A Method for Analysing Complex Markovian Models

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Erlang's Formula

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Arrival rate

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Arrival rate

Number of circuits

$$l_j = E\left(\sum_{r\in\mathcal{R}} \lambda_{jr}\nu_r \prod_{i\in r-\{j\}} (1 - l_i)\,,\; c_j\right)$$

$$j = 1, 2, \ldots, J$$

Erlang fixed point approximation

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Effective arrival rate at link j

Erlang fixed point approximation

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$$j = 1, 2, \ldots, J$$

Number of circuits on link j

The problem

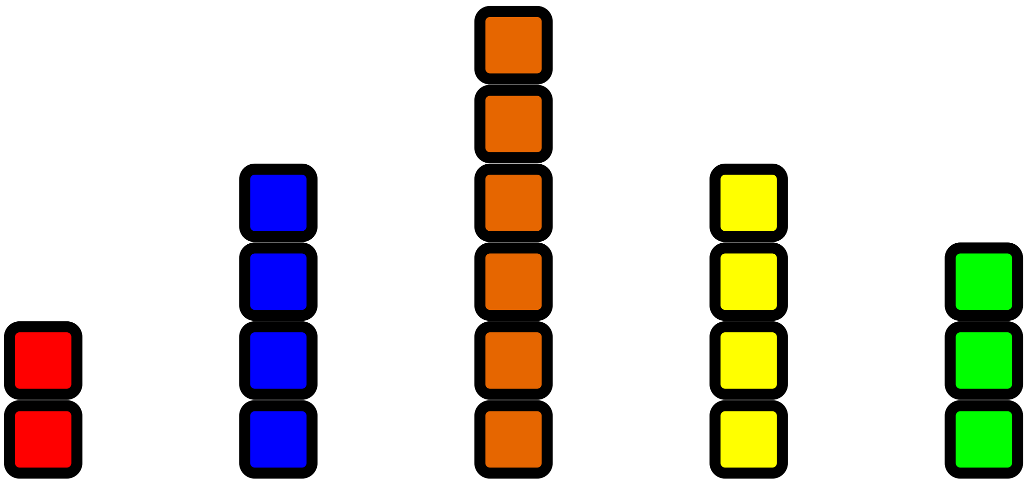
Consider any large-scale stochastic system whose natural state description is Markovian, yet its behaviour (equilibrium or time-dependent behaviour) is difficult to analyze.

Can we find an alternative state description, together with an approximating transition structure, that can be analyzed more simply?

Our goal is to approximate quantities of interest and to assess the quality of the approximation.

Competition for resources

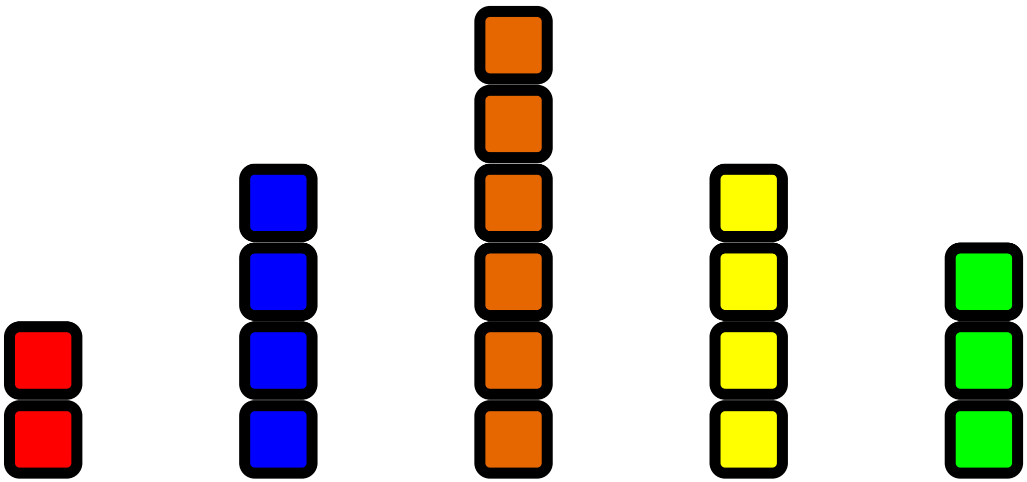
A collection of resources of different types and differing amounts (capacities)



$c_1 = 2$ $c_2 = 4$ $c_3 = 6$ $c_4 = 4$ $c_5 = 3$ (Capacity)

Competition for resources

Customers of different types arrive as independent Poisson streams and request groups of resources



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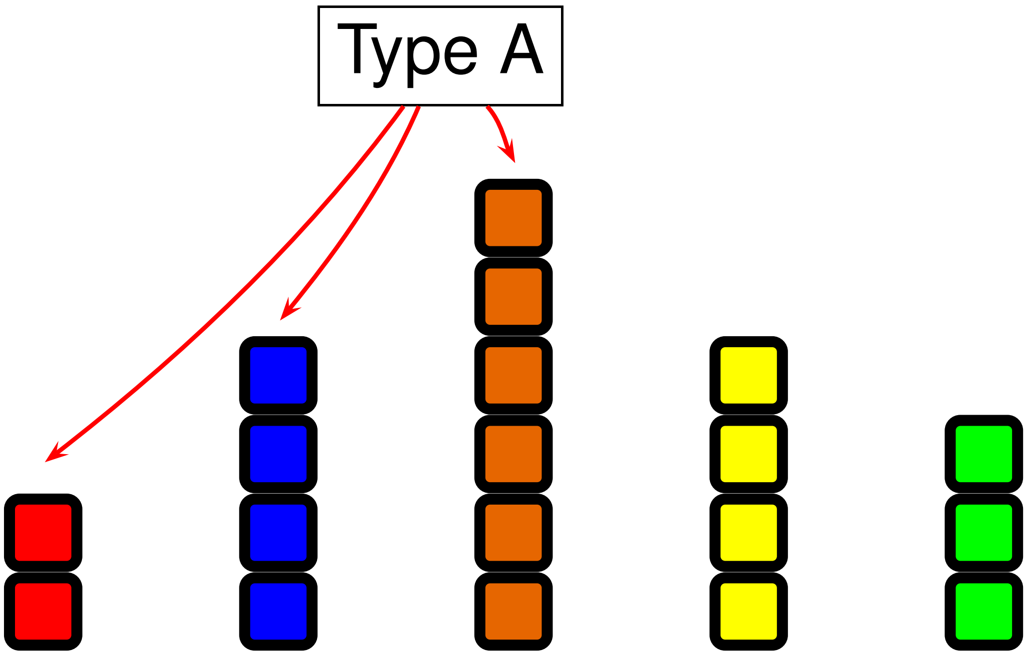
Customers are identified by *which* resources they require



