Diffusion Approximations for a Circuit Switching Network with Random Alternative Routing

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In this paper I shall consider a model for the simplest kind of dynamic routing in a circuit-switched network, namely Random Alternative Routing: if a call cannot be carried on a first-choice route, then a second-choice route is chosen at random from a fixed set of alternatives. This kind of routing can give rise to several modes of behaviour. For example, the simple model I shall consider can exhibit bistability; the system fluctuates between a low-blocking state, where calls are accepted readily, and a high-blocking state, where the likelihood of a call being accepted can be quite low. I shall describe a method which enables one to assess the stability of the two states. In particular, it allows one to obtain qualitative estimates of the time for which these states persist.

Keywords: *Circuit-switching, Networks, Random Alternative Routing.*

1 INTRODUCTION

Recently, Gibbens et al. [3] introduced a simple model which helps to explain why circuit-switched networks with Random Alternative Routing can exhibit bistable behaviour. Such bistability can have serious implications for the performance of the network, for, in the high-blocking state, a situation can persist where large numbers of calls use alternative routes, which generally demand greater link occupancy than do first-choice routes and, thus, new calls are likely to be blocked frequently. The persistence of the high-blocking state is brought about because, even when calls are accepted, they are allotted first-choice routes rather rarely. It is of interest, therefore, to have a tool for determining the time it takes for the system to relax to the low-blocking state, where new calls are accepted more readily, and then to determine the time for which the low-blocking state persists. In this paper I shall describe a method by which one can model the random fluctuations of the system about its various states, either stable or unstable. For example, the method allows one to show that, as the number of links becomes large, the distribution of the time it takes to leave a region containing a stable equilibrium is, asymptotically, negative exponential.

2 THE MODEL: A SYMMETRIC FULLY CONNECTED NETWORK

There are N nodes connected to one another and, thus, a total of $K = \frac{1}{2}N(N-1)$ links (circuit groups). The links are assumed to have the same number of circuits, C, and calls between any two given nodes, a and b, arrive according to a Poisson process with rate $\nu > 0$; the arrival streams are assumed to be independent. If a call is offered to the link connecting a and b, and there is at least one free circuit on that link, then the call is connected and is held for a negative exponentially distributed period with mean 1. If there are no free circuits on the direct link, a third node, c, is chosen at random from the remaining nodes and an attempt is made to connect the call on the route via c. If there is spare capacity on each of the two links comprising the alternative route, the call is connected and holds one circuit from each link for a period that is negative exponentially distributed with mean 1. The call is blocked, and then lost, if it cannot be accommodated on the alternative route. Call lengths are assumed to be mutually independent, and independent of the arrival process.

Although the usual model of this network is a finite-state irreducible Markov chain, its state description is rather complicated and its equilibrium behaviour cannot be analysed simply. For this reason, Gibbens et al. [3] proposed a simplified model which does not respect the graph structure of the network, but whose behaviour for large N is a good approximation to that of the original model. The simplified description of the network, which I shall refer to as the GHK (Gibbens-Hunt-Kelly) model, differs in two ways. In cases when a call cannot be connected on a direct link, two *links* are chosen *at random* from the re-

maining $K - 1$ links. If there is spare capacity on each of these links, one circuit is held on each link, but, now, the two holding times are *independent* negative exponentially distributed random variables with mean 1. Thus, in contrast to the original model, the two circuits are released at *different* times (with probability 1). Gibbens et al. justify their approximation via simulation results based on two summary statistics, an estimator for the link blocking probability and the total circuit utilization. They also provide evidence which confirms that these statistics are sufficient to summarize the behaviour of both models.

