QUASISTATIONARY DISTRIBUTIONS FOR LEVEL-DEPENDENT QUASI-BIRTH-AND-DEATH PROCESSES

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ABSTRACT

In this paper we provide a complete quasistationary analysis for the class of level-dependent, discrete-time quasi-birth-and-death processes (QBDs) for which level zero has been collapsed into an absorbing state.

We show that the form of a quasistationary distribution depends upon whether the eigenvalue of a certain matrix is equal to one or less than one.

Furthermore, we show how to calculate the convergence norm α of such a QBD and observe that the QBD is α-recurrent in the first case mentioned above and α-transient in the second case. The further classification of an α-recurrent QBD as α-positive or α-null depends on whether the convergence radius α_2 of the modified QBD in which level one is collapsed into an absorbing state is strictly greater than, or equal to, α. In the first of these cases the QBD is α-positive, while in the second case the QBD may be α-positive or α-null.

Key words: Quasi-birth-and-death process, Quasistationary distributions, Limiting conditional distributions, β-invariant measures.

1 INTRODUCTION

Suppose that we are given a discrete-time Markov chain $(X(n);\ n \in \mathbb{Z}_+)$ with a countable state space consisting of a single absorbing state 0 and an irreducible class $\mathbb{C}$ from which state 0 can be reached.

The communicating class $\mathbb{C}$ of $(X(n))$ is, of course, transient. However, in many physical systems well-modelled by such Markov chains, the absorbing state is reached at an extremely slow rate and it is observed that the system appears to settle down into an equilibrium over the states of $\mathbb{C}$. Examples occur in the modelling of wildlife population processes (see [15, 19]), the modelling of autocatalytic chemical reactions (see [7, 8, 22]) and in reliability theory (see [10, 23]). In such applications we often observe that a physical system cannot recover if all the individuals have died, all reactant has disappeared or a system of machines has completely broken down, and yet the system does appear to exist in a state of equilibrium before such an event occurs.

The notion of a *quasistationary distribution* has proved to be a potent tool in modelling this behaviour. It is a distribution over the transient communicating class of an absorbing Markov chain defined as follows. Let P be the transition matrix of $(X(n))$ and ${}_1P$ be the restriction of P to $\mathbb{C}$. A probability distribution $\boldsymbol{\pi} = (\pi_x,\ x \in \mathbb{C})$ is a *quasistationary distribution* if, for all $x \in \mathbb{C}$ and $n \geq 1$, the state probabilities $p_x(n) = \Pr(X(n) = x)$ of the chain with initial distribution $p_x(0) = \pi_x$ satisfy $p_x(n)/(1 - p_0(n)) = \pi_x$. Thus, if the initial distribution is a quasistationary distribution $\boldsymbol{\pi}$, the state probabilities of the chain conditioned on non-absorption by time n are also given by $\boldsymbol{\pi}$ (see van Doorn and Schrijner [31]).

It follows immediately that $\boldsymbol{\pi}$ is a quasistationary distribution if and only if $\boldsymbol{\pi}$ is a solution of

$$\boldsymbol{m} = \beta \boldsymbol{m}\,{}_1P \tag{1.1}$$

where $\beta^{-1} = 1 - \sum_{x \in \mathbb{C}} \pi_x P_{x0}$.

For any real number $\beta > 0$, a nontrivial, nonnegative vector $\boldsymbol{m}$ that satisfies (1.1) is called a *β-invariant measure* (on $\mathbb{C}$ for ${}_1P$). Under appropriate conditions (see, for example, Kesten [14] and Seneta and Vere-Jones [28]) a normalised β-invariant measure is a quasistationary distribution, and the most obvious method for finding quasistationary distributions is to attempt to solve equation (1.1) and then check these conditions. This is essentially the approach that we take in this paper.

Let α be the common radius of convergence of the series $\sum_{n=0}^{\infty} \delta^n P_{ij}^{(n)}$, $i, j \in \mathbb{C}$, taken as a function of δ. When $\sum_{n=0}^{\infty} \alpha^n P_{ij}^{(n)}$ converges, the chain is called *α-transient*; otherwise it is called *α-recurrent*. The α-recurrent case can further be split into the *α-null* and *α-positive* cases according to whether $\lim_{n\to\infty} \alpha^n P_{ij}^{(n)}$ is equal to zero or greater than zero. It is known that no β-invariant measure can exist for $\beta > \alpha$ and, for many Markov chains, there exist β-invariant measures for all $\beta \in [1, \alpha]$.

An α-invariant measure $\boldsymbol{m}$ may admit a *limiting-conditional interpretation* in the sense that, for all $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \Pr(X(n) = x | X(n) \in \mathbb{C}, X(0) = z) = \frac{m_x}{\sum_{y\in\mathbb{C}} m_y}, \qquad x \in \mathbb{C}. \tag{1.2}$$

This limit exists under a variety of conditions (see, for example, Kesten [14] and Seneta and Vere-Jones [28]). When it does exist, for any *atomic* initial distribution $\boldsymbol{p} = (p_z(0))$ (that is one concentrated on a single state $z \in \mathbb{C}$), the distribution $p_x(n)/(1 - p_0(n))$ of the Markov chain $X(n)$ at time n, conditional on non-absorption, converges to $m_x/\sum_{y\in\mathbb{C}} m_y$.

In this paper we shall investigate the characterisation of quasistationary distributions for discrete-time Markov chains $(X(n))$ which are level-dependent quasi-birth-and-death processes (QBDs) with level 0 collapsed into an absorbing state. For such chains, $\mathbb{C}$ can be written in the form $\{(k, j) : k \geq 1,\ 1 \leq j \leq M_k\}$ and P takes the block-partitioned form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ A_2^{(1)}\boldsymbol{e} & A_1^{(1)} & A_0^{(1)} & 0 & 0 & \cdots \\ 0 & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & 0 & \cdots \\ 0 & 0 & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \cdots \\ 0 & 0 & 0 & A_2^{(4)} & A_1^{(4)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.3}$$

In (1.3), and throughout, $\boldsymbol{e}$ denotes a column vector of ones of the appropriate size.

A level-dependent QBD can thus be regarded as a two-dimensional Markov chain, with one dimension corresponding to subsets of states called *levels*, with level k defined by $l(k) = \{(k, j) : 1 \leq j \leq M_k\}$ for $k \geq 1$ and level 0 being the absorbing state, and the second (finite) dimension corresponding to the *phase* in each level. The only transitions from states in level k which have non-zero probability are those which move to states in level $k + 1$ (whose transition probabilities are recorded in $A_0^{(k)}$), level k (whose transition probabilities are recorded in $A_1^{(k)}$) and level $k-1$ (whose transition probabilities are recorded in $A_2^{(k)}$)

The scalar special case of a QBD occurs when there is only one phase at each level. The quasistationary properties of this Markov chain, the discrete-time birth-and-death process on $\mathbb{Z}_+$, have been studied extensively by van Doorn and Schrijner (see [31]). The class of QBDs has far greater modelling power than does the class of birth-and-death processes.

In a queueing context the phase dimension is often used to denote the stage of arrival and service processes, information about the queueing discipline or information about other customers in the system. In models of other systems, it can be used to track the state of an environment, of other interfering processes or of modulating processes. Thus, many more real systems can be accurately modelled with QBDs than with birth-and-death processes. This motivates the need for tractable methods of analysis for QBDs.

In Bean et al. [2], we studied the *level-independent* case where the matrices $A_i = A_i^{(k)}$ are the same for all k. We proved that the limiting-conditional distribution exists under certain natural conditions. We were also able to characterize the convergence norm α in terms of the maximal eigenvalue $\chi(z)$ of the matrix $A(z) = A_0 + zA_1 + z^2A_2$ for $0 < z \leq 1$. More recently in Bean, Pollett and Taylor [3], again in the level-independent case, we found finite β-invariant measures for all $\beta \leq \alpha$. These are all quasistationary distributions.

The level-dependent case is considerably more complicated than the level-independent case, and different techniques are needed to characterize the convergence norm α and to identify the β-invariant measures for all $\beta \leq \alpha$. We approach the problem using a south-east-corner truncation procedure, in which we consider the QBD truncated to levels ℓ and above. In the context of continuous-time birth-and-death processes, such truncations have been used previously by Karlin and McGregor [13]. The alternative, north-west-corner truncation approach has also been used by many authors (for recent developments see Tweedie [30]).

Specifically, we define a family of (substochastic) matrices $({}_\ell P,\ \ell \geq 1)$ with ${}_\ell P$ obtained by deleting rows and columns of P corresponding to the levels up to and including $\ell - 1$. That is,

$$
{}_\ell P = \begin{pmatrix} A_1^{(\ell)} & A_0^{(\ell)} & 0 & 0 & \cdots \\ A_2^{(\ell+1)} & A_1^{(\ell+1)} & A_0^{(\ell+1)} & 0 & \cdots \\ 0 & A_2^{(\ell+2)} & A_1^{(\ell+2)} & A_0^{(\ell+2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},
$$

and, in particular,

$$
{}_1 P = \begin{pmatrix} A_1^{(1)} & A_0^{(1)} & 0 & 0 & \cdots \\ A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & 0 & \cdots \\ 0 & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \cdots \\ 0 & 0 & A_2^{(4)} & A_1^{(4)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$

We shall denote the convergence norm of ${}_{\ell}P$ by α_ℓ, for $\ell = 1, 2, \ldots$. In this notation $\alpha_1 = \alpha$.

We establish two intriguing dichotomies:

1. the Perron-Frobenius eigenvalue $\eta_1(\alpha)$ of the matrix $U^{(1)}(\alpha)$, defined in equation (3.4) below, is either less than one or equal to one, and

2. the sequence α_ℓ is either constant or $\alpha_1 < \alpha_2$.

The first dichotomy enables us to distinguish the form of the α-invariant measure from a choice of two. One of these forms is β-invariant for $\beta = \alpha$ in α-recurrent QBDs. The second of these forms is β-invariant for $\beta < \alpha$ in all QBDs and for $\beta = \alpha$ in α-transient QBDs. The first dichotomy also tells us whether the QBD is α-recurrent or α-transient. The second dichotomy helps us to distinguish an α-positive process from an α-null process. These results will be proved in Section 7. Our main theorem, Theorem 16, gives the form of the α-invariant measure in each of the possible cases.

Preliminary results are given in Sections 5 and 6. In Section 5 we present the two families of β-invariant measures. In Section 6, we investigate the relationship between the sequence $\{\alpha_\ell\}$ and the eigenvalues of a sequence of matrices $\{U^{(\ell)}(\delta)\}$. In Section 4 we present three examples which motivate our approach and which highlight the salient features of QBDs with lower-truncated transition matrices, while in Sections 2 and 3 we present background results on β-invariant measures and level-dependent QBDs, respectively.

