{i \in S} \pi_i q_{ij} = \pi_j \sum_{i \in S} q_{ji} \qquad (j \in S).$$

Suppose that Q is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i q_{ij} = \pi_j q_{ji}$.

Let A be the symmetric matrix with entries $a_{ij} = \sqrt{\pi_i} q_{ij} / \sqrt{\pi_j}$. It is orthogonally similar to a diagonal matrix $D = \mathrm{diag}\{d_0, d_1, \ldots, d_N\}$: $A = MDM^{\top}$ ..., et cetera, ... leading to the *spectral solution* of $P'(t) = QP(t)$ (BE):

$$p_{ij}(t) = \pi_j \sum_{k=0}^{N} e^{d_k t} \mathcal{Q}_i^{(k)} \mathcal{Q}_j^{(k)}, \qquad \text{where } \mathcal{Q}_i^{(k)} = \frac{M_{ik}}{\sqrt{\pi_i}}.$$

General symmetric Markov chains - some functional analysis

Let $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ be a collection of strictly positive numbers and suppose that P is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$ $(i, j \in S)$.

[5] Kendall, D.G (1959) Unitary dilations of one-parameter semigroups of Markov transition operators, and the corresponding integral representations for Markov processes with a countable infinity of states. *Proc. London Math. Soc.* 9, 417–431.

General symmetric Markov chains - some functional analysis

Let $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ be a collection of strictly positive numbers and suppose that P is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$ $(i, j \in S)$.

Define $T_t : \ell_2 \to \ell_2$ by

$$(T_t x)_j = \sum_{i \in S} x_i (\pi_i/\pi_j)^{1/2} p_{ij}(t) \qquad (i \in S,\ x \in \ell_2).$$

[5]Kendall, D.G (1959) Unitary dilations of one-parameter semigroups of Markov transition operators, and the corresponding integral representations for Markov processes with a countable infinity of states. *Proc. London Math. Soc.* 9, 417–431.

General symmetric Markov chains - some functional analysis

Let $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ be a collection of strictly positive numbers and suppose that P is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$ $(i, j \in S)$.

Define $T_t : \ell_2 \to \ell_2$ by

$$(T_t x)_j = \sum_{i \in S} x_i (\pi_i/\pi_j)^{1/2} p_{ij}(t) \qquad (i \in S,\ x \in \ell_2).$$

Then $(T_t,\ t \geq 0)$ is a semigroup which is self adjointing $\langle T_t x, y \rangle = \langle x, T_t y \rangle$.

[5] Kendall, D.G (1959) Unitary dilations of one-parameter semigroups of Markov transition operators, and the corresponding integral representations for Markov processes with a countable infinity of states. *Proc. London Math. Soc.* 9, 417–431.

General symmetric Markov chains - some functional analysis

Let $\boldsymbol{\pi} = (\pi_j,\ j \in S)$ be a collection of strictly positive numbers and suppose that P is weakly symmetric with respect to $\boldsymbol{\pi}$: $\pi_i p_{ij}(t) = \pi_j p_{ji}(t)$ $(i, j \in S)$.

Define $T_t : \ell_2 \to \ell_2$ by

$$(T_t x)_j = \sum_{i \in S} x_i (\pi_i/\pi_j)^{1/2} p_{ij}(t) \qquad (i \in S,\ x \in \ell_2).$$

Then $(T_t,\ t \geq 0)$ is a semigroup which is self adjointing $\langle T_t x, y\rangle = \langle x, T_t y\rangle$.

Kendall used a result of Riesz and Sz.-Nagy on the spectral representation of self-adjoint semigroups to show that there is a finite signed measure γ_{ij} with support $[0, \infty)$ such that

$$p_{ij}(t) = (\pi_j/\pi_i)^{1/2} \int_{[0,\infty)} e^{-tx}\, d\gamma_{ij}(x).$$

Furthermore, γ_{ii} is a probability measure.

[5] Kendall, D.G (1959) Unitary dilations of one-parameter semigroups of Markov transition operators, and the corresponding integral representations for Markov processes with a countable infinity of states. *Proc. London Math. Soc.* 9, 417–431.

General symmetric Markov chains - speculation

In can be seen from the definition of the birth-death polynomials $\boldsymbol{\mathcal{Q}} = (\mathcal{Q}_n,\ n \in S)$,

$$\begin{aligned} \mathcal{Q}_0(x) &= 1 \\ -x\mathcal{Q}_0(x) &= -(\lambda_0 + \mu_0)\mathcal{Q}_0(x) + \lambda_0 \mathcal{Q}_1(x) \\ -x\mathcal{Q}_n(x) &= \mu_n \mathcal{Q}_{n-1}(x) - (\lambda_n + \mu_n)\mathcal{Q}_n(x) + \lambda_n \mathcal{Q}_{n+1}(x), \end{aligned}$$

and the form of transition rate matrix restricted to $S = \{0, 1, \dots\}$,

$$Q = \begin{pmatrix} -(\lambda_0+\mu_0) & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

that $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}(x)$ as a column vector satisfies $Q\boldsymbol{\mathcal{Q}} = -x\boldsymbol{\mathcal{Q}}$ ($\boldsymbol{\mathcal{Q}}(x)$ is an x-invariant vector for Q), and $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}(x)$, where $\mathcal{R}_j(x) = \pi_j \mathcal{Q}_j(x)$, as a row vector satisfies $\boldsymbol{\mathcal{R}}Q = -x\boldsymbol{\mathcal{R}}$ ($\boldsymbol{\mathcal{R}}(x)$ is an x-invariant measure for Q).

General symmetric Markov chains - speculation

One might speculate that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x) \mathcal{Q}_j(x) \, d\psi(x) \qquad (i, j \geq 0, \ t \geq 0)$$

holds more generally under weak symmetry ($\pi_i q_{ij} = \pi_j q_{ji}$) for a function system $\boldsymbol{\mathcal{Q}} = (\mathcal{Q}_n, \ n \in S)$ (necessarily orthogonal with respect to ψ) satisfying $Q\boldsymbol{\mathcal{Q}} = -x\boldsymbol{\mathcal{Q}}$.

General symmetric Markov chains - speculation

One might speculate that

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-tx} \mathcal{Q}_i(x)\mathcal{Q}_j(x)\, d\psi(x) \qquad (i,j \geq 0,\ t \geq 0)$$

holds more generally under weak symmetry ($\pi_i q_{ij} = \pi_j q_{ji}$) for a function system $\boldsymbol{\mathcal{Q}} = (\mathcal{Q}_n,\ n \in S)$ (necessarily orthogonal with respect to ψ) satisfying $Q\boldsymbol{\mathcal{Q}} = -x\boldsymbol{\mathcal{Q}}$.

It might perhaps be too much to expect that

$$p_{ij}(t) = \int_0^\infty e^{-tx} \mathcal{Q}_i(x)\mathcal{R}_j(x)\, d\psi(x) \qquad (i,j \geq 0,\ t \geq 0)$$

holds with just $\boldsymbol{\pi} Q = 0$ for function systems $\boldsymbol{\mathcal{Q}} = (\mathcal{Q}_n,\ n \in S)$ and $\boldsymbol{\mathcal{R}} = (\mathcal{R}_n,\ n \in S)$ satisfying $Q\boldsymbol{\mathcal{Q}} = -x\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{R}}Q = -x\boldsymbol{\mathcal{R}}$, and, of necessity,

$$\int_0^\infty \mathcal{Q}_i(x)\mathcal{R}_j(x)\, d\psi(x) = \delta_{ij} \qquad (i,j \geq 0).$$