The Markovian description of the simplified model is as follows: Since the number of links, K, shall vary, it will be necessary to make the dependence on K explicit in the notation. If one writes $n_j^{(K)}(t)$ for the number of links with j circuits in use at time t, then, under the above assumptions, $(n^{(K)}(t),\ t \geq 0)$, where $n^{(K)} = (n_0^{(K)}, n_1^{(K)}, \ldots, n_C^{(K)})$, is a continuous-time Markov chain which takes values in

$$S^{(K)} = \left\{ n \in \{0, 1, \ldots K\}^{C+1} : \sum_{i=0}^{C} n_i = K \right\}$$

and which has transition rates, $Q^{(K)} = (q^{(K)}(n, n'),\ n, n' \in S^{(K)})$, given by

$$q^{(K)}(n, n + e_{j+1} - e_j) = \nu n_j, \qquad 0 \leq j \leq C - 1,$$

$$q^{(K)}(n, n + e_{j-1} - e_j) = j n_j, \qquad 1 \leq j \leq C,$$

and

$$q^{(K)}(n, n + e_{i+1} - e_i + e_{j+1} - e_j) = \nu n_C \frac{n_i n_j}{\binom{K}{2}},$$
$$0 \leq i < j \leq C - 1,$$

$$q^{(K)}(n, n + 2(e_{j+1} - e_j)) = \nu n_C \frac{\binom{n_j}{2}}{\binom{K}{2}},$$
$$0 \leq j \leq C - 1,$$

where e_i is the unit vector with 1 as its ith entry.

Gibbens et al. [3] proved a functional law of large numbers for the simplified model, and this has recently been shown to be valid for the original model, subject to certain natural constraints on the initial state of the network (see [2]). In particular, they considered the behaviour of $X^{(K)} = (X_0^{(K)}, X_1^{(K)}, \ldots, X_C^{(K)})$, where

$$X_j^{(K)}(t) = \frac{n_j^{(K)}(t)}{K}$$

is the *proportion* of links with j circuits in use at time t. The process $(X^{(K)}(t),\ t \geq 0)$ is itself a Markov chain, but

one that takes values in a lattice which, for every K, is contained in

$$U = \left\{ X \in [0, 1]^{C+1} : \sum_{i=0}^{C} X_i = 1 \right\}. \qquad (1)$$

Gibbens et al. showed that if, in the limit as $K \to \infty$,

$$X^{(K)}(0) \Rightarrow x_0 \quad \text{in} \quad U,$$

then

$$X^{(K)}(\cdot) \Rightarrow X(\cdot, x_0) \quad \text{in} \quad D_U[0, \infty),$$

where $(X(t, x_0),\ t \geq 0)$ is a deterministic process with initial point $X(0, x_0) = x_0$; here $\Rightarrow$ denotes weak convergence and $D_U[0, \infty)$ is the space of all sample paths on $[0, \infty)$. By studying the behaviour of $X(t, x_0)$ in the limit as $t \to \infty$, they were able to demonstrate the remarkable fact that the model can exhibit bistable behaviour for C large enough and for a narrow range of values of the ratio ν/C.

The law of large numbers establishes that, when K is large, the sample behaviour of the model can be approximated by the deterministic trajectory $X(\cdot, x_0)$. However, it does not provide information concerning the random fluctuations about this trajectory. For this reason, I shall propose an analogous functional central limit theorem that establishes a diffusion approximation for $X^{(K)}(\cdot)$ which is valid over finite intervals of time. This will be made possible by observing that the GHK model is *asymptotically density dependent*, a notion which I introduced earlier in connection with a study of Markovian models for a population of searching insect parasites ([6]). I shall first recall this notion and then state two functional limit laws for asymptotically density dependent Markov chains which are appropriate for analysing the GHK model.

3 ASYMPTOTIC DENSITY DEPENDENCE

In the present context we can restrict our attention to Markov chains over a *finite* state space, though all of the results presented in this section hold more generally. Further technical conditions are required to deal with infinite-state processes; details of this can be found in [6].