As mentioned above, a discrete-time birth-and-death process is a special case of a quasi-birth-and-death process. Our results therefore apply to level-dependent birth-and-death processes. The calculation of β-invariant measures for these processes was studied in van Doorn and Schrijner [31] and Schrijner [26] using a method analogous to that used in continuous-time by Karlin and McGregor [11, 12] (see also Anderson [1]).

This method involves the derivation of a set of orthogonal polynomials, called birth-and-death polynomials, and the analysis of their orthogonalising measure. The supremum of the support of the orthogonalising measure is equal to $1/\alpha$, and for $1/\alpha \leq 1/\beta < 1$, the β-invariant measure can be written in terms of the birth-and-death polynomials evaluated at $1/\beta$. This method works well if the birth-and-death polynomials are a standard set of orthogonal polynomials whose orthogonalising measure is known, but can be more difficult to apply in other cases.

Our method, applied to the scalar case, works in a different manner. The convergence norm α is characterised as the supremum over δ such that $U^{(0)}(\delta)$, defined in equation (3.4) below, is less than or equal to one. This can be calculated using the numerical method discussed in Section 6. For $\beta \leq \alpha$ a β-invariant measure can be expressed in terms of the functions $R^{(k)}(\beta)$ and $G^{(k)}(\beta)$ defined in equations (3.1) and (3.2). These are analogues of the familiar matrices R and G used in the analysis of QBDs (see Neuts [20, 21]) and can be calculated using similar algorithms. Combining these features, we have a numerical method which can be used to calculate the quasistationary distributions of an arbitrary birth-and-death process.

2 β-INVARIANT MEASURES

In order to make the statement of our results concise, we shall assume that ${}_{\ell}P$ is irreducible for all $\ell \geq 0$, a property which we shall call **total irreducibility**. The transitions between states in the transient class of a level-dependent QBD are recorded in the matrix ${}_{1}P$. Hence the transient class of a totally irreducible process is irreducible. On the other hand, an irreducible process can be rendered reducible by truncation, and so the class of totally irreducible processes is a proper subset of the class of processes for which ${}_{1}P$ is irreducible.

With suitably redefined concepts, our results can be generalised to the case where ${}_{\ell}P$ may be reducible for $\ell \geq 2$.

For $\delta \in \mathbb{R}_{+}$, let ${}_{\ell}N_{ij}(\delta)$ be defined by

$$ {}_{\ell}N_{ij}(\delta) = \sum_{n=0}^{\infty} \delta^{n}\, {}_{\ell}P_{ij}^{(n)} \ , \tag{2.1} $$

where ${}_{\ell}P_{ij}^{(n)}$ is the (i, j)th entry of the n-step transition matrix generated from ${}_{\ell}P$.

Theorem 6.1 of Seneta [27] states that, for each ℓ, and a given value of δ, either ${}_{\ell}N_{ij}(\delta)$ is finite for all (i, j) or ${}_{\ell}N_{ij}(\delta)$ is infinite for all (i, j). Thus we can define the convergence radius associated with ${}_{\ell}P$ as

$$ \alpha_{\ell} = \sup\{\delta : {}_{\ell}N_{ij}(\delta) \text{ is finite}\}. \tag{2.2} $$

There are two possibilities for the behaviour of ${}_{\ell}N_{ij}(\delta)$ at $\delta = \alpha_{\ell}$. In the case where (2.1) diverges for $\delta = \alpha_{\ell}$, ${}_{\ell}P$ is said to be α_{ℓ}-recurrent (either positive or null), while in the case where (2.1) converges for $z = \alpha_{\ell}$, ${}_{\ell}P$ is said to be α_{ℓ}-transient (see Vere-Jones [32]). Denote the intersection of the interval of convergence of the series ${}_{\ell}N_{ij}(\delta)$ and $\mathbb{R}_+$ by $\mathcal{I}_\ell \subset \mathbb{R}_+$ and observe that $\mathcal{I}_\ell = [0, \alpha_\ell)$ if ${}_{\ell}P$ is α_ℓ-recurrent and $\mathcal{I}_\ell = [0, \alpha_\ell]$ if ${}_{\ell}P$ is α_ℓ-transient.

A β-invariant measure for ${}_{\ell}P$ is a nontrivial, nonnegative vector $\boldsymbol{m}$ such that $\boldsymbol{m} = \beta \boldsymbol{m}_{\ell}P$. For a level-dependent QBD, this can be written

$$\boldsymbol{m}_\ell = \beta \left[\boldsymbol{m}_\ell A_1^{(\ell)} + \boldsymbol{m}_{\ell+1} A_2^{(\ell+1)} \right], \tag{2.3}$$

and, for $k \geq \ell$,

$$\boldsymbol{m}_{k+1} = \beta \left[\boldsymbol{m}_k A_0^{(k)} + \boldsymbol{m}_{k+1} A_1^{(k+1)} + \boldsymbol{m}_{k+2} A_2^{(k+2)} \right], \tag{2.4}$$

where the M_k-vector $\boldsymbol{m}_k$ is the restriction of $\boldsymbol{m}$ to level k.

3 THE MATRICES $N^{(k)}(\delta)$, $R^{(k)}(\delta)$ AND $G^{(k)}(\delta)$

Let $N^{(k)}(\delta)$ denote the $M_k \times M_k$ matrix whose (s,t)th entry is ${}_{k}N_{(k,s)(k,t)}(\delta)$ as defined in (2.1). Further, define the $M_{k-1} \times M_k$ matrix

$$R^{(k)}(\delta) = \delta A_0^{(k-1)} N^{(k)}(\delta) \tag{3.1}$$

and the $M_k \times M_{k-1}$ matrix

$$G^{(k)}(\delta) = \delta N^{(k)}(\delta) A_2^{(k)}. \tag{3.2}$$

The families of matrices $\{R^{(k)}(\delta)\}$ and $\{G^{(k)}(\delta)\}$ were discussed in Ramaswami [24]. They have a physical interpretation in terms of taboo probabilities which we give below.

Let $\left[R_{i,j}^{(k)}\right]^{(n)}$ be the probability that the process visits (k, j) at time point n with level $k-1$ taboo at the time points $1, 2, \ldots, n$, given that it starts in state $(k-1, i)$ at time point 0. Then, the (i,j)th entry of $R^{(k)}(\delta)$ is $\sum_{n=1}^{\infty} \left[R_{i,j}^{(k)}\right]^{(n)} \delta^n$. Also, let $\left[G_{i,j}^{(k)}\right]^{(n)}$ be the probability that the process first visits level $k-1$ in state $(k-1, j)$ at time point n, given that it starts in state (k, i) at time point 0. Then, the (i,j)th entry of $G^{(k)}(\delta)$ is $\sum_{n=1}^{\infty} \left[G_{i,j}^{(k)}\right]^{(n)} \delta^n$.

The matrix $N^{(k)}(\delta)$ can be written as

$$N^{(k)}(\delta) = \sum_{n=0}^{\infty} \left[U^{(k)}(\delta)\right]^n , \tag{3.3}$$

where $U^{(k)}(\delta)$ is given by (see Ramaswami [24, equation (3.31)])

$$U^{(k)}(\delta) = \delta\left[A_1^{(k)} + R^{(k+1)}(\delta)A_2^{(k+1)}\right], \quad (3.4)$$

$$= \delta\left[A_1^{(k)} + A_0^{(k)}G^{(k+1)}(\delta)\right]. \quad (3.5)$$

Let $\left[U_{i,j}^{(k)}\right]^{(n)}$ be the probability that the process visits (k, j) at its next visit to level k with level $k-1$ taboo, given that it starts in state (k, i), and that this visit occurs at time point n. Then, the (i, j)th entry of $U^{(k)}(\delta)$ is $\sum_{n=0}^{\infty}\left[U_{i,j}^{(k)}\right]^{(n)}\delta^n$.

The property of total irreducibility implies that $U^{(k)}(\delta)$ is irreducible for all $k \geq 1$. This implies that $N^{(k)}(\delta)$ is elementwise positive. Also, from equations (2.1), (3.1), (3.2) and (3.4) it is clear that, for all $k \geq 1$, $N^{(k)}(\delta)$, $R^{(k)}(\delta)$, $G^{(k)}(\delta)$ and $U^{(k)}(\delta)$ are increasing in δ.

In Lemma 1 below, and throughout, we say that a matrix is finite if all its entries are finite.

Lemma 1

(i) For all $k \geq 1$, $N^{(k)}(\delta)$ is finite iff $\delta \in \mathcal{I}_k$.

(ii) For all $k \geq 2$, $R^{(k)}(\delta)$ is finite iff $\delta \in \mathcal{I}_k$.

(iii) For all $k \geq 1$, $G^{(k)}(\delta)$ is finite iff $\delta \in \mathcal{I}_k$.

Proof: These results are a simple consequence of the definitions of $\mathcal{I}_k$, $N^{(k)}(\delta)$, $R^{(k)}(\delta)$ and $G^{(k)}(\delta)$. ■

By total irreducibility and the definition of the convergence radius, either all entries of $N^{(k)}(\delta)$ are finite, or they are all infinite. It follows from equations (3.1) and (3.2) that if $R^{(k)}(\delta)$ or $G^{(k)}(\delta)$ are not finite, then each element is either zero or infinite.

Lemma 2 *If $\delta \in \mathcal{I}_\ell$ then $N^{(k)}(\delta)$ is finite for all $k \geq \ell$.*

Proof: Assume, for a given value of δ and $k \geq \ell$, that $N^{(k)}(\delta)$ is not finite. Then, if $k \geq \ell$, $N_{ij}^{(\ell)}(\delta)$ consists of a sum of nonnegative terms, including a term of the form $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s\in l(k)}\sum_{t\in l(k)}\,{}_{\ell}P_{(\ell,i)(k,s)}^{(m)}\;N_{st}^{(k)}(\delta)\;{}_{\ell}P_{(k,t)(\ell,j)}^{(n)}\delta^{m+n}$, which, by total irreducibility, must be nonzero for each pair (i, j). Therefore, $N^{(\ell)}(\delta)$ is not finite and so by Lemma 1 $\delta \notin \mathcal{I}_\ell$. ■

Corollary 3 *If $k \geq \ell$ then $\alpha_k \geq \alpha_\ell$.*

Proof: By the above lemma, $\delta \in \mathcal{I}_\ell$ implies finiteness of $N^{(k)}(\delta)$ for all $k \geq \ell$. Therefore, $\mathcal{I}_\ell \subset \mathcal{I}_k$ and so $\alpha_k \geq \alpha_\ell$ for all $k \geq \ell$. ■

A particular consequence of Lemmas 1 and 2 is the following corollary.