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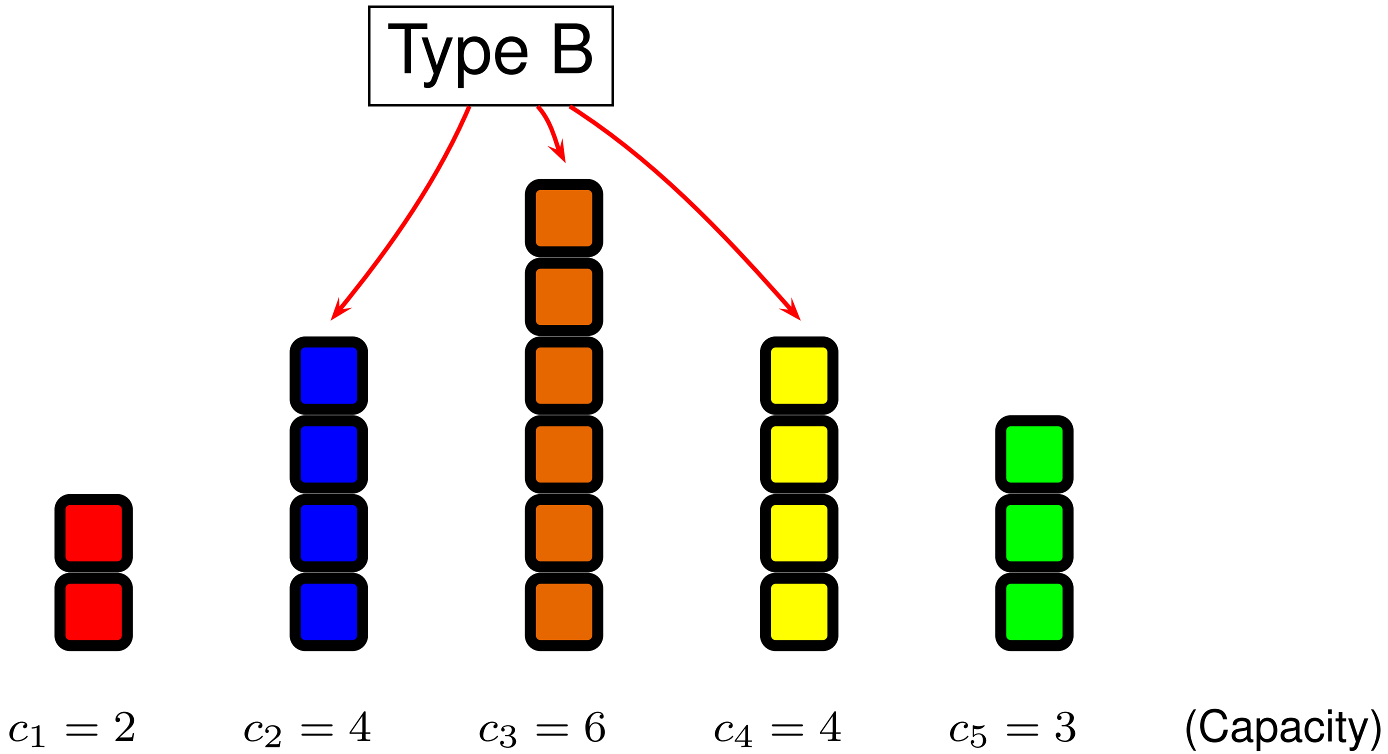

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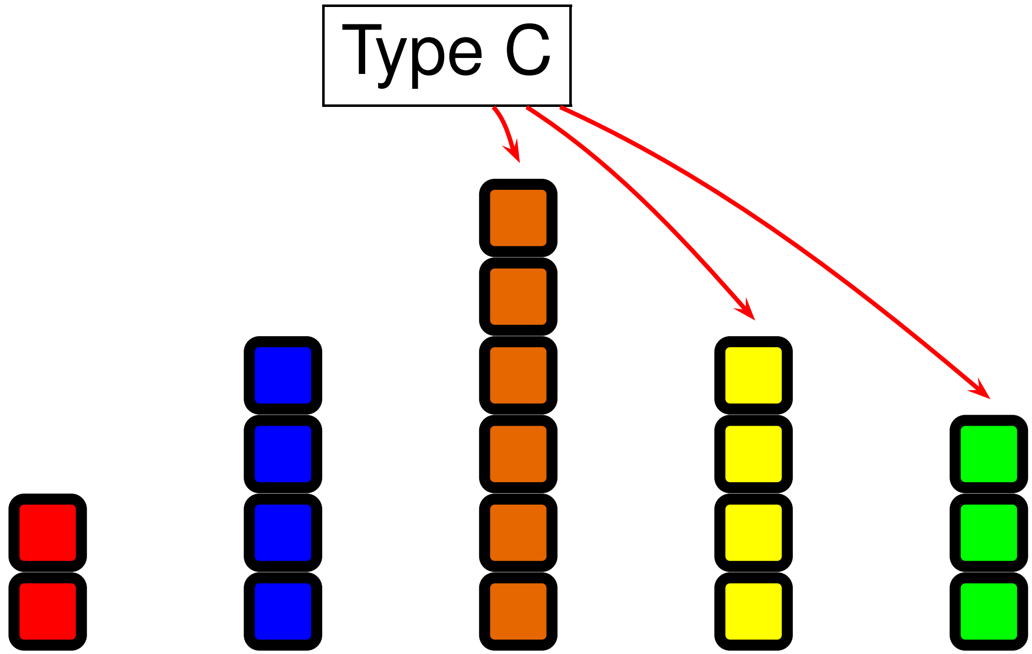