Let $\{n^{(K)}(\cdot)\}$ be a family of continuous-time Markov chains, indexed by $K > 0$, and suppose that $n^{(K)}(\cdot)$ takes values in $S^{(K)}$, a finite subset of Z^J, and has transition rates $Q^{(K)} = (q^{(K)}(n, n'),\ n, n' \in S^{(K)})$.

Definition. Suppose that there exists an open set $U \subseteq R^J$ and a family, $\{f^{(K)},\ K > 0\}$, of continuous functions, with $f^{(K)} : U \times Z^J \to R$, such that

$$q^{(K)}(n, n + l) = K f^{(K)}\left(\frac{n}{K}, l\right), \qquad l \neq 0. \quad (2)$$

Then, the family of Markov chains is *asymptotically density dependent* if, in addition, there exists a function, $F : U \to R^J$, such that $\{F^{(K)}\}$, given by

$$F^{(K)}(x) = \sum_l l f^{(K)}(x, l), \qquad x \in U,$$

converges to F on U.

This definition of density dependence differs from that introduced by Kurtz [5]. His definition requires only that there exists a continuous function, $f : R^J \times Z^J \to R$, such that

$$q^{(K)}(n, n+l) = K f\left(\frac{n}{K}, l\right), \qquad l \neq 0.$$

Thus, an asymptotically density dependent family of Markov chains is density dependent if $f^{(K)}$ (and hence $F^{(K)}$) does not depend on K. Roughly speaking, a family is density dependent if the transition rates of the corresponding "density process", $X^{(K)}(\cdot)$, defined by

$$X^{(K)}(t) = \frac{n^{(K)}(t)}{K}, \qquad t \geq 0,$$

depend on the present state, n, only through the density n/K; an *asymptotically* density dependent family is one which exhibits this property in the limit as $K \to \infty$. Thus, there is a natural way to associate with this process a density dependent deterministic process which, for large K, is "tracked" by the process. Indeed, a straightforward formal argument based on the Kolmogorov forward differential equations for the state probabilities, shows that, for large K,

$$\frac{d}{dt} \mathsf{E} X^{(K)}(t) \simeq \mathsf{E} F^{(K)}(X^{(K)}(t)), \qquad t > 0.$$

Thus one might expect this deterministic process, call it $X(\cdot)$, to satisfy

$$\frac{d}{dt} X(t) = F(X(t)), \qquad t > 0.$$

The following "law of large numbers" establishes that, under appropriate conditions, the density process *does* track a deterministic process; see [6] for details.

Theorem 3.1. Suppose that $f^{(K)}(\cdot, l)$ is bounded on U, for each l and K, that F is Lipschitz continuous on U and that $\{F^{(K)}\}$ converges uniformly to F on U. Then, if

$$\lim_{K \to \infty} X^{(K)}(0) = x_0, \qquad (3)$$

we have that

$$\lim_{K \to \infty} \Pr\left(\sup_{s \leq t} \left| X^{(K)}(s) - X(s, x_0) \right| > \epsilon\right) = 0,$$

for all $t > 0$ and for all $\epsilon > 0$, where $X(\cdot, x)$ is the unique trajectory satisfying

$$X(0, x) = x,$$

$$X(s, x) \in U, \qquad 0 \leq s \leq t,$$
$$\frac{\partial}{\partial s} X(s, x) = F(X(s, x)).$$

Remark. The technical conditions are usually satisfied in most practical situations. Condition (3) stipulates that the density process should begin close to the initial value, x_0, of the deterministic trajectory and, then, the conclusion of the theorem is that the density process converges (uniformly in probability) over any finite time interval, to that trajectory.

The following "central limit law" establishes that, for large K, the fluctuations about the deterministic path follow a Gaussian diffusion, provided that certain "second-order" conditions are satisfied; again see [6] for details.