Corollary 4 *If $\delta \in \mathcal{I}_\ell$ then $R^{(k)}(\delta)$ is finite for $k \geq \ell + 1$ and $G^{(k)}(\delta)$ is finite for $k \geq \ell$.*

Ramaswami [24, Corollary 3.4 and Theorem 4.3] stated the following results (with a typographical error that does not restrict the value of δ when it should).

Theorem 5 *If $\delta \in \mathcal{I}_\ell$ then*

(i) the family of matrices $\left\{R^{(k)}(\delta),\ k \geq \ell + 1\right\}$ is the minimal nonnegative solution to the family of equations

$$R^{(k)}(\delta) = \delta \left[A_0^{(k-1)} + R^{(k)}(\delta)A_1^{(k)} + R^{(k)}(\delta)R^{(k+1)}(\delta)A_2^{(k+1)}\right]. \tag{3.6}$$

(ii) the family of matrices $\left\{G^{(k)}(\delta),\ k \geq \ell\right\}$ is the minimal nonnegative solution to the family of equations

$$G^{(k)}(\delta) = \delta \left[A_2^{(k)} + A_1^{(k)}G^{(k)}(\delta) + A_0^{(k)}G^{(k+1)}(\delta)G^{(k)}(\delta)\right]. \tag{3.7}$$

In Bean et al. [2], for the level-independent case, we presented an extension of the algorithm in Latouche and Ramaswami [18] to evaluate the matrix $R(\delta) \equiv R^{(k)}(\delta)$. In Bright and Taylor [5] an extension of the algorithm in [18] was presented to evaluate the family $\left\{R^{(k)}(1), k \geq \ell + 1\right\}$. Using very similar techniques to those in [2] we can similarly extend the algorithm in [5] to find the family $\left\{R^{(k)}(\delta), k \geq \ell + 1\right\}$ for $\delta \in \mathcal{I}_\ell$. We give the details below where we define the empty matrix product to be the identity matrix and tacitly assume in any matrix product that the matrices are ordered with the first instance of the dummy variable on the left and the last instance on the right.

The matrices in the family $\left\{R^{(k)}(\delta), k \geq \ell + 1\right\}$ are given by

$$R^{(k)}(\delta) = \sum_{\nu=0}^{\infty} H_k^{\nu} \prod_{\mu=0}^{\nu-1} L_{k+2^{\nu-\mu}}^{\nu-1-\mu}, \qquad k \geq \ell + 1, \tag{3.8}$$

where H_k^ν and L_k^ν are $M_k \times M_{k+2^\nu}$ and $M_k \times M_{k-2^\nu}$ matrices, defined recursively by

$$
\begin{aligned}
H_k^0 &= \delta A_0^{(k)} \sum_{n=0}^{\infty} \left(\delta A_1^{(k+1)}\right)^n, & (3.9)\\
L_k^0 &= \delta A_2^{(k)} \sum_{n=0}^{\infty} \left(\delta A_1^{(k-1)}\right)^n, & (3.10)\\
H_k^{\nu+1} &= H_k^\nu H_{k+2^\nu}^\nu \sum_{n=0}^{\infty} \left[H_{k+2^{\nu+1}}^\nu L_{k+3.2^\nu}^\nu + L_{k+2^{\nu+1}}^\nu H_{k+2^\nu}^\nu\right]^n, & (3.11)\\
L_k^{\nu+1} &= L_k^\nu L_{k-2^\nu}^\nu \sum_{n=0}^{\infty} \left[H_{k-2^{\nu+1}}^\nu L_{k-2^\nu}^\nu + L_{k-2^{\nu+1}}^\nu H_{k-3.2^\nu}^\nu\right]^n. & (3.12)
\end{aligned}
$$

It is a consequence of the fact that $\delta \in \mathcal{I}_\ell$, that the infinite series in (3.9) to (3.12) converge, and so (3.9) to (3.12) can be written

$$
\begin{aligned}
H_k^0 &= \delta A_0^{(k)} \left[I - \delta A_1^{(k+1)}\right]^{-1}, & (3.13)\\
L_k^0 &= \delta A_2^{(k)} \left[I - \delta A_1^{(k-1)}\right]^{-1}, & (3.14)\\
H_k^{\nu+1} &= H_k^\nu H_{k+2^\nu}^\nu \left[I - \left(H_{k+2^{\nu+1}}^\nu L_{k+3.2^\nu}^\nu + L_{k+2^{\nu+1}}^\nu H_{k+2^\nu}^\nu\right)\right]^{-1}, & (3.15)\\
L_k^{\nu+1} &= L_k^\nu L_{k-2^\nu}^\nu \left[I - \left(H_{k-2^{\nu+1}}^\nu L_{k-2^\nu}^\nu + L_{k-2^{\nu+1}}^\nu H_{k-3.2^\nu}^\nu\right)\right]^{-1} & (3.16)
\end{aligned}
$$

which is more suitable for computational purposes.

An algorithm based on equations (3.8) and (3.13) to (3.16) is computationally efficient and easily implemented along the lines presented in Bright and Taylor [5], or as slightly improved in Thorne [29].

As noted in [5], if we require the family $\left\{R^{(k)}(\delta), N \geq k \geq \ell+1\right\}$ then it is most efficient to determine $R^{(N)}(\delta)$ first and then, by using one matrix inversion per step, recursively generate each of the remaining matrices.

A similar expression for the matrices $\left\{G^{(k)}(1), k \geq \ell\right\}$ was given in Ramaswami and Taylor [25]. Thorne [29] showed how this expression can be used as the basis for a computationally efficient algorithm. This algorithm can also be extended, as above, to an algorithm for $\left\{G^{(k)}(\delta), k \geq \ell\right\}$ for all $\delta \in \mathcal{I}_\ell$.

4 BIRTH-AND-DEATH PROCESS EXAMPLES

In Corollary 3 we proved that the sequence $\{\alpha_\ell,\ \ell \geq 1\}$ is an increasing sequence. However, it is easy to construct examples where it is not **strictly** increasing.

In this section we provide some birth-and-death process examples which serve to illustrate the types of behaviour that can occur.

For each of these cases the transition matrix P is of the form of equation (1.3), where

- the 1×1 matrices $A_0^{(k)}$ are given by p_k for all $k \geq 1$,
- the 1×1 matrices $A_1^{(k)}$ are 0 for all $k \geq 1$,
- the 1×1 matrices $A_2^{(k)}$ are given by $q_k \equiv 1 - p_k$ for all $k \geq 1$.

Example 1 Consider the level-independent birth-and-death process with $p_k = p < 1/2$ and $q_k = 1 - p$ for all $k \geq 1$. Truncating this process at any level gives a process identical to the original which, of course, has an identical convergence radius. Thus $\alpha_k = \alpha_\ell$ for all k and ℓ.

Example 2 Consider a birth-and-death process in which $0 \leq p_1 \leq p_2 \leq \cdots \leq p_{\ell-1} \leq p < 1/2$ and $p_k = p$ for all $k \geq \ell$. Let ${}_kX(t)$ be the state of the process at time t, with the levels up to and including $k-1$ removed. Then stochastic comparison arguments (see Daley [6]) show that if $m' \geq m$ and

$$P({}_{m'}X(0) > m' + k) \geq P({}_mX(0) > m + k) \quad \text{for all} \quad k \geq 0$$

then

$$P({}_{m'}X(n) > m' + k) \geq P({}_mX(n) > m + k) \quad \text{for all} \quad k \geq 0, \quad n \geq 0. \tag{4.1}$$

Inequality (4.1) implies that

$$\lim_{n\to\infty} P({}_{m'}X(n) > m' + k)^{-1/n} \leq \lim_{n\to\infty} P({}_mX(n) > m + k)^{-1/n} \tag{4.2}$$

so that, if we can apply Corollary 18 in Appendix A, we have $\alpha_{m'} \leq \alpha_m$. Corollary 3 shows that if $m' \geq m$ then $\alpha_{m'} \geq \alpha_m$ and so it must be the case that $\alpha_{m'} = \alpha_m$ for all $m', m \geq 1$.

To justify application of Corollary 18 in Appendix A, we need to show that $\alpha_{m'} > 1$ and $\alpha_m > 1$. This can be done as follows.

The process ${}_\ell X(n)$ is the homogeneous birth-and-death process with $p_{k,k+1} = p$ and $p_{k,k-1} = 1 - p$. By Anderson [1, page 170], $\alpha_\ell = 1/(2\sqrt{p(1-p)})$ which, since $p < 1/2$, is strictly greater than 1 and so, by Corollary 18, for all $k \geq 0$, $\alpha_\ell = \lim_{n\to\infty} [P({}_\ell X(n) > \ell + k)]^{-1/n}$. Then equation (4.2) implies, for all $1 \leq m \leq \ell$, that

$$\lim_{n\to\infty} P({}_mX(n) > m + k)^{-1/n} \geq \lim_{n\to\infty} [P({}_\ell X(n) > \ell + k)]^{-1/n} = \alpha_\ell > 1. \tag{4.3}$$

By equation (A.2) it is always true that

$$\alpha_m \geq \lim_{n\to\infty} P({}_mX(n) > m+k)^{-1/n} \tag{4.4}$$

and so $\alpha_m \geq \alpha_\ell > 1$. This justifies the application of Corollary 18 for any $m \geq 1$. Thus, we can use equation (4.2) and Corollary 18 to show that if $m' \geq m$ then $\alpha_{m'} \leq \alpha_m$.

Example 3 In contrast to Example 2, if we take $1 \geq p_1 \geq p_2 \geq \cdots \geq p_{\ell-1} \geq p$ and $p_j = p$ for all $j > \ell$, it is possible that there may be an $m < m'$ for which $\alpha_m < \alpha_{m'}$. It is easy to construct numerical examples where this is true. For example, if we choose $\ell = 2$, $p = 0.2$ and $p_1 = 0.5$, then $\alpha_1 = 1.2247$ and $\alpha_2 = 1.25$.

In summary, we have identified the following kinds of behaviour:

(i) for a level-independent birth-and-death process the sequence $\{\alpha_\ell\}$ is constant,

(ii) it is possible for the sequence $\{\alpha_\ell\}$ to be constant even if the birth-and-death process is level-dependent, and

(iii) it is also possible that $\alpha_1 < \alpha_2$ in a level-dependent birth-and-death process.

We shall see in the next sections that similar results hold true for quasi-birth-and-death processes.

5 TWO FAMILIES OF β-INVARIANT MEASURES

By total irreducibility, the matrix $U^{(k)}(\delta)$ defined in (3.4) is irreducible. Therefore, by Seneta [27, Theorem 1.5], for $\delta \in \mathcal{I}_{k+1}$ there is a real, simple eigenvalue $\eta_k(\delta)$ of $U^{(k)}(\delta)$ which is equal to its spectral radius. This is the *Perron-Frobenius eigenvalue* of $U^{(k)}(\delta)$.