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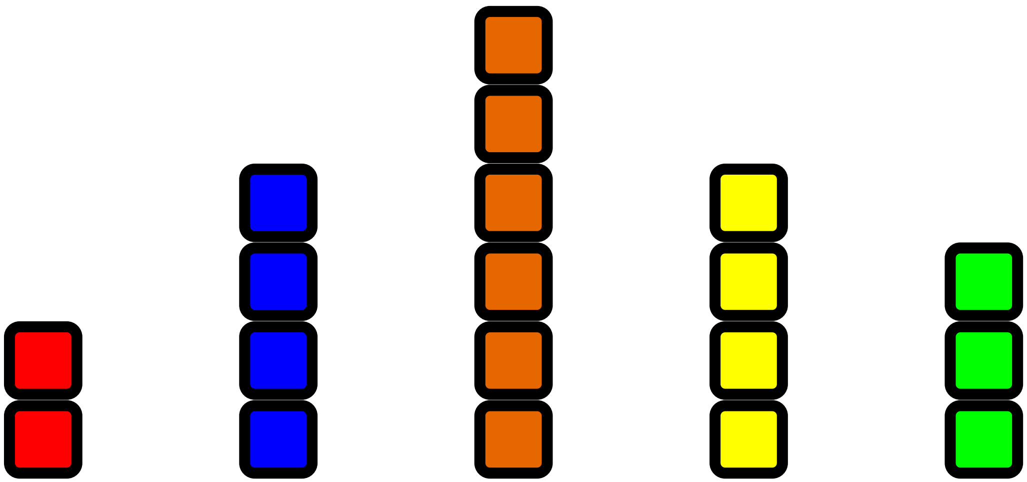
Competition for resources

Resources are captured and held for a random period and released simultaneousness



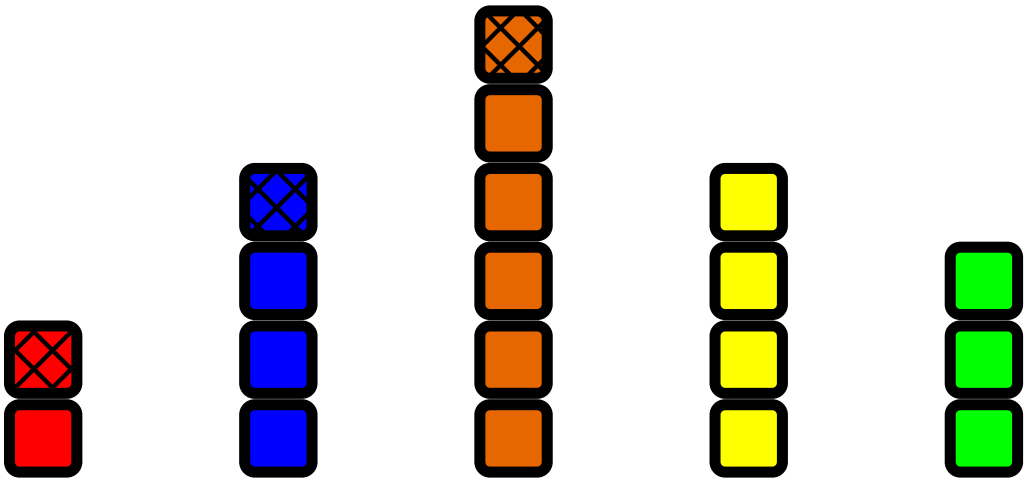
$u_1 = 0$ $u_2 = 0$ $u_3 = 0$ $u_4 = 0$ $u_5 = 0$ (Usage)

Competition for resources

Type A $(n_A = 0)$

Type B $(n_B = 0)$

Type C $(n_C = 0)$



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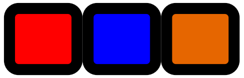
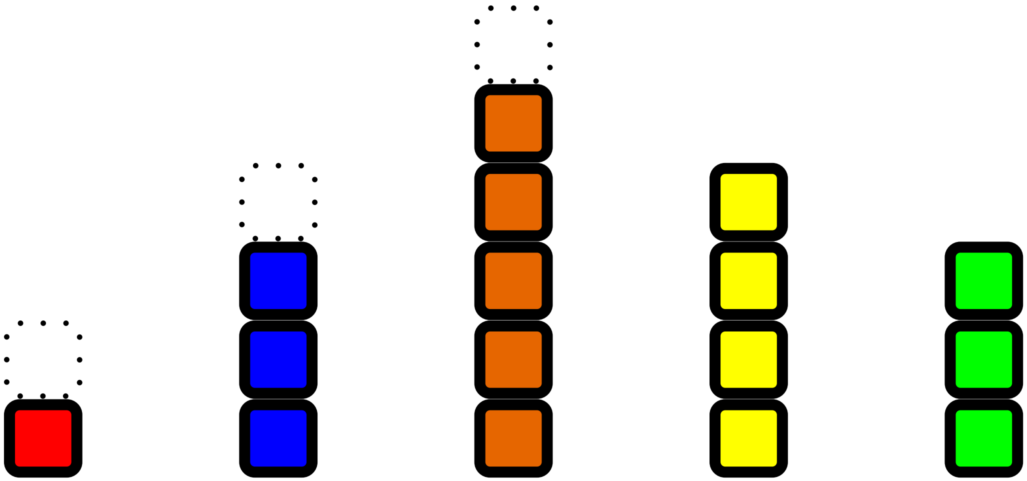
Competition for resources