Theorem 3.2. Suppose that $f^{(K)}(\cdot, l)$ is bounded on U, for each l and K, that F is Lipschitz continuous on U and has uniformly continuous first partial derivatives, and that

$$\lim_{K \to \infty} \sup_{x \in U} \sqrt{K} |F^{(K)}(x) - F(x)| = 0. \qquad (4)$$

Suppose, also, that the sequence $\{G^{(K)}\}$, where $G^{(K)}$ is a $J \times J$ matrix with elements

$$g_{ij}^{(K)}(x) = \sum_l l_i l_j f^{(K)}(x, l), \qquad x \in U,$$

converges uniformly to G, where G is uniformly continuous on U. Then, provided

$$\lim_{K \to \infty} \sqrt{K} \left(X^{(K)}(0) - x_0 \right) = z, \qquad (5)$$

the family of processes $\{Z^{(K)}(\cdot)\}$, defined by

$$Z^{(K)}(s) = \sqrt{K} \left(X^{(K)}(s) - X(s, x_0) \right), \qquad 0 \leq s \leq t,$$

converges weakly in $D_U[0, t]$ to a Gaussian diffusion, $Z(\cdot)$, with initial value $Z(0) = z$ and with characteristic function, $\psi = \psi(s, \theta)$, which satisfies

$$\frac{\partial \psi}{\partial s}(s, \theta) = -\frac{1}{2} \sum_{j,k} \theta_j g_{jk}(X(s, x_0)) \theta_k \psi(s, \theta)$$
$$+ \sum_{j,k} \theta_j \frac{\partial F_j}{\partial x_k}(X(s, x_0)) \frac{\partial \psi}{\partial \theta_k}(s, \theta). \qquad (6)$$

Remark. Condition (4) strengthens the condition that $\{F^{(K)}\}$ converges uniformly to F to ensure this happens at the correct rate, while Condition (5) provides the initial value of the diffusion.

4   A FUNCTIONAL CENTRAL LIMIT THEOREM FOR THE GHK MODEL

The GHK model is clearly asymptotically density dependent, for one can define U by (1) and $f^{(K)} : U \times Z^{(C+1)} \to R$, $K \geq 1$, by

$$f^{(K)}(x, e_{j+1} - e_j) = \nu x_j, \qquad 0 \leq j \leq C-1,$$

$$f^{(K)}(x, e_{j-1} - e_j) = jx_j, \qquad 1 \leq j \leq C,$$

$$f^{(K)}(x, e_{i+1} - e_i + e_{j+1} - e_j) = 2\nu \left(\frac{K}{K-1} \right) x_C x_i x_j,$$
$$i > j, 0 \leq i, j \leq C - 1,$$

$$f^{(K)}(x, 2(e_{j+1} - e_j)) = \nu \left(\frac{K}{K-1} \right) x_C x_j \left(x_j - \frac{1}{K} \right),$$
$$0 \leq j \leq C - 1,$$

so that (2) is satisfied. It is clear that $f^{(K)}(\cdot, l)$ is bounded on U, for each l and K, and that, as $K \to \infty$, $f^{(K)}$ converges uniformly on U to f, given by

$$f(x, e_{j+1} - e_j) = \nu x_j, \qquad 0 \leq j \leq C - 1,$$

$$f(x, e_{j-1} - e_j) = jx_j, \qquad 1 \leq j \leq C,$$

$$f(x, e_{i+1} - e_i + e_{j+1} - e_j) = 2\nu x_C x_i x_j,$$
$$i > j, 0 \leq i, j \leq C - 1,$$

$$f(x, 2(e_{j+1} - e_j)) = \nu x_C x_j^2, \qquad 0 \leq j \leq C - 1,$$

and so the corresponding sequence $\{F^{(K)}\}$, defined by $F^{(K)}(x) = \sum_l l f^{(K)}(x, l)$, $x \in U$, converges uniformly on U to F, given by $F(x) = \sum_l l f(x, l)$, $x \in U$. On evaluating this latter summation, one finds that

$$F(x) = (H^T + \lambda(x)V^T)x,$$

where $\lambda(x) = 2\nu x_C (1 - x_C)$, and H and R are $(C+1) \times (C+1)$ matrices, given by