In this section we shall present two families of β-invariant measures for ${}_\ell P$. One of these satisfies equations (2.3) and (2.4) when $\eta_\ell(\beta) = 1$. In the next section we shall see that this occurs when $\beta = \alpha_\ell$ and the QBD is α_ℓ-recurrent, that is $\mathcal{I}_\ell$ is of the form $[0, \alpha_\ell)$. The second family of β-invariant measures satisfies equations (2.3) and (2.4) for all $\beta \in \mathcal{I}_\ell$. In Section 6, we shall see that $\eta_\ell(\beta) < 1$ for all such β. We shall also see that when $\eta_\ell(\beta) > 1$, $\beta > \alpha_\ell$, and so the two previously-mentioned cases when $\eta_\ell(\beta) = 1$ and $\eta_\ell(\beta) < 1$ are the only two cases in which a β-invariant measure can exist.

Lemma 6 *Assume that $\beta \geq 1$ is such that $\eta_\ell(\beta) = 1$ and let $\boldsymbol{x}$ be the left eigenvector of $U^{(\ell)}(\beta)$ associated with the eigenvalue 1. Then $\boldsymbol{m} = (\boldsymbol{m}_\ell, \boldsymbol{m}_{\ell+1}, \ldots)$ defined by*

$$\boldsymbol{m}_k = \boldsymbol{x} \prod_{n=\ell+1}^{k} R^{(n)}(\beta) \tag{5.1}$$

is a β-invariant measure for ${}_\ell P$.

Proof: The proof follows by direct substitution of equation (5.1) into equations (2.3) and (2.4), bearing in mind equations (3.4) and (3.6). ∎

If $\eta_\ell(\beta) < 1$ then a β-invariant measure is more difficult to find. We first need the following lemma.

Lemma 7 *If $\beta \in \mathcal{I}_\ell$ then there exists a sequence of stochastic vectors $\{\boldsymbol{x}_k, k \geq \ell\}$ and a sequence of positive numbers $\{\rho_k, k \geq \ell\}$ such that, for all $k \geq \ell$,*

$$\boldsymbol{x}_{k+1} G^{(k+1)}(\beta) = \rho_k \boldsymbol{x}_k. \tag{5.2}$$

Proof: The proof of this lemma follows in an identical way to that of Theorem 3.1 of Latouche, Pearce and Taylor [17] except that we consider the family of matrices $\left\{G^k(\beta), k \geq \ell\right\}$ instead of $\left\{\hat{G}^k(1), k \geq 1\right\}$. The requirement that $\beta \in \mathcal{I}_\ell$, together with Corollary 4, is sufficient to guarantee that $G^{(k)}(\beta)$ is finite for all $k \geq \ell$. ∎

Theorem 8 *If $\beta \in \mathcal{I}_\ell$ then a β-invariant measure for ${}_\ell P$ is given by*

$$\boldsymbol{m} = (\boldsymbol{m}_\ell, \boldsymbol{m}_{\ell+1}, \ldots)$$

where for all $k \geq \ell$

$$\boldsymbol{m}_k = \boldsymbol{x}_k \left(\prod_{n=\ell}^{k-1} \rho_n^{-1}\right) \sum_{\nu=\ell}^{k} \left(\left[\prod_{\mu=0}^{k-\nu-1} G^{(k-\mu)}(\beta) \right] N^{(\nu)}(\beta) \left[\prod_{\mu=0}^{k-\nu-1} R^{(\nu+1+\mu)}(\beta) \right] \right), \tag{5.3}$$

and $\boldsymbol{x_k}$ and the ρ_n satisfy (5.2).

Proof: In order to simplify the proof, let us first write out explicitly $\boldsymbol{m}_{k+1}$, $\boldsymbol{m}_k$ and $\boldsymbol{m}_{k-1}$, for a particular value of $k > \ell$. Thus we have, by appropriate applications of equation (5.2),

$$\begin{aligned}
\boldsymbol{m}_{k+1} &= \boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k+1}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k+1-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right),\\
\boldsymbol{m}_{k} &= \boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right),\\
\boldsymbol{m}_{k-1} &= \boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k-1}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k-1-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right).
\end{aligned}$$

Substitute these expressions into equation (2.4) with $k > \ell$. The right-hand side is then

$$\begin{aligned}
&\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k-1}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k-1-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right)\\
&\qquad\times\beta\left[A_0^{(k-1)}+R^{(k)}(\beta)A_1^{(k)}+R^{(k)}(\beta)R^{(k+1)}(\beta)A_2^{(k+1)}\right]\\
&\qquad+\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)G^{(k+1)}(\beta)N^{(k)}(\beta)\beta A_1^{(k)}\\
&\qquad+\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)G^{(k+1)}(\beta)N^{(k)}(\beta)R^{(k+1)}(\beta)\beta A_2^{(k+1)}\\
&\qquad+\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)N^{(k+1)}(\beta)\beta A_2^{(k+1)},
\end{aligned}$$

which, by equations (3.6) and (3.2), can be written as

$$\begin{aligned}
&\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k-1}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right)\\
&+\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)G^{(k+1)}(\beta)N^{(k)}(\beta)\left[\beta A_1^{(k)}+\beta R^{(k+1)}(\beta)A_2^{(k+1)}+\left(N^{(k)}(\beta)\right)^{-1}\right],\\
=\ &\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k-1}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right)\\
&+\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)G^{(k+1)}(\beta)N^{(k)}(\beta),\\
=\ &\boldsymbol{x}_{k+1}\left(\prod_{n=\ell}^{k}\rho_n^{-1}\right)\sum_{\nu=\ell}^{k}\left(\left[\prod_{\mu=0}^{k+1-\nu-1}G^{(k+1-\mu)}(\beta)\right]N^{(\nu)}(\beta)\left[\prod_{\mu=0}^{k-\nu-1}R^{(\nu+1+\mu)}(\beta)\right]\right)\\
=\ &\boldsymbol{m}_k.
\end{aligned}$$

Here the first equality holds by applying equations (3.4) and (3.3), since $\beta \in \mathcal{I}_\ell$ implies that $N^{(k)}(\beta)$ is finite and equal to $\left(I - U^{(k)}(\beta)\right)^{-1}$.

Now assume that $k = \ell$. Then $\boldsymbol{m}_\ell$ and $\boldsymbol{m}_{\ell+1}$ are given by,

$$
\begin{aligned}
\boldsymbol{m}_{\ell+1} &= \boldsymbol{x}_{\ell+1}\rho_\ell^{-1}\sum_{\nu=\ell}^{\ell+1}\left(G^{(\ell+1)}(\beta)N^{(\ell)}(\beta)R^{(\ell+1)}(\beta) + N^{(\ell+1)}(\beta)\right), \\
\boldsymbol{m}_\ell &= \boldsymbol{x}_{\ell+1}\rho_\ell^{-1}G^{(\ell+1)}(\beta)N^{(\ell)}(\beta)
\end{aligned}
$$

Substitute these expressions into equation (2.3). The right-hand side is then given by

$$
\begin{aligned}
&\boldsymbol{x}_{\ell+1}\rho_\ell^{-1}\left[G^{(\ell+1)}(\beta)N^{(\ell)}(\beta)\beta A_1^{(\ell)} + G^{(\ell+1)}(\beta)N^{(\ell)}(\beta)R^{(\ell+1)}(\beta)\beta A_2^{(\ell+1)} + N^{(\ell+1)}(\beta)\beta A_2^{(\ell+1)}\right], \\
&\qquad = \boldsymbol{x}_{\ell+1}\rho_\ell^{-1}G^{(\ell+1)}(\beta)N^{(\ell)}(\beta)\left[\beta A_1^{(\ell)} + \beta R^{(\ell+1)}(\beta)A_2^{(\ell+1)} + \left(N^{(\ell)}(\beta)\right)^{-1}\right], \\
&\qquad = \boldsymbol{x}_{\ell+1}\rho_\ell^{-1}G^{(\ell+1)}(\beta)N^{(\ell)}(\beta), \\
&\qquad = \boldsymbol{m}_\ell.
\end{aligned}
$$

Again the first equality holds by applying equation (3.2) and reorganising the expression and the second equality holds by applying equations (3.4) and (3.3) in the same way as above. ■

The question of whether the β-invariant measures given in equations (5.1) and (5.3) are actually computable is relevant. As mentioned in Section 3, the families of matrices $R^{(k)}(\beta)$ and $G^{(k)}(\beta)$ can be calculated using a modification of the algorithms in [2, 18, 5, 29], so that the expression (5.1) can be computed and the question as to whether (5.3) can be computed reduces to the question of whether we can evaluate the $N^{(k)}(\beta)$, $\boldsymbol{x}_k$ and the ρ_k.

Since $\beta \in \mathcal{I}_\ell$, we know that the matrix $N^{(k)}(\beta)$ is finite for all $k \geq \ell$ and so equation (3.3) can be rewritten as

$$
\begin{aligned}
N^{(k)}(\beta) &= \left[I - U^{(k)}(\beta)\right]^{-1}, \\
&= \left[I - \beta A_1^{(k)} - \beta R^{(k+1)}(\beta)A_2^{(k+1)}\right]^{-1}, \\
&= \left[I - \beta A_1^{(k)} - \beta A_0^{(k)}G^{(k+1)}(\beta)\right]^{-1},
\end{aligned}
$$

which is easily evaluated from $R^{(k+1)}(\beta)$ or $G^{(k+1)}(\beta)$.

In practice, we will only ever calculate the β-invariant measure on a finite number of levels, say K levels. Therefore, we only need a single vector $\boldsymbol{x}_K$ and the family of scalars $\{\rho_k, \ell \leq k \leq K\}$. To see this, repeatedly apply equation (5.2) to the expression for $\boldsymbol{m}_k$ in order to replace $\boldsymbol{x}_k$ by $\boldsymbol{x}_K$.

The proof of Lemma 7 in Latouche, Pearce and Taylor [17] suggests a numerical scheme for the calculation of $\boldsymbol{x}_K$ and the ρ_k. Choose an integer, say $J > K$, and any stochastic vector $\boldsymbol{x}_J^{(J)}$ and recursively apply equation (5.2) to determine a family of vectors $\left\{\boldsymbol{x}_k^{(J)}, K \leq k \leq J\right\}$ that satisfies equation (5.2). By increasing the value of J we create a sequence of stochastic vectors $\boldsymbol{x}_K^{(J)}$ which must have a convergent subsequence with limit $\boldsymbol{x}_K$. This vector can then be used to generate the family of scalars $\{\rho_k, \ell \leq k \leq K\}$, as required.

6 THE SPECTRAL RADIUS OF $U^{(k)}(\delta)$

In this section we shall investigate the relationship between $\eta_k(\delta)$ and the values of α_k and α_{k+1}. This relationship is fundamental to our major results, which will be given in Section 7.