Type A $(n_A = 1)$



Type B $(n_B = 0)$

Type C $(n_C = 0)$



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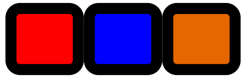
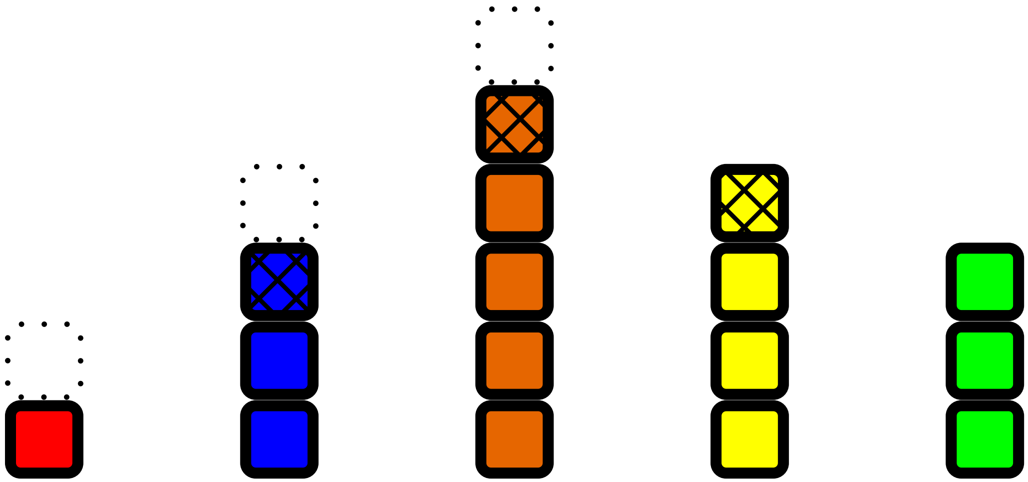
Type A $(n_A = 1)$ 

Type B $(n_B = 0)$

Type C $(n_C = 0)$



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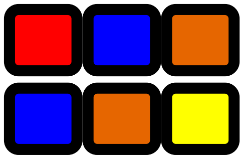

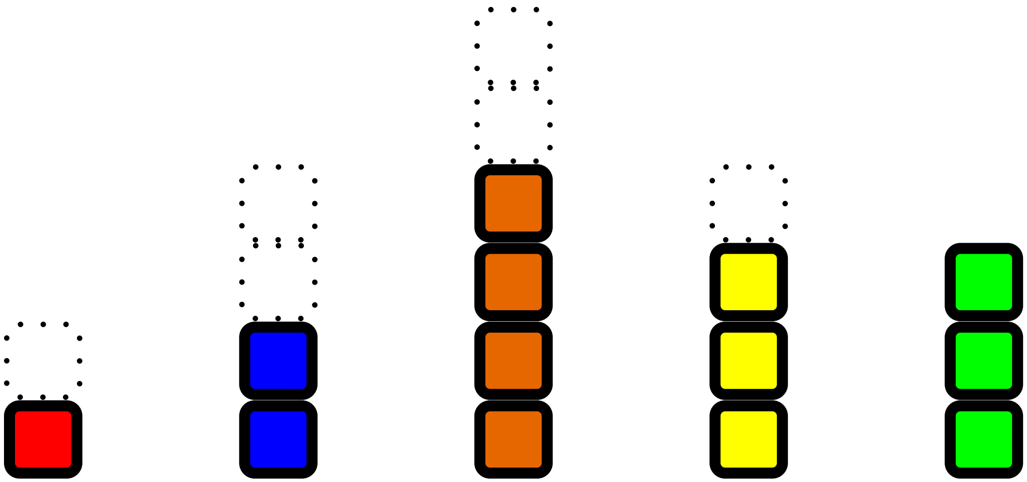

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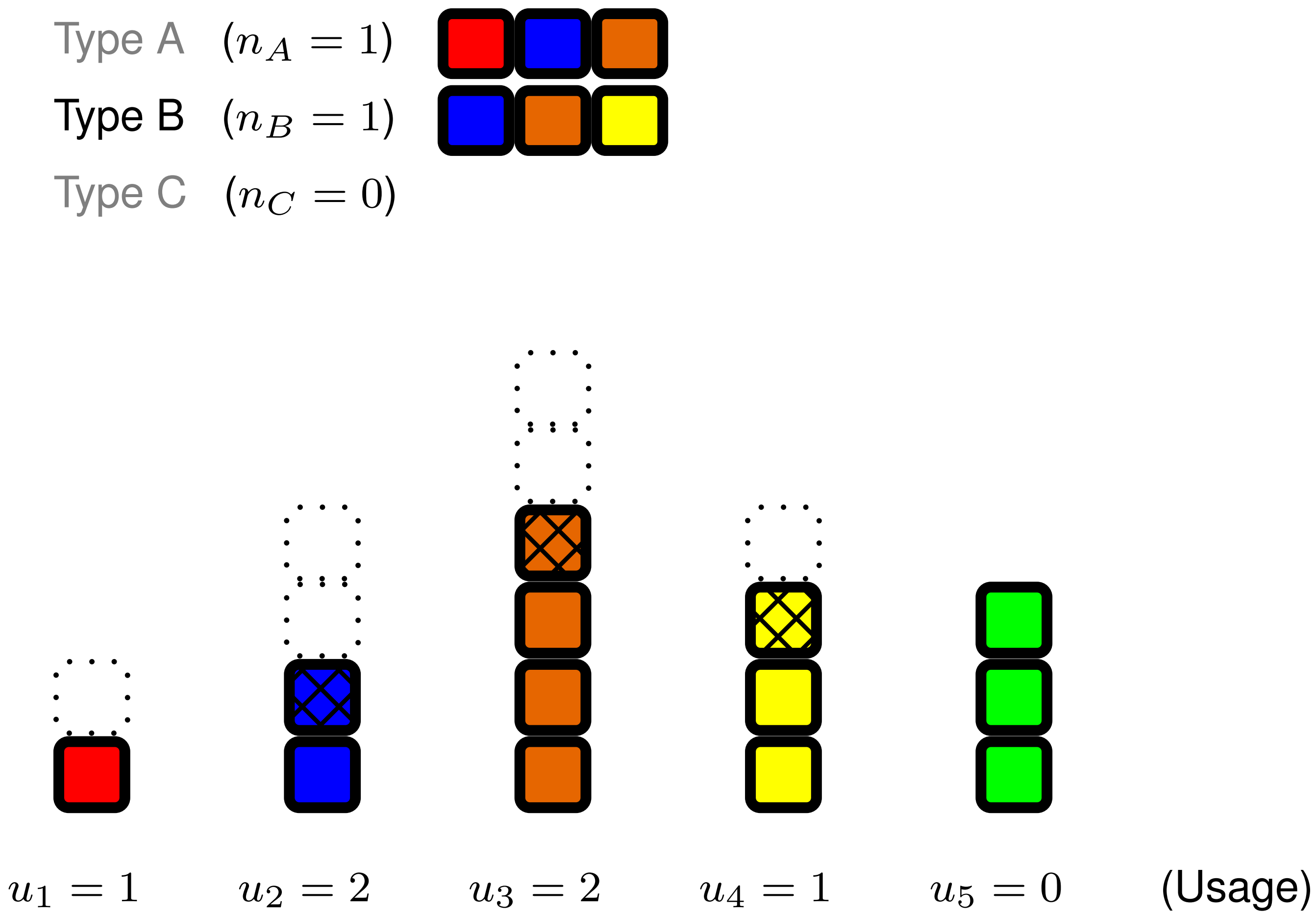