$$H = \begin{pmatrix} -\nu & \nu & 0 & \cdots & 0 & 0 \\ 1 & -(\nu+1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\nu+C-1) & \nu \\ 0 & 0 & 0 & \cdots & -C & C \end{pmatrix}$$

and

$$V = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus, the appropriate deterministic model to consider is

$$\frac{d}{dt} X_0(t) = X_1(t) - (\nu + \lambda(t))X_0(t),$$

$$\frac{d}{dt} X_j(t) = (\nu + \lambda(t))X_{j-1}(t) - (\nu + \lambda(t) + j)X_j(t)$$
$$+ (j+1)X_{j+1}(t), \qquad 1 \leq j \leq C - 1, \quad (7)$$

$$\frac{d}{dt} X_C(t) = (\nu + \lambda(t))X_{C-1}(t) - CX_C(t),$$

where $\lambda(t) = \lambda(X(t))$. One might have expected this kind of law of motion to govern the limit proportions, for notice that H is the transition-rate matrix of an Erlang loss system with C circuits and with Poisson traffic offered at rate ν, and so if λ were identically zero, (7) would comprise the forward equations for the state probabilities of such a system. As it is, $\lambda(t)$ gives the additional arrival rate at time t due to overflowing calls. As Gibbens et al. point out, (7) admits a unique solution, $X(\cdot, x_0)$, for each given initial point, $X(0, x_0) = x_0$; this follows from the fact that F is Lipschitz continuous on U.

Using Theorem 3.1, we have the following version of the law of large numbers for the GHK model:

Theorem 4.1. In the GHK model, let $X_j^{(K)}(t)$ be the proportion of links with j circuits in use at time t and define $(X^{(K)}(t), \ t \geq 0)$ by $X^{(K)} = (X_0^{(K)}, X_1^{(K)}, \ldots, X_C^{(K)})$. Then, if

$$\lim_{K \to \infty} X^{(K)}(0) = x_0,$$

we have that

$$\lim_{K \to \infty} \Pr \left(\sup_{s \leq t} \left| X^{(K)}(s) - X(s, x_0) \right| > \epsilon \right) = 0,$$

for all $t > 0$ and for all $\epsilon > 0$, where $X(\cdot, x_0)$ is the unique solution to (7) such that $X(0, x_0) = x_0$.

The theorem allows us to conclude, for example, that $\{X^{(K)}(s)\}$ converges in probability to $X(s, x_0)$ and, since for each s, $X^{(K)}(s)$ is uniformly bounded, dominated convergence implies that

$$\lim_{K \to \infty} \mathsf{E} X^{(K)}(s) = X(s, x_0),$$

over all finite time intervals.

The additional conditions of Theorem 3.2 are also satisfied, and so one can establish an analogous central limit law. We have already seen that $f^{(K)}(\cdot, l)$ is bounded on U, for each l and K, and that $\{F^{(K)}\}$ converges uniformly to F. Now, it is easy to show that the matrix of first partial derivatives, $\nabla F = [\partial F_i / \partial x_j]$, is uniformly continuous. Further, the second-order condition (4) is satisfied because

$$|F_j^{(K)}(x) - F_j(x)| = \frac{1}{K-1}(2\nu - \lambda(x))\Delta x_j,$$

where

$$\Delta x_j = \begin{cases} -x_0, & j = 0, \\ x_{j-1} - x_j, & j = 1, 2, \ldots, C-1, \\ x_{C-1}, & j = C. \end{cases}$$