Lemma 9 *The function $\eta_k(\cdot)$ is a well defined, continuous and strictly increasing function on $\mathcal{I}_{k+1}$.*

Proof: If $\delta \in \mathcal{I}_{k+1}$ then $R^{(k+1)}(\delta)$ is finite by Lemma 1 and so, by its definition (3.4), $U^{(k)}(\delta)$ is also finite. Since $U^{(k)}(\delta)$ is irreducible its Perron-Frobenius eigenvalue, $\eta_k(\delta)$, is well defined.

By its definition (3.1), it is clear that $R^{(k+1)}(\delta)$ is nonnegative and increasing in δ with at least one entry strictly increasing in δ. Therefore, $A_1^{(k)} + R^{(k+1)}(\delta)A_2^{(k+1)}$ is also nonnegative and strictly increasing in δ. This implies that its maximal eigenvalue is strictly increasing [27, Theorem 1.5(e)]. Therefore, $\eta_k(\delta)$, the maximal eigenvalue of $U^{(k)}(\delta) = \delta\left[A_1^{(k)} + R^{(k+1)}(\delta)A_2^{(k+1)}\right]$, must be strictly increasing.

To prove continuity, note that $\eta_k(\delta)$ is the maximal root of a polynomial whose coefficients are polynomials in δ. Since it is known that the Perron-Frobenius eigenvalue is a simple root (see, for example, Seneta [27, Theorem 1.5]), the implicit function theorem gives that $\eta_k(\delta)$ is continuous. $\blacksquare$

Theorem 10 *Assume $\delta \in \mathcal{I}_{k+1}$, then,*

(i) If $\eta_k(\delta) < 1$ then $\delta \leq \alpha_k$.

(ii) If $\eta_k(\delta) \geq 1$ then $\delta \geq \alpha_k$.

(iii) If $\eta_k(\delta) = 1$ then $\delta = \alpha_k$.

Proof:

(i) If $\eta_k(\delta) < 1$ then $\lim_{n\to\infty} \left[U^{(k)}(\delta)\right]^n \to 0$ and so the series $\sum_{n=0}^{\infty} \left[U^{(k)}(\delta)\right]^n$ converges [27, Lemma B.1]. Consequently, by equation (3.3), $N^{(k)}(\delta)$ is finite and so $\delta \leq \alpha_k$.

(ii) If $\eta_k(\delta) \geq 1$ then the series $\sum_{n=0}^{\infty} \left[U^{(k)}(\delta)\right]^n$ does not converge. Consequently, $N^{(k)}(\delta)$ is not finite and so $\delta \geq \alpha_k$.

(iii) If δ is such that $\eta_k(\delta) = 1$ then $\delta \geq \alpha_k$, as shown in (ii).

Since $\delta \in \mathcal{I}_{k+1}$, $R^{(k+1)}(\delta)$ is finite and, because $\eta_k(\delta) = 1$, Lemma 6 applies and we can construct a δ-invariant measure for ${}_kP$. Anderson [1, Lemma 5.2.4 (3)] states that if there exists a δ-(sub)invariant measure for ${}_kP$ then $\delta \leq \alpha_k$. This implies that $\delta = \alpha_k$.

$\blacksquare$

Theorem 11

(i) If $\delta < \alpha_k$ then $\eta_k(\delta) < 1$.

(ii) If $\alpha_k < \delta < \alpha_{k+1}$ then $\eta_k(\delta) > 1$.

(iii) If $\alpha_k = \delta < \alpha_{k+1}$ then $\eta_k(\delta) = 1$.

Proof:

(i) If $\delta < \alpha_k$, then $N^{(k)}(\delta)$ is finite and so $\sum_{n=0}^{\infty} \left[U^{(k)}(\delta)\right]^n$ converges which implies that $\eta_k(\delta) < 1$.

(ii) If $\alpha_k < \delta < \alpha_{k+1}$ then $N^{(k)}(\delta)$ is not finite. Therefore, the series $\sum_{n=0}^{\infty} \left[U^{(k)}(\delta)\right]^n$ does not converge and so $\eta_k(\delta) \geq 1$. Theorem 10(iii) then implies that $\eta_k(\delta) \neq 1$ and so $\eta_k(\delta) > 1$.

(iii) This is a simple consequence of the continuity of η_k and parts (i) and (ii) of this theorem.

$\blacksquare$

Remark 1 *Theorem 11 does not tell us anything about the value of $\eta_k(\delta)$ when $\alpha_k = \delta = \alpha_{k+1}$. In fact, since $N^{(k+1)}(\delta)$, and hence $R^{(k+1)}(\delta)$ and $U^{(k)}(\delta)$, might diverge at $\delta = \alpha_k$ (they would if the process were α_k-recurrent), we cannot yet say whether $\eta_k(\alpha_k)$ is well-defined in this case. In the theorem below, we show that $\eta_k(\alpha_k)$ is well-defined and less than or equal to one.*

Theorem 12

(i) $\lim_{\delta \uparrow \alpha_k} \eta_k(\delta) \leq 1$.

(ii) If $\alpha_k = \alpha_{k+1}$ *then* $\lim_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) < 1$.

(iii) If $\alpha_k = \alpha_{k+1}$ *then* $N^{(k+1)}(\alpha_{k+1})$ *is finite and so* $\alpha_{k+1} \in \mathcal{I}_{k+1}$.

(iv) If $\alpha_k = \alpha_{k+1}$ *then* $\eta_k(\alpha_k)$ *is well-defined and* $\eta_k(\alpha_k) \leq 1$.

Proof: See Appendix B.

To illustrate the three possible types of behaviour of α_1, α_2 and $\eta_1(\alpha_1)$, consider the following birth-and-death-process examples.

Example 4 Within the setting of Example 2, consider the case when $\ell = 2$, that is a birth-and-death process with $p_k = p$ for $k \geq 2$ and where p_1 may not be equal to p. It is easy to show the following:

(i) If $p_1 > 2p$ then $\alpha_1 < \alpha_2$ and $\eta_1(\alpha_1) = 1$.

(ii) If $p_1 < 2p$ then $\alpha_1 = \alpha_2$ and $\eta_1(\alpha_1) < 1$.

(iii) If $p_1 = 2p$ then $\alpha_1 = \alpha_2$ and $\eta_1(\alpha_1) = 1$.

The details of this section can be summarised in the following theorem which gives a simple expression for the convergence radius α_ℓ of the process ${}_\ell P$.

Theorem 13 *The convergence radius α_ℓ of the process ${}_\ell P$ is given by*

$$\alpha_\ell = \sup\{\delta \leq \alpha_{\ell+1} : \eta_\ell(\delta) \leq 1\}. \tag{6.1}$$

Proof: Corollary 3 states that $\alpha_\ell \leq \alpha_{\ell+1}$ and so the only two cases we need to consider are $\alpha_\ell < \alpha_{\ell+1}$ and $\alpha_\ell = \alpha_{\ell+1}$.

Theorem 11(iii) shows that if $\alpha_\ell < \alpha_{\ell+1}$ then $\eta_\ell(\alpha_\ell) = 1$. Since Lemma 9 states that $\eta_\ell(\cdot)$ is a strictly increasing function on $\mathcal{I}_{\ell+1}$, equation (6.1) must hold in this case.

Theorem 12(iv) shows that if $\alpha_\ell = \alpha_{\ell+1}$ then $\eta_\ell(\alpha_\ell) \leq 1$. Moreover, from Lemma 1 the matrix $R^{(\ell+1)}(\delta)$, and hence $U_\ell(\delta)$ and $\eta_\ell(\delta)$, are not finite for $\delta > \alpha_\ell = \alpha_{\ell+1}$. Hence equation (6.1) holds in this case as well. ■

We now present some ideas on how the results of this section can be used to identify the value of α_k. Since we are searching for a single value on the real number line, a bisection search is very attractive. It turns out that there are three conditions we need to understand to develop the search. All decisions are based on the value of $\eta_k(\delta)$ which involves calculation of the matrix $R^{(k+1)}(\delta)$ via the explicit expression (3.8) and the algorithm discussed in Section 3, and then calculation of the matrix $U^{(k)}(\delta)$ via expression (3.4).

1. If $\eta_k(\delta) < 1$ then $\delta \leq \alpha_k$ by Theorem 10 (ii).

2. If $\eta_k(\delta) > 1$ then $\alpha_k < \delta \leq \alpha_{k+1}$ by Theorem 10 (iii) and (iv).

3. If $\eta_k(\delta)$ does not exist (equivalently $R^{(k+1)}(\delta)$ is not finite) then $\delta \geq \alpha_{k+1} \geq \alpha_k$ by Lemma 9. In the above algorithm this can manifest itself in one of two ways: either at least one of the terms in the summation in equation (3.8) is divergent or all terms are convergent but the summation is itself divergent.

   If one of the terms is divergent, then this must be caused by the series $\sum_{n=0}^{\infty} X^n$, in either equation (3.11) or (3.12), being divergent. This can occur only if the Perron-Frobenius eigenvalue of X is at least 1. This is easy to detect. In this case we can conclude that $\eta_k(\delta)$ does not exist, and $\delta \geq \alpha_k$.

   If, instead, all terms are convergent but the series itself is divergent then the above test will not work. However, consider the partial sums

$$S_K^{(k)} = \sum_{\nu=0}^{K} H_k^\nu \prod_{\mu=0}^{\nu-1} L_{k+2^{\nu-\mu}}^{\nu-1-\mu}, \quad K = 1, 2, \ldots.$$

These partial sums are increasing in K (as all terms are nonnegative) and tending to ∞. Let

$$V_K^k = \delta\left[A_1^{(k)} + S_K^{(k+1)}A_2^{(k+1)}\right].$$

Now the sequence $V_K^{(k)}$ is increasing in K without bound. Hence, we know that there must exist K_0 such that the Perron-Frobenius eigenvalue of $V_K^{(k)}$ is greater than 1 for all $K > K_0$. This is again easy to detect. In this case we can again conclude that $\eta_k(\delta)$ does not exist and $\delta \geq \alpha_k$. In fact, an efficient implementation of this search algorithm will use this test in the case discussed in 2 above as well.

7 THE MAIN THEOREM

The examples of Section 4 motivate us to investigate the possible relationships within the set $\{\alpha_k\}$.

Theorem 14

(i) If $\alpha_k < \alpha_{k+1}$ then $\alpha_{k-1} < \alpha_k$.

(ii) Either $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_k$ for all $k \geq 1$.

Proof:

(i) If $\alpha_k < \alpha_{k+1}$ then, by Theorem 11(iii), $\eta_k(\alpha_k) = 1$. Theorem 12 (ii) and continuity of $\eta_k(\cdot)$ then implies (under a shift in the index) that $\alpha_{k-1} < \alpha_k$.