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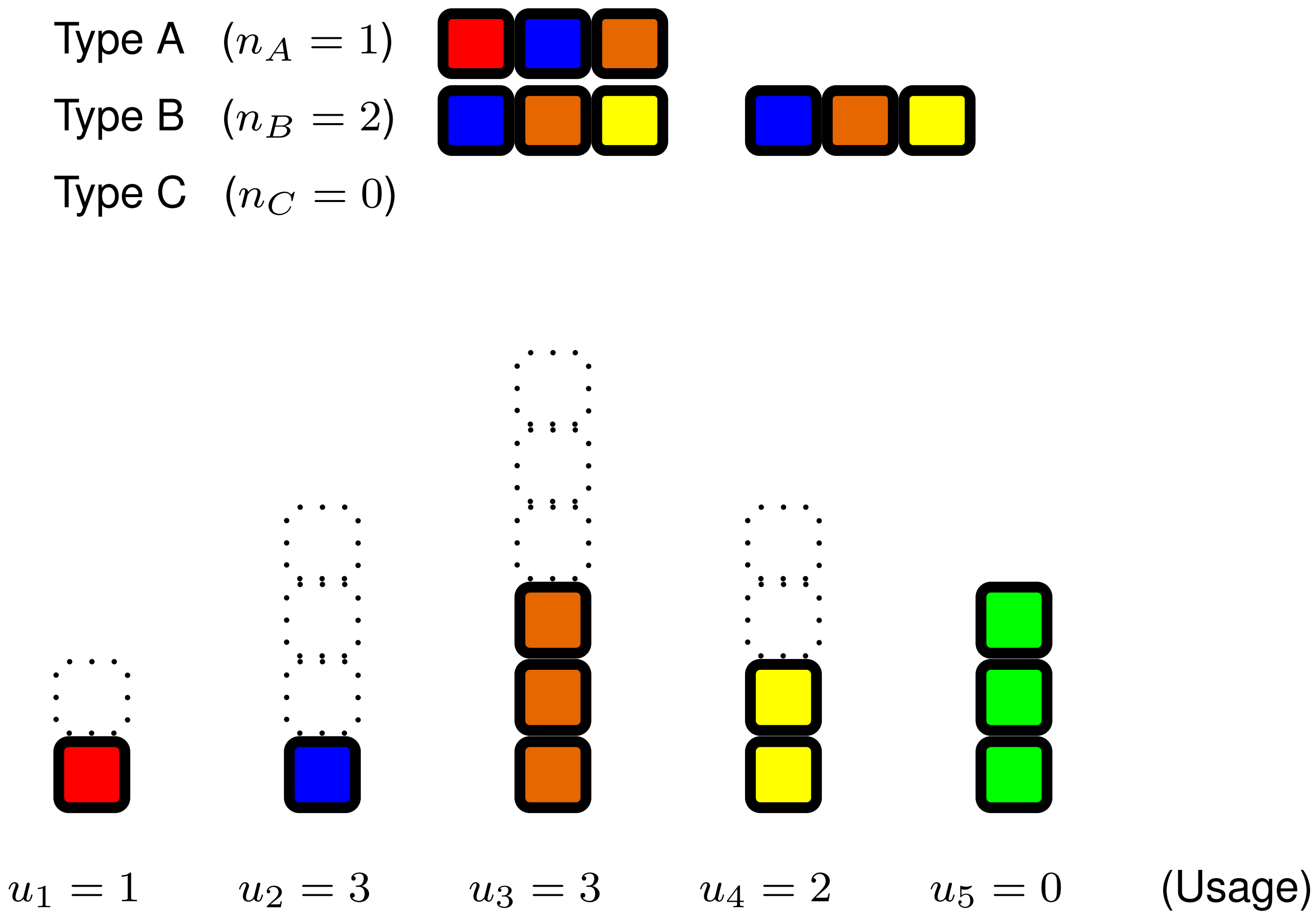