Also, a suitable sequence of covariance matrices, $\{G^{(K)}\}$, can be constructed from $\{f^{(K)}\}$ and it is easy to see that this sequence converges uniformly to the matrix G with elements

$$g_{ij}(x) = \sum_l l_i l_j f(x, l), \qquad x \in U.$$

Although the precise arithmetical evaluation of G is tedious, it is clear that G is uniformly continuous on U. This follows from the definition of G and the fact that, for each l, $f(\cdot, l)$ is uniformly continuous on U. Thus, provided (5) holds, a diffusion approximation is justified. Using expression (6), one can obtain the mean and variance of $Z(s)$ and, thus, an approximate formula for the mean and variance of $X^{(K)}(s)$ which is valid for large K. If one puts $B_s = \nabla F(X(s, x_0))$, then

$$\mathsf{E} Z(s) = M_s z,$$

where

$$M_s = \exp\left(\int_0^s B_u du\right).$$

On the other hand, the covariance matrix, Σ_s, of $Z(s)$ is given by

$$\Sigma_s = M_s \left(\int_0^s M_u^{-1} G(X(u, x_0))(M_u^{-1})^T du\right) M_s^T.$$

It follows that $X^{(K)}(s)$ has an approximate normal distribution with

$$\mathsf{Cov}\, X^{(K)}(s) \simeq K^{-1} \Sigma_s,$$

and, a "working" approximation for the mean, obtained by setting z *equal to* $\sqrt{K}\left(X^{(K)}(0) - x_0\right)$ (c.f. (5)), is given by

$$\mathsf{E} X^{(K)}(s) \simeq X(s, x_0) + M_s(X^{(K)}(0) - x_0).$$

Observe that the mean and variance of the numbers of circuits in use at time s are both of order K.

In the important special case where x_0 is chosen as an equilibrium point of the deterministic model, one can be far more precise in specifying the approximating diffusion. The equilibrium points of (7) have been studied extensively in [3]. Gibbens et al. showed that if $x_0 = (p_0, p_1, \ldots p_C)$ is an equilibrium point it must be of the form given by

$$p_j = \frac{\xi^j}{j!}\left(\sum_{i=0}^C \frac{\xi^i}{i!}\right)^{-1}, \qquad 0 \le j \le C,$$

where ξ solves

$$\xi = \nu + 2\nu E(\xi, C)\left(1 - E(\xi, C)\right). \tag{8}$$

The quantity $E(\xi, C)$, given by

$$E(\xi, C) = \frac{\xi^C}{C!}\left(\sum_{i=0}^C \frac{\xi^i}{i!}\right)^{-1},$$

is Erlang's formula for the loss probability of a single link with C circuits and with Poisson traffic offered at rate ξ. It is usually more convenient to calculate the solutions to (8) by setting $b = E(\xi, C)$ and solving the equation

$$b = E\left(\nu + 2\nu b(1 - b), C\right); \tag{9}$$

this transformation of (8) shows that b could have been obtained as the celebrated Erlang Fixed Point of the model (see, for example, [4]). For C sufficiently small, equation (9) has a unique solution and the corresponding equilibrium point is stable. However, if C is large enough, there can be two or even three solutions depending, then, on the magnitude of the ratio ν/C, and these give rise to two, respectively three, equilibrium points. In the case of two equilibrium points, one is stable and the other unstable, while in the case of three, two are stable and the other is unstable.

The following central limit law shows that the random fluctuations about any given equilibrium point, x_0, can be approximated by an Ornstein-Uhlenbeck process. It should be emphasised that x_0 need not be stable. Indeed, the approximation is appropriate for describing the fluctuations about the unstable equilibria.