(ii) By recursively applying part (i), if there exists k for which $\alpha_k < \alpha_{k+1}$, then $\alpha_1 < \alpha_2$. The only other possibility occurs when $\alpha_k = \alpha_{k+1}$ for all k and so $\alpha_1 = \alpha_k$ for all $k \geq 1$. ■

Definition 1

*(i) If $\alpha_1 = \alpha_k$ for all $k \geq 1$, we say that the process is **interior determined**.*

*(ii) If $\alpha_1 < \alpha_2$ we say that the process is **boundary determined**.*

The terminology in Definition 1 is motivated by the fact that, roughly speaking, the convergence radius of an interior determined process is given by the transition probabilities at high levels of the process, whilst the convergence radius of a boundary determined process is given by the transition probabilities at low levels of the process. The birth-and-death processes discussed in Examples 1 and 2 are interior determined, whilst the one in Example 3 is boundary determined.

Lemma 15 *If $\eta_1(\alpha) = 1$ then the process ${}_1P$ is α-recurrent. Further, the process ${}_1P$ is α-positive if $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ is finite and α-null otherwise.*

Proof: See Appendix B.

We can now present the main theorem of this paper. It gives an expression for the α-invariant measure and states the α-classification for the process.

Theorem 16

(1) If $\eta_1(\alpha) = 1$ then the α-invariant measure is given by $\boldsymbol{m} = (\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots)$ as in equation (5.1) with $\ell = 1$. The process is α-recurrent.

(a) If the process is boundary determined, then it is α-positive and the α-invariant measure also has a limiting-conditional interpretation in the sense of equation (1.2).

(b) If the process is interior determined, then it is

(i) α-positive if $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ is finite, and

(ii) α-null otherwise.

(2) If $\eta_1(\alpha) < 1$, then the process is interior determined and an α-invariant measure is given by $\boldsymbol{m} = (\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots)$ as in equation (5.3) with $\ell = 1$. The process is α-transient.

Proof:

(1) Since $\eta_1(\alpha) = 1$, Lemma 6 shows that the α-invariant measure is as stated in equation (5.1) and Lemma 15 implies that the process is α-recurrent.

(a) Since the process is boundary determined, that is, $\alpha = \alpha_1 < \alpha_2$, we have that $\alpha \in \mathcal{I}_2$. Therefore, Corollary 4 implies that $R^{(k)}(\alpha)$ is finite for all $k \geq 2$ and so the α-invariant measure in equation (5.1) is finite.

Further, it is clear that $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ is finite and so the process is α-positive by Lemma 15. The fact that the α-invariant measure then has a limiting conditional interpretation follows from basic theory, for example Seneta and Vere-Jones [28].

(b) Since the process is interior determined, we have that $\alpha = \alpha_2$. Theorem 12(iii) then implies that $\alpha \in \mathcal{I}_2$. Therefore, Corollary 4 implies that $R^{(k)}(\alpha)$ is finite for all $k \geq 2$ and so the α-invariant measure in equation (5.1) is finite.

The process ${}_2P$ is α-transient because $\alpha \in \mathcal{I}_2$ and so $N^{(2)}(\alpha)$ is finite. The α-classification follows from Lemma 15.

(2) Since $\eta_1(\alpha) < 1$, Theorem 11(iii) shows that $\alpha_1 = \alpha_2$ and so the process is interior determined.

The argument in the proof of Theorem 10(ii) shows that the matrix $N^{(1)}(\alpha)$ is finite. Therefore, by definition, the process is α-transient. Consequently, we know that $\alpha \in \mathcal{I}_1$ and so Lemma 2, Corollary 4 and Theorem 8 show that an α-invariant measure is as stated in equation (5.3) and is finite.

■

Remark 2

(i) With respect to the classification in (1)(b) of Theorem 16, two examples of α-null interior-determined birth-and-death processes are given in Section 8. Therefore there certainly exist QBDs of the type defined in (1)(b)(ii).

The authors have been unable to construct a QBD of the type defined in (1)(b)(i), and do not know whether such a process exists. Such a process would have to have an α-transient truncation ${}_2P$ with $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ finite. We approached the construction problem by trying to find such a truncation, and found that, although we could certainly construct a non-QBD Markov chain that is α-transient and has finite $\dfrac{dN(\alpha)}{d\alpha}$ we were unable to construct a QBD example with these properties. Whether such a process exists is an open question.

(ii) *The general issue of how Theorem 16 compares with the classical orthogonal polynomial analysis applied in the birth-and-death process special case is non-trivial. It will be discussed in a forthcoming paper [4]. We can note here that Theorem 2.6 of Schrijner [26, page 34] states that the process is α-positive if and only if $1/\alpha$ is a point of positive mass of the corresponding orthogonalising measure. This observation may prove useful in distinguishing between the cases in (1)(b)(i) and (1)(b)(ii) of Theorem 16.*

(iii) *The α-invariant measures for processes that are α-null or α-transient may or may not have a limiting-conditional interpretation in the sense of equation (1.2). The best way to answer this question is to use the results in Kesten [14]. Theorem 2 of [14] showed that an α-invariant measure is a limiting-conditional distribution for a discrete-time Markov chain provided that jumps satisfy a boundedness condition and that the chain is uniformly irreducible and uniformly aperiodic. These conditions are defined in detail in [14].*

The conditions are not always satisfied in our context: counter examples in which the number of phases increase with the level are easy to find. To obtain a limiting-conditional interpretation for an α-invariant measure, Kesten's conditions will need to be checked in each particular case.

(iv) *In Section 1 we stated that, under certain conditions, a normalised α-invariant measure has the limiting conditional interpretation equation (1.2). One of the necessary conditions for this is that absorption must occur with probability 1 from any starting state in $\mathbb{C}$. If this is not the case, it is possible that an alternative limiting conditional interpretation*

$$\lim_{n\to\infty} \Pr(X(n) = x | X(n) \in \mathbb{C}, X(0) = z, X(n+r) = 0 \text{ for some } r \geq 1) = \frac{m_x a_x}{\sum_{y\in\mathbb{C}} m_y a_y}, \tag{7.1}$$

may hold, in which a_x is the probability that $X(n)$ ever reaches the absorbing state given that $X(0) = x$. In order to evaluate the right hand side of (7.1), in addition to $\boldsymbol{m}$, we require a knowledge of the vector $\boldsymbol{a}_k$ of absorption probabilities from level k. It follows easily from the discussion after equation (3.2) that these are given by

$$\boldsymbol{a}_k = \prod_{n=0}^{k-1} G^{(k-n)}(1). \tag{7.2}$$

The sequence of matrices $G^{(k)}(1)$ can be calculated using the algorithm mentioned at the end of Section 3.

8 BIRTH-AND-DEATH PROCESS EXAMPLE

In this section we present the full analysis of a birth-and-death process example. We choose the example of van Doorn and Schrijner [31, page 140] and modify the notation to suit the notation of this paper. In a similar manner to the examples that we considered in Section 4, we take the transition matrix P to be of the form of equation (1.3), where

- the 1×1 matrices $A_0^{(j)}$ are given by p for all $j \geq 2$,
- the 1×1 matrices $A_1^{(j)}$ are 0 for all $j \geq 2$,
- the 1×1 matrices $A_2^{(j)}$ are given by $q \equiv 1 - p$ for all $j \geq 2$,
- $A_0^{(1)} = p_1$ and $A_1^{(1)} = r_1$ are arbitrary,
- and $A_2^{(1)} = 1 - p_1 - r_1$.

We observe that the process ${}_2P$ is level-independent and so by the methods in [2, 3] we have that

$$\alpha_2 = \frac{1}{2\sqrt{pq}}\,. \tag{8.1}$$

We can therefore deduce that

$$R^{(3)}(\delta) = \frac{1 - \sqrt{1 - 4\delta^2 pq}}{2\delta q} \quad \text{for} \quad \delta \leq \alpha_2.$$

Now let us turn our attention to ${}_1P$. Using the above expression for $R^{(3)}(\delta)$ we can deduce from equation (3.6) that

$$R^{(2)}(\delta) = \frac{\delta p_1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\delta^2 pq}} \quad \text{for} \quad \delta \leq \alpha_2. \tag{8.2}$$

Since

$$U^{(1)}(\delta) = \delta \left(r_1 + R^{(2)}(\delta) q \right) \tag{8.3}$$

is a 1×1 matrix, $U^{(1)}(\delta)$ is equal to $\eta_1(\delta)$. Theorem 13 then says that we can identify α_1 as

$$\alpha_1 = \sup \left\{ \delta \leq \alpha_2 \ : \ U^{(1)}(\delta) \leq 1 \right\}. \tag{8.4}$$

Substitution of (8.2) and (8.1) into equation (8.3) gives us

$$U^{(1)}(\alpha_2) = \frac{r_1}{2\sqrt{pq}} + \frac{p_1}{2p} \,.$$

Corollary 3 and Theorem 10 can be used to show that

- if $U^{(1)}(\alpha_2) > 1$ then $\alpha_2 > \alpha_1$,
- if $U^{(1)}(\alpha_2) = 1$ then $\alpha_2 = \alpha_1$, and
- if $U^{(1)}(\alpha_2) < 1$ then $\alpha_2 = \alpha_1$.

We shall use these facts to investigate the three cases considered in van Doorn and Schrijner.

- Case 1: $r_1 = 0$
  - If $p_1 < 2p$ then $U^{(1)}(\alpha_2) < 1$ and so $\alpha_2 = \alpha_1$ and the process is interior determined. Theorem 16(2) implies that the process is α_2-transient.
  - If $p_1 = 2p$ then $U^{(1)}(\alpha_2) = 1$ and so $\alpha_2 = \alpha_1$ and the process is interior determined. Theorem 16(1) implies that the process is α_2-recurrent. According to Theorem 16(1)(b) the process could be α_2-null or α_2-positive. To distinguish between these cases, consider the Derman–Vere-Jones transformation [26] of the process ${}_1P$; that is, the process represented by the matrix T on page 20 of Vere-Jones [32]. This is easily seen to be the birth-and-death process with probabilities $p_{i,i+1} = p_{i,i-1} = 1/2$, which is null-recurrent. It follows from [32, page 21] that ${}_1P$ is α_2-null.
  - If $p_1 > 2p$ then $U^{(1)}(\alpha_2) > 1$ and so $\alpha_2 > \alpha_1$ and the process is boundary determined. Solution of equation (8.4) tells us that $\alpha_1 = \sqrt{\frac{p_1-p}{p_1^2 q}}$ and $U^{(1)}(\alpha_1) = 1$. Theorem 16(1)(a) implies that the process is α_1-positive.
- Case 2: $p_1 = 2p$
  - If $r_1 = 0$ we have exactly the second case above.
  - If $r_1 > 0$, then $U^{(1)}(\alpha_2) > 1$ and so $\alpha_2 > \alpha_1$ and the process is boundary determined. Solution of equation (8.4) tells us that $\alpha_1 = 1/\sqrt{4pq + r_1^2}$ and $U^{(1)}(\alpha_1) = 1$. Theorem 16(1)(a) implies that the process is α_1-positive.