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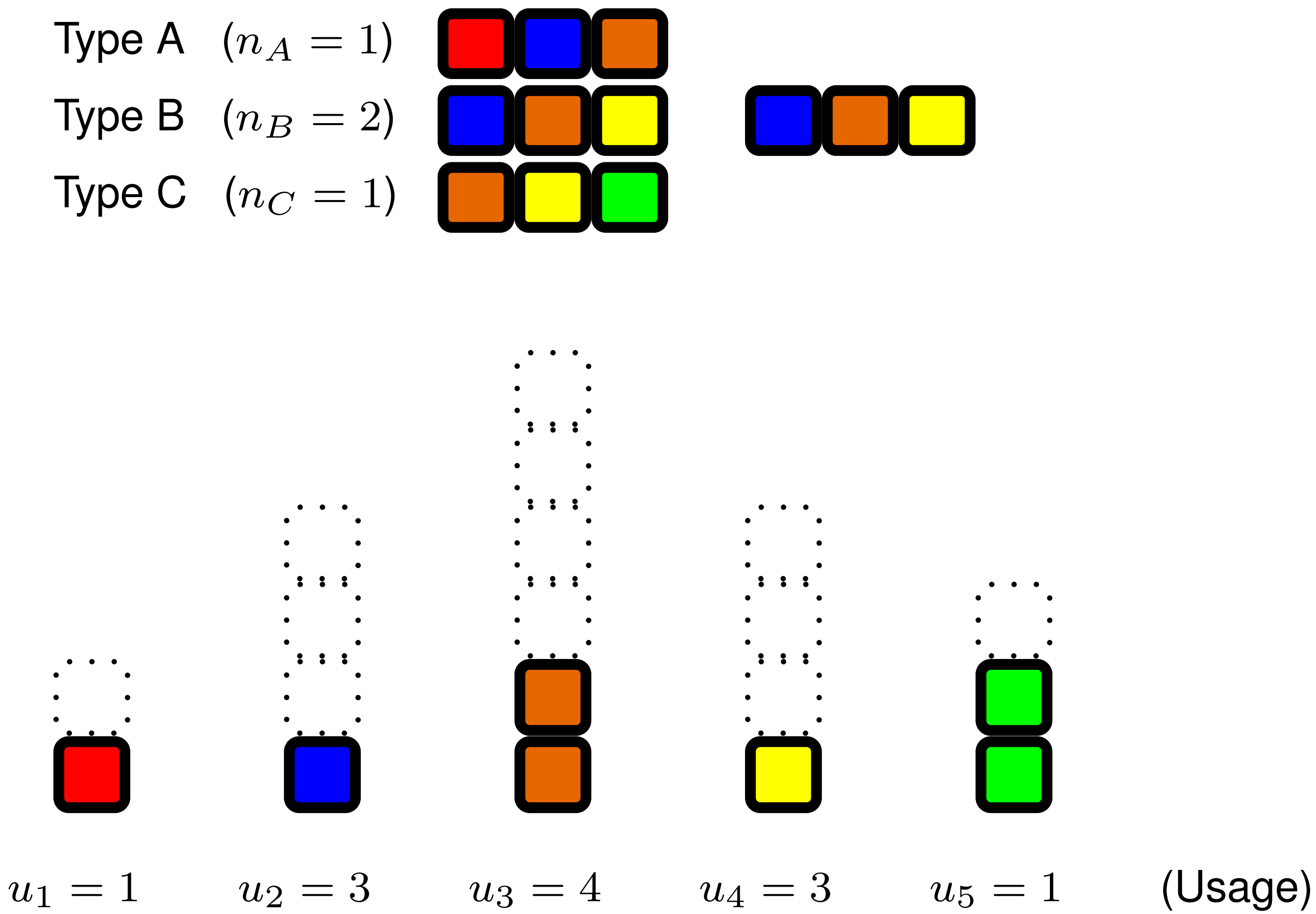

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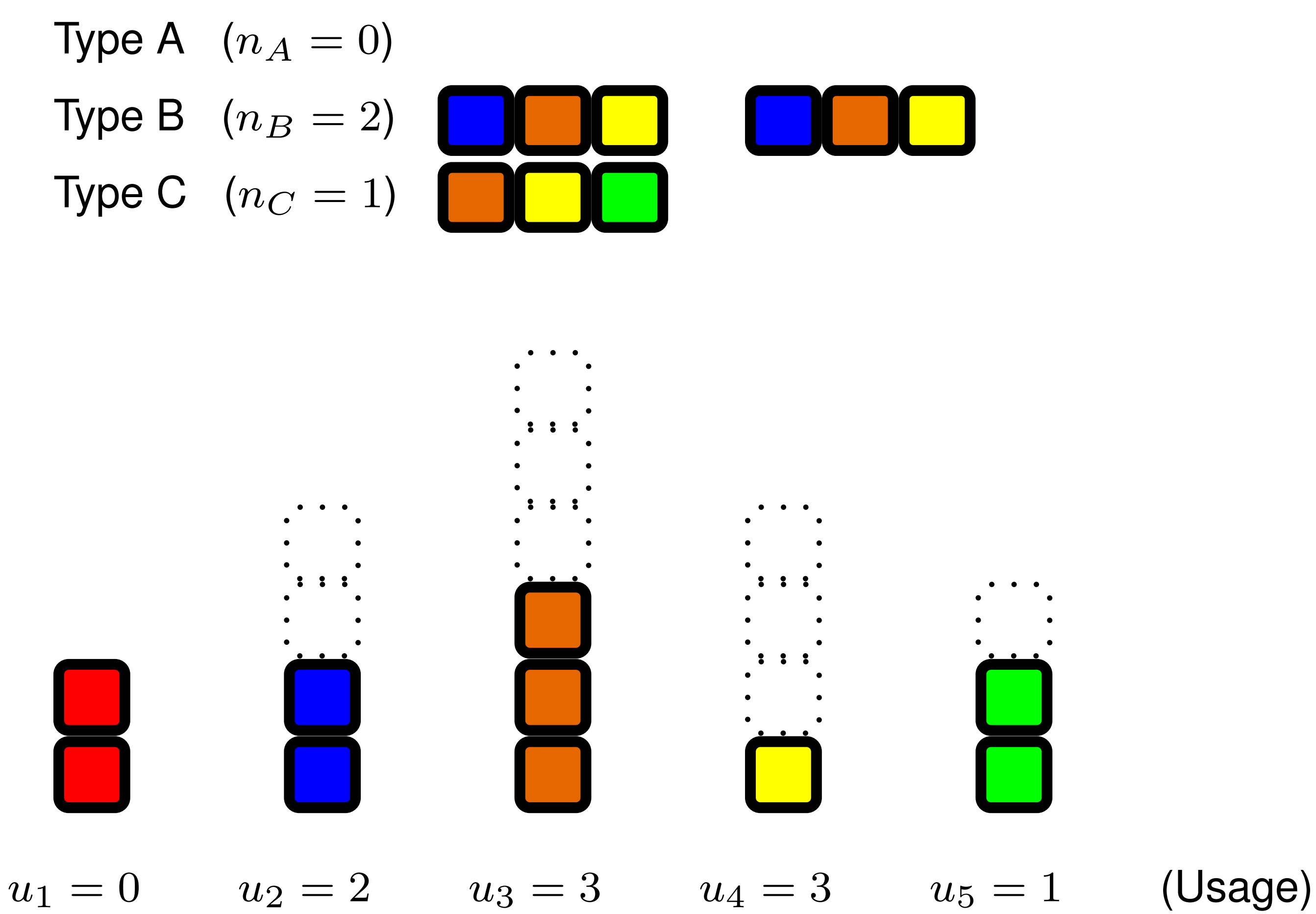