Theorem 4.2. Let x_0 be an equilibrium point of (7). Then, if

$$\lim_{K \to \infty} \sqrt{K}\left(X^{(K)}(0) - x_0\right) = z,$$

the family of processes $\{Z^{(K)}(\cdot)\}$, defined by

$$Z^{(K)}(s) = \sqrt{K}\left(X^{(K)}(s) - x_0\right), \qquad 0 \le s \le t,$$

converges weakly in $D_U[0, t]$ to an Ornstein-Uhlenbeck process, $Z(\cdot)$, with local drift matrix $B = \nabla F(x_0)$, local covariance matrix $G = G(x_0)$, and with initial value $Z(0) = z$. In particular, $Z(s)$ is normally distributed with mean

$$\mu_s = e^{Bs} z$$

and covariance matrix

$$\Sigma_s = \int_0^s e^{Bu} G e^{B^T u} du = \Sigma - e^{Bs}\Sigma e^{B^T s},$$

where Σ, the stationary covariance matrix, satisfies

$$B\Sigma + \Sigma B^T + G = 0.$$

One can conclude that, for K large, $X^{(K)}(s)$ has an approximate normal distribution and an approximation for the mean and the covariance matrix of $X^{(K)}(s)$ is given by

$$\mathsf{E} X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0)$$

and

$$\mathsf{Cov}\, X^{(K)}(s) \simeq K^{-1}\left(\Sigma - e^{Bs}\Sigma e^{B^T s}\right).$$

Although the diffusion approximation obtained using Theorem 3.2 is likely to provide a more accurate estimate of the distribution of $X^{(K)}(s)$, for s small, the Ornstein-Uhlenbeck approximation has the advantage that the approximate formulae for the mean and covariance are specified explicitly. Further, as we shall see, this approximation allows one to estimate the times for which the two stable states persist.

The special case $C = 1$, where there is only one circuit available on each link, is exceedingly simple to analyse. Set

$$F(x) = 1 - (\nu + 1)x - 2\nu(1 - x)x^2, \qquad x \in (0, 1),$$

and

$$G(x) = 1 + (\nu - 1)x + 4\nu(1 - x)x^2 \qquad x \in (0, 1).$$

Then, it can be shown that F has a unique zero, x_0, on $(0, 1)$, for all values of $\nu > 0$, this being a stable equilibrium point of the deterministic model, $dx/dt = F(x)$. If $X^{(K)}(s)$ is the proportion of links with no circuits in use at time s, then by virtue of the Ornstein-Uhlenbeck approximation, $X^{(K)}(s)$ has an approximate normal distribution with

$$\mathsf{E} X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0),$$

where $B = F'(x_0) = 6\nu x_0^2 - 4\nu x_0 - (\nu + 1)(< 0)$, and

$$\mathsf{Var}\, X^{(K)}(s) \simeq K^{-1} \frac{G(x_0)}{2B}(e^{2Bs} - 1).$$

The magnitude of B, and hence the stability of x_0, increases as ν becomes large, but the stationary variance, $G(x_0)/(-2B)$, increases from 0 to a maximum around $\nu = 0.5$ and then decreases to 0.

In cases where $C > 1$, it is convenient (see [1]) to employ a change of coordinates. If, as is the case envisaged here, the eigenvalues of $B = \nabla F(x_0)$ are real, an appropriate transformation is given by

$$W^{(K)}(s) = A Z^{(K)}(s),$$

where the rows of A are the left-eigenvectors of B. Since the column sums of B are all equal to 0, B has a zero eigenvalue, and so one of the components of $W^{(K)}$, say $W_0^{(K)}$, is identically zero, since because $\sum_{i=0}^{C} X_i^{(K)} = 1$, we have that $\sum_{i=0}^{C} Z_i^{(K)} = 0$. The sequence $\{W^{(K)}(\cdot)\}$, where for convenience $W^{(K)} = (W_1^{(K)}, W_2^{(K)}, \ldots W_C^{(K)})$, converges weakly to an Ornstein-Uhlenbeck process, $W(\cdot)$, whose individual components are, themselves, Ornstein-Uhlenbeck processes. Its local drift matrix is $D = \mathrm{diag}\,(\eta_1, \eta_2, \ldots \eta_C)$, where $\eta_1, \eta_2, \ldots \eta_C$ are the non-zero eigenvalues of B, and its local covariance matrix, S, is obtained from the matrix AGA^T by deleting the zeroth row and column. In particular, $W(s)$ has a properly C-dimensional normal distribution with

$$\mathsf{E} W_i(s) = w_i e^{\eta_i s},$$
$$\mathsf{Var}\, W_i(s) = \frac{S_{ii}}{2\eta_i}(e^{2\eta_i s} - 1)$$

and

$$\mathsf{Cov}\,(W_i(s),\ W_j(s)) = \frac{S_{ij}}{\eta_i + \eta_j}(e^{(\eta_i + \eta_j)s} - 1),$$

for $i = 1, 2 \ldots, C$, where $w = Az$.