- Case 3: $p_1 = p$
  - If $r_1 < \sqrt{pq}$ then $U^{(1)}(\alpha_2) < 1$ and so $\alpha_2 = \alpha_1$ and the process is interior determined. Theorem 16(2) implies that the process is α_2-transient.
  - If $r_1 = \sqrt{pq}$ then $U^{(1)}(\alpha_2) = 1$ and so $\alpha_2 = \alpha_1$ and the process is interior determined. Theorem 16(1)(b) implies that the process is α_2-recurrent. A similar argument to that in Case 1 with $p_1 = 2p$ can be used to show that the process is α_2-null.
  - If $r_1 > \sqrt{pq}$ then $U^{(1)}(\alpha_2) > 1$ and so $\alpha_2 > \alpha_1$ and the process is boundary determined. Solution of equation (8.4) tells us that $\alpha_1 = 1/(r_1 + pq/r_1)$ and $U^{(1)}(\alpha_1) = 1$. Theorem 16(1)(a) implies that the process is α_1-positive.

9 CONCLUSION

In this paper we have shown how to calculate the convergence radius α for an arbitrary level-dependent quasi-birth-and-death-process. For all $\beta \leq \alpha$ we have also given an expression for a β-invariant measure. This expression can take one of two forms depending on whether $\eta(\beta) < 1$ or $\eta(\beta) = 1$. Both the convergence radius and the β-invariant measure are computable. The results are summarised in our main theorem, Theorem 16.

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APPENDIX A

Theorem 17 *Consider a birth-and-death process with a unique absorbing state* 0 *and an irreducible communicating class* $\mathcal{C} = \{1, 2, \ldots\}$. *If the convergence radius* α *is greater than* 1*, then it is given by*

$$\frac{1}{\alpha} = \lim_{n\to\infty} \left[P(X(n) \in \mathcal{A})\right]^{\frac{1}{n}}, \tag{A.1}$$

for any set $\mathcal{A} \subset \mathcal{C}$.

Proof: Schrijner [26, Equation (2.16), page 19] shows that

$$\lim_{n\to\infty} [P(X(n) = j|X(0) = i)]^{\frac{1}{n}} = \frac{1}{\alpha},$$

for all $i, j \in \mathcal{C}$, since $(X(n))$ is transient. Clearly, for any $\mathcal{A} \subset \mathcal{C}$ and some $j \in \mathcal{A}$,

$$\lim_{n\to\infty} [P(X(n) \in \mathcal{A}|X(0) = i)]^{\frac{1}{n}} \geq \lim_{n\to\infty} [P(X(n) = j|X(0) = i)]^{\frac{1}{n}} = \frac{1}{\alpha}.$$

Therefore,

$$\lim_{n\to\infty} [P(X(n) \in \mathcal{A}|X(0) = i)]^{\frac{1}{n}} \geq \frac{1}{\alpha}. \tag{A.2}$$

Under the hypothesis of this theorem we can always find a finite α-invariant measure $\boldsymbol{m}$ [26, Corollary 3.1, page 42]. It can be easily shown by induction that, for all $n \geq 1$,

$$\alpha^n \sum_i m_i P(X(n) = j|X(0) = i) = m_j.$$

Therefore,

$$P(X(n) = j|X(0) = i) \leq \left(\frac{1}{\alpha}\right)^n \frac{m_j}{m_i}.$$

For any $\mathcal{A} \subset \mathcal{C}$, it follows that

$$P(X(n) \in \mathcal{A}|X(0) = i) \leq \left(\frac{1}{\alpha}\right)^n \frac{1}{m_i} \sum_{j\in\mathcal{A}} m_j \ ,$$

and so

$$[P(X(n) \in \mathcal{A}|X(0) = i)]^{\frac{1}{n}} \leq \left[\left(\frac{1}{\alpha}\right)^n \frac{1}{m_i} \sum_{j\in\mathcal{A}} m_j\right]^{\frac{1}{n}}.$$

Since we know that $\sum_{j\in\mathcal{A}} m_j < \infty$ for all $\mathcal{A} \subset \mathcal{C}$, we have that

$$\lim_{n\to\infty} [P(X(n) \in \mathcal{A}|X(0) = i)]^{\frac{1}{n}} \leq \frac{1}{\alpha},$$

and the proof is complete. ■

Corollary 18 *In the notation of Example 2 of Section 4, for any $m \geq 1$ and $k \geq 0$, if $\alpha_m > 1$ then it is given by*

$$\alpha_m = \lim_{n\to\infty} [P({}_mX(n) > m + k)]^{-\frac{1}{n}}. \tag{A.3}$$

APPENDIX B

Theorem 12

(i) $\lim_{\delta \uparrow \alpha_k} \eta_k(\delta) \leq 1$.

(ii) If $\alpha_k = \alpha_{k+1}$ *then* $\lim_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) < 1$.

(iii) If $\alpha_k = \alpha_{k+1}$ *then* $N^{(k+1)}(\alpha_{k+1})$ *is finite and so* $\alpha_{k+1} \in \mathcal{I}_{k+1}$.

(iv) If $\alpha_k = \alpha_{k+1}$ *then* $\eta_k(\alpha_k)$ *is well-defined and* $\eta_k(\alpha_k) \leq 1$.

Proof:

(i) We shall assume that $\lim_{\delta \uparrow \alpha_k} \eta_k(\delta) > 1$ (including ∞) and attempt to derive a contradiction. The matrix $U^{(k)}(1)$ is irreducible and strictly substochastic. Therefore $\eta_k(1) < 1$. By Lemma 9, $\eta_k(\cdot)$ is continuous and strictly increasing on $\mathcal{I}_k$. Therefore, there exists $\gamma \in (1, \alpha_k)$ such that $\eta_k(\gamma) = 1$. However, by Theorem 10(iii) this means that $\gamma = \alpha_k$ which is a contradiction. Therefore, $\lim_{\delta \uparrow \alpha_k} \eta_k(\delta) \leq 1$.

(ii) Part (i) of this theorem implies that $\lim_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) \leq 1$. Therefore, we need only show that if $\alpha_k = \alpha_{k+1}$ then $\lim_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) \neq 1$ to complete the proof. We shall prove this result by proving the contrapositive, that $\lim_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) = 1$ implies that $\alpha_k < \alpha_{k+1}$.

Since $\lim_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) = 1$, for each $0 < \epsilon < 1$ there exists a $\delta_\epsilon < \alpha_{k+1}$ such that, for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$, $\eta_{k+1}(\delta) > 1 - \epsilon$. Therefore, for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$, the Perron-Frobenius eigenvalue of $N^{(k+1)}(\delta) = \sum_{n=0}^{\infty} \left[U^{(k+1)}(\delta)\right]^n$ is greater than

$$\sum_{n=0}^{\infty} (1 - \epsilon)^n = \frac{1}{\epsilon}.$$

Theorem 1.5 of Seneta [27] says that the maximum row sum of $N^{(k+1)}(\delta)$ is greater than or equal to $1/\epsilon$ and so there must exist an entry (i, j) such that $N_{i,j}^{(k+1)}(\delta) > 1/\epsilon M_{k+1}$. (Recall that M_{k+1} is the number of phases in level $k + 1$).

Now we shall show that we can choose ϵ in such a way that there exists a phase l and an integer $N_0 \in \mathbb{Z}_+$ with

$$\left[\left(U^{(k)}(\delta)\right)^{N_0+1}\right]_{l,l} > 1$$

for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$. This can then be used to show that $\left[N^{(k)}(\delta)\right]_{l,l}$ diverges for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$ and hence that $\alpha_k < \alpha_{k+1}$. In order to construct our argument, we will need to bound δ_ϵ from below. Since α_{k+1} could be as low as 1, we shall choose ϵ so that $\delta_\epsilon > 1/2$.

Irreducibility implies that there exists a number $p_1 > 0$ and states (k, l) and $(k+1, r)$ such that $\left[A_0^{(k)}\right]_{l,r} = p_1$ and a number $p_2 > 0$ and states $(k+1, s)$ and (k, t) such that $\left[A_2^{(k+1)}\right]_{s,t} = p_2$. Total irreducibility implies that there exists $N_1 \in \mathbb{Z}_+$ and $p_3 > 0$ such that

$$\left[\left(U^{(k+1)}(1/2)\right)^{N_1}\right]_{r,i} = p_3$$

and $N_2 \in \mathbb{Z}_+$ and $p_4 > 0$ such that

$$\left[\left(U^{(k+1)}(1/2)\right)^{N_2}\right]_{j,s} = p_4.$$

Therefore, since $U^{(k+1)}(\delta)$ is increasing in δ, for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$,

$$\begin{aligned}
& U_{l,t}^{(k)}(\delta) \\
= \; & \delta \left[A_1^{(k)} + \delta A_0^{(k)} N^{(k+1)}(\delta) A_2^{(k+1)}\right]_{l,t}, \\
\geq \; & \delta^2 \left[A_0^{(k)} \left[\sum_{n=0}^{\infty} \left(U^{(k+1)}(\delta)\right)^n\right] A_2^{(k+1)}\right]_{l,t}, \\
\geq \; & \delta^2 \left[A_0^{(k)} \left(U^{(k+1)}(\delta)\right)^{N_1} \left[\sum_{n=0}^{\infty} \left(U^{(k+1)}(\delta)\right)^n\right] \left(U^{(k+1)}(\delta)\right)^{N_2} A_2^{(k+1)}\right]_{l,t}, \\
= \; & \delta^2 \left[A_0^{(k)} \left(U^{(k+1)}(\delta)\right)^{N_1} N^{(k+1)}(\delta) \left(U^{(k+1)}(\delta)\right)^{N_2} A_2^{(k+1)}\right]_{l,t}, \\
> \; & \left(\frac{1}{2}\right)^2 \left[A_0^{(k)}\right]_{l,r} \left[\left(U^{(k+1)}(\frac{1}{2})\right)^{N_1}\right]_{r,i} \left[N^{(k+1)}(\delta)\right]_{i,j} \left[\left(U^{(k+1)}(\frac{1}{2})\right)^{N_2}\right]_{j,s} \left[A_2^{(k+1)}\right]_{s,t}, \\
> \; & \left(\frac{1}{2}\right)^2 p_1 p_3 \frac{1}{M_{k+1}\epsilon} p_4 p_2.
\end{aligned}$$

Total irreducibility further implies that there exists an integer $N_0 > 0$ and $p_5 > 0$ such that $\left[\left(U^{(k)}(1/2)\right)^{N_0}\right]_{t,l} = p_5$ and so $\left[\left(U^{(k)}(\delta)\right)^{N_0}\right]_{t,l} > p_5$. Therefore,

$$\left[\left(U^{(k)}(\delta)\right)^{N_0+1}\right]_{l,l} > \frac{p_1 p_2 p_3 p_4 p_5}{4 M_{k+1}\epsilon}$$

for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$.