Competition for resources

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Type B $(n_B = 3)$

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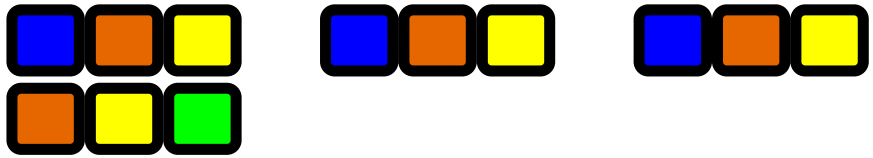

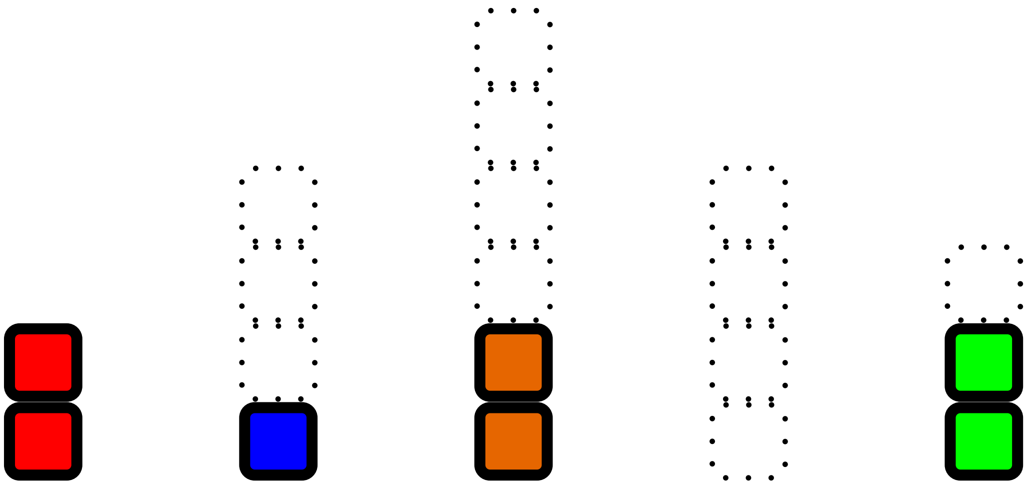

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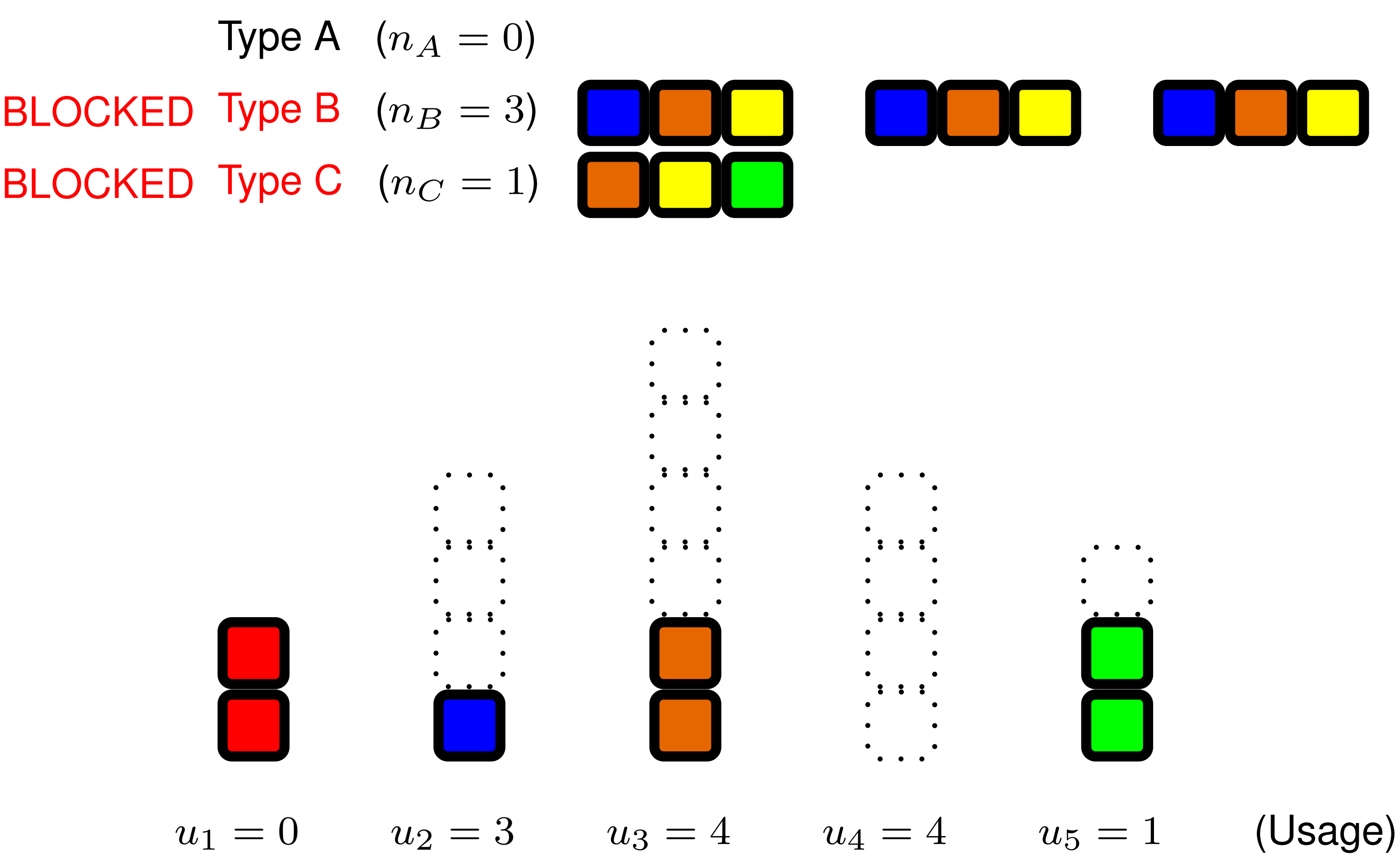
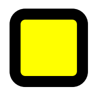


Type-B and Type-C customers are now blocked because there is no more 

Competition for resources

Let $\mathcal{R}$ be the set of customer types and let $\mathbf{n} = (n_r, r \in \mathcal{R})$, where n_r is the number of type-r customers in the system.