The change of coordinates allows us to use some powerful results of Barbour [1], which establish asymptotic results on the time of first exit of $X^{(K)}(\cdot)$ from a region containing x_0. For example, suppose that x_0 is a stable equilibrium point and let $\tau(K, c_K)$ be the time when $W^{(K)}(\cdot)$ first crosses the contour

$$\left\{ w \in R^C : \sum_{i=1}^{C} \sqrt{\frac{2T_{ii}}{w_i^2}} \exp\left(\frac{w_i^2}{2T_{ii}}\right) = \frac{\exp(c_K^2)}{c_K} \right\}, \quad (10)$$

where T, the stationary covariance matrix of $W(\cdot)$, has elements $T_{ij} = -S_{ij}/(\eta_i + \eta_j)$ and $\{c_K\}$ is a sequence of real numbers such that $c_K \to \infty$ as $K \to \infty$; as Barbour notes, to order c_K^{-1}, this contour delimits the rectangle

$$\left\{ w \in R^C : |w_i| \leq c_K \sqrt{(2T_{ii})},\ i = 1, 2, \ldots, C \right\}.$$

Then, Theorem 3 of [1] states that if $c_K = o(K^{\frac{1}{8}})$, the random variable

$$-\tau(K, c_K) \frac{2}{\sqrt{\pi}} \eta c_K \exp(-c_K^2),$$

where $\eta = \sum_{i=1}^{C} \eta_i$, converges weakly to a unit-mean negative exponential random variable as $K \to \infty$. Thus, provided $c_K = o(K^{\frac{1}{8}})$, the time at which $W^{(K)}(\cdot)$ first crosses the contour (10) is of order $c_K^{-1} \exp(c_K^2)$.

The result for the $C = 1$ case is more straightforward. Using Theorem 1(iii) of [1], one can see that the time that $X^{(K)}(\cdot)$ first leaves the interval

$$\left\{ x\ :\ |x - x_0| \leq K^{-\frac{1}{2}} c_K \right\}$$

is of order

$$\frac{1}{-2Bc_K} \sqrt{\frac{\pi G(x_0)}{-B}} e^{-Bc_K^2/G(x_0)},$$

whenever $c_K = o(K^{\frac{1}{8}})$. Hence, it is asymptotically larger than any power of K if, for example, $c_K = O(K^{\frac{1}{8}}/\log K)$.

5 SUMMARY

I have established a *law of large numbers* and a *central limit law* for a simple model of a circuit-switching network with random alternative routing. The law of large numbers dictates that the state probabilities, in particular, the blocking probabilities (or, indeed, any other performance measure), can be accurately approximated by a specified deterministic process, when the number of switching nodes is reasonably large. The central limit law allows one to assess the degree of random fluctuation about the deterministic process. I have demonstrated that the fluctuations have an approximate (multivariate) normal distribution whose parameters are determined explicitly. Both approximations are valid over any given time period. For example, they are valid during periods, well before equilibrium is reached,

where explicit analytical formulae for the state probabilities are *unavailable*.

In addition, I have provided some qualitative results concerning the distribution of the time taken for the process to escape from a region containing a deterministic equilibrium state. These results are useful in instances where the system possesses multiple stable states, for example, a high and a low blocking state "separated" by an unstable state; I have demonstrated that the two stable states persist for an exponentially distributed amount of time.

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