Now choose ϵ such that $\epsilon < \dfrac{p_1 p_2 p_3 p_4 p_5}{4M_{k+1}}$ and so for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$,

$$\left[\left(U^{(k)}(\delta)\right)^{N_0+1}\right]_{l,l} > 1.$$

Therefore, for all $\delta \in (\delta_\epsilon, \alpha_{k+1})$,

$$\left[N^{(k)}(\delta)\right]_{l,l} \geq \sum_{n=0}^{\infty} \left[\left(U^{(k)}(\delta)\right)^{(N_0+1)n}\right]_{l,l} = \infty,$$

so $\alpha_k \leq \delta_\epsilon$ and hence $\alpha_k < \alpha_{k+1}$.

(iii) Since $\lim\limits_{\delta \uparrow \alpha_{k+1}} \eta_{k+1}(\delta) = t < 1$, the increasing nature of $\eta_{k+1}(\cdot)$ (see Lemma 9) implies, for each $\delta < \alpha_{k+1}$, that $\eta_{k+1}(\delta) < t$. Therefore, for all $\delta < \alpha_{k+1}$ the Perron-Frobenius eigenvalue of $N^{(k+1)}(\delta) = \sum_{n=0}^{\infty} \left[U^{(k+1)}(\delta)\right]^n$ is less than

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t} = s, \text{ say.}$$

Theorem 1.5 of Seneta [27] says that the minimum row sum of $N^{(k+1)}(\delta)$ is less than or equal to s and so there must exist an entry (i,j) such that $N^{(k+1)}_{i,j}(\delta) \leq s/M_{k+1}$. Since, for all $\delta < \alpha_{k+1}$, the entry $N^{(k+1)}_{i,j}(\delta)$ is a power series, with nonnegative coefficients and radius of convergence α_{k+1}, which is bounded above by the finite constant $\dfrac{s}{M_{k+1}}$, the theorem on page 178 of Knopp [16] implies that $N^{(k+1)}_{i,j}(\alpha_{k+1})$ is finite. Theorem 6.1 of Seneta then implies that all entries of the matrix $N^{(k+1)}(\alpha_k)$ are finite and so $\alpha_{k+1} \in \mathcal{I}_{k+1}$.

(iv) By part (iii) of this theorem, $\alpha_{k+1} \in \mathcal{I}_{k+1}$ and so Lemma 1 shows that $R^{(k+1)}(\alpha_{k+1})$ is finite. Therefore, equation (3.4) implies that $U^{(k)}(\alpha_{k+1}) = U^{(k)}(\alpha_k)$ is finite and so its irreducibility is sufficient for $\eta_k(\alpha_k)$ to be well-defined. Part (i) of this theorem and the continuity of $\eta_k(\cdot)$ gives $\eta_k(\alpha_k) \leq 1$. ■

Lemma 15 *If $\eta_1(\alpha) = 1$ then the process ${}_1P$ is α-recurrent. Further, the process ${}_1P$ is α-positive if $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ is finite and α-null otherwise.*

Proof: Since $\eta_1(\alpha) = 1$, it is clear that the series $N^{(1)}(\alpha) = \sum_{n=1}^{\infty} \left[U^{(1)}(\alpha)\right]^n$ does not converge and so $N^{(1)}(\alpha)$ is not finite. Therefore, the process is α-recurrent.

The key relationship in the remainder of this proof is the equation which gives $U^{(1)}(\alpha)$ in terms of $N^{(2)}(\alpha)$. From equations (3.1) and (3.4), this is

$$U^{(1)}(\alpha) = \alpha A_1^{(1)} + \alpha^2 A_0^{(1)} N^{(2)}(\alpha) A_2^{(2)}. \tag{B.1}$$

If $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ is finite then

$$\frac{dU^{(1)}(\alpha)}{d\alpha} = A_1^{(1)} + 2\alpha A_0^{(1)} N^{(2)}(\alpha) A_2^{(2)} + \alpha^2 A_0^{(1)} \frac{dN^{(2)}(\alpha)}{d\alpha} A_2^{(2)}, \tag{B.2}$$

and so $\dfrac{dU^{(1)}(\alpha)}{d\alpha}$ is also finite. Otherwise, $\dfrac{dU^{(1)}(\alpha)}{d\alpha}$ is infinite.

Let $f_{ii}^{(n)}$ be the probability that, given the process starts in state i, it is in state i at time n and has not visited state i in between. Seneta [27, Definition 6.2, page 202] states that an α-recurrent process is α-positive if $\mu_i(\alpha) \equiv \sum_n n f_{ii}^{(n)} \alpha^n < \infty$, and α-null if not.

If level 1 consists of only one phase, then $f_{11}^{(n)} = \left[U^{(1)}\right]^{(n)}$, as defined after equation (3.5), and $\mu_1(\alpha) = \alpha \dfrac{dU^{(1)}(\alpha)}{d\alpha}$. Thus in this case the result is proved.

The difficulty when there is more than one phase lies in the fact that $\left[U_{i,i}^{(1)}\right]^{(n)}$ records the probability that the process visits $(1, i)$ *at its next visit to level* 1 with level 0 taboo, given that it starts in state $(1, i)$, and that this visit occurs at time point n. In calculating $f_{(1,i)(1,i)}^{(n)}$ in terms of $\left[U_{i,i}^{(1)}\right]^{(n)}$ we need to take account of sample paths which first visit level 1 in a phase other than i.

Let $F_{ii}(\alpha) = \sum_{n=0}^{\infty} f_{(1,i)(1,i)}^{(n)} \alpha^n$ and so

$$\mu_i(\alpha) = \alpha \frac{dF_{ii}(\alpha)}{d\alpha}. \tag{B.3}$$

Now $F_{ii}(\alpha)$ can be written as

$$\begin{aligned} F_{ii}(\alpha) &= U_{i,i}^{(1)}(\alpha) + \sum_{j \neq i} U_{i,j}^{(1)}(\alpha) U_{j,i}^{(1)}(\alpha) + \sum_{j_1 \neq i} \sum_{j_2 \neq i} U_{i,j_1}^{(1)}(\alpha) U_{j_1,j_2}^{(1)}(\alpha) U_{j_2,i}^{(1)}(\alpha) + \cdots \\ &= \sum_{t=1}^{\infty} \sum_{\substack{j_0 = i, \\ j_t = i, \\ j_s \neq i,\ s=1,2,\ldots,t-1}} \prod_{s=1}^{t} U_{j_{s-1},j_s}^{(1)}(\alpha), \end{aligned}$$

and so

$$\frac{dF_{ii}(\alpha)}{d\alpha} = \sum_{t=1}^{\infty} \sum_{\substack{j_0=i,\\ j_t=i,\\ j_s\neq i,\ s=1,2,\ldots,t-1}} \sum_{r=1}^{t} \left[\prod_{s=1}^{r-1} U^{(1)}_{j_{s-1}j_s}(\alpha)\right] \frac{dU^{(1)}_{j_{r-1}j_r}(\alpha)}{d\alpha} \prod_{s=r+1}^{t} U^{(1)}_{j_{s-1}j_s}(\alpha)\ . \tag{B.4}$$

If $\dfrac{dN^{(2)}(\alpha)}{d\alpha}$ and thus $\dfrac{dU^{(1)}(\alpha)}{d\alpha}$ is not finite, then total irreducibility implies, via equations (B.2), (B.3) and (B.4), that $\mu_i(\alpha)$ is not finite and so the process is α-null.

Now assume $\dfrac{dN^{(2)}(\alpha)}{d\alpha} < \infty$ and consider the stochastic matrix $\hat{U}$ whose (i,j)th entry is given by

$$\hat{U}_{ij} = \frac{x_j U^{(1)}_{j,i}(\alpha)}{x_i}, \tag{B.5}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_{M_1})$ is the left-eigenvector of $U^{(1)}(\alpha)$ with eigenvalue 1. This is a transition matrix on the finite state space $\{1, 2, \ldots, M_1\}$ and so the expected first return time for state i

$$E[\hat{F}_i] \equiv \sum_{t=1}^{\infty} t \sum_{\substack{j_0=i,\\ j_t=i,\\ j_s\neq i,\ s=1,2,\ldots,t-1}} \prod_{s=1}^{t} \hat{U}_{j_{s-1},j_s} < \infty. \tag{B.6}$$

Substitution of equation (B.5) into this expression gives

$$E[\hat{F}_i] = \sum_{t=1}^{\infty} t \sum_{\substack{j_0=i,\\ j_t=i,\\ j_s\neq i,\ s=1,2,\ldots,t-1}} \prod_{s=1}^{t} U^{(1)}_{j_s,j_{s-1}}(\alpha) < \infty. \tag{B.7}$$

Observe now that, because $\dfrac{dN^{(2)}(\alpha)}{d\alpha} < \infty$, there exists a number $K < \infty$ such that

$$\frac{dU^{(1)}_{ij}(\alpha)}{d\alpha} \leq K U^{(1)}_{ij}(\alpha), \tag{B.8}$$

uniformly in i and j. This follows because M_1 is finite and $U^{(1)}_{ij}(\alpha) = 0$ implies that $\dfrac{dU^{(1)}_{ij}(\alpha)}{d\alpha} = 0$.

Substitution of this inequality into equation (B.3), using equation (B.4), gives

$$\begin{aligned}
\mu_1(\alpha) &= \alpha \frac{dF_{ii}(\alpha)}{d\alpha}, \\
&= \alpha \sum_{t=1}^{\infty} \sum_{\substack{j_0=i, \\ j_t=i, \\ j_s \neq i,\ s=1,2,\ldots,t-1}} \sum_{r=1}^{t} \prod_{s=1}^{r-1} U^{(1)}_{j_{s-1}j_s}(\alpha) \frac{dU^{(1)}_{j_{r-1}j_r}(\alpha)}{d\alpha} \prod_{s=r+1}^{t} U^{(1)}_{j_{s-1}j_s}(\alpha), \\
&\leq \alpha K \sum_{t=1}^{\infty} t \sum_{\substack{j_0=i, \\ j_t=i, \\ j_s \neq i,\ s=1,2,\ldots,t-1}} \prod_{s=1^t} U^{(1)}_{j_{s-1}j_s}(\alpha) \\
&= \alpha K E[\hat{F}_i],
\end{aligned}$$

which is finite by equation (B.7). Therefore, if $\frac{dN^{(2)}(\alpha)}{d\alpha}$ is finite, the process ${}_1P$ is α-positive. ■

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