Let $\mathbf{c} = (c_j, j = 1, \ldots, J)$ be the resource capacities, and $\Lambda = (\lambda_{jr})$ be the $J \times \mathcal{R}$ design matrix with $\lambda_{jr} = 1$ if resource j is used by type-r customers. The set of all states is then $S = \{\mathbf{n} \in \mathrm{Z}_+^{\mathcal{R}} : \Lambda\mathbf{n} \leq \mathbf{c}\}$.

If type-r customers arrive at rate ν_r (independent Poisson streams) and hold sets of resources for independent exponentially distributed times (with unit mean say), then $(\mathbf{n}_t, t \geq 0)$ is a Markov chain with transition rates:

$$q(\mathbf{n}, \mathbf{n} + \mathbf{e}_r) = \nu_r \qquad q(\mathbf{n}, \mathbf{n} - \mathbf{e}_r) = n_r$$

(here $\mathbf{e}_r$ is the unit vector with a 1 as its r-th entry).

Competition for resources

The chain has stationary distribution

$$p(\mathbf{n}) = B \prod_{r \in \mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}, \qquad \mathbf{n} \in S,$$

where $B = B(\mathbf{c})$ is a normalizing constant.

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The chance that an arriving type-r customer is blocked is $b_r = 1 - B(\mathbf{c})/B(\mathbf{c} - \Lambda \mathbf{e}_r)$.

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However, this cannot (usually) be computed in polynomial time.

Expected rates

We have a large-scale stochastic system whose natural state description is Markovian, yet its behaviour (equilibrium or time-dependent behaviour) is difficult to analyze.

Idea (Peter Taylor, 1996). Find an alternative state description, together with an approximating transition structure, that can be analyzed more simply. For this description, *impose* a Markovian assumption: the rates of transition are given by the expected rates of the corresponding transitions of the original chain:

$$q'(u,v) = \mathrm{E}_p\left(\sum_{\mathbf{m}\in A(\mathbf{n}(t))} q(\mathbf{n}(t),\mathbf{m})\right),$$

where $\mathbf{n} \to A(\mathbf{n})$ is all transitions out of $\mathbf{n}$ that give rise to a $u \to v$ transition in the new structure.

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States $\mathbf{n}$ for which the new state is u

$$q'(u,v) = \mathrm{E}_p\left(\sum_{\mathbf{m}\in A(\mathbf{n}(t))} q(\mathbf{n}(t),\mathbf{m}) \,\middle|\, \mathbf{n}(t) \in \underline{A}_u\right),$$

where $\mathbf{n} \to A(\mathbf{n})$ is all transitions out of $\mathbf{n}$ that give rise to a $u \longrightarrow v$ transition in the new structure.

A general fixed point method

Modified idea.

$$q'(u,v) = \mathrm{E}_{\pi^{(0)}}\left(\sum_{\mathbf{m}\in A(\mathbf{n}(t))} q(\mathbf{n}(t),\mathbf{m}) \,\middle|\, \mathbf{n}(t) \in \underline{A}_u\right),$$

where $\pi^{(0)}$ is our best guess for the stationary distribution of the new Markov chain.

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Now evaluate the stationary distribution $\pi^{(1)}$ of the new chain using transition rates $q'(u,v)$.

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Repeat the procedure.

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Hopes

- $\pi^{(n)}$ will converge (to π say)
- π will provide good estimates of quantities of interest
- π will provide the best estimate of a particular quantity of interest among members of a class of distributions (for example, product-form distributions)
- to delimit conditions under which the approximations are good

Application to the resource model

Focus on the *usage* $\mathbf{u} = (u_j, j = 1, \ldots, J)$. The process $(\mathbf{u}_t, t \geq 0)$ is not (usually) Markovian. Let

$$\pi(\mathbf{u}) = \prod_{j=1}^{J} \pi_j(u_j), \qquad \pi_j(u) = \frac{a_j^u}{u!} \left(\sum_{v=0}^{c_j} \frac{a_j^v}{v!} \right)^{-1} (u = 0, \ldots, c_j)$$

where the a_j's are to be determined. Then,

$$q'(\mathbf{u}, \mathbf{u} + \mathbf{e}_k) = \mathrm{E}_\pi \left(\textstyle\sum_{r \in \mathcal{R}: k \in r} \nu_r \prod_{i \in r - \{k\}} 1_{\{U_i < c_i\}} \middle| U_k = u_k \right).$$

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$$\Pr_\pi \Big(U_i < c_i, \, \forall i \in r - \{k\} \,\Big|\, U_k = u_k \Big).$$

Application to the resource model

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$$q'(\mathbf{u}, \mathbf{u} + \mathbf{e}_k) = \sum_{r \in \mathcal{R}} \lambda_{kr} \nu_r \prod_{i \in r - \{k\}} (1 - l_i),$$

where $l_i = \pi_i(c_i)$.

Application to the resource model

Similarly,

$$q'(\mathbf{u}, \mathbf{u} - \mathbf{e}_k) = \mathrm{E}_\pi \left(u_k 1_{\{U_k = u_k\}} \middle| U_k = u_k \right) = u_k.$$

Application to the resource model

Similarly,

$$q'(\mathbf{u}, \mathbf{u} - \mathbf{e}_k) = \mathrm{E}_\pi \left(u_k 1_{\{U_k = u_k\}} \middle| U_k = u_k \right) = u_k.$$

The limiting set of resource blocking probabilities $\mathbf{l} = (l_j, j = 1, \ldots, J)$ will satisfy

$$l_j = E\left(\sum_{r \in \mathcal{R}} \lambda_{jr} \nu_r \prod_{i \in r - \{j\}} (1 - l_i)\,,\ c_j \right),$$

where

$$E(a, c) = \frac{a^c}{c!} \left(\sum_{v=0}^{c} \frac{a^v}{v!} \right)^{-1}. \quad \text{(}\textit{Erlang's formula}\text{)}$$

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- In the case of resource systems, it gives the widely used, and highly accurate, Erlang fixed point approximation
- Plenty of scope for applications (for example, queueing networks with blocking)
- Plenty of scope for mathematical developments (for example, fixed point